ON CONJECTURES OF FOULKES, SIEMONS AND WAGNER AND STANLEY

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Abstract

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n. An *unordered* λ*-tabloid* is a partition of the set $\{1, 2, \ldots, n\}$ into r pairwise disjoint sets of sizes $\lambda_1, \ldots, \lambda_r$. Let F denote the field of complex numbers and G the symmetric group of $\{1, 2, \ldots, n\}$. Define H^λ to be the permutation module of FG whose basis is the set of unordered λ-tabloids. Foulkes conjectured in [13] that there exists an injective FG-homomorphism $H^{(b^a)} \to H^{(a^b)}$ when $a \leq b$. Independently Siemons and Wagner [27] and Stanley [29] generalized this conjecture to ask if there exists an injective map $H^\lambda \to H^{\lambda'}$. In this thesis we investigate these conjectures.

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Contents

Chapter 1

Introduction and Overview

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n. An *unordered* λ*-tabloid* is a partition of the set $\{1, \ldots, n\}$ into r pairwise disjoint sets of sizes $\lambda_1, \ldots, \lambda_r$. Let F denote the field of complex numbers and let G denote the symmetric group of the set $\{1, 2, \ldots, n\}$. Define H^λ to be the permutation module of FG whose basis is the set of unordered λ-tabloids. A long standing open problem is Foulkes' Conjecture made in [13] that there exists an injective FG-homomorphism $H^{(b^a)} \hookrightarrow H^{(a^b)}$ if $a \leq b$.

By explicitly computing the composition factors (see Example 9 on Page 140 of Macdonald's Book [20]) one can easily see that this conjecture holds when $a = 2$ and b is arbitrary. Dent and Siemons [11] showed that the conjecture holds when $a = 3$ and $b \geq a$ by producing a linearly independent subset of $Hom_{FG}(S^\lambda, H^{(a^b)})$ of size equal to the dimension of $Hom_{FG}(S^\lambda, H^{(b^a)})$ for each irreducible FG-module S^λ that appears in $H^{(b^a)}$. Briand [3] claimed to prove Foulkes' Conjecture is true when $a \leq 4$ and $b \geq a$ using diagonally symmetric functions although it is our understanding ([4] and [5]) that this proof contains a flaw and should be refined to $a \leq 3$. In [6] and [7] Brion uses arguments from algebraic geometry to show that Foulkes' Conjecture is true when b is large compared to a.

In [2] Black and List defined a map $\psi_{b^a} : H^{(b^a)} \rightarrow H^{(a^b)}$ and conjectured that it was injective. Coker [9], Dent [10], Doran [12] and Pylyavskyy [23] then showed that the map ψ_{b^2} is injective. If $\psi_{b^2}^T$ denotes the adjoint of ψ_{b^2}, Coker and Dent independently showed that all the eigenvalues of $\psi_{b^2} \circ \psi_{b^2}^T$ are non-zero. In [12] Doran observed that for each irreducible FG-module S^λ, the FG-homomorphism ψ_{b^a} induces an F-linear map $\hat{\psi}_{b^a} : Hom_{FG}(S^\lambda, H^{b^a}) \to Hom_{FG}(S^\lambda, H^{b^a})$ by $\theta_T \mapsto \theta_T \circ \psi_{b^a}$. Using this he went on to show that ψ_{b^2} is injective. Pylyavskyy directly showed that the map ψ_{b^2} is injective using an inductive argument. Of particular interest to us are the computational results of Jacob's Thesis [15] and the paper by Müller and Neunhöffer [22]. In [15], Jacob proved that the Black and List map is injective when $a = b = 2$, $a = b = 3$ and $a = b = 4$. Müller and Neunhöffer [22] then showed that it is injective when $a = b = 5$.

Independently, Siemons and Wagner [27] and Stanley [29] (SWS) extended the Black and List map in a natural way to an FG-homomorphism $\psi_\lambda : H^\lambda \to H^{\lambda'}$ and conjectured this to be injective iff λ dominates its conjugate. We call this the *standard map*. When $\lambda = (r, 1^s)$ is a hook then the standard map is the map between the layers M_r and M_{r+1} in the Boolean algebra of a set of size $r + s$. This is well known (see Proposition 5.4.7 of Sagan's book [24]) to be injective iff $r > s$ and so the SWS conjecture holds in this case. In [23] Pylyavskyy shows that the standard map of $(6, 2, 2, 1, 1)$ is not injective and refines the conjecture to say the standard map has maximal rank. In [28] Sivek shows that this new conjecture is also false and fails for large classes of partitions, the smallest being $(4, 3, 3)$.

Finally, in [30] Vessenes generalizes Foulkes' Conjecture to conjecture that there exists an injective map $H^{(b^a)} \to H^{(d^c)}$ when $a \leq c$ and $b \geq d$. The main result of [30] is then to prove that this conjecture holds when $a = 2$. In the setting of algebraic geometry Abdesselam and Chipalkatti [1] then proved that a given map $\psi : H^{(b^2)} \to H^{(d^c)}$ is injective when $d, c \geq 2$.

We now turn to our own efforts. This thesis has five chapters after the present introductory one. The important results of Chapter 2 are Proposition 2.3.1 and Theorem 2.3.4 on pages 11 and 12. These results allow us to produce homomorphisms between permutation modules with wanton abandon and view these as compositions of other homomorphisms. In Chapter 3 we look at spaces of homomorphisms between tabloid spaces. We define the important ϵ-map and prove that it is injective. We also prove the following result:

THEOREM 3.4.2 (page 33): Let $V \cong S^\nu$ be an irreducible submodule in the kernel of the standard map of (b^a). Then ν has at least three parts.

Chapter 4 is devoted to understanding when the standard map is injective. This leads to the following definition of a *good column* of a Young diagram on page 44. We say that the node λ_{ij} is *good* if the hook h_{ij} has an arm at least as long as its leg. We then say that a column is *good* if every node in it is good. We can now state the main theorem of the thesis

THEOREM 4.1.15 (page 45): *Let λ be a partition and μ the partition obtained by removing a good column from λ. Suppose the standard map of μ is injective. Then the standard map of λ is injective.*

When combined with the results of Jacob, Theorem 4.1.15 shows that Foulkes' Conjecture holds when $a \leq 4$. That is

THEOREM 4.1.17 (page 45): *Let $a \leq 4$ and $a \leq b$. Then the standard map $\psi_{b^a} : H^{(b^a)} \to H^{(a^b)}$ is injective.*

In Chapter 5 we look at when the standard map is not injective. Inspired by the definition of a good column, on page 62 we define $A(i,j)$ to be the hook whose arm includes the ith highest removable node and whose leg includes the jth highest removable node. We say that a partition is *good* if all the $A(i,j)$ have arm at least as long as their leg. Most of this chapter is then devoted to proving:

THEOREM 5.2.4 (page 63): *Suppose that the standard map of λ is injective. Then λ is good.*

In Chapter 6 we collect together all the results in the thesis to prove Theorem 6.1.3, which shows that the standard map controls the existence of injective maps $H^\lambda \to H^{\lambda'}$ when λ has at most three parts. However when λ has four or more parts the situation is more complicated as Theorem 6.3.14 shows

THEOREM 6.1.3 (page 76): *Let λ be a partition with at most three parts. Then the following are equivalent:*

(i) There exists an injective map $H^\lambda \to H^{\lambda'}$.

(ii) The partition λ is good.

(iii) The standard map of λ is injective.

THEOREM 6.3.14 (page 86): *Let $\mu = (a, b+1, b^2)$ with $a \geq 2b+2$. Then the standard map of μ is not injective but there exists an injective map $H^\mu \to H^{\mu'}$.*

Theorem 5.2.4 tells us that a partition with an injective standard map is neccesarily good, while Theorem 6.1.3 says that being good is sufficient for three part partitions to have an injective standard map. It is natural ask whether the property of being good is always sufficent. Sivek's Lemma 5.1.7 states that adding rows to a partition with non-injective standard map yields a partition with non-injective standard map. Hence when combined with the result of Müller and Neunhöffer that $\psi_{(5^5)}$ is non-injective we see that one can produce many partitions that are good but have non-injective standard map. However something can be saved using our techniques. Let $B(i,j)$ denote the unique hook whose arm lies in the highest row of length λ_i and whose leg contains the removable node in a row of length λ_j. In Subsection 4.2.2 we strengthen the definition of good by saying that $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$ is *extremely good* if the arm of each $B(i,j)$ is at least as long as its leg and for each i the standard map of $(\lambda_i - \lambda_{i+1})^{m_i}$ is injective and prove:

THEOREM 4.2.21 (page 55): *Let λ be extremely good. Then the standard map of λ is injective.*

An interesting special case of Theorem 4.2.21 is the following:

COROLLARY 4.2.22 (page 55): *Let $\lambda \vdash n$. Suppose all the parts of λ are distinct. Then the standard map of λ is injective.*

Thus the answer to the SWS conjecture seems to lie somewhere between good and extremely good.

Chapter 2

General results

2.1 Notation and definitions

A *composition* of a positive integer n is a finite sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\sum_{i=1}^{r} \lambda_i = n$. A *partition* of n is a composition λ such that $\lambda_i \geq \lambda_{i+1}$ for each i. We write $\lambda \vdash n$ to identify λ as a partition of n. If a partition has a parts of equal length b then we write (b^a) instead of $(b, b, \ldots, b)$. For example we write $(5^3, 2, 1)$ in place of $(5, 5, 5, 2, 1)$.

The *Young diagram* $[\lambda]$ of a partition λ is the left aligned array of nodes such that the ith row has λ_i nodes. For example the Young diagram of $(5^3, 2, 1)$ is

$$[\lambda] = \begin{array}{lllll} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & & & \\ \circ & & & & \end{array} \qquad .$$

The node in the ith row and jth column of $[\lambda]$ is denoted by $\lambda_{i,j}$. The hook $h_{i,j}$ consists of the node $\lambda_{i,j}$ together with the $\lambda_i - j$ nodes to the right of it (the *arm*) and the $\lambda_j - i$ nodes below it (*the leg*). The *arm length* $a_{i,j}$ of $h_{i,j}$ is $\lambda_i - j$ and the *leg length* $l_{i,j}$ is $\lambda_j - i$.

EXAMPLE 2.1.1 *Let* $\lambda = (5^3, 2, 1)$. *Then* $a_{2,1} = 3 = l_{2,1}$ *and* $h_{2,1}$ *is illustrated below.*

$$\begin{array}{ccccc} \circ & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ & \circ \\ \circ & \bullet & \circ & \circ & \circ \\ \circ & \bullet & & & \\ \circ & & & & \end{array} \quad .$$

If $\lambda \vdash n$ define the *conjugate partition* $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ by letting λ'_i be the number of parts of λ of length at least i. Informally, the ith row of the Young diagram of λ' is the ith column of the Young diagram of λ. For example, $(5^3, 2, 1)' = (5, 4, 3^3)$.

A *tableau* of shape λ (or λ-tableau) is a filling without repeats of the Young diagram of λ with the numbers $1, 2, \ldots, n$. The *primary λ-tableau* is the λ-tableau whose first row is $1, 2, 3, \ldots, \lambda_1$, second row $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ and so on. The *conjugate tableau* t' of the λ-tableau t is the λ'-tableau whose ith row read left to right is the ith column of t read top to bottom.

EXAMPLE 2.1.2 *Let* $\lambda = (5^3, 2, 1)$. *Below* s *and* t *are two* λ*-tableaux, with* t *the primary* $(5^3, 2, 1)$*-tableau. Also* t' *is the* λ'*-tableau that is the conjugate of* t.

$$t = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & & & & \end{array} \quad , \qquad s = \begin{array}{ccccc} 2 & 10 & 3 & 4 & 12 \\ 6 & 7 & 8 & 9 & 1 \\ 11 & 5 & 14 & 13 & 15 \\ 16 & 17 & & & \\ 18 & & & & \end{array} \quad ,$$

$$t' = \begin{array}{ccccc} 1 & 6 & 11 & 16 & 18 \\ 2 & 7 & 12 & 17 & \\ 3 & 8 & 13 & & \\ 4 & 9 & 14 & & \\ 5 & 10 & 15 & & \end{array} \quad .$$

The set of all λ-tableaux is denoted $\mathcal{F}^\lambda$. Define an equivalence relation $\sim$ on $\mathcal{F}^\lambda$ by $s \sim t$ iff for each i the ith rows of s, t are equal as sets. The $\sim$-equivalence class that contains t is the λ-*tabloid* $\{t\}$. The set of all λ-tabloids is denoted by $\mathcal{M}^\lambda$. To distinguish between tabloids and tableaux we draw lines between the rows of a tabloid. We regard the rows of tabloids as sets, so we draw them subject to the convention that the elements are written in increasing order from left to right.

Example 2.1.3 *The tableaux s, t in Example 2.1.2 correspond to the following tabloids*

$$\{t\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} \quad , \qquad \{s\} = \begin{array}{ccccc} \hline 2 & 3 & 4 & 10 & 12 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 5 & 11 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} \quad .$$

Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_u)$ be partitions. Then we write $\mu \cup \nu$ to denote the composition $(\mu_1, \ldots, \mu_s, \nu_1, \ldots, \nu_u)$. The $\mu \cup \nu$-tabloid $\{t\}$ is an ordered $(s+u)$-tuple of sets $(X_1, \ldots, X_{s+u})$. We call these sets *classes.* We say that the first s-many classes are *rows* and the last u-many classes are *columns.* As befits their names when we draw $\mu \cup \nu$-tabloids we draw rows horizontally and columns vertically. For motivation as to why we would want to adopt such a convention we refer the reader to Chapter 4.

Example 2.1.4 *Let $\mu = (3^3)$ and $\nu = (5, 4)$. Then $\mu \cup \nu = (3^3, 5, 4)$ and below is a $\mu \cup \nu$-tabloid.*

$$\begin{array}{|c|c|ccc} \cline{3-5} 1 & 2 & 3 & 4 & 5 \\ \cline{3-5} 6 & 7 & 8 & 9 & 10 \\ \cline{3-5} 11 & 12 & 13 & 14 & 15 \\ \cline{3-5} 16 & 17 & & & \\ 18 & \multicolumn{1}{c|}{} & & & \end{array} \quad .$$

Throughout this thesis we are interested in the symmetric group. Thus unless otherwise stated G will denote the symmetric group S_n. Our G acts regularly on $\mathcal{F}^\lambda$: For $t \in \mathcal{F}^\lambda$ and $g \in G$ let tg be the tableau obtained from t by replacing each node i with ig. For example if $g = (1, 2, 10)(5, 12)(13, 14)$ then $s = tg$ in Example 2.1.2. Now define an action of G on $\mathcal{M}^\lambda$ by letting $\{t\}g := \{tg\}$. The point stabilizer $G_{\{t\}}$ is isomorphic to the direct product of symmetric groups $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_r}$. A *Young subgroup* S_λ is a subgroup of G isomorphic to $G_{\{t\}}$.

Throughout this thesis F will denote a field of characteristic zero. If X is a set we let FX denote the vector space over F with basis X. We let M^λ denote $F\mathcal{M}^\lambda$.

Let t be a λ-tableau and t' its conjugate λ'-tableau. Define

$$k_t := \sum_{g \in G_{\{t'\}}} sgn(g)g \in FG.$$

Then define the λ-*polytabloid* $e_t := \{t\}k_t$. The *Specht module* S^λ is the subspace of M^λ which is spanned by the set of all λ-polytabloids. The importance of these modules is seen in the following result which is Theorem 4.12 on page 16 of James' book [16].

THEOREM 2.1.5 *The set* $\{S^\lambda \mid \lambda \vdash n\}$ *is a complete set of non-isomorphic irreducible* FG*-modules.*

□

2.2 The space H^λ

The results of this thesis concern a submodule $H^\lambda \subseteq M^\lambda$ which we now introduce. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ let $S_{\lambda^*} \subseteq S_r$ be the set of all $\pi \in S_r$ such that $\lambda_{i\pi} = \lambda_i$ for all $i = 1, \ldots, r$. We say that S_{λ^*} is the *twist group* of λ. Our notation suggests that S_{λ^*} is a Young subgroup. Indeed, if λ has m_i parts of length λ_i for each i then $S_{\lambda^*} \cong \times_i S_{m_i}$. For example if $\lambda = (5^3, 2, 1)$ then $\lambda^* = (3, 1, 1)$ and so

$$S_{\lambda^*} \cong S_3 \times S_1 \times S_1 \cong S_3.$$

The twist group S_{λ^*} acts on $\mathcal{M}^\lambda$ by permuting the rows of equal length in all possible ways. That is, the ith row of $\{t\}\pi$ is the $i\pi^{-1}$th row of $\{t\}$. For example, let $\{s\}$ and $\{t\}$ be the $(5^3, 2, 1)$-tabloids in Example 2.1.3. If $\sigma = (12), \pi = (123) \in S_3 \cong S_{\lambda^*}$ we have

$$
\{t\}\sigma = \begin{array}{ccccc} \hline 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 2 & 3 & 4 & 5 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} \quad , \qquad \{s\}\pi = \begin{array}{ccccc} \hline 5 & 11 & 13 & 14 & 15 \\ \hline 2 & 3 & 4 & 10 & 12 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} \quad .
$$

LEMMA 2.2.1 *The actions of G and S_{λ^*} on M^λ commute.*

Proof: Let $g \in G$ and $\pi \in S_{\lambda^*}$. Let x be in the ith row of $\{t\}$ and let $xg^{-1} = y$ with y in the jth row of $\{t\}$. Then x is in the jth row of $\{t\}g$ and so the $j\pi^{-1}$th row of $\{t\}g\pi$. Similarly x is in the $i\pi^{-1}$th row of $\{t\}\pi$ and y in the $j\pi^{-1}$th row of $\{t\}\pi$. Then x is in the $j\pi^{-1}$ row of $\{t\}\pi g$. □

Define a linear map θ on M^λ by $\{t\}\theta = \sum_{\pi \in S_{\lambda^*}} \{t\}\pi$. We define H^λ to be the image of this map. By Lemma 2.2.1 we know that θ is an FG-homomorphism and so H^λ is an FG-submodule of M^λ.

We now describe a group $A = A_{\{t\}} \subseteq G_{\{t\}'}$ whose action on $\{t\}$ agrees with that of the twist group. Let $t_{i,j}$ denote the jth smallest element in the ith row of $\{t\}$ and let $\pi \in S_{\lambda^*}$. Define $a_\pi \in G$ by $(t_{i,j})a_\pi = t_{(i\pi^{-1},j)}$ for all j. We define $A := \{a_\pi \mid \pi \in S_{\lambda^*}\}$ and it is easy to see that we have $A \cong S_{\lambda^*}$ and $\{t\}a_\pi = \{t\}\pi$. For example if $\{t\}$ is the by now familiar tabloid in Example 2.1.3 we have

$$
\begin{aligned}
a_{(1,2)} &= (1,6)(2,7)(3,8)(4,9)(5,10) \quad \textit{and} \\
a_{(123)} &= (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10).
\end{aligned}
$$

Suppose that $g = (t_{i,j_1}, t_{i,j_2}, \ldots, t_{i,j_r})$ is a permutation of the ith row of $\{t\}$. Then

$$
\begin{aligned}
(a_\pi)^{-1} g a_\pi &= (t_{i,j_1}a_\pi, t_{i,j_2}a_\pi, \ldots, t_{i,j_r}a_\pi) \\
&= (t_{i\pi^{-1},j_1}, t_{i\pi^{-1},j_2}, \ldots, t_{i\pi^{-1},j_r}).
\end{aligned}
$$

Thus $a_\pi^{-1} g a_\pi$ is a permutation of the (π^{-1})th row of $\{t\}$ and so A normalizes $G_{\{t\}}$. Conversely let $g \in N_G(G_{\{t\}})$. As $g^{-1}G_{\{t\}}g = G_{\{t\}g}$ we see that $g \in N_G(G_{\{t\}})$ iff $G_{\{t\}g} = G_{\{t\}}$. Then $G_{\{t\}} = G_{\{s\}}$ iff $\{s\} = \{t\}$ for $\pi \in S_{\lambda^*}$. That is $g \in N_G(G_{\{t\}})$ iff $\{t\}g = \{t\}a_\pi$. Thus $N_G(G_{\{t\}}) = G_{\{t\}} \rtimes A$ and when G acts on M^λ we the argument above shows that $N_G(G_{\{t\}})$ is the stabilizer of $\sum_{\pi \in S_{\lambda^*}} \{t\}\pi$. Hence we have proved the following;

LEMMA 2.2.2 *(i)* $N_G(G_{\{t\}}) = G_{\{t\}} \rtimes A$.

(ii) $H^\lambda = 1_{N_\lambda}^G$.

□

REMARK 2.2.3 *An immediate corollary of Lemma 2.2.2 (ii) is that H^λ is a transitive permutation module of FG.*

2.3 Constructions of homomorphisms

In this section we give a canonical construction of FG-homomorphisms whose domain is a transitive permutation module. Together with the important Theorem 2.3.4 this allow us to easily define FG-homomorphisms $M^\lambda \to M^\mu$ and view these homomorphisms as compositions of simpler homomorphisms of the same type.

We begin with a very general construction. Let G be any finite group and let V and W be FG-modules. Let $Hom_F(V, W)$ denote the space of F-homomorphisms from V to W. The algebra FG acts on $Hom_F(V, W)$ by $v\phi^g = vg^{-1}\phi g$. The space fixed by this action is $Hom_{FG}(V, W)$. As F has characteristic zero we see that $Hom_{FG}(V, W)$ is the image of the map $\phi \mapsto \sum_{g \in G} \phi^g$. Suppose $V = FA$ is a transitive permutation module and let $a \in A$ and $w \in W$. Define the F-linear map $\phi : FA \to W$ by $a \mapsto w$ and $a_1 \mapsto 0$ for $a_1 \neq a$. Then $\hat{\phi} := \sum_{g \in G} \phi^g$ is an

FG-homomorphism. As $a_1\phi = 0$ for $a_1 \neq a$ we have

$$
\begin{aligned}
a\hat{\phi} &= \sum_{g\in G} ag^{-1}\phi g \\
&= \sum_{g\in G_a} a\phi g \\
&= \sum_{g\in G_a} wg.
\end{aligned}
$$

Now suppose that $w \in W$ such that $wg = w$ for all $g \in G_a$. Then $w = \sum_{g\in G} w_1 g$ for some $w_1 \in W$. Hence if $\psi : FA \to W$ is given by $a\psi = \frac{1}{|G_a|}w$ and $a_1\psi = 0$ for $a_1 \neq a$ we see that $a\hat{\psi} = w$. Finally suppose that $a\phi_1 = a\phi_2$. Then $a\hat{\phi}_1 = a\hat{\phi}_2$ and as FA is cyclic $\hat{\phi}_1$ and $\hat{\phi}_2$ agree on all points of FA. Hence we have proved the following:

PROPOSITION 2.3.1 *Let FA be a transitive FG-permutation module generated by a and W any other FG-module with $w \in W$. Then there exists an FG-homomorphism $\phi : FA \to W$ with $a\phi = w$ iff $wg = w$ for all $g \in G_a$. Furthermore, this FG-homomorphism is unique.*

□

We are of course interested in the case $G = S_n$. Throughout we shall use Proposition 2.3.1 to define homomorphisms $\phi : M^\lambda \to M^\mu$ in the following way. Fix a λ-tabloid $\{t_\lambda\}$ and μ-tabloid $\{t_\mu\}$. By Proposition 2.3.1 it suffices to prescribe ϕ on just $\{t_\lambda\}$. We force the criterion of Proposition 2.3.1 by letting $\{t_\lambda\}\phi = \sum_{g\in G_{\{t_\lambda\}}} \{t_\mu\}g$. The next lemma is elementary but useful:

LEMMA 2.3.2 *Let $\phi : M^\lambda \to M^\mu$ be the map defined by $\{t_\lambda\}\phi = \sum_{h\in G_{\{t_\lambda\}}} \{t_\mu\}h$. Then $\{t_\lambda\}g\phi = \sum_{k\in G_{\{t_\lambda\}g}} \{t_\mu\}gk$ for all $g \in G$.*

Proof: Since the stabilizer of $\{t_\lambda\}g$ is equal to $g^{-1}G_{\{t_\lambda\}}g$ we have

$$\sum_{k \in G_{\{t_\lambda\}g}} \{t_\mu\}gk = \sum_{h \in G_{\{t_\lambda\}}} \{t_\mu\}g(g^{-1}hg)$$
$$= \sum_{h \in G_{\{t_\lambda\}}} \{t_\mu\}hg.$$

□

Crucial to this thesis will be our ability to control the composition of two maps defined using Proposition 2.3.1. For the remainder of this section we fix some notation. Let $\lambda, \mu, \nu \vdash n$ and fix tabloids $\{t_\lambda\} \in \mathcal{M}^\lambda, \{t_\mu\} \in \mathcal{M}^\mu$ and $\{t_\nu\} \in \mathcal{M}^\nu$. Using Proposition 2.3.1 define two maps $\phi : M^\lambda \to M^\mu$ and $\theta : M^\mu \to M^\nu$ by $\{t_\lambda\}\phi = \sum_{g \in G_{\{t_\lambda\}}} \{t_\mu\}g$ and $\{t_\mu\}\theta = \sum_{g \in G_{\{t_\mu\}}} \{t_\nu\}g$.

LEMMA 2.3.3 *Let* $v = \sum_{h \in G_{\{t_\mu\}}} \{t_\nu\}h$. *Then we have* $\{t_\lambda\}\phi \circ \theta = \sum_{g \in G_{\{t_\lambda\}}} vg$.

Proof: As θ is an FG-homomorphism we have

$$\{t_\lambda\}\phi \circ \theta = \sum_{g \in G_{\{t_\lambda\}}} \{t_\mu\}g\theta$$
$$= \sum_{g \in G_{\{t_\lambda\}}} \{t_\mu\}\theta g$$
$$= \sum_{g \in G_{\{t_\lambda\}}} \sum_{h \in G_{\{t_\mu\}}} \{t_\nu\}hg$$
$$= \sum_{g \in G_{\{t_\lambda\}}} vg,$$

with $v = \sum_{h \in G_{\{t_\mu\}}} \{t_\nu\}h$. □

THEOREM 2.3.4 *Let* $\lambda, \mu, \nu \vdash n$ *and suppose that* $\{t_\lambda\} \in \mathcal{M}^\lambda$, $\{t_\mu\} \in \mathcal{M}^\mu$ *and* $\{t_\nu\} \in \mathcal{M}^\nu$ *have the property that every row of* $\{t_\mu\}$ *is a subrow of at least one of* $\{t_\lambda\}$ *or* $\{t_\nu\}$. *Let* ϕ *and* θ *be as before and let* $\psi : M^\lambda \to M^\nu$ *be the map defined by* $\{t_\lambda\}\psi = \sum_{g \in G_{\{t_\lambda\}}} \{t_\nu\}g$. *Then* $\phi \circ \theta = |G_{\{t_\mu\}}|\psi$.

Proof: By Lemma 2.3.3 we have

$$\{t_\lambda\}\phi \circ \theta = \sum_{g \in G_{\{t_\lambda\}}} \sum_{x \in G_{\{t_\mu\}}} \{t_\nu\}xg.$$

Let μ have r-many parts. Clearly we have that $G_{\{t_\mu\}} = \times_{i=1}^{r} S_{\mu_i}$, where S_{μ_i} is the symmetric group on the elements in the ith row of $\{t_\mu\}$. Without loss suppose that the rows of length $\mu_1, \ldots, \mu_p$ are subrows of rows of $\{t_\lambda\}$ and the remaining rows of $\{t_\mu\}$ are subrows of $\{t_\nu\}$. Thus we can write $G_{\{t_\mu\}} = H \times K$ where $H = \times_{i=p+1}^{r} S_{\mu_i}$ and $K = \times_{i=1}^{p} S_{\mu_i}$. In particular we have $H \subseteq G_{\{t_\nu\}}$ and $K \subseteq G_{\{t_\lambda\}}$. Therefore we have

$$\begin{aligned} \{t_\lambda\}\phi \circ \theta &= \sum_{g \in G_{\{t_\lambda\}}} \sum_{h \in H} \sum_{k \in K} \{t_\nu\}hkg && (1) \\ &= |H| \sum_{g \in G_{\{t_\lambda\}}} \sum_{k \in K} \{t_\nu\}kg && (2) \\ &= |H||K| \sum_{g \in G_{\{t_\lambda\}}} \{t_\nu\}g && (3) \\ &= |G_{\{t_\mu\}}|\{t_\lambda\}\psi. \end{aligned}$$

Here (2) follows from (1) as $H \subseteq G_{\{t_\nu\}}$ and (3) follows from (2) as $K \subseteq G_{\{t_\lambda\}}$. $\square$

2.4 The standard map

In this section we introduce the standard map $\psi_\lambda : M^\lambda \to M^{\lambda'}$. This map was first defined by Black and List in [2] for the partitions (b^a). Independently Siemons and Wagner [27] and Stanley [29] extended this definition to an arbitrary partition. The rest of this thesis will then be devoted to applying the tools we have developed so far to the study of ψ_λ. For the λ-tabloid $\{t\}$ let $\{t\}'$ be the λ'-tabloid whose ith row consists of the ith smallest element of each row of $\{t\}$.

Example 2.4.1 *If $\{t\}$ is the primary $(5^3, 2, 1)$-tabloid we have*

$$\{t\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} , \qquad \{t\}' = \begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & & & & \end{array} .$$

Definition 2.4.2 *The standard map $\psi_\lambda : M^\lambda \to M^{\lambda'}$ is the FG-homomorphism given by*

$$\psi_\lambda : \{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}' g.$$

Lemma 2.4.3 *The image $(M^\lambda)\psi$ is a submodule of $H^{\lambda'}$.*

Proof: As in Subsection 2.2, for $\pi \in S_{(\lambda')^*}$ define $a_\pi \in G_{\{t\}}$ by $a_\pi : \{t\}_{i,j} \mapsto \{t\}_{i\pi^{-1},j}$. Then $\{t\}'\pi = \{t\}' a_\pi$. Hence $\{t\}'\pi \in (\{t\}')^{G_{\{t\}}}$. As the action of the twist group commutes with that of G we have

$$\sum_{g \in G_{\{t\}}} \{t\}' g \pi = \sum_{g \in G_{\{t\}}} \{t\}' \pi g = \sum_{g \in G_{\{t\}}} \{t\}' g.$$

□

Lemma 2.4.4 *Let $\pi \in S_{\lambda^*}$. Then $\{t\}\pi\psi_\lambda = \{t\}\psi_\lambda$.*

Proof: Let a_π be the unique element of $G_{\{t\}'}$ such that $\{t\}\pi = \{t\} a_\pi$. Hence

$$\begin{aligned} \{t\}\pi\psi &= \{t\} a_\pi \psi \\ &= \sum_{g \in G_{\{t\} a_\pi}} \{t\}' a_\pi g \\ &= \sum_{g \in G_{\{t\}}} \{t\}' g \\ &= \{t\}\psi. \end{aligned}$$

□

We are now able to show that we are only interested in the action of the standard map on H^λ. The *augmentation ideal* $\Delta(FS_{\lambda^*}) \subseteq FS_{\lambda^*}$ is the set of all elements $\sum_{\pi \in S_{\lambda^*}} a_\pi \pi$ such that $\sum_{\pi \in S_{\lambda^*}} a_\pi = 0$. Define H^λ_Δ to be the subspace of M^λ spanned by the elements $\{t\}\sigma$ where $\sigma \in \Delta(FS_{\lambda^*})$. Then clearly $FS_{\lambda^*} = 1_{S_{\lambda^*}} \bigoplus \Delta(FS_{\lambda^*})$ and these two summands contain no common composition factor and so it easily follows by orthogonality of idempotents that $M^\lambda = H^\lambda \bigoplus H^\lambda_\Delta$. Of course one could take this further and for an irreducible FS_{λ^*}-module V with character χ define H^λ_V to be the image of the map $\theta_\chi : M^\lambda \to M^\lambda$ given by $\{t\} \mapsto \sum_{\pi \in S_{\lambda^*}} \{t\}\chi(\pi^{-1})\pi$. Then we can write $M^\lambda = \bigoplus_V H^\lambda_V$.

Proposition 2.4.5 *We have the inclusion $H^\lambda_\Delta \subseteq ker(\psi_\lambda)$.*

Proof: By Lemma 2.4.4 we have $(\{t\}\pi)\psi_\lambda = (\{t\})\psi_\lambda$. Let $\sum_\pi a_\pi\{t\}\pi \in H^\lambda_\Delta$. Then

$$\left(\sum_{\pi \in S_{\lambda^*}} a_\pi\{t\}\pi\right)\psi_\lambda = \sum_{\pi \in S_{\lambda^*}} a_\pi(\{t\}\pi)\psi_\lambda \tag{4}$$

$$= (\sum_\pi a_\pi)(\{t\})\psi_\lambda \tag{5}$$

$$= 0 \cdot (\{t\})\psi_\lambda \tag{6}$$

$$= 0.$$

Here (4) is due to the linearity of the standard map. Second (5) follows from (4) by Lemma 2.4.4. Finally (6) follows from (5) as $\sum_\pi a_\pi \in \Delta(FS_{\lambda^*})$. □

Corollary 2.4.6 *Suppose that V is an FS_{λ^*}-module that does not contain the trivial S_{λ^*}-module. Then we have $H^\lambda_V \subseteq ker(\psi_\lambda)$.*

Proof: Let χ_V denote the character of V and let $[\cdot,\cdot]$ denote the usual inner product of S_{λ^*}-characters. Then we have

$$\sum_{\pi \in S_{\lambda^*}} \chi_V(\pi^{-1}) = [\chi_V, 1_{S_{\lambda^*}}] = 0.$$

Hence $\sum_{\pi \in S_{\lambda^*}} \chi_V(\pi^{-1})\pi \in \Delta(FS_{\lambda^*})$. □

We now state the computational results of Jacob's thesis [15] and Müller and Neunhöffer's paper [22]. In pages $95 - 100$ of her thesis, Jacob proves the following result;

PROPOSITION 2.4.7 *The standard maps of the partitions* $(2^2), (3^3)$ *and* (4^4) *are injective.*

Proof (sketch): As it is an FG-map the standard map ψ yields a map $\hat{\psi} : End_{FG}(H^{(a^a)}) \to End_{FG}(H^{(a^a)})$ given by $\theta \mapsto \theta \circ \psi$. As we are working in characteristic zero it is easy to see that the map $\hat{\psi}$ is injective iff the map ψ is injective. The map $\hat{\psi}$ is much easier to deal with computationally since the dimension of $End_{FG}(H^{(a^a)})$ is much smaller than that of $H^{(a^a)}$. Jacob then uses a computer program to produce the matrix of $\hat{\psi}$ when $a = 2, 3, 4$ and then computes the eigenvalues of this map, all of which are non-zero. □

Müller and Neunhöffer use same technique to prove:

PROPOSITION 2.4.8 *The standard map of* $H^{(5^5)}$ *is not injective.*

2.5 Lifting homomorphisms

2.5.1 General results

For $m < n$ we regard the symmetric group S_m as the subgroup of S_n that fixes $m+1, m+2, \ldots, n$. In this section we demonstrate a method of lifting FS_m-homomorphisms to FS_n-homomorphisms.

We begin by reviewing the definition of the exterior tensor product. Let H be a group and V some FH-module. Let K be a second group and U some FK-module. Given bases $\{v_i\}_i$ and $\{u_j\}_j$ of V and U respectively define the *exterior tensor product* $V \otimes U$ to be the vector space whose basis is the set of ordered pairs $\{v_i \otimes u_j\}_{i,j}$. Make $V \otimes U$ into an $F(H \times K)$ module by letting $v \otimes u(h, k) = vh \otimes uk$.

Let $\phi : V \to W$ be an FH-homomorphism and define $\phi^* : V \otimes U \to W \otimes U$ by $\phi^* : v \otimes u \mapsto v\phi \otimes u$. We say that ϕ^* *extends* ϕ.

LEMMA 2.5.1 *We have $ker(\phi^*) = ker(\phi) \otimes U$. In particular ϕ^* is injective iff ϕ is injective.*

Proof: Suppose $y = \sum_{i,j} a_{i,j} v_i \otimes u_j \in V \otimes U$ and define $x_j = \sum_i a_{i,j} v_i$. This gives $y = \sum_j x_j \otimes u_j$ and so $y\phi^* = \sum_j x_j\phi \otimes u_j$. As the u_j are linearly independent this gives $y\phi^* = 0$ iff $x_j\phi = 0$ for each j, which in turn holds iff $y \in ker(\phi) \otimes U$. □

Let $H \subseteq G$ and let V be an FH-module. Then FG acts on the induced module $V^G = V \bigotimes_{FH} FG$ by $(v \otimes g)g_1 = v \otimes gg_1$. If $\{g_i\}$ is a complete set of coset representatives of H in G we may write $V^G = \bigoplus_i V \otimes g_i$. To ease notation we shall write vg in place of $v \otimes g$ and we identify the space $V \cdot 1 = V \otimes 1$ with V. Let $\phi : V \to W$ be an FH-homomorphism. The *induced homomorphism* $\phi^G : V^G \to W^G$ is defined to be the FG-homomorphism given by

$$v\phi^G = \frac{1}{|G_v|} \sum_{g \in G_v} v\phi g \quad for\ v \in V\ and$$

$$vg_1\phi^G = v\phi g_1 \quad \forall\ g_1 \in G.$$

LEMMA 2.5.2 *For all $v_1 \in V$ we have $v_1\phi^G = v_1\phi$.*

Proof: For $v_1 \in V$ we have $G_{v_1} = H_{v_1}$ and this gives

$$v_1\phi^G = \frac{1}{|H_{v_1}|} \sum_{h \in H_{v_1}} v_1\phi h = v_1\phi.$$

□

LEMMA 2.5.3 *We have $ker(\phi^G) = ker(\phi)^G$. In particular ϕ^G is injective iff ϕ is injective.*

Proof: For $y = \sum_g v_g g \in V^G$ with $v_g \in V$ we have $y\phi^G = \sum_g v_g \phi g$ with $v_g \phi g \in W \cdot g$. Hence the $v_g \phi g$ are linearly independent and so $y\phi^G = 0$ iff $v_g \in ker(\phi)$ for each g, which in turn holds iff $y \in ker(\phi)^G$. □

COROLLARY 2.5.4 *With notation as in Lemmas 2.5.3 and 2.5.1 we have* $ker(\phi^{*G}) = (ker(\phi) \otimes U)^G$. *In particular* ϕ^{*G} *is injective iff* ϕ *is injective.*

Proof: By first Lemma 2.5.3 and then Lemma 2.5.1 we have

$$ker(\phi^{*G}) = ker(\phi^*)^G = (ker(\phi) \otimes U)^G.$$

□

We now apply the results of this section to the homomorphisms constructed in Section 2.3. Let FA and FB be transitive permutation modules of FH and let W be some other FH-module. Fix $a \in A$ and $w \in W$. Define $\phi : FA \to W$ to be the unique map that satisfies $a\phi = \sum_{h \in H_a} wh$.

THEOREM 2.5.5 *The map* ϕ^{*G} *is a scalar multiple of the unique map* $(FA \otimes FB)^G \to (W \otimes FB)^G$ *given by*

$$a \otimes b \mapsto \sum_{g \in G_{a\otimes b}} (w \otimes b)g.$$

Proof: Call the unique map θ. We have $G_{a\otimes b} = (H \times K)_{a\otimes b} = H_a \times K_b$. Hence

$$\begin{aligned} a \otimes b\theta &= \sum_{g \in G_{a\otimes b}} (w \otimes b)g \\ &= \sum_{h \in H_a} \sum_{k \in K_b} (w \otimes b)(h, k) \\ &= \sum_{h \in H_a} \sum_{k \in K_b} wh \otimes bk \\ &= |K_b| \sum_{h \in H_a} wh \otimes b & (7) \\ &= |K_b| a \otimes b\phi^{*G}. & (8) \end{aligned}$$

Here (8) follows from (7) by Theorem 2.5.4. The module $(FA \otimes FB)^G$ is cyclic with generator (a, b) and hence (8) implies that $\theta = |K_b|\phi^{*G}$. □

2.5.2 Lifting homomorphisms $M^\mu \to M^{\mu^\dagger}$

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and fix $1 \leq p \leq r$. Define two new partitions $\mu = (\lambda_1, \ldots, \lambda_p) \vdash n_1$ and $\nu = (\lambda_{p+1}, \ldots, \lambda_r) \vdash n_2$. Fix the λ-tabloid $\{t\}$ to be the tabloid whose first row is the set $\{1, 2, \ldots, \lambda_1\}$, whose second is $\{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\}$ and so on. Then let the μ-tabloid $\{t^X\}$ consist of the top p rows of $\{t\}$ and let the ν-tabloid $\{t^Y\}$ consist of the bottom $(r-p)$ rows of $\{t\}$. Using Proposition 2.3.1 define $\varphi_1 : M^\lambda \to (M^\mu \otimes M^\nu)^G$ by $\{t\} \mapsto \{t^X\} \otimes \{t^Y\}$. Then φ_1 is an FG-isomorphism with inverse given by $\{t^X\} \otimes \{t^Y\} \mapsto \{t\}$.

Example 2.5.6 *Let* $\lambda = (6, 4, 3, 2) \vdash 15$ *with* $\mu = (6, 4) \vdash 10$ *and* $\nu = (3, 2) \vdash 5$. *Then with the notation above*

$$\{t\} = \begin{array}{cccccc} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & 10 & & \\ \cline{1-4} 11 & 12 & 13 & & & \\ \cline{1-3} 14 & 15 & & & & \\ \cline{1-2} \end{array} , \quad \{t^X\} = \begin{array}{cccccc} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 7 & 8 & 9 & 10 & & \\ \cline{1-4} \end{array} ,$$

$$\{t^Y\} = \begin{array}{ccc} \hline 11 & 12 & 13 \\ \hline 14 & 15 & \\ \cline{1-2} \end{array} .$$

If $\mu^\dagger$ is some other partition of n_1 let $\lambda^\dagger$ denote the composition obtained by adding the parts of ν to the bottom of those of $\mu^\dagger$. For a $\mu^\dagger$-tabloid $\{s^X\}$ let $\{s\}$ denote the $\lambda^\dagger$-tabloid obtained from $\{s^X\}$ by adding the rows of $\{t^Y\}$ to the bottom of those of $\{s^X\}$. Define $\varphi_2 : (M^{\mu^\dagger} \otimes M^\nu)^G \to M^{\lambda^\dagger}$ by $(\{s^X\} \otimes \{t^Y\}) \mapsto \{s\}$. Then φ_2 is an FG-isomorphism with inverse given by $\{s\} \mapsto (\{s^X\} \otimes \{t^Y\})$.

Example 2.5.7 *Let* $\mu^\dagger = (5^2)$ *and* $\nu = (3, 2)$. *Then* $\lambda^\dagger = (5^2, 3, 2)$. *With the notation above*

$$\{s^X\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 7 & 8 & 9 & 10 & 6 \\ \hline \end{array} \quad , \quad \{t^Y\} = \begin{array}{ccc} \hline 11 & 12 & 13 \\ \hline 14 & 15 & \\ \cline{1-2} \end{array} \quad ,$$

$$\{s\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 7 & 8 & 9 & 10 & 6 \\ \hline 11 & 12 & 13 & & \\ \cline{1-3} 14 & 15 & & & \\ \cline{1-2} \end{array} \quad .$$

THEOREM 2.5.8 *Let $\phi : M^{\mu} \to M^{\mu^{\dagger}}$ be the FS_{n_1}-homomorphism defined by*

$$\{t^X\} \mapsto \sum_{g \in S_{n_1\{t^X\}}} \{s^X\}g.$$

Then $\varphi_1 \circ \phi^{*G} \circ \varphi_2$ is a scalar multiple of the unique FG-homomorphism $M^{\lambda} \to M^{\lambda^{\dagger}}$ given by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{s\}g$.

Proof: We have $(\{t\})\varphi_1 = (\{t^X\} \otimes \{t^Y\})$. By Theorem 2.5.5 and as $G_{(\{t^X\} \otimes \{t^Y\})} = G_{\{t\}}$ there is a scalar α such that

$$\begin{aligned} (\{t^X\} \otimes \{t^Y\})\phi^{*G} &= \alpha \sum_{g \in G_{(\{t^X\} \otimes \{t^Y\})}} (\{s^X\} \otimes \{t^Y\})g \\ &= \alpha \sum_{g \in G_{\{t\}}} (\{t^X\} \otimes \{t^Y\})g. \end{aligned}$$

Finally $((\{s^X\} \otimes \{t^Y\})g)\varphi_2 = \{s\}g$. Hence we have

$$\{t\}\varphi_1 \circ \phi^{*G} \circ \varphi_2 = \alpha \sum_{g \in G_{\{t\}}} \{s\}g.$$

□

COROLLARY 2.5.9 *The map φ_1 is an isomorphism from $ker(\varphi_1 \circ \phi^{*G} \circ \varphi_2)$ onto $(ker(\phi^{*G}) \otimes M^{\nu})^G$.*

□

2.5.3 Lifting the standard map

Let $\lambda = \mu \cup \nu$ be a composition and $\{t\}$ some λ-tabloid. Recall the convention of Section 2.1; we call the μ-classes of $\{t\}$ rows and draw them horizontally and we call the ν-classes of $\{t\}$ columns and draw them vertically. The twist group of μ acts on λ-tabloids by permuting rows. For a λ-tabloid $\{t\}$ let $\{t^X\}$ denote the μ-tabloid that consists of the rows of $\{t\}$ and let $\{t^Y\}$ denote the ν-tabloid that consists of the columns of $\{t\}$.

DEFINITION 2.5.10 *Let H^λ_μ be the subspace of M^λ spanned by the elements $\sum_{\pi \in S_{\mu^*}} \{t\}\pi$.*

Define $\varphi_1 : M^\lambda \to (M^\mu \otimes M^\nu)^G$ by $\{t\} \mapsto (\{t^X\} \otimes \{t^Y\})$. Then φ_1 is an FG-isomorphism with inverse given by $(\{t^X\} \otimes \{t^Y\}) \mapsto \{t\}$. Further, for $\pi \in S_{\mu^*}$ we have $\{t\}\pi \mapsto (\{t^X\}\pi \otimes \{t^Y\})$. Hence φ_1 is an isomorphism between M^λ_μ and $(H^\mu \otimes M^\nu)^G$.

EXAMPLE 2.5.11 *Let $\mu = (4^3, 1)$ and $\nu = (5)$. Then $\lambda = (4^3, 1, 5)$. Below $\{t\}$ is a $(4^3, 1, 5)$-tabloid, with $\{t^X\}$ and $\{t^Y\}$ the corresponding μ- and ν-tabloids described above.*

$$\{t\} = \left|\begin{array}{c|cccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & & & & \end{array}\right. ,$$

$$\{t^X\} = \begin{array}{cccc} \hline 2 & 3 & 4 & 5 \\ \hline 7 & 8 & 9 & 10 \\ \hline 12 & 13 & 14 & 15 \\ \hline 17 & & & \\ \hline \end{array} , \qquad \{t^Y\} = \left|\begin{array}{c} 1 \\ 6 \\ 11 \\ 16 \\ 18 \end{array}\right| .$$

Now let $\lambda^\dagger = \mu' \cup \nu$ and as usual let $\{t^X\}'$ denote the μ'-tabloid that is conjugate to $\{t^X\}$. Let $\{s\}$ denote the $\lambda^\dagger$-tabloid obtained by adding the columns of $\{t^X\}'$ to the right of those of $\{t^Y\}$. Define the FG-isomorphism $\varphi_2 : (M^{\mu'} \otimes M^\nu)^G \to M^{\lambda^\dagger}$ by $(\{t^X\}' \otimes \{t^Y\}) \mapsto \{s\}$.

EXAMPLE 2.5.12 *Let $\mu = (3^3, 1)$ and $\nu = (5)$. Suppose that $\{t^X\}$ and $\{t^Y\}$ are as in Example 2.5.11. Then below are $\{t^X\}'$ and $\{s\}$.*

$$\{t^X\}' = \begin{array}{|c|c|c|c|} 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 \\ 12 & 13 & 14 & 15 \\ 17 & & & \end{array} \quad , \quad \{s\} = \begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & & & & \end{array} \quad .$$

THEOREM 2.5.13 *The map $\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2$ is a scalar multiple of the unique FG-homomorphism $H^\lambda \to M^{\lambda^\dagger}$ given by*

$$\sum_{\pi \in S_{\mu^*}} \{t\}\pi \mapsto \sum_{g \in G_{\{t\}}} \{s\}g.$$

Proof: By Remark 2.2.3 we know that H^λ is a transitive permutation module of FG and as a result all the results of Section 2.3 concerning constructions of homomorphisms can be applied to it. We have

$$\sum_{\pi \in S_{\mu^*}} \{t\}\pi\varphi_1 = (\sum_{\pi \in S_{\mu^*}} \{t^X\}\pi \otimes \{t^Y\}).$$

By first Theorem 2.5.5 and then as $G_{(\{t^X\}\otimes\{t^Y\})} = G_{\{t\}}$ there is a scalar α such that

$$\begin{aligned}(\sum_{\pi \in S_{\mu^*}} (\{t^X\}\pi \otimes \{t^Y\})\psi_\mu|^{*G} &= \alpha \sum_{g \in G_{(\{t^X\}\otimes\{t^Y\})}} (\{t^X\}' \otimes \{t^Y\})g \\ &= \alpha \sum_{g \in G_{\{t\}}} (\{t^X\} \otimes \{t^Y\})g.\end{aligned}$$

Finally as $((\{t^X\}' \otimes \{t^Y\})g)\varphi_2 = (\{t^X\}' \otimes \{t^Y\})\varphi_2 g = \{s\}g$ we have

$$(\sum_{\pi \in S_{\mu^*}} \{t\}\pi)\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2 = \alpha \sum_{g \in G_{\{t\}}} \{s\}g.$$

□

Corollary 2.5.14 *The map φ_1 is an isomorphism from $ker(\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2)$ into $(ker(\psi_\mu|^{*G}) \otimes M^\nu)^G$.*

□

Chapter 3

Generalized tabloids and Hom-spaces

The tabloid spaces M^λ appear in the books of James [16] and Sagan [24]. In this chapter we generalize these spaces by looking at tabloids $\{T\}$ with repeated entries. Let ν be a partition of n. Then we let $M^{\lambda,\nu}$ denote the space of all tabloids of shape λ that contain ν_1 copies of 1 and ν_2 copies of 2 and so on. We first identify these spaces with $Hom_{FG}(M^\nu, M^\lambda)$ by producing a homomorphism $\phi_{\{T\}}$ for each $\{T\}$ that is the adjoint of the homomorphism $\theta_{\{T\}}$ found in the books of James and Sagan. We then use these spaces to study the homomorphisms $\theta_{\{U\}}$ for a tabloid $\{U\}$ of shape λ and content μ in the following way. As $\theta = \theta_{\{U\}} : M^\lambda \to M^\mu$ is an FG-homomorphism we obtain a map $\theta(\nu) : Hom_{FG}(M^\nu, M^\lambda) \to Hom_{FG}(M^\nu, M^\mu)$ by $\phi_{\{T\}} \mapsto \phi_{\{T\}} \circ \theta$. We then relate the rank of $\theta(\nu)$ with the irreducible submodules of the kernel of θ. Finally we look at two important examples. First we define a natural map $\epsilon : M^{(x,y)} \to M^{(x-1,y+1)}$ and show that it is injective. Second we look at the action of the twist group S_{λ^*} on $M^{\lambda,\nu}$. We then use the techniques above to study the standard map. We show that bad partitions have a non-injective standard map and that the kernel of the standard map $\psi_{b^a}|$ for $b \geq a$ contains no irreducible submodules isomorphic to a Specht module $S^{(x,y)}$ for all two part partitions (x, y).

3.1 Definitions and basic results

Recall that an *ordinary* λ*-tableau* is a filling of the Young diagram $[\lambda]$ with the numbers $1, 2, \ldots, n$. We define an equivalence class $\sim$ on the set $\mathcal{F}^\lambda$ of all λ-tableau by writing $t \sim s$ if s can be obtained from t by permuting elements in a row. The λ*-tabloid* $\{t\}$ is the $\sim$-equivalence class that contains the tableau t. For example $\{t\}$ is a $(4, 2)$-tabloid:

$$\{t\} = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & & \\ \cline{1-2} \end{array} \quad .$$

A *generalized tableau* of shape λ and content ν (or (λ, ν)-tableau) is a filling of the Young diagram of λ with ν_1 copies of 1 and ν_2 copies of 2 and so on. Hence an ordinary λ-tableau is a $(\lambda, 1^n)$-tableau. Letting $\mathcal{F}^{\lambda,\nu}$ denote the set of all generalized tableaux, we define an equivalence relation $\sim$ on $\mathcal{F}^{\lambda,\nu}$ by writing $T \sim S$ if S can be obtained from T by permuting elements in a row. A *generalized tabloid* of shape λ and content ν (or (λ, ν)-tabloid) is then a $\sim$-equivalence class. The vector space whose basis is the set of all (λ, ν)-tabloids is denoted $M^{\lambda,\nu}$. For example $\{T\}$ is a $(4, 2), (5, 1)$-tabloid:

$$\{T\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & & \\ \cline{1-2} \end{array} \quad .$$

We follow the books of James [16] and Sagan [24] to produce a homomorphism $\theta_{\{T\}} : M^\lambda \to M^\nu$ for each (λ, ν)-tabloid $\{T\}$. Let $\{t\}$ be a λ-tabloid and $\{T\}$ a (λ, ν)-tabloid, without loss we may assume t and T are row-standard. Let t_{ij} and T_{ij} denote the jth smallest entry of the ith row of t and T respectively. Then we define $(t)T$ to be the ν-tableau whose ath row consists of those t_{ij} such that $T_{ij} = a$. Finally let $\{r\} = \{(t)T\}$. We now define $\theta_{\{T\}} : M^\lambda \to M^\nu$ to be the homomorphism given by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{r\}g$. For example we have

$$\{t\} = \begin{array}{cccc} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \cline{1-2} \end{array} \quad , \quad \{T\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & & \\ \cline{1-2} \end{array} \quad , \quad \{r\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & & & & \\ \cline{1-1} \end{array} \quad .$$

Less standard is the construction of the adjoint $\phi_{\{T\}}$ of $\theta_{\{T\}}$ which we now use to identify $M^{\lambda,\nu}$ with $Hom_{FG}(M^{\nu}, M^{\lambda})$. Let $\{r\}$ be a ν-tabloid and $\{T\}$ a (λ, ν)-tabloid. Without loss we can assume r and T are row-standard. Define $(r)T$ to be the λ-tableau obtained from T by replacing the first (reading left to right then top to bottom) copy of 1 with the first entry of the first row of r and more generally replacing the jth copy of i with the jth entry of the ith row of r. Hence if $\{r\}$ is the primary ν-tabloid and $\{T\}$ is a (λ, ν)-tabloid then we define $\phi_{\{T\}} : M^{\nu} \to M^{\lambda}$ by $\{r\} \mapsto \sum_{g \in G_{\{r\}}} \{t\}g$ where $\{t\} = (\{r\})\{T\}$. For an example we have

$$\{r\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & & & & \\ \cline{1-1} \end{array} \quad , \quad \{T\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & & \\ \cline{1-2} \end{array} \quad ,$$

$$r = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & & & & \end{array} \quad , \quad T = \begin{array}{cccc} 1 & 1 & 1 & 2 \\ 1 & 1 & & \end{array} \quad , \quad (r)T = \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & & \end{array} \quad .$$

THEOREM 3.1.1 *The set* $\{\phi_{\{T\}} \mid \{T\} \in M^{\lambda,\nu}\}$ *is a basis of* $Hom_{FG}(M^{\nu}, M^{\lambda})$.

Proof: Let $\{r\}$ be the primary λ-tabloid. Let $B = (b_{ij})$ be a matrix of positive integers such that the ith row sum is ν_i and the jth column sum is λ_j. Let X_B denote the set of all λ-tabloids whose jth row contains b_{ij} elements of the ith row of $\{r\}$. Then M^{λ} is a union of the X_B. Let $\{T\}$ be the (λ, ν)-tabloid with b_{ij} copies of j in its ith row. Then clearly the ith row of $\{r\}$ intersects the jth row of $\{t\}$ with size $b_{i,j}$ iff the ith row of $\{r\}$ intersects the jth row of $\{t\}g$ with size $b_{i,j}$ for all i, j. Hence every tabloid involved in $\{r\}\phi_{\{T\}}$ lies in X_B. Similarly the ith row of $\{r\}$ intersects the jth row of $\{t_1\}$ with size $b_{i,j}$ for all i, j iff there exists $g \in G_{\{r\}}$ such that $\{t\}g = \{t_1\}$. Hence every element of X_B is involved in $\{r\}\phi_{\{T\}}$. $\square$

COROLLARY 3.1.2 *The set* $\{\theta_{\{T\}} \mid \{T\} \in M^{\lambda,\nu}\}$ *is a basis of* $Hom_{FG}(M^{\lambda}, M^{\nu})$.

As a result of Theorem 3.1.1 we will freely identify $Hom_{FG}(M^{\nu}, M^{\lambda})$ with $M^{\lambda,\nu}$. Let $\{U\}$ be a (λ, μ)-tabloid and let a_{ij} denote the number of copies of j in the ith row of $\{U\}$. To ease notation let $\theta = \theta_{\{U\}}$. Then for the λ-tabloid $\{t\}$ we have $\{t\}\theta = \sum \{s_1\}$ where the sum is over all μ-tabloids $\{s_1\}$ whose jth row contains a_{ij}

elements from the ith row of $\{t\}$. Then θ defines a map $\theta(\nu) : M^{\lambda,\nu} \to M^{\mu,\nu}$ by $\phi_T \mapsto \phi_T \circ \theta$. Hence we have

$$\begin{aligned}\{r\}\phi_{\{T\}} \circ \theta &= \sum_{h \in G_{\{r\}}} \{t\}h\theta \\ &= \sum_{h \in G_{\{r\}}} \sum_{\{s_1\}} \{s_1\}h \\ &= \sum_{\{S_1\}} \{r\}\phi_{\{S_1\}} \ .\end{aligned}$$

Where $\{S_1\}$ is the (μ, ν)-tabloid obtained from $\{s_1\}$ by replacing each entry x of $\{s_1\}$ with row number of $\{r\}$ in which x lies.

PROPOSITION 3.1.3 *We have $\theta(\nu) : \{T\} \mapsto \sum\{S_1\}$ where the sum is over all $\{S_1\}$ whose jth row contains a_{ij} entries from the ith row of $\{T\}$ counting multiplicities.*

Proof: Let $x_1, \ldots, x_{a_{ij}}$ be the elements that lie in the ith row of $\{t\}$ and the jth row of $\{s_1\}$. If x_k lies in the ath row of r then it corresponds to an a in the ith row of $\{t\}$ and an a in the jth row of $\{s_1\}$. □

If $\lambda, \mu \vdash n$ we write $\lambda \geq \mu$ if for each k the inequality $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$ holds. We call $\geq$ the *dominance ordering* of partitions of n. The relation between θ and $\theta(\nu)$ is described in the following theorem which as a corollary has the result that motivated Müller and Neunhöffer's paper [22].

THEOREM 3.1.4 *The map $\theta(\nu)$ is injective iff $ker(\theta)$ contains no irreducible submodule $U \cong S^\eta$ such that η dominates ν.*

Proof: Let $\eta \geq \nu$ and suppose $S^\eta \cong U \subseteq ker(\theta)$. By Young's rule M^ν contains a submodule isomorphic to U and so there exists a non-zero FG-map $\phi : M^\nu \to M^\lambda$ whose image is U. Hence ϕ lies in the kernel of $\theta(\nu)$. Conversely let $0 \neq \phi \in ker(\theta(\nu))$. Then the image of ϕ is a non-zero submodule of M^λ which lies in the kernel of θ. □

As a corollary we have the following well known result that one can find as Proposition 4.4 on page 57 of the book [21] by Mathas.

Corollary 3.1.5 *Let $\nu = \lambda$. Then $M^{\lambda,\lambda}$ is the FG-endomorphism algebra of M^{λ}. Hence θ is injective iff $\theta(\lambda)$ is injective.*

$\square$

3.2 The ϵ-map

Let (x, y) be a composition of n with $x \geq 1$ and let $\{U\}$ be the $(x, y), (x-1, y+1)$-tabloid whose top row consists of $x-1$ copies of 1 and a single copy of 2 and whose bottom row consists of y copies of 2. To ease notation we write $\epsilon = \theta_{\{U\}} : M^{(x,y)} \to M^{(x-1,y+1)}$. For example if $x = 4$ and $y = 2$ we have

$$\{U\} = \begin{array}{l} \hline 1\;\;1\;\;1\;\;2 \\ \hline 2\;\;2 \\ \hline \end{array} \quad .$$

If $\nu = (x, y)$, then $\epsilon(\nu)$ sends the $(x, y), (x, y)$-tabloid $\{T\}$ to the sum of all tabloids $\{S\}$ obtained by moving an element from the top row of $\{T\}$ into the bottom row. For example

$$\begin{aligned}
\epsilon(\nu) : \begin{array}{l} \hline 1\;\;1\;\;2\;\;2 \\ \hline 1\;\;1 \\ \hline \end{array} &\to 2 \cdot \begin{array}{l} \hline 1\;\;2\;\;2 \\ \hline 1\;\;1\;\;1 \\ \hline \end{array} + 2 \cdot \begin{array}{l} \hline 1\;\;1\;\;2 \\ \hline 1\;\;1\;\;2 \\ \hline \end{array} \quad , \\
\epsilon(\nu) : \begin{array}{l} \hline 1\;\;1\;\;1\;\;2 \\ \hline 1\;\;2 \\ \hline \end{array} &\to 3 \cdot \begin{array}{l} \hline 1\;\;1\;\;2 \\ \hline 1\;\;1\;\;2 \\ \hline \end{array} + \begin{array}{l} \hline 1\;\;1\;\;1 \\ \hline 1\;\;2\;\;2 \\ \hline \end{array} \quad , \\
\epsilon(\nu) : \begin{array}{l} \hline 1\;\;1\;\;1\;\;1 \\ \hline 2\;\;2 \\ \hline \end{array} &\to 4 \cdot \begin{array}{l} \hline 1\;\;1\;\;1 \\ \hline 1\;\;2\;\;2 \\ \hline \end{array} \quad .
\end{aligned}$$

Lemma 3.2.1 *(i) The space $M^{(x,y),(x,y)}$ has dimension $min(x+1, y+1)$.*

(ii) The space $M^{(x-1,y+1),(x,y)}$ has dimension $min(x, y+1)$.

Proof: Define $\{T_i\}$ to be the $(x, y), (x, y)$-tabloid that has i copies of 2 in its top row. Similarly define $\{S_j\}$ to be the $(x-1,y+1),(x,y)$-tabloid with j copies of 2 in the top row. Clearly the $\{T_i\}$ and $\{S_j\}$ form bases of $M^{(x,y),(x,y)}$ and $M^{(x-1,y+1),(x,y)}$ respectively. The result now follows. □

The next result is well known and has a myriad of proofs, see for example results $2.2-2.4$ on pages $393-394$ of Siemons paper [25] or Proposition 5.4.7 on page 210 of Sagans book [24].

THEOREM 3.2.2 *The map ϵ is injective if $x > y$ and surjective if $x \leq y$.*

Proof: Let $\{T_i\}$ and $\{S_j\}$ be as in the proof of Lemma 3.2.1. Then consider $\epsilon(\nu)$ as a matrix with rows indexed by $\{T_i\}$ and columns by $\{S_j\}$ we see it contains non-zero entries in the entries (i,i) and $(i-1,i)$ entries and these are the only non-zero entries. Hence $\epsilon(\nu)$ has maximal rank. The result now follows from Lemma 3.2.1.□

Now let $\{t\}$ be the primary λ-tabloid. Fix $i < j$ and let λ^- be the composition obtained by subtracting one from λ_i and adding one to λ_j. Let $\{t\}^-$ be the λ^--tabloid obtained from $\{t\}$ by moving the largest element from the ith row into the jth row. Define a map $\epsilon(\lambda,\lambda^-) : M^\lambda \to M^{\lambda^-}$ by

$$\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^- g.$$

The main result of this section is the following:

THEOREM 3.2.3 *The map $\epsilon(\lambda,\lambda^-)$ is injective iff $\lambda_i > \lambda_j$.*

Proof: By Theorem 2.5.8 the map $\epsilon(\lambda,\lambda^-)$ is the lift of ϵ to λ and so by Theorem 3.2.2 is injective iff $\lambda_i > \lambda_j$. □

Recall the definition of the dominance ordering of partitions on page 27. An interesting corollary of Theorem 3.2.3 is the following generalization of Theorem 1 in the Paper [19] by Livingstone and Wagner. Other proofs can be found in the Papers of Liebler and Vitale [18] and White [31].

Theorem 3.2.4 *Let $\lambda \geq \mu$ be partitions of n. Then there is an injective FG-homomorphism $M^\lambda \hookrightarrow M^\mu$.*

To prove Theorem 3.2.4 we need a lemma:

Lemma 3.2.5 *Let $\lambda \geq \mu$. Then there is a chain of partitions $\lambda = \lambda^0 > \cdots > \lambda^k = \mu$ such that λ^{i+1} is obtained from λ^i by moving a single node from one row of the Young diagram of λ^i to a lower row.*

Proof: Suppose $\lambda > \mu$ and let λ_{i_1} be the first part of λ such that $\lambda_{i_1} > \mu_{i_1}$. Similarly let λ_{j_1} be the first part of λ such that $\lambda_{j_1} < \mu_{j_1}$. Then let λ^1 be the partition obtained by moving the right most node of the i_1st row of λ into the j_1st row. Clearly $\lambda > \lambda^1 \geq \mu$. By induction there is such a sequence $\lambda^1 > \lambda^2 > \cdots > \lambda^k = \mu$ between λ^1 and μ. Hence $\lambda > \lambda^1 > \lambda^2 > \cdots > \lambda^k = \mu$ is such a sequence between λ and μ. □

Proof Of Theorem 3.2.4: Let $\lambda^1 > \lambda^2 > \cdots > \lambda^k = \mu$ be the chain of partitions obtained by Lemma 3.2.5. Then by Theorem 3.2.3 for each i the map ϵ_{λ^i} is injective. Hence the composition

$$\epsilon_\lambda \circ \epsilon_{\lambda^1} \circ \cdots \circ \epsilon_{\lambda^k}$$

is an injective map from M^λ to M^μ. □

3.3 The action of the twist group

The action of the twist group S_{λ^*} on M^λ commutes with the action of G. Hence S_{λ^*} acts on $Hom_{FG}(M^\nu, M^\lambda)$ by post composition, that is $\theta\pi = \theta \circ \pi$. The twist group also acts on $M^{\lambda,\nu}$ by permuting rows. Our first result shows that these two actions agree

Lemma 3.3.1 *With the notation above $\phi_{\{T\}\pi} = \phi_{\{T\}} \circ \pi$.*

Proof: Let $\{t_0\} = \{r\}\{T\}$ and $\{t_1\} = \{r\}(\{T\}\pi)$. Then

$$\begin{aligned}\{r\}\theta_{\{T\}} \circ \pi &= \sum_{g \in G_{\{r\}}} \{t_0\}g\pi \\ &= \sum_{g \in G_{\{r\}}} \{t_0\}\pi g.\end{aligned}$$

Hence it is enough to show $\{t_0\}\pi$ and $\{t_1\}$ are in the same $G_{\{r\}}$ orbit. The number of elements in the ith row of $\{r\}$ in the jth row of $\{t_0\}\pi$ is the number of i's in the $(j\pi^{-1})$th row of $\{T_0\}$ which is the number of i's in the jth row of $\{T_0\}\pi$ which is the number of elements in the ith row of $\{r\}$ in the jth row of $\{t_1\}$. Hence $\{t_0\}\pi$ and $\{t_1\}$ are in the same $G_{\{r\}}$ orbit. □

The subspace of $Hom_{FG}(M^\nu, M^\lambda)$ fixed by the action of S_{λ^*} is $Hom_{FG}(M^\nu, H^\lambda)$. We let $H^{\lambda,\nu}$ denote the subspace of $M^{\lambda,\nu}$ fixed by S_{λ^*}. The natural basis of $H^{\lambda,\nu}$ is the set $\mathcal{H}^{\lambda,\nu}$ that consists of the elements $\sum_{\pi \in S_{\lambda^*}} \{T\}\pi$. As a result of Lemma 3.3.1 we will freely associate $H^{\lambda,\nu}$ with $Hom_{FG}(M^\nu, H^\lambda)$. We have the following analogue of Theorem 3.1.4 whose proof follows verbatim from Theorem 3.1.4;

THEOREM 3.3.2 *Let $\theta : H^\lambda \to H^\mu$ be an FG-homomorphism. Then $\theta(\nu)$ is injective iff $ker(\theta)$ contains no irreducible submodule $U \cong S^\eta$ such that η dominates ν.*

□

3.4 The standard map $\psi_{(b^a)}(x, y)$

We now apply the results of Section 3.1 to the map $\theta(\nu)$ where $\theta = \psi_\lambda$ is the standard map and $\nu = (x, y)$ is a two part partition of n. We remark that Theorem 3.4.2 is called Hermite reciprocity in Theorem 5.4.34 on page 227 of James and Kerber's book [17] and that the combinatorics of their proof is very similar to ours, although they interpret the results in terms of Polya theory.

Let $\{T\}$ be a (λ, ν)-tabloid and without loss assume that T is a row-standard tableau. Then define the conjugate tabloid $\{T\}'$ to be the tabloid whose ith column contains the ith smallest entry from each row of $\{T\}$. For example we have

$$\{T\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 \\ \hline \end{array} \quad , \quad \{T\}' = \left|\begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{array}\right| \quad .$$

Write ψ in place of $\psi_\lambda(x, y)$. By Proposition 3.1.3 we have that $\{T\}\psi = \sum\{S\}$ where the sum runs over all tabloids $\{S\}$ such that each column of $\{S\}$ contains one element from each row of $\{T\}$. We say that an $(b^a), (x, y)$-tabloid has *shape* $\alpha \vdash y$ if one row contains α_1 copies of 1 and one row contains α_2 copies of 1 and so on. For example the $(4^3), (7, 5)$-tabloid below has shape $(4, 2, 1)$.

$$\begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 \\ \hline \end{array} \quad .$$

We now refine the dominance ordering to a linear order. Let λ and μ be partitions of n. We write $\lambda >_L \mu$ if the first non-zero difference $\lambda_i - \mu_i$ is positive. This is called the *reverse lexicographic ordering* of partitions. Thus $\lambda = (n)$ is the maximal element in this ordering as $\lambda_1 - \mu_1$ will be positive for all choices of μ and (1^n) is the smallest. If λ dominates μ then the first time that $\lambda_i \neq \mu_i$ we must have $\lambda_i \geq \mu_i$ and so $\lambda >_L \mu$. Note that the converse is false as $(5, 2, 1) >_L (4, 4)$ but $(5, 2, 1)$ does not dominate $(4, 4)$.

LEMMA 3.4.1 *Let $\{T\}$ be a $(b^a), (x, y)$-tabloid of shape α. Suppose $\{S\}$ is an $(a^b), (x, y)$-tabloid involved in $\{T\}\psi$ of shape β. Then $\beta' \geq_L \alpha$.*

Proof: To construct the 1st column of $\{T\}'$ we pick an element from each row of $\{T\}$ following the rule that we pick a 1 if possible. To construct the 2nd column we pick an element from each row of $\{T\}$ that we have not already picked, again following the rule that we pick a 1 if possible. In the same way we construct all the columns of $\{T\}'$. Now suppose that $\{S\}$ is a tabloid involved in $\{T\}\psi$ with shape β. To construct the 1st column of $\{S\}$ we pick one element from each row of $\{T\}$

and to construct the 2nd column we pick one element from each row that we have not already chosen. In the same way we construct all columns of $\{S\}$. It easy to see that $(\beta')_1$ is the number of columns of $\{S\}$ that contain one copy of 1. As we choose copies of 1 in preference in $\{T\}'$ then this is at least as big as the same number in $\{T\}'$ which is $\alpha_1 = (\alpha'')_1$. Similarly $(\beta')_i$ is the number of columns of $\{S\}$ that contain i copies of 1 and again as we always choose copies of one in preference in $\{T\}'$ it is easy to see that the first time that $(\beta')_i \neq \alpha_i$ we must have that $(\beta')_i > \alpha_i$. Hence $\beta' \geq_L \alpha$. □

THEOREM 3.4.2 *Let (x, y) be a partition of ab. Then the kernel of the standard map $\psi_{(b^a)}|$ contains no irreducible submodules isomorphic to $S^{(x,y)}$.*

Proof: Let M be the matrix of $\psi_{(b^a)}(x, y)|$. The columns and rows of M are indexed by the sets $\mathcal{H}^{(b^a),(x,y)}$ and $\mathcal{H}^{(a^b),(x,y)}$ respectively. Order the columns by the linear order $>_L$ by writing α left of γ if $\alpha >_L \gamma$. Order the rows by putting the row indexed by $\{S_1\}$ above that of $\{S_2\}$ iff $(\alpha^1)' >_L (\alpha^2)'$, where α_i is the shape of $\{S_i\}$. By Lemma 3.4.1 we see that M is upper triangular and so $\psi_{(b^a)}(x, y)|$ injective. Hence by Theorem 3.3.2 the kernel of $\psi_{(b^a)}$ contains no Specht modules isomorphic to $S^{(x,y)}$. □

3.5 The standard map of (a, b^c)

Let $\lambda = (a, b^c) \vdash n$ with $a - b < c$ and $\lambda \geq \lambda'$. The aim of this subsection is to show that there is no injective map $H^\lambda \to H^{\lambda'}$. We do this by showing that if $r = a-b+1$ and $\nu = (n - r, r)$ then $H^{\lambda,\nu}$ has larger dimension than $H^{\lambda',\nu}$. Let $x \leq r$ and let $\alpha = (\alpha_1, \ldots, \alpha_r) \vdash c - x$. Then we define $\{T_{\alpha,x}\}$ to be the (λ, ν)-tabloid with x copies of 2 in its top row, α_1 copies of 2 in its bottom row, α_2 copies of two in its second to bottom row and so on. As $a \leq c$ we see that this definition gives a unique tabloid. We let $X^{\lambda,\nu}$ denote the set of all these tabloids. Similarly we define the (λ', ν)-tabloid $\{S_{\beta,y}\}$ to be the (λ', ν)-tabloid whose right most y columns of size one contain a copy of 2, whose rightmost column of size c contains β_1 copies of two, whose second-to-rightmost column contains β_2 copies of two and so on. We let $X^{\lambda',\nu}$ denote the set of all these tabloids.

Lemma 3.5.1 *With the notation above we have*

$$H^{\lambda,\nu} =< \sum_{\pi\in S_{\lambda^*}} \{T\}\pi \mid \{T\} \in X^{\lambda,\nu} >_F \quad \textit{and}$$
$$H^{\lambda',\nu} =< \sum_{\sigma\in S_{(\lambda')^*}} \{S\}\sigma \mid \{S\} \in X^{\lambda',\nu} >_F .$$

Proof: Let $\{T\} \in \mathcal{M}^{\lambda,\nu}$. Then there exists some $\pi \in S_{\lambda^*}$ such that the number of copies of 2 in the rows of length b of $\{T\}\pi$ increases. Hence $\{T\}\pi = \{T_{\alpha,x}\}$ for some α and x. □

We now look at the motivating example of $\lambda = (4,3,3)$ and $\nu = (8,2)$. Below are all (λ,ν) and their conjugate (λ',ν)-tabloids. Note that $\{T_{\phi,2}\}' = \{T_{(1),1}\}'$

$$\{T_{(2),0}\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & \\ \hline 1 & 2 & 2 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \\ 1 & 2 & 2 & \end{array} = \{S_{(1^2),0}\} \quad ,$$

$$\{T_{(1^2),0}\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & \\ \hline 1 & 1 & 2 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & \\ 1 & 1 & 2 & \end{array} = \{S_{(1^2),0}\} \quad ,$$

$$\{T_{\phi,2}\} = \begin{array}{cccc} \hline 1 & 1 & 2 & 2 \\ \hline 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & \end{array} = \{S_{(1),1}\} \quad ,$$

$$\{T_{(1),1}\} = \begin{array}{cccc} \hline 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 1 & \\ \hline 1 & 1 & 2 & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & \\ 1 & 1 & 2 & \end{array} = \{S_{(1),1}\} \quad .$$

Theorem 3.5.2 *Let $\lambda = (a, b^c) \vdash n$ with $a - b < c$ and $\lambda \geq \lambda'$. Then there is no injective map $H^\lambda \to H^{\lambda'}$.*

Proof: Let $r = a - b + 1$ and let $\nu = (n - r, r)$. Importantly we have $c \geq r$. Define a map $\varphi : X_\lambda \to X_{\lambda'}$ by $\{T_\alpha, x\} \mapsto \{S_{\alpha',x}\}$. We show that φ is surjective but not injective. Firstly it is clear that $\{T_{\phi,c}\}' = \{T_{(1),c-1}\}'$. Hence φ is not injective. Let $\{S_{\beta,b}\}$ be a (λ', ν)-tabloid. As $c \geq r \geq x$ the Young diagram of β fits inside the (c^b) rectangle at the bottom of $[\lambda']$ and so the Young diagram of β' fits inside the (b^c)-rectangle. Hence the (λ, ν)-tabloid $\{T_{\beta',b}\}$ exists and $\{T_{\beta',b}\}\psi = \{S_{\beta,b}\}$. Thus φ is surjective. Hence by Lemma 3.5.1 the dimension of $H^{\lambda,\nu}$ is strictly larger than that of $H^{\lambda',\nu}$. □

Corollary 3.5.3 *Let $\lambda = (c, b^a)$. Suppose that $a \leq b < c$ and $c - b < a$. Then the standard map is not injective.*

□

Chapter 4

Injective standard maps

In Section 2.4 we defined the standard map. In this chapter we look at when this map is injective. This chapter consists of two sections. In Section 4.1 we describe a method of adding a column to a partition that preserves the injectivity of the standard map. We then generalize this idea in Section 4.2 where we show how to add several columns simultaneously to a partition whilst preserving the injectivity of the standard map.

4.1 Column removal and the modules M^{λ^i}

In this section we introduce the key idea of the chapter. Let λ be a partition and μ^i the partition obtained by removing the first i columns from λ. In Subsection 4.1.1 we describe a method of viewing ψ_λ as a composition of more managable maps ϕ_i. In Subsection 4.1.2 we then show that the maps ϕ_i work well with the action of the twist groups S_{λ^i} and $S_{(\mu^i)^*}$. In Subsection 4.1.3 we relate compositions of some of the ϕ_i with the standard map ψ_{μ^i}. Finally in Subsection 4.1.4 we give a criterion for the remaining ϕ_i to be injective.

4.1.1 An important sequence of tabloids

We define a sequence of tabloids $\{t\}^i$ which will act as intermediate steps between $\{t\}$ and $\{t\}'$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition and μ^i be the partition obtained by removing the first i columns from λ. Let $\nu^i = (\lambda_1', \lambda_2', \ldots, \lambda_i')$, where λ_j' is the jth part of λ'. Then define the composition $\lambda^i = \mu^i \cup \nu^i$. Note that $\lambda^0 = \lambda$ and $\lambda^r = \lambda'$. Recall the convention of Section of 2.1; for a λ^i-tabloid we call the μ^i-classes rows and the ν^i-classes columns. Define $\{t\}^i$ to be the λ^i-tabloid whose jth row consists of the $\lambda_j - i$ largest elements of the jth row of $\{t\}$ and whose kth column consists of the kth smallest element of each row of $\{t\}$.

Example 4.1.1 *Let $\lambda = (5^3, 2, 1)$ and suppose $\{t\}$ is the primary λ-tabloid. Then*

$$\{t\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 \\ \cline{1-2} 18 \\ \cline{1-1} \end{array} = \{t\}^0 \quad , \qquad \{t\}^1 = \left| \begin{array}{c|cccc} 1 & 2 & 3 & 4 & 5 \\ \cline{2-5} 6 & 7 & 8 & 9 & 10 \\ \cline{2-5} 11 & 12 & 13 & 14 & 15 \\ \cline{2-5} 16 & 17 \\ \cline{2-2} 18 & \end{array} \right. ,$$

$$\{t\}^2 = \left| \begin{array}{c|c|ccc} 1 & 2 & 3 & 4 & 5 \\ \cline{3-5} 6 & 7 & 8 & 9 & 10 \\ \cline{3-5} 11 & 12 & 13 & 14 & 15 \\ \cline{3-5} 16 & 17 & \\ 18 & \end{array} \right. , \qquad \{t\}^3 = \left| \begin{array}{c|c|c|cc} 1 & 2 & 3 & 4 & 5 \\ \cline{4-5} 6 & 7 & 8 & 9 & 10 \\ \cline{4-5} 11 & 12 & 13 & 14 & 15 \\ \cline{4-5} 16 & 17 & \\ 18 & \end{array} \right. ,$$

$$\{t\}^4 = \left| \begin{array}{c|c|c|c|c} 1 & 2 & 3 & 4 & 5 \\ \cline{5-5} 6 & 7 & 8 & 9 & 10 \\ \cline{5-5} 11 & 12 & 13 & 14 & 15 \\ \cline{5-5} 16 & 17 & \\ 18 & \end{array} \right. , \qquad \{t\}^5 = \left| \begin{array}{c|c|c|c|c|} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & \\ 18 & \end{array} \right. = \{t\}' \quad .$$

In view of Theorem 2.3.4 the reason for studying the $\{t\}^i$ is the following:

Lemma 4.1.2 *Every class of $\{t\}^i$ is a subclass of $\{t\}$ or $\{t\}^{i+1}$.*

Proof: The jth row of $\{t\}^i$ is obtained from that of $\{t\}^{i-1}$ by removing the j smallest elements. The kth column of $\{t\}^i$ is equal to that of $\{t\}^{i+1}$. □

Define a map $\phi_i : M^{\lambda^i} \to M^{\lambda^{i+1}}$ by

$$\phi_i : \{t\}^i \mapsto \sum_{g \in G_{\{t\}^i}} \{t\}^{i+1} g.$$

PROPOSITION 4.1.3 *The map ψ_λ is a scalar multiple of the composition of maps $\phi_0 \circ \phi_1 \circ \cdots \circ \phi_{\lambda_1 - 1}$.*

Proof: Define $\varphi_i : M^\lambda \to M^{\lambda^i}$ by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^i g$ and induct on the hypothesis that φ_i factors as a scalar multiple of $\phi_0 \circ \cdots \circ \phi_{i-1}$. When $i = 1$ we have $\varphi_1 = \phi_0$. By Lemma 4.1.2 every class of $\{t\}^i$ is a subclass of $\{t\}$ or $\{t\}^{i+1}$. Thus by Theorem 2.3.4 we have that φ_{i+1} is a scalar multiple of $\varphi_i \circ \phi_i$. By induction φ_i is a scalar multiple of $\phi_0 \circ \cdots \circ \phi_{i-1}$ and so φ_{i+1} is a scalar multiple of $\phi_0 \circ \cdots \circ \phi_i$. □

4.1.2 The action of the twist groups

The twist group $S_{(\mu^i)^*}$ acts naturally on M^{λ^i} by permuting rows. Define an FG-homomorphism $\theta_{\mu^i} : M^{\lambda^i} \to M^{\lambda^i}$ by

$$\theta_{\mu^i} : \{t\} \mapsto \sum_{\pi \in S_{(\mu^i)^*}} \{t\}\pi$$

and let $H^{\lambda^i}_{\mu^i}$ denote the image of θ_{μ^i}. Recall the group A from Subsection 2.2. In studying how the twist group interacts with ϕ_0 it will be very useful to have a description of the action of S_{λ^*} in terms of the action of G.

LEMMA 4.1.4 *With the notation above we have*

$$\sum_{a \in A} \{t\} a = \sum_{\pi \in S_{\lambda^*}} \{t\}\pi \quad \textit{and}$$

$$\sum_{a \in A} \{t\}^i a = |S_{\lambda^*} : S_{(\mu^i)^*}| \sum_{\sigma \in S_{(\mu^i)^*}} \{t\}^i \sigma.$$

Proof: The jth row of $\{t\}a_\pi$ is the set $\{(t_{j,k})a_\pi\}_{k\geq 1} = \{t_{j\pi^{-1},k}\}_{k\geq 1}$. This is exactly the jth row of $\{t\}\pi$. Hence $\{t\}a_\pi = \{t\}\pi$. Hence $\sum_{\pi\in S_{\lambda^*}}\{t\}\pi = \sum_{a\in A}\{t\}a$. Second, write $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$. Then for some $s \leq r$ we have

$$S_{\lambda^*} = \times_{j=1}^{r} S_{m_j},$$
$$S_{\mu^{i*}} = \times_{j=1}^{s} S_{m_j}.$$

Thus we may write

$$S_{\lambda^*} = S_{\mu^{i*}} \times (\times_j^s S_{m_j}). \tag{9}$$

Let $\pi \in S_{(\mu^i)^*}$. The kth row of $\{t\}^i a_\pi$ is the set $\{(t_{j,k})a_\pi\}_{k>i} = \{t_{j\pi^{-1},k}\}_{k>i}$ which is the kth row of $\{t\}^i\pi$. Hence $\{t\}^1 a_\pi = \{t\}\pi$. To ease notation let $K = (\times_j^s S_{m_j})$. By definition, for $\sigma \in K$ we have $\{t\}_{j,k}\sigma = \{t\}_{j,k}$ if $j \leq s$. Hence the j row of $\{t\}^i\sigma$ is equal to the jth row of $\{t\}^i$. The kth column of $\{t\}^i a_\sigma$ is the set $\{(t_{(j)\sigma,k})\}_j$ which is the kth column of $\{t\}^i$. Thus a_σ fixes $\{t\}^i$. Hence we have

$$\sum_{a\in A}\{t\}^i a = \sum_{\pi\in S_{\lambda^*}}\{t\}^i a_\pi \tag{10}$$
$$= \sum_{\pi_1\in S_{\mu^*}}\sum_{\sigma\in K}\{t\}^i a_\sigma a_{\pi_1} \tag{11}$$
$$= |K| \sum_{\pi_1\in S_{\mu^*}}\{t\}^i a_{\pi_1} \tag{12}$$
$$= |K| \sum_{\pi_1\in S_{\mu^*}}\{t\}^i \pi_1 \tag{13}$$
$$= |S_{\lambda^*} : S_{(\mu^i)^*}| \sum_{\pi_1\in S_{\mu^*}}\{t\}^i \pi_1. \tag{14}$$

Above (10) follows from definition of the group A. Second (11) follows from (10) by (9). Third (12) follows from (11) as a_σ fixes $\{t\}^i$ for $\sigma \in K$. Fourth (13) follows from (12) by definition of the group A. Finally (14) follows from (13) by (9). $\square$

PROPOSITION 4.1.5 *Define* $\varphi_i : M^\lambda \to M^{\lambda^i}$ *by* $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^i g$. *Then the image* $(H^\lambda)\varphi_i$ *is a submodule of* $H_{\mu^i}^{\lambda^i}$.

Proof: Let $\sum_{\pi \in S_{\lambda^*}} \{t\}\pi \in H^\lambda$. Then

$$\left(\sum_{\pi \in S_{\lambda^*}} \{t\}\pi\right) \varphi_i = \left(\sum_{a \in A} \{t\}a\right) \varphi_i \tag{15}$$

$$= \sum_{a \in A} \sum_{g \in G_{\{t\}a}} \{t\}^i a g \tag{16}$$

$$= \sum_{a \in A} \sum_{g \in G_{\{t\}}} \{t\}^i a g \tag{17}$$

$$= \sum_{g \in G_{\{t\}}} \left(\sum_{a \in A} \{t\}^i a\right) g \tag{18}$$

$$= |S_{\lambda^*} : S_{(\mu^i)^*}| \cdot \sum_{g \in G_{\{t\}}} \sum_{\pi \in S_{\mu^*}} \{t\}^i \pi g. \tag{19}$$

Here (15) is true by Lemma 4.1.4. Secondly (16) follows from (15) by Lemma 2.3.2. To see that (17) follows from (16) note that we have $\{t\}^i a = \{t\}^i \pi$ for some $\pi \in S_{\lambda^*}$. Hence $G_{\{t\}a} = G_{\{t\}\pi} = G_{\{t\}}$. Fourthly (18) follows from (17) by swapping the order of the two sums. Fifthly (19) follows from (18) by Lemma 4.1.4. By Lemma 2.2.1 the actions of G and S_{λ^*} commute and so

$$\sum_{\pi \in S_{\mu^*}} \{t\}^i \pi g = \sum_{\pi \in S_{\mu^*}} \{t\}^i g \pi$$

is a basis element of $H_{\mu^i}^{\lambda^i}$. Hence the right hand side of (19) is an element of $H_{\mu^i}^{\lambda^i}$ and so

$$\left(\sum_{\pi \in S_{\lambda^*}} \{t\}^i \pi\right) \varphi_i$$

is an element of $H_{\mu^i}^{\lambda^i}$. □

4.1.3 Induction

In this section we use the results of Subsection 4.1.2 to prove an inductive result concerning the standard map. Let $\theta : M^{\lambda^1} \to M^{\lambda'}$ denote the map defined by $\{t\}^1\theta = \sum_{g \in G_{\{t\}^1}} \{t\}'g$.

PROPOSITION 4.1.6 *The standard map ψ_λ is a scalar multiple of the composition $\phi_0 \circ \theta$.*

Proof: Every row of $\{t\}^1$ is a subrow of $\{t\}$ and every column of $\{t\}^1$ is a column of $\{t\}'$. The result now follows from Theorem 2.3.4. □

We now introduce some notation. Let $\psi|$ and $\phi_0|$ denote the restriction of ψ and ϕ_0 respectively to H^λ. Similarly let $\theta|$ denote the restriction of θ to $H_{\mu^1}^{\lambda^1}$. Throughout this section we let μ denote the partition obtained by removing the left most column of λ.

LEMMA 4.1.7 *The map $\theta|$ is injective iff $\psi_\mu|$ is injective.*

Proof: Recall the construction of ψ_μ^{*G} from Section 2.5. Fix a λ-tabloid $\{t\}$ and let $\{t\}^1$ be as above. Define $\{t^X\}$ to be the μ-tabloid that consists of the rows of $\{t\}^1$ and let $\{t^Y\}$ be the tabloid that consists of the single column of $\{t\}^1$. Define the FG-isomorphism $\varphi : M^\lambda \to (M^\mu \otimes M^{(\lambda_1')})^G$ by $\{t\}^1 \mapsto (\{t^X\}, \{t^Y\})$. Then notice that we can obtain $\{t\}'$ by adding the columns of $\{t^X\}'$ to the right of $\{t^Y\}$. Hence define the FG-isomorphism $\varphi_2 : (M^{\mu'} \otimes M^{(\lambda_1')})^G \to M^{\lambda'}$ by $(\{t^X\}', \{t^Y\}) \mapsto \{t\}'$. Thus by Theorem 2.5.13 the map $\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2$ is a scalar multiple of the map $H^{\lambda^1} \to M^{\lambda'}$ obtained by $\sum_{\pi \in S_{\mu^*}} \{t\}^1\pi \mapsto \sum_{g \in G_{\{t\}^1\}}} \{t\}'g$. This is the definition of

$\theta|$. Hence $\theta|$ is a scalar multiple of $\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2$ which is injective iff $\psi_\mu|$ is injective by Corollary 2.5.14. $\square$

THEOREM 4.1.8 *Let λ be a partition and μ the partition obtained by removing the left-most column. Suppose that both ϕ_0 and $\psi_\mu|$ are injective. Then $\psi_\lambda|$ is injective.*

Proof: By Proposition 4.1.3 we have $\psi_\lambda = \phi_0 \circ \theta$. By Lemma 4.1.5 the image $(H^\lambda)\phi_0$ is a submodule of $H^{\lambda^1}_{\mu^1}$. Hence $\psi_\lambda| = \phi_0| \circ \theta|$. The map $\phi_0|$ is injective by hypothesis. By hypothesis the map $\psi_\mu|$ is injective and so by Lemma 4.1.7 this means the map $\theta|$ is injective. Hence $\psi_\lambda|$ is injective. $\square$

4.1.4 Good nodes and columns

Theorem 4.1.8 shows us that it is important to understand when the map ϕ_0 is injective. In this section we give such a criterion for the map ϕ_0 to be injective. We study the map ϕ_0 in the same way as we studied the standard map. We again define a sequence of tabloids $\{t\}^{(i)}$ which will act as intermediates between $\{t\}$ and $\{t\}^1$. We then use Theorem 2.3.4 to view the map ϕ_0 as a composition of maps $\phi_{(i)}$. The maps $\phi_{(i)}$ will turn out to be equal to the maps $\epsilon(\lambda^{(i)}, \lambda^{(i+1)})$ of Section 3.2, where $\lambda^{(i)}$ is the shape of $\{t\}^{(i)}$. Hence using Theorem 3.2.3 we will be able to prove the main result of this section, Theorem 4.1.15.

Let $\{t\}$ be a λ-tabloid. For $0 \leq j \leq \lambda'_1$ define $\{t\}^{(i)}$ inductively by letting $\{t\}^{(0)} = \{t\}$ and obtaining $\{t\}^{(i+1)}$ from $\{t\}^{(i)}$ by moving the smallest element from the $(\lambda'_1 - i)$th row into the unique column of $\{t\}^{(i)}$.

EXAMPLE 4.1.9 *Letting $\{t\}$ be the primary $(5^3, 2, 1)$-tabloid gives us*

$$\{t\}^{(0)} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & & & & \\ \cline{1-1} \end{array} = \{t\} \quad , \quad \{t\}^{(1)} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 9 & 8 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} \multicolumn{1}{|c|}{18} & & & & \end{array} \quad ,$$

$$\{t\}^{(2)} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline \multicolumn{1}{|c|}{16} & 17 & & & \\ \cline{2-2} \multicolumn{1}{|c|}{18} & & & & \end{array} , \quad \{t\}^{(3)} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline \multicolumn{1}{|c|}{11} & 12 & 13 & 14 & 15 \\ \cline{2-5} \multicolumn{1}{|c|}{16} & 17 & & & \\ \cline{2-2} \multicolumn{1}{|c|}{18} & & & & \end{array} ,$$

$$\{t\}^{(4)} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{|c|}{6} & 7 & 8 & 9 & 10 \\ \cline{2-5} \multicolumn{1}{|c|}{11} & 12 & 13 & 14 & 15 \\ \cline{2-5} \multicolumn{1}{|c|}{16} & 17 & & & \\ \cline{2-2} \multicolumn{1}{|c|}{18} & & & & \end{array} , \quad \{t\}^{(5)} = \begin{array}{|c|cccc} 1 & 2 & 3 & 4 & 5 \\ \cline{2-5} 6 & 7 & 8 & 9 & 10 \\ \cline{2-5} 11 & 12 & 13 & 14 & 15 \\ \cline{2-5} 16 & 17 & & & \\ \cline{2-2} 18 & & & & \end{array} = \{t\}^1 .$$

Lemma 4.1.10 *Every class of $\{t\}^{(i)}$ is a subclass of $\{t\}$ or $\{t\}^{(i+1)}$.*

Proof: Every row of $\{t\}^{(i)}$ is a subrow of $\{t\}$. Every column of $\{t\}^{(i)}$ is a subcolumn of $\{t\}^{(i+1)}$. □

Let $\{t\}$ be a λ-tabloid. Let $\lambda^{(i)}$ denote the shape of $\{t\}^{(i)}$. Then using Proposition 2.3.1 define a map $\phi_{(i)} : M^{\lambda^{(i)}} \to M^{\lambda^{(i+1)}}$ by

$$\{t\}^{(i)} \mapsto \sum_{g \in G_{\{t\}^{(i+1)}}} \{t\}^{(i)} g.$$

Proposition 4.1.11 *The map ϕ_0 is a scalar multiple of the composition of maps*

$$\phi_{(0)} \circ \phi_{(1)} \circ \cdots \circ \phi_{(\lambda'_1 - 1)}.$$

Proof: Define $\rho_i : M^\lambda \to M^{\lambda^{(i)}}$ by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^{(i)} g$. We induct on the hypothesis that ρ_i is a scalar multiple of the composition $\phi_{(0)} \circ \cdots \circ \phi_{(i-1)}$. When $i = 1$ we have $\rho_1 = \phi_{(0)}$. By Lemma 4.1.10 every class of $\{t\}^{(i)}$ is a subclass of $\{t\}$ or $\{t\}^{(i+1)}$. Hence by Theorem 2.3.4 we see that ρ_{i+1} is a scalar multiple of $\rho_i \circ \phi_{(i)}$. By induction ρ_i is a scalr multiple of $\phi_{(0)} \circ \cdots \circ \phi_{(i-1)}$ and so the result holds for ρ_{i+1}.□

Proposition 4.1.12 *The map $\phi_{(i)}$ is injective iff the length of the $(\lambda'_1 - i + 1)$th part of λ is greater than i.*

Proof: The map $\phi_{(i)}$ is defined by

$$\phi_{(i)} : \{t\}^{(i)} \mapsto \sum_{g \in G_{\{t\}^{(i)}}} \{t\}^{(i+1)}.$$

We obtain $\{t\}^{(i+1)}$ from $\{t\}^{(i)}$ by moving the smallest element of the $(\lambda_1' - i + 1)$th row into the column. Hence $\phi_{(i)}$ is equal to the map $\epsilon(\lambda^{(i)}, \lambda^{(i+1)})$ of Section 3.2. The result now follows from Theorem 3.2.3. □

For a partition λ recall the definition of the hook $h_{i,j}$ of $[\lambda]$ from Section 2.1. We say that the node $\lambda_{i,j}$ is *good* if the arm of the hook $h_{i,j}$ is at least as long as its leg. We say that a column of $[\lambda]$ is *good* if all the nodes in it are good.

EXAMPLE 4.1.13 *Let* $\lambda = (5^3, 2, 1)$. *Below we have replaced the node* $\lambda_{i,j}$ *in* $[\lambda]$ *with* $a_{i,j} - l_{i,j}$. *Thus the good nodes are exactly those whose entry is non-negative. Hence in this example the first three columns are good.*

$$\begin{array}{ccccc} 0 & 0 & 0 & -1 & -2 \\ 1 & 1 & 1 & 0 & -1 \\ 2 & 2 & 2 & 1 & 0 \\ 0 & 0 & & & \\ 0 & & & & \end{array}.$$

THEOREM 4.1.14 *Suppose that the left-most column of the Young diagram* $[\lambda]$ *is good. Then the homomorphism* ϕ_0 *is injective.*

Proof: To ease notation we let $r_i := \lambda_1' - i$. The number of elements in the r_ith row of $\{t\}^{(i)}$ is equal to the number of nodes right of $\lambda_{r_i,1}$ in the Young diagram of λ. The number of elements in the column of $\{t\}^{(i)}$ is the number of nodes below $\lambda_{r_i,1}$ in the Young diagram of λ. As the first column of λ is good there are strictly more elements in the r_ith row of $\{t\}^{(i)}$ than in the column of $\{t\}^{(i)}$. Hence by Theorem 4.1.12 the map $\phi_{(i)}$ is injective. Hence the composition below is injective

$$\phi_{(0)} \circ \phi_{(1)} \circ \cdots \circ \phi_{(\lambda_1')} = r\phi_0.$$

□

THEOREM 4.1.15 *Let λ be a partition whose left-most column is good. Let μ be the partition obtained by removing this column. Suppose $\psi_\mu|$ is injective. Then $\psi_\lambda|$ is injective.*

Proof: As the left most column is good the map ϕ_0 is injective by Theorem 4.1.14. The map $\psi_\mu|$ is injective by hypothesis. The map $\psi_\lambda|$ is now injective by Theorem 4.1.8. □

THEOREM 4.1.16 *Suppose that $\psi_{a^a}|$ is injective. Then $\psi_{b^a}|$ is injective for all b such that $b \geq a$. In particular Foulkes' Conjecture holds for these values of a and b.*

Proof: Let $a \leq b$. Let $\lambda_{i,j}$ be a node in the ith column. Then if $i \leq b - a$ there are at least a nodes to the right of $\lambda_{i,j}$ and at most a nodes below it. Hence the first $b - a$ columns of λ are good. The map $\psi_{a^a}|$ is injective by assumption. The result now follows from Theorem 4.1.15. □

Proposition 2.4.7 shows us that if $a \leq 4$ then the map $\psi_{a^a}|$ is injective. Thus we have the following confirmation of Foulkes' Conjecture for partitions with at most four parts:

THEOREM 4.1.17 *Let $a \leq b$ and $a \leq 4$. Then the map $\psi_{b^a}|$ is injective. In particular Foulkes' Conjecture holds for all $b \geq a$.*

□

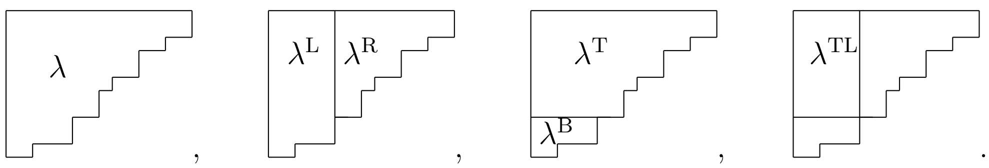
4.2 Block removal

In Section 4.1 we saw how we can add a column to a partition whilst preserving the injectivity of its standard map. In this section we stretch these ideas to their limit and show how to remove a block of c columns simultaneously. The ideas are similar to those in Section 4.1 and as a result we shall at times be terse in those proofs which copy ideas from Section 4.1 and advise the reader to skip this section on the first reading.

Let λ be a partition and let c be an integer such that the cth and $(c+1)$th columns of $[\lambda]$ have different lengths and suppose further that $\lambda_{p+1} = c$ and $\lambda_p > c$. We can now define five natural subpartitions $\lambda^L, \lambda^R, \lambda^T, \lambda^B, \lambda^{TL}$ of λ:

$$\begin{aligned}
\lambda^L &= (c^p, \lambda_{p+1}, \ldots, \lambda_r),\\
\lambda^R &= (\lambda_1 - c, \lambda_2 - c, \ldots, \lambda_p - c),\\
\lambda^T &= (\lambda_1, \lambda_2, \ldots, \lambda_p),\\
\lambda^B &= (\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_r),\\
\lambda^{TL} &= (c^p).
\end{aligned}$$

The partitions defined above can (and we believe should) be viewed pictorially as subdiagrams of $[\lambda]$.



As is Section 4.1 we will show that the standard map of λ is a scalar multiple of a composition of other maps. To define these other maps we need to cut a λ-tabloid into smaller tabloids. Let $\{t\}$ be a λ-tabloid. Define $\{t\}^T$ to be the tabloid that consists of the top p rows of $\{t\}$. We let $\{t\}^B$ be the λ^B-tabloid that consists of the bottom $(r-p)$ rows of $\{t\}$. Let $\{t\}^L$ denote the λ^L-tabloid whose jth row consists of the smallest c-many elements from the jth row of $\{t\}$ for each j. We let $\{t\}^R$ be the tabloid obtained from $\{t\}^T$ by removing the c-many smallest elements from each row.

Example 4.2.1 *Let* $\lambda = (5^3, 2^2)$. *Then* $\lambda^L = (2^5)$, $\lambda^R = (3^3)$, $\lambda^T = (5^3)$, $\lambda^B = (2^2)$ *and* $\lambda^{TL} = (2^3)$. *Further, if* $\{t\}$ *is the primary* λ*-tabloid we have:*

$$\{t\} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & 19 & & & \\ \cline{1-2} \end{array} ,$$

$$\{t\}^T = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline \end{array} , \quad \{t\}^B = \begin{array}{cc} \hline 16 & 17 \\ \hline 18 & 19 \\ \hline \end{array} ,$$

$$\{t\}^L = \begin{array}{cc} \hline 1 & 2 \\ \hline 6 & 7 \\ \hline 11 & 12 \\ \hline 16 & 17 \\ \hline 18 & 19 \\ \hline \end{array} , \qquad \{t\}^R = \begin{array}{ccc} \hline 3 & 4 & 5 \\ \hline 8 & 9 & 10 \\ \hline 13 & 14 & 15 \\ \hline \end{array} .$$

Given tabloids $\{s\}$ and $\{t\}$ we write $\{t\} \cup \{s\}$ to denote the tabloid obtained by adding the rows of $\{s\}$ to the bottom of those of $\{t\}$. For example in Example 4.2.1 we have $\{t\} = \{t\}^T \cup \{t\}^B$. As we did in Section 4.1.1 we define a sequence of tabloids $\{t\}^{[0]}, \{t\}^{[1]}, \{t\}^{[2]}, \{t\}^{[3]}$ by first letting $\{t\}^{[0]} := \{t\}$. Then let $\{t\}^{[1]} := \{t\}^T \cup (\{t\}^B)'$ and $\{t\}^{[2]} := \{t\}^R \cup (\{t\}^L)'$. Finally set $\{t\}^{[3]} := \{t\}'$.

Example 4.2.2 *Let $\lambda = (5^3, 2^2)$ and suppose $\{t\}$ is the primary λ-tabloid as in Example 4.2.1. Then we have:*

$$\{t\}^{[0]} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ \cline{1-2} 18 & 19 & & & \\ \cline{1-2} \end{array} , \quad \{t\}^{[1]} = \begin{array}{|c|c|ccc} \hline \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & 3 & 4 & 5 \\ \hline \multicolumn{1}{c}{6} & \multicolumn{1}{c}{7} & 8 & 9 & 10 \\ \hline \multicolumn{1}{c}{11} & \multicolumn{1}{c}{12} & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ 18 & 19 & & & \end{array} ,$$

$$\{t\}^{[2]} = \begin{array}{|c|c|ccc} \cline{3-5} 1 & 2 & 3 & 4 & 5 \\ \cline{3-5} 6 & 7 & 8 & 9 & 10 \\ \cline{3-5} 11 & 12 & 13 & 14 & 15 \\ \cline{3-5} 16 & 17 & & & \\ 18 & 19 & & & \end{array} , \quad \{t\}^{[3]} = \begin{array}{|c|c|c|c|c|} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & 19 & & & \end{array} .$$

For $i = 0, 1, 2$ let $\lambda^{[i]}$ denote the shape of the tabloid $\{t\}^{[i]}$. Define maps $\phi_{[i]} : M^{\lambda^{[i]}} \to M^{\lambda^{[i+1]}}$ by

$$\phi_{[i]} : \{t\}^{[i]} \mapsto \sum_{g \in G_{\{t\}^{[i]}}} \{t\}^{[i+1]} g.$$

LEMMA 4.2.3 *For $0 \leq i \leq 2$ every row or column of $\{t\}^{[i]}$ is a subrow or column of $\{t\}^{[0]}$ or $\{t\}^{[i+1]}$.*

Proof: First we have $\{t\}^{[1]} := \{t\}^T \cup (\{t\}^B)'$. Hence any row of $\{t\}^{[1]}$ is a row of $\{t\} = \{t\}^{[0]}$. The jth column of $(\{t\}^B)'$ consists of the j smallest element (if it exists) or each of the rows of $\{t\}^B$. The jth column of $(\{t\}^L)'$ consists of the jth smallest element of each row of $\{t\}$. As every row of $\{t\}^B$ is a row of $\{t\}$, this gives us that the jth column of $(\{t\}^B)'$ is a subcolumn of the jth column of $(\{t\}^L)'$. Hence any column of $\{t\}^{[1]}$ is a subcolumn of $\{t\}^{[2]}$. Now consider $\{t\}^{[2]} := \{t\}^R \cup (\{t\}^L)'$. The jth row of $\{t\}^R$ is obtained from the jth row of $\{t\}^B$ by removing the smallest λ_{i-1}-many elements. Hence the jth row of $\{t\}^R$ is a subrow of the jth row of $\{t\}^T$. Finally the jth column of $(\{t\}^L)'$ consists of the jth smallest (if it exists) element of each row of $\{t\}^L$. Now the jth column of $\{t\}'$ consists of the jth smallest element of each row of $\{t\}$. Hence as every row of $\{t\}^L$ is a subrow of $\{t\}$ we see that the jth column of $(\{t\}^L)'$ is a column of $\{t\}'$. Hence every row or column of $\{t\}^{[2]}$ is a subrow or subcolumn of $\{t\}^{[0]}$ or $\{t\}^{[3]}$. $\square$

PROPOSITION 4.2.4 *The map ψ_λ is a scalar multiple of the composition $\phi_{[0]} \circ \phi_{[1]} \circ \phi_{[2]}$.*

Proof: Define $\psi_i : M^\lambda \to M^{\lambda^{[i]}}$ by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^{[i]} g$. We induct on the hypothesis that ψ_i is a scalar multiple of the composition $\phi_{[0]} \circ \cdots \circ \phi_{[i-1]}$. When $i = 1$ we have $\psi_1 = \phi_{[0]}$. By Lemma 4.2.3 every class of $\{t\}^{[i]}$ is a subclass of $\{t\}$ or $\{t\}^{[i+1]}$. Hence by Theorem 2.3.4 we see that ψ_{i+1} is a scalar multiple of $\psi_i \circ \phi_{[i]}$. By induction ψ_i is a scalar multiple of $\phi_{[0]} \circ \cdots \circ \phi_{[i-1]}$ and so the result holds for ψ_{i+1}. $\square$

LEMMA 4.2.5 *(i) The map $\phi_{[0]}|$ is injective iff the map $\psi_{\lambda^B}|$ is injective.*

(ii) The map $\phi_{[2]}|$ is injective iff the map $\psi_{\lambda^R}|$ is injective.

Proof: (*i*) Define an FG-isomorphism $\varphi_1 : M^\lambda \to (M^{\lambda^B} \otimes M^{\lambda^T})^G$ by $\{t\} \mapsto \{t\}^B \otimes \{t\}^T$. Notice that we obtain $\{t\}^{[1]}$ by adding the columns of $(\{t\}^B)'$ to the bottom of $\{t\}^T$ and so we can define an FG-isomorphism $\varphi_2 : (M^{(\lambda^B)'} \otimes M^{\lambda^T})^G \to M^{\lambda^1}$ by $(\{t\}^B)' \otimes \{t\}^T \mapsto \{t\}^{[1]}$. Hence by Theorem 2.5.13 the map $\varphi_1| \circ \psi_{\lambda^B}|^{*G} \circ \varphi_2$ is a scalar multiple of the map $H^\lambda \to M^{\lambda^{[1]}}$ given by

$$\sum_{\pi \in S_{\lambda^B *}} \{t\}\pi \mapsto \sum_{g \in G_{\{t\}}} \{t\}^{[1]} g.$$

This is the definition of $\phi_{[0]}|$ and so the result follows from Corollary 2.5.9

(ii) Define the FG-isomorphism $\varphi_1 : M^{\lambda^{[2]}} \to (M^{\lambda^L} \otimes M^{\lambda^R})^G$ by $\{t\}^{[2]} \mapsto \{t\}^L \otimes \{t\}^R$. Then notice that $\{t\}'$ can be obtained by adding the columns of $(\{t\}^R)'$ to the right of $\{t\}^L$ and thus by Theorem 2.5.13 then map $\varphi_1| \circ \psi_{\lambda^R}|^{*G} \circ \varphi_2$ is a scalar multiple of the map $H^{\lambda^{[2]}} \to M^{\lambda'}$ given by

$$\sum_{\pi \in S_{\lambda^R *}} \{t\}\pi \mapsto \sum_{g \in G_{\{t\}}} \{t\}' g.$$

Again this is the definition of $\phi_2|$ and so once more the result follows from Corollary 2.5.9. □

Theorem 4.2.6 *Suppose that the maps $\psi_{\lambda^R}|$, $\psi_{\lambda^B}|$ and $\phi_{[1]}$ are all injective. Then ψ_λ is injective.*

Proof: By Proposition 4.2.4 we have $\psi_\lambda = \phi_{[0]} \circ \phi_{[1]} \circ \phi_{[2]}$. Hence $\psi_\lambda|$ is a scalar multiple of

$$(\phi_{[0]} \circ \phi_{[1]} \circ \phi_{[2]})| = (\phi_{[0]}| \circ \phi_{[1]}) \circ \phi_{[2]}.$$

By Proposition 4.1.5 we have $(H^\lambda)\phi_{[0]} \circ \phi_{[1]} \subseteq (H^{\lambda^R} \otimes M^{\lambda^L})^G$. Hence

$$(\phi_{[0]}| \circ \phi_{[1]}) \circ \phi_{[2]} = (\phi_{[0]}| \circ \phi_{[1]}) \circ \phi_{[2]}|.$$

By hypothesis $\psi_{\lambda^R}|$ and $\psi_{\lambda^L}|$ are injective and so by Lemma 4.2.5 the maps $\phi_{[0]}|$ and $\phi_{[2]}|$ are injective. By hypothesis the map $\phi_{[1]}$ is injective. Hence $\psi_\lambda|$ is a scalar multiple of a composition of three injective maps and so injective. □

4.2.1 Two more sequences of tabloids

Theorem 4.2.6 shows that if the standard maps of λ^R and λ^B are injective then to show that the standard map $\psi_\lambda|$ is injective it suffices to show that the map ϕ_1 is injective. In this subsection we address this issue.

For $0 \leq i \leq \lambda_{i-1}$ define tabloids $\{t\}^{[1.i]}$ inductively by letting $\{t\}^{[1.0]} := \{t\}^{[1]}$ and obtaining $\{t\}^{[1.i+1]}$ from $\{t\}^{[1.i]}$ by moving the smallest element from each row of $\{t\}^{[1.i]}$ into the ith column.

Example 4.2.7 *Letting $\lambda = (5^3, 2^2)$ and $\{t\}^{[1]}$ as in Example 4.2.2 gives us the following tabloids;*

$$\{t\}^{[1.0]} = \begin{array}{|cc|ccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline 16 & 17 & & & \\ 18 & 19 & & & \end{array} = \{t\}^{[1]} \quad , \quad \{t\}^{[1.1]} = \begin{array}{|c|c|cccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & 19 & & & \end{array} ,$$

$$\{t\}^{[1.2]} = \begin{array}{|c|c|ccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & & & \\ 18 & 19 & & & \end{array} = \{t\}^{[2]} \quad .$$

Let $\lambda^{[1.i]}$ denote the shape of the tabloid $\{t\}^{[1.i]}$ and define maps $\phi_{[1.i]} : M^{\lambda^{[1.i]}} \to M^{\lambda^{[1.i+1]}}$ by

$$\phi_{[1.i]} : \{t\}^{[1.i]} \mapsto \sum_{g \in G_{\{t\}^{[1.i]}}} \{t\}^{[1.i+1]} g.$$

Again the combinatorial significance of the tabloids $\{t\}^{[1.i]}$ lies in the following lemma from which Proposition 4.2.9 follows by Theorem 2.3.4.

Lemma 4.2.8 *Every row of $\{t\}^{[1.i]}$ is a subrow of either $\{t\}^{[1.0]}$ or $\{t\}^{[1.i+1]}$.*

Proof: The jth row of $\{t\}^{[1.i]}$ is obtained from the jth row of $\{t\}^{[1.0]}$ by removing the i smallest elements. Hence every row of $\{t\}^{[1.i]}$ is a subrow of $\{t\}^{[1.0]}$. The jth column of $\{t\}^{[1.i]}$ is equal to the jth column of $\{t\}^{[1.i+1]}$ if $j \neq i$. The ith column of

$\{t\}^{[1.i+1]}$ is obtained from the ith column of $\{t\}^{[1.i]}$ by adding the smallest element from each row of $\{t\}^{[1.i]}$. Hence every column of $\{t\}^{[1.i]}$ is a subcolumn of $\{t\}^{[1.i+1]}$.

□

PROPOSITION 4.2.9 *The map $\phi_{[1]}$ is a scalar multiple of the composition $\phi_{[1.1]} \circ \phi_{[1.2]} \circ \cdots \circ \phi_{[1.r]}$.*

Proof: Define $\rho_i : M^\lambda \to M^{\lambda^{[1.i]}}$ by $\{t\} \mapsto \sum_{g \in G_{\{t\}^{[1]}}} \{t\}^{[1.i]} g$. We induct on the hypothesis that ρ_i is a scalar multiple of the composition $\phi_{[1.0]} \circ \cdots \circ \phi_{[1.i-1]}$. When $i = 1$ we have $\rho_1 = \phi_{[1.0]}$. By Lemma 4.2.8 every class of $\{t\}^{[1.i]}$ is a subclass of $\{t\}^{[1]}$ or $\{t\}^{[1.i+1]}$. Hence by Theorem 2.3.4 we see that ρ_{i+1} is a scalar multiple of $\rho_i \circ \phi_{[i]}$. By induction ρ_i is a scalar multiple of $\phi_{[0]} \circ \cdots \circ \phi_{[i-1]}$ and so the result holds for ρ_{i+1}. □

Define tabloids $\{t\}^{[1.i.j]}$ by letting $\{t\}^{[1.i.0]} = \{t\}^{[1.i]}$ and obtaining $\{t\}^{[1.i.j+1]}$ from $\{t\}^{[1.i.j]}$ by moving the smallest element of the $(j+1)$st lowest row into the ith column.

EXAMPLE 4.2.10 *Let $\lambda = (5^3, 2)$ and suppose $\{t\}^{[1.1]}$ is as in Example 4.2.7. Then we have:*

$$\{t\}^{[1.1.0]} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ \hline \multicolumn{1}{|c|}{16} & \multicolumn{1}{c|}{17} & & & \\ \multicolumn{1}{|c|}{18} & \multicolumn{1}{c|}{19} & & & \end{array} = \{t\}^{[1.1]} \quad , \quad \{t\}^{[1.1.1]} = \begin{array}{c|cccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & 9 & 10 \\ \hline \multicolumn{1}{|c|}{11} & 12 & 13 & 14 & 15 \\ \cline{2-5} \multicolumn{1}{|c|}{16} & \multicolumn{1}{c|}{17} & & & \\ \multicolumn{1}{|c|}{18} & \multicolumn{1}{c|}{19} & & & \end{array} ,$$

$$\{t\}^{[1.1.2]} = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{|c|}{6} & 7 & 8 & 9 & 10 \\ \cline{2-5} \multicolumn{1}{|c|}{11} & 12 & 13 & 14 & 15 \\ \cline{2-5} \multicolumn{1}{|c|}{16} & \multicolumn{1}{c|}{17} & & & \\ \multicolumn{1}{|c|}{18} & \multicolumn{1}{c|}{19} & & & \end{array} , \qquad \{t\}^{[1.1.3]} = \begin{array}{|c|cccc} \cline{2-5} 1 & 2 & 3 & 4 & 5 \\ \cline{2-5} 6 & 7 & 8 & 9 & 10 \\ \cline{2-5} 11 & 12 & 13 & 14 & 15 \\ \cline{2-5} 16 & \multicolumn{1}{c|}{17} & & & \\ 18 & \multicolumn{1}{c|}{19} & & & \end{array} = \{t\}^{[1.2.0]} \quad .$$

Now define maps $\phi_{[1.i.j]} : M^{[1.i.j]} \to M^{[1.i.j]}$ by

$$\phi_{[1.i.j]} : \{t\}^{[1.i.j]} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^{[1.i.j]} g.$$

Lemma 4.2.11 *Every row of* $\{t\}^{[1.i.j]}$ *is a subrow of either* $\{t\}^{[1.i]}$ *or* $\{t\}^{[1.i.j+1]}$.

Proof: If the kth row of $\{t\}^{[1.i.j]}$ is not the $[(\lambda^T)'_i - j]$th row then the kth row of $\{t\}^{[1.i.j]}$ is equal to the kth row of $\{t\}^{[1.i]}$. The $[(\lambda^T)'_i - j]$th row of $\{t\}^{[1.i.j]}$ is obtained from the kth row of $\{t\}^{[1.i]}$ by removing the smallest element. Hence every row of $\{t\}^{[1.i.j]}$ is a subrow of $\{t\}^{[1.i]}$. If $k \neq j$ the kth column of $\{t\}^{[1.i.j+1]}$ is equal to the kth column of $\{t\}^{[1.i.j]}$. The ith column of $\{t\}^{[1.i.j+1]}$ is obtained from the ith column of $\{t\}^{[1.i.j]}$ by adding the smallest element from the last row of $\{t\}^{[1.i.j]}$. Hence every column of $\{t\}^{[1.i.j]}$ is a subcolumn of $\{t\}^{[1.i.j+1]}$. □

Proposition 4.2.12 *The map* $\phi_{[1.i]}$ *is a scalar multiple of the composition*

$$\phi_{[1.i.0]} \circ \phi_{[1.i.1]} \circ \cdots \circ \phi_{[1.i.r]}.$$

Proof: Define $\rho_j : M^{\lambda^{[1.i]}} \to M^{\lambda^{[1.i.j]}}$ by $\{t\}^{[1.i]} \mapsto \sum_{g \in G_{\{t\}^{[1.i]}}} \{t\}^{[1.i.j]} g$. We induct on the hypothesis that ρ_i is a scalar multiple of the composition $\phi_{[1.0]} \circ \cdots \circ \phi_{1.i-1]}$. When $i = 1$ we have $\rho_1 = \phi_{[1.i.0]}$. By Lemma 4.2.11 every class of $\{t\}^{[1.i.j]}$ is a subclass of $\{t\}^{[1.i]}$ or $\{t\}^{[1.i.j+1]}$. Hence by Theorem 2.3.4 we see that ρ_{i+1} is a scalar multiple of $\rho_i \circ \phi_{[1.i.j])}$. By induction ρ_i is a scalar multiple of $\phi_{[1.0]} \circ \cdots \circ \phi_{[1.i-1]}$ and so the result holds for ρ_{i+1}. □

Proposition 4.2.13 *The map* $\phi_{[1.i.j]}$ *is injective iff the arm of the hook* $h_{i,j}$ *is at least as long as its leg.*

Proof: The map $\phi_{[1.i.j]}$ is defined by

$$\phi_{[1.i.j]} : \{t\}^{[1.i.j]} \mapsto \sum_{g \in G_{\{t\}^{[1.i.j]}}} \{t\}^{[1.i.j+1]}.$$

We obtain $\{t\}^{[1.i.j+1]}$ from $\{t\}^{[1.i.j]}$ by moving the smallest element of the last row into the ith column. Hence $\phi_{[1.i.j]}$ is equal to the map $\epsilon(\lambda^{[1.i.j]}, \lambda^{[1.i.j+1]})$ of Section 3.2. The result now follows from Theorem 3.2.3. □

Recall that we say the node $\lambda_{i,j}$ is *good* if the arm of the hook $h_{i,j}$ is at least as long as its leg. We say that λ^{TL} is *good* if all its nodes are good (as nodes of λ).

LEMMA 4.2.14 *Let λ^{TL} be good. Then the map ϕ_1 is injective.*

Proof: Every node in λ^{TL} is good and so every map $\phi_{[1.i.j]}$ is injective by Proposition 4.2.13. Hence by Proposition 4.2.12 each map $\phi_{[1.i]}$ is injective. Thus by Proposition 4.2.9 the map $\phi_{[1]}$ is injective. □

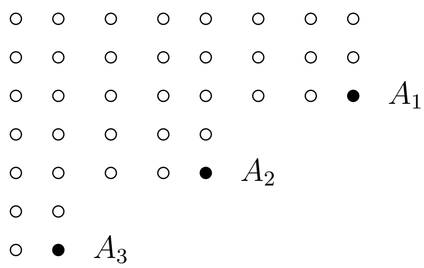
THEOREM 4.2.15 *Let the standard maps of $\lambda^R|$ and $\lambda^B|$ be injective and let λ^{TL} be good. Then the standard map of λ is injective.*

Proof: As λ^{TL} is good by Lemma 4.2.14 the map $\phi_{[1]}$ is injective. Hence the result follows from Theorem 4.2.6. □

4.2.2 Very good and extremely good partitions

In this section we introduce very good and extremely good partitions. We then use the results of Section 4.2 to prove that the standard maps of extremely good partitions are injective. We begin as always with some definitions. A node of the Young diagram of $[\lambda]$ is *removable* if there is no node immediately right or immediately below it. Let A_i denote the ith highest removable node.

EXAMPLE 4.2.16 *Let $\lambda = (8^3, 5^2, 2^2)$. The Young diagram $[\lambda]$ has three removable nodes. They are labelled A_1, A_2, A_3 and coloured black.*



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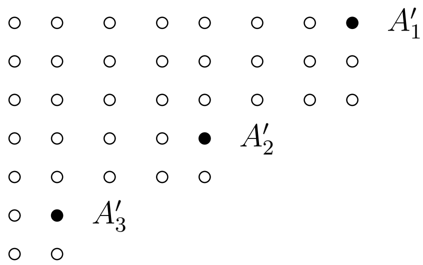
Let $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$ be a partition with the λ_i distinct. Let A_i' denote the rightmost node in the first row of length λ_i in $[\lambda]$.

Example 4.2.17 *Let $\lambda = (8^3, 5^2, 2^2)$. The nodes A_1', A_2' and A_3' are labelled below and coloured black.*



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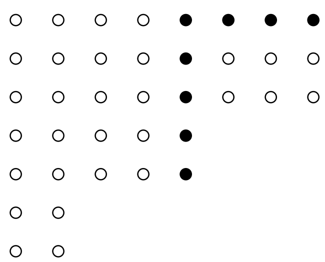
Let $i < j$ and define $B(i,j)$ to be the unique hook whose arm contains the node A_i' and whose leg includes the node A_j (We do mean A_j here). The hook $B(i,j)$ is *good* if its arm is at least as long as its leg. The partition λ is *very good* if for each i, j the hook $B(i,j)$ is good. Note that a partition is very good iff for all j the hooks $B(1,j)$ are good. We adopt the convention that the partitions (b^a) are very good since they have no hooks $B(i,j)$. We say that λ is *extremely good* if it is very good and for $1 \leq i \leq r-1$ the standard map of $(\lambda_i - \lambda_{i+1})^{m_i}$ is injective and the standard map of $(\lambda_r^{m_r})$ is injective.

Example 4.2.18 *Let $\lambda = (8^3, 5^2, 2^2)$. On the left is the hook $B(1,2)$ and on the right $B(1,3)$.*



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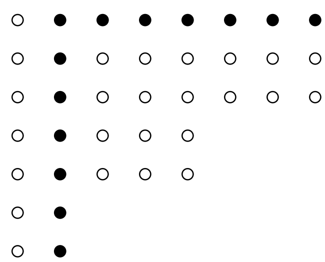

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Lemma 4.2.19 *Let λ be very good. Then all the nodes of λ^{TL} are good.*

Proof: Consider the node $\lambda_{x,y}$ that lies in λ^{TL}. Suppose that there exists a node λ_{x_1,y_1} such that $x_1 \leq x$ and $y_1 \geq y$. Then it is easy to see that if $h_{x,y}$ is good then h_{x_1,y_1} is good. Now let (x,y) lie in a row of length λ_i and column of length λ_p'. Then as $\lambda_{x,y}$ lies in λ^{TL} we have $i \leq s$. It is easy to see that there exists a unique removable node that lies in a column of length λ_p', namely A_{r-p}. Further as $\lambda_{x,y}$ lies in λ^{TL} we see that A_{r-p} lies in λ^B and so $r-p>i$. Hence it makes sense to consider the hook $B(i,r-p)$. Let λ_{x_1,y_1} denote the node which at the arm and leg of $B(i,j)$ meet. Then (x_1,y_1) lies in the first row of length λ_i and rightmost column of length λ'_p. Hence $x_1 \leq x$ and $y_1 \geq y$. □

LEMMA 4.2.20 *Let* $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$ *be extremely good. Then* λ^R *is extremely good.*

Proof: Let $B_\lambda(i,j)$ and $B_{\lambda^R}(i,j)$ denote the hooks $B(i,j)$ in $[\lambda]$ and $[\lambda^R]$ respectively. Then clearly $B_\lambda(i,j) = B_{\lambda^R}(i,j)$. As λ is extremely good, for each $i \leq r-2$ the standard map of $(\lambda_i - \lambda_{i+1})^{m_i}$ is injective and the standard map of $(\lambda_{r-1} - \lambda_r)^{m_{r-1}}$ is injective. Hence λ^R is very good. □

THEOREM 4.2.21 *Let* λ *be extremely good. Then the standard map of* λ *is injective.*

Proof: Proceed by induction on r. If $r=1$ then $\lambda = (\lambda_r^{m_r})$ and an extremely good partition of this shape is injective by definition. By induction the standard map of λ^R is injective. As λ is extremely good the standard map of λ^B is injective and by Lemma 4.2.19 all nodes in λ^{TL} are good. Hence the standard map of λ is injective by Theorem 4.2.15. □

COROLLARY 4.2.22 *Suppose all the parts of* λ *are distinct. Then the standard map of* λ *is injective.*

Proof: If all the parts of λ are distinct then the hooks $B(i,j)$ all have leg length one and arm length at least one. □

REMARK 4.2.23 *Note that Theorem 4.2.15 is a much more useful result than Theorem 4.2.21. In particular there are many partitions (e.g.* $(8,8,7,3,3,3)$*) that are good but not extremely and whose standard map one can show to be injective using Theorem 4.2.15.*

Chapter 5

Non-injective standard maps

In [23] Pylyavskyy showed that some partitions have a non-injective standard map. In [28] Sivek showed that if μ had a non-injective standard map then inserting rows into μ produced another partition whose standard map was not injective. In Section 5.1 we reprove Sivek's results again using the techniques of Chapter 2 and 4. Secondly in Section 5.2 we use Sivek's lemma to obtain a strong necessary condition for $\psi_\lambda|$ to be injective. In Section 5.3 we look at the relationship between the standard map of λ and those of the branches of λ.

5.1 The results of Sivek

In [28] Sivek shows that if a partition μ admits a non-injective standard map then inserting a row into μ yields a partition whose standard map is not injective. Sivek then uses this result to show that every partition can be embedded into a partition whose standard map is not injective. In this section we use the tools we have developed so far to give new proofs of these results.

Suppose $\mu = (\mu_1, \dots, \mu_s)$ is a partition of m. For $n > m$ let λ be the partition of n obtained from μ by adding the part $(n - m)$ to μ and then rearranging the parts so they weakly decrease. For a μ-tabloid $\{t\}$ let $\{t\}^+$ be the λ-tabloid obtained from $\{t\}$ by adding the row $\{m + 1, m + 2, \dots, n\}$ to $\{t\}$ in such a way that the lengths of the rows of $\{t\}^+$ weakly decrease and this new row is the highest of length $(n-m)$. Similarly for $v \in M^\mu$ let $v^+ \in M^\lambda$ be the vector obtained by adding the row $\{m+1, m+2, \ldots, n\}$ to every tabloid involved in v.

Example 5.1.1 *Let $n = 10$ and $m = 5$ with $\mu = (2, 1^3)$. Then $\lambda = (5, 2, 1^3)$ and below we have $\{t\}$ and $\{t\}^+$.*

$$\{t\} = \begin{array}{l} \hline 1\ \ 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}\ , \qquad \{t\}^+ = \begin{array}{l} \hline 6\ \ 7\ \ 8\ \ 9\ \ 10 \\ \hline 1\ \ 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}\ .$$

Let η be the composition $(n-m) \cup \mu'$ and let $(\{t\}')^+$ be the η-tabloid obtained by adding the row $\{m+1, m+2, \ldots, n\}$ to the top of $\{t\}'$.

Example 5.1.2 *Again let $\mu = (2, 1^3)$, $n = 10$ and let $\{t\}$ be as in Example 5.1.1. Then $\mu' = (4, 1), \eta = (5, 4, 1)$ and we have:*

$$\{t\}^+ = \begin{array}{l} \hline 6\ \ 7\ \ 8\ \ 9\ \ 10 \\ \hline 1\ \ 2 \\ \hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}\ , \qquad (\{t\}')^+ = \begin{array}{l} \hline 6\ \ 7\ \ 8\ \ 9\ \ 10 \\ \hline \begin{array}{|c|c|} 1 & 2 \\ 3 & \\ 4 & \\ 5 & \end{array} \end{array}\ ,$$

$$(\{t\}^+)' = \begin{array}{|c|c|c|c|c|} 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & & & \\ 3 & & & & \\ 4 & & & & \\ 5 & & & & \end{array}\ .$$

Define two maps $\phi : M^\lambda \to M^\eta$ and $\theta : M^\eta \to M^{\lambda'}$ by

$$\phi : \{t\}^+ \mapsto \sum_{g \in G_{\{t\}^+}} (\{t\}')^+,$$

$$\theta : (\{t\}')^+ \mapsto \sum_{g \in G_{\{t\}'^+}} (\{t\}^+)'.$$

Notice that the row $\{n+1, \ldots, n+m\}$ of $(\{t\}')^+$ is equal to a row of $\{t\}^+$ and every column of $(\{t\}')^+$ is a sub-column of a column of $\{t\}'$. Thus Theorem 2.3.4 gives us:

LEMMA 5.1.3 *The standard map ψ_λ is a scalar multiple of $\phi \circ \theta$.*

The twist group S_{μ^*} is a subgroup of S_{λ^*}. Define the submodule H^λ_μ to be the subspace of M^λ spanned by the sums $\sum_{\pi \in S_{\mu^*}} \{t\}\pi$. Let $\phi|_{H \times M}$ be the restriction of ϕ to H^λ_μ and let $\psi_\lambda|_H$ denote the restriction of ψ_λ to H^λ.

LEMMA 5.1.4 *There exists an FG-isomorphism $\varphi_1^{-1} : (ker(\psi_\mu|_H) \times M^{(n-m)})^G \rightarrow ker(\phi|_{H \times M})$ such that $(v, \{t^Y\})\varphi_1^{-1} = v^+$ where $\{t^Y\}$ is the one part tabloid $\{m+1, \ldots n\}$.*

Proof: First define the FG-isomorphism $\varphi_1 : M^\lambda \rightarrow (M^\mu \times M^{(n-m)})^G$ by $\{t\}^+ \mapsto (\{t^X\}, \{t^Y\})$ where $\{t^X\}$ is the μ-tabloid obtained by removing the top row of length $(n-m)$ from $\{t\}$ and $\{t^Y\}$ is the tabloid that consists of this row. Now notice that we can obtain $(\{t\}')^+$ by adding the columns of $\{t^X\}'$ to the bottom of $\{t^Y\}$. Hence we can define the FG-isomorphism $\varphi_2 : (M^{\mu'} \times M^{(n-m)})^G \rightarrow M^\eta$ by $(\{t^X\}', \{t^Y\}) \mapsto (\{t\}')^+$. Thus by Theorem 2.5.13 the map $(\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2)|_{H \times M}$ is a scalar multiple of the map $H^\lambda_\mu \rightarrow M^\eta$ given by

$$\sum_{\pi \in S_{\mu^*}} \{t\}^+\pi \mapsto \sum_{g \in G_{\{t\}^+}} (\{t\}')^+ g.$$

Hence $(\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2)_{H \times M}$ is a scalar multiple of $\phi|_{H \times M}$. Thus by Corollary 2.5.9 we have the isomorphism $\varphi^{-1} : (ker(\psi_\mu|^{*G}) \times M^{(n-m)})^G \rightarrow ker(\phi|_{H \times M})$ given by $(\{t^X\}, \{t^Y\}) \mapsto \{t_\lambda\}^+$. Hence if $v \in ker(\psi_\mu|^{*G})$ then we have

$$(v, \{t^Y\})\varphi_1^{-1} = v^+ \in ker(\varphi_1 \circ \psi_\mu|^{*G} \circ \varphi_2) = ker(\phi|_{H \times M}).$$

$\square$

LEMMA 5.1.5 *Let $0 \neq v \in H^\mu$. Then $\sum_{\pi \in S_{\lambda^*}} v^+\pi \neq 0$.*

Proof: First suppose that no part of μ is equal to $(n-m)$. Then $S_{\lambda^*} = S_{\mu^*}$ and so $\sum_{\pi \in S_{\lambda^*}} v = |S_{\lambda^*}|v$. So suppose that a part of μ has length $(n-m)$. Without loss assume $\mu_1 = (n-m)$. Thus the row $\{m+1, \ldots, n\}$ appears as the top row in every tabloid that appears in v^+. Let σ_j be a set of coset representatives of S_{μ^*} in S_{λ^*}. Then

$$\sum_{\pi \in S_{\lambda^*}} v^+\pi = |S_{\mu^*}| \sum_i v^+\sigma_i.$$

Let $v^+ = \sum_j a_j\{t_j\}$ with $a_1 \neq 0$. For a contradiction suppose $\sum_i v^+\sigma_i = 0$. Then for $\sigma_j \neq 1$ we have $\{t_1\}\sigma_j = \{t_i\}$ for some i with $a_i \neq 0$. But then $\{m+1, \ldots, n\}$ appears as a row other than the first in $\{t_i\}$. This is a contradiction as we have assumed the row $\{m+1, \ldots, n\}$ is the first row in each tabloid that appears in v^+. Thus $\sum_{\pi \in S_{\lambda^*}} v^+\pi \neq 0$. □

LEMMA 5.1.6 *Let $v \in ker(\psi_\lambda)$ and $\pi \in S_{\lambda^*}$. Then $v\pi \in ker(\psi_\lambda)$.*

Proof: Let $v = \sum_{g \in G} a_g\{t\}g \in ker(\psi_\lambda)$. Then we have

$$\begin{aligned} \{t\}g\pi\psi_\lambda &= \{t\}\pi g\psi_\lambda &(20)\\ &= \{t\}\pi\psi_\lambda g &(21)\\ &= \{t\}\psi_\lambda g &(22)\\ &= \{t\}g\psi_\lambda. &(23) \end{aligned}$$

Above (20) follows as the actions of S_{λ^*} and G commute. Second (21) follows from (20) as ψ_λ is an FG-homomorphism. Third (22) follows from (21) by Lemma 2.4.4.

Finally (23) follows from (22) again as ψ_λ is an FG-homomorphism. Hence

$$(v\pi)\psi_\lambda = \left(\sum_{g\in G} a_g\{t\}g\pi\right)\psi_\lambda \tag{24}$$
$$= \sum_{g\in G} a_g(\{t\}g\pi\psi_\lambda) \tag{25}$$
$$= \sum_{g\in G} a_g(\{t\}g\psi_\lambda) \tag{26}$$
$$= \left(\sum_{g\in G} a_g\{t\}g\right)\psi_\lambda \tag{27}$$
$$= 0. \tag{28}$$

Above (24) follows by definition of v. Second (25) follows from (24) by linearity of the standard map. Third (26) follows from (25) by (23). Fourth (27) follows from (26) by linearity of ψ_λ. Finally (28) follows from (27) as $v \in ker(\psi_\lambda)$ □

The main result of this section is the following theorem which first appeared as Lemma 2.1 in Sivek's Paper [28].

THEOREM 5.1.7 (SIVEK'S LEMMA) *Let λ be a partition obtained by inserting a part into μ. Suppose that $\psi_\mu|_{H^\mu}$ has a nonzero kernel. Then $\psi_\lambda|_{H^\lambda}$ has a nonzero kernel.*

Proof: Let $0 \neq v \in ker(\psi_\mu|_{H^\mu})$. Then by Lemma 5.1.4 we have $v^+ \in ker(\phi|_{H\times M})$. By Lemma 5.1.3 we have $\psi_\lambda|_{H\times M}$ is a scalar multiple of $\phi|_{H\times M} \circ \theta$. Hence $v^+ \in ker(\psi_\lambda|_{H\times M})$. By Lemma 5.1.6, for each twist element π we have $v^+\pi \in ker(\psi_\lambda|_{H\times M})$. Hence

$$\sum_{\pi\in S_{\lambda^*}} v^+\pi \in ker(\psi_\lambda|_{H\times M}).$$

As $\sum_{\pi\in S_{\lambda^*}} \pi$ projects M^λ onto H^λ this gives

$$\sum_{\pi\in S_{\lambda^*}} v^+\pi \in ker(\psi_\lambda|_{H^\lambda}).$$

Finally by Lemma 5.1.5 the element $\sum_{\pi \in S_{\lambda^*}} v^+\pi \neq 0$. Hence the kernel of $\psi_\lambda|_{H^\lambda}$ is non-zero. □

Remark 5.1.8 *Note that if μ does not have a part of length $n - m$ then the proof of Sivek's Lemma becomes much simpler as we no longer need to project the kernel of $\phi|_{H \times M}$ onto H^λ. In fact in this case the result follows almost immediately from Lemma 5.1.4.*

Definition 5.1.9 *A node of a Young diagram is bad if it is not good and has a node to its right.*

Lemma 5.1.10 *Let λ dominate λ'. Suppose that a partition λ has a bad node in the first column of $[\lambda]$. Then $\psi_\lambda|_{H^\lambda}$ is not injective.*

Proof: As $\lambda \geq \lambda'$ we have $\lambda_1 \geq (\lambda')_1$. Hence the bad node $\lambda_{i,j}$ is not in a row of length λ_1. Let μ be the partition obtained from λ by removing the rows of λ so that $\lambda_{i,j}$ is in the top row μ. Then as $\mu_{1,1}$ is a bad node we have $\mu_1 < \mu'_1$ and so $\mu \ngeq \mu'$. Thus $\psi_\mu|_{H^\mu}$ has a non-zero kernel. Hence $\psi_\lambda|_{H^\lambda}$ has a non-zero kernel by Sivek's Lemma 5.1.7. □

We can now prove a second result that appeared in Sivek's paper [28].

Theorem 5.1.11 ([28], Theorem 2.3) *Let μ be a partition that dominates its conjugate. Then there exists a partition λ obtained from μ by adding at most one row and/or one column to μ that also dominates its conjugate and is such that $\psi_\lambda|_{H^\lambda}$ is not injective.*

Proof: Let μ have r parts. Add a column of length $\mu_r + r + 1$ to the front of μ. Call this new partition ν. Add a row of length $\mu_r + r + 1$ to the top of ν and call this partition λ. It is easy to see that λ dominates its conjugate and that the node $\lambda_{2,r}$ is bad. Hence by Lemma 5.1.10 the standard map of λ is not injective. □

5.2 Good hooks

Recall the definition of the removable nodes A_i that we made in Subsection 4.2.2. Let $i < j$ and define $A(i,j)$ to be the unique hook whose arm contains the node A_i and whose leg includes the node A_j.

EXAMPLE 5.2.1 *Let $\lambda = (7^3, 4^2, 2^2)$. On the left is the hook $A(1,2)$ and on the right $A(1,3)$.*

$$\begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \bullet & \bullet & \bullet & \bullet \\
\circ & \circ & \circ & \bullet & & & \\
\circ & \circ & \circ & \bullet & & & \\
\circ & \circ & & & & & \\
\circ & \circ & & & & &
\end{array} \quad , \quad
\begin{array}{ccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\circ & \bullet & \circ & \circ & & & \\
\circ & \bullet & \circ & \circ & & & \\
\circ & \bullet & & & & & \\
\circ & \bullet & & & & &
\end{array} \quad .$$

We denote the arm and leg lengths of $A(i,j)$ by $a(i,j)$ and $l(i,j)$ respectively. We say that $A(i,j)$ is *good* if its arm length is greater than its leg length. We say that λ is *good* if all $A(i,j)$ are good. A partition is *bad* if it is not good.

EXAMPLE 5.2.2 *Recall the definition of a very good partition from Section 4.2.2. Clearly if the hook $B(i,j)$ is good then the hook $A(i,j)$ is good. Hence an extremely good partition is good.*

LEMMA 5.2.3 *The partition λ is good iff for each i the hook $A(i,i+1)$ is good.*

Proof: If λ is good then the hooks $A(i,i+1)$ are good. Conversely suppose that each $A(i,i+1)$ is good. Let $j < k$ and consider the hook $A(j,k)$. Then it is clear that we have

$$a(j,k) = \sum_{j \leq i < k} a(i,i+1),$$

$$l(j,k) = \sum_{j \leq i < k} l(i,i+1).$$

As each $A(i,i+1)$ is good we have $a(i,i+1) \geq l(i,i+1)$. Hence $a(j,k) \geq l(j,k)$ and so $A(j,k)$ is good. □

THEOREM 5.2.4 *Let λ be a partition of n. Suppose that $\psi_\lambda|$ is injective. Then λ is good.*

Proof: Suppose that $A(i_1, j_1)$ is not good. By Lemma 5.2.3 there exists some i with $i_1 \leq i \leq j_1$ such that $A(i,i+1)$ is not good. Let $\mu = (\lambda_i, \lambda_{i+1}^{m_{i+1}})$. As $A(i,i+1)$ is not good we have $\lambda_i - \lambda_{i+1} < m_{i+1}$. By Corollary 3.5.3 the standard map of μ is not injective. Note that λ is obtained by adding rows to μ. Hence by Sivek's Lemma the standard map of λ is not injective. □

5.3 Restriction to S_{n-1}

Recall the definition of the removable node A_i of $[\lambda]$ from Section 5.2. Define λ^{-i} to be the partition whose Young diagram is obtained by removing A_i from $[\lambda]$. We shall say that λ^{-i} is a *branch* of λ.

EXAMPLE 5.3.1 *Let $\lambda = (5, 3^2, 1)$. Then $\lambda^{-2} = (5, 3, 2, 1)$. Below left is $[\lambda]$ with A_2 in black and right is $[\lambda^{-2}]$.*

$$[\lambda] = \begin{array}{lllll} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & \\ \circ & \circ & \bullet & A_2 & \\ \circ & & & & \end{array} \quad , \quad [\lambda^{-2}] = \begin{array}{lllll} \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & & \\ \circ & \circ & & & \\ \circ & & & & \end{array} \quad .$$

In this short section we consider what can be said when we restrict to the subgroup S_{n-1}. Subsection 5.3.1 is devoted to studying the restriction of H^λ to S_{n-1}. The main result is Theorem 5.3.9. In Subsection 5.3.2 we ask what can be said about the relationship between the standard maps ψ_λ and $\psi_{\lambda^{-i}}$. Here the main results are Theorems 5.3.17 and 5.3.20. Finally in Subsection 5.3.3 we study the restriction of the kernel of the standard map to S_{n-1} when λ has at most two removable nodes.

Throughout this section $\lambda = (\lambda_1, \ldots, \lambda_r)$ will be a partition with r-many parts and q-many distinct parts. That is $\lambda = (f_1^{m_1}, \ldots, f_q^{m_q})$ for some f_j and m_j. Further we regard $H := S_{n-1}$ as the subgroup of elements of G that fix n. We start by considering the restriction of H^λ to H.

5.3.1 Branching the module H^λ

In Section 2.2 we defined the module H^λ as the image of the map θ_1. In this subsection, in order to emphasize the role of λ, we alter our notation slightly and write θ_λ in place of θ_1. By definition the module H^λ has basis consisting of elements of the shape $\{t\}\theta_\lambda$. Let V_i denote the subspace of H^λ spanned by those $\{t\}\theta_\lambda$ such that n lies in a row of length f_i in $\{t\}$. We now describe a nice set of representatives in the spaces V_i. Let $\{t\}$ denote the primary λ-tabloid and let y_i denote the largest element in the lowest row of length f_i of $\{t\}$. Define $\omega_i := (n, y_i)$ and $\{t\}^i := \{t\}\omega_i$.

Example 5.3.2 *Let $\lambda = (5, 3^2, 1) \vdash 12$. With the notation above, $y_1 = 5$, $y_2 = 11$, $y_3 = 12$. This gives $\omega_1 = (5, 12)$, $\omega_2 = (11, 12)$ and $\omega_3 = (12, 12) = 1$. Hence we have*

$$\{t\} = \{t\}^3 = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & & \\ \cline{1-3} 9 & 10 & 11 & & \\ \cline{1-3} 12 & & & & \\ \cline{1-1} \end{array} \quad , \quad \{t\}^1 = \{t\}\omega_1 = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 12 \\ \hline 6 & 7 & 8 & & \\ \cline{1-3} 9 & 10 & 11 & & \\ \cline{1-3} 5 & & & & \\ \cline{1-1} \end{array} \quad ,$$

$$\{t\}^2 = \{t\}\omega_2 = \begin{array}{ccccc} \hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 & & \\ \cline{1-3} 9 & 10 & 12 & & \\ \cline{1-3} 11 & & & & \\ \cline{1-1} \end{array} \quad .$$

Recall that the twist groups S_{λ^*} and $S_{(\lambda^{-i})^*}$ are subgroups of S_r. To ease notation let $T := S_{\lambda^*} \cap S_{(\lambda^{-i})^*}$. Let the kth part of λ be the last part of length f_i. Define $Stab_{\lambda^*}(k)$ and $Stab_{(\lambda^{-i})^*}(k)$ to be the subgroups of S_{λ^*} and $S_{(\lambda^{-i})^*}$ respectively that fix the kth row.

Lemma 5.3.3 *The groups $Stab_{\lambda^*}(k)$ and $Stab_{(\lambda^{-i})^*}(k)$ are subgroups of T.*

Proof: Let $\pi \in S_r$ and suppose that $k\pi = k$. For $1 \leq x, y \leq r$ and $x, y \neq k$ we have $\lambda_x = (\lambda^{-i})_x$ and $\lambda_y = (\lambda^{-i})_y$. Thus $\lambda_{x\pi} = \lambda_y$ iff $(\lambda^{-i})_{x\pi} = (\lambda^{-i})_y$ for all $1 \leq x, y \leq r$. Hence $\pi \in S_{\lambda^*} \cap S_{(\lambda^{-i})^*} = T$. $\square$

Let $c_1, \ldots, c_p, k$ index the parts of λ of length f_i and let $\alpha_j = (c_j, k)$. Thus $\{\alpha_j\}_j$ is a complete set of coset representatives of $Stab_{\lambda^*}(k)$ in S_{λ^*}. Similarly let $k, d_1, \ldots d_q$ index the parts of length $f_i - 1$ in λ^{-i} and $\beta_j = (d_j, k)$. Then $\{\beta_j\}_j$ is a complete set of coset representatives of $Stab_{(\lambda^{-i})^*}(k)$ in $S_{(\lambda^{-i})^*}$.

LEMMA 5.3.4 *We have the equalities* $Stab_{\lambda^*}(k) = T = Stab_{(\lambda^{-i})^*}(k)$.

Proof: Note that $(\lambda^{-i})_{c_j} = f_i \neq f_i - 1 = (\lambda^{-i})_k$. Hence $\alpha_j \notin S_{(\lambda^{-i})^*}$ for each j. Every element of S_{λ^*} can be written $h\alpha_j$ for some α_j and some $h \in Stab_{\lambda^*}(k)$. By Lemma 5.3.3 we have $Stab_{\lambda^*}(k) \subseteq T$. Thus $h\alpha_j \in T$ iff $\alpha_j = 1$. Hence $T = Stab_{\lambda^*}(k)$. Similarly $\beta_j \notin S_{\lambda^*}$ for each j and for $h\beta_j \in S_{(\lambda^{-i})^*}$ we have that $h\beta_j \in T$ iff $\beta_j = 1$. Hence $T = Stab_{(\lambda^{-i})^*}(k)$. $\square$

LEMMA 5.3.5 *The set* $\{\alpha_j\}_j$ *is a complete set of coset representatives of* T *in* S_{λ^*}.

Proof: The $\{\alpha_j\}_j$ are a complete set of coset representatives of $Stab_{\lambda^*}(k)$ in S_{λ^*} and by Lemma 5.3.4 we have $S_{\lambda^*} \cap S_{(\lambda^{-i})^*} = Stab_{\lambda^*}(k)$. $\square$

Let M_j denote the subspace of M^λ spanned by those tabloids which contain n in the jth row of length f_i. Clearly M_j is a transitive FH-permutation module. Let $\{s\}^i$ denote the λ^{-i}-tabloid obtained by removing n from $\{t\}^i$. Using Proposition 2.3.1 define $\epsilon_j : M^{\lambda^{-i}} \to M_j$ to be the unique FH-homomorphism that satisfies $\epsilon_j : \{s\}^i \mapsto \{t\}^i\alpha_j$.

LEMMA 5.3.6 *For* $\pi \in T$ *we have* $\{s\}^i\pi\epsilon_j = \{t\}^i\pi\alpha_j$.

Proof: Before proving the result recall from Subsection 2.2 the construction of the elements a_π. Let $\{s\}^i_{l,m}$ denote the mth smallest element of the lth row of $\{s\}^i$. Then for $\pi \in T$ we define a_π to be the unique element of H defined by

$$a_\pi : \{s\}^i_{l,m} \mapsto \{s\}^i_{l\pi^{-1},m}.$$

Then importantly we have $\{s\}\pi = \{s\}a_\pi$. As $\pi \in T$ we have $k\pi = k$ by Lemma 5.3.4. Thus as $\{t\}_{l,m} = \{s\}_{l,m}$ for all $l \neq k$ we have

$$a_\pi : \{t\}^i_{l,m} \mapsto \{t\}^i_{l\pi^{-1},m}.$$

In particular

$$a_\pi : \{t\}^i_{(l)\alpha_j^{-1},m} \mapsto \{t\}^i_{(l)\alpha_j^{-1}\pi^{-1},m}.$$

Now we are ready to prove our lemma. First we have

$$\{s\}^i \pi\epsilon_j = \{s\}^i a_\pi \epsilon_j \tag{29}$$

$$= \{s\}^i \epsilon_j a_\pi \tag{30}$$

$$= \{t\}^i \alpha_j a_\pi. \tag{31}$$

Here (29) is true as $\{s\}\pi = \{s\}a_\pi$. Second (30) follows from (29) as ϵ_j is an FH-homomorphism. Third (31) follows from (30) by definition of ϵ_j. The lth row of $\{t\}^i\pi\alpha_j$ is the $(l)(\pi\alpha_j)^{-1}$th row of $\{t\}^i$, that is

$$\left(\{t\}^i \pi\alpha_j\right)_{l,m} = \{t\}^i_{(l)\alpha_j^{-1}\pi^{-1},m}. \tag{32}$$

Next the lth row of $\{t\}^i\alpha_j$ is the $(l)\alpha_j^{-1}$th row of $\{t\}^i$. Thus we have

$$\left(\{t\}^i \alpha_j\right)_{l,m} = \{t\}^i_{(l)\alpha_j^{-1},m}.$$

Which in turn gives;

$$\left(\{t\}^i \alpha_j a_\pi\right)_{l,m} = \{t\}^i_{(l)\alpha_j^{-1}\pi^{-1},m}.$$

Hence for all l the lth row of $\{t\}^i\pi\alpha_j$ is equal to the lth row of $\{t\}^i\alpha_j a_\pi$. Hence by first (31) and then (32) we have

$$\{s\}^i\pi\epsilon_j = \{t\}^i\alpha_j a_\pi = \{t\}^i\pi\alpha_j.$$

□

Now define a map $\theta_{\lambda,\lambda^{-i}} : M^{\lambda^{-i}} \to M^{\lambda^{-i}}$ by $\{s\}\theta_{\lambda,\lambda^{-i}} = \sum_{\pi\in T}\{s\}\pi$. We denote the image of $\theta_{\lambda,\lambda^{-i}}$ by $H_T^{\lambda^{-i}}$.

LEMMA 5.3.7 *The map ϵ_j is an FH-isomorphism between $M^{\lambda^{-i}}$ and M_j. Further $(H_T^{\lambda^{-i}})\epsilon_j \subseteq M_j$.*

Proof: The first part is well known. For the second part, let $\pi \in T$. By Lemma 5.3.6 we have $\{s\}^i\pi\epsilon_j = \{t\}^i\pi\alpha_j$. By Lemma 5.3.4 we have that π fixes k. Hence $\pi \circ \alpha_j$ sends k to j and so $\{t\}^i\pi\alpha_j \in M_j$ and so $\{s\}^i\pi\epsilon_j \in M_j$. □

Now define a map $\epsilon_{\lambda,\lambda^{-i}} : M^{\lambda^{-i}} \to M^\lambda$ by $\epsilon_{\lambda,\lambda^{-i}} : \{s\}^i \mapsto \sum_j\{t\}^i\alpha_j$. Thus we have $\epsilon_{\lambda,\lambda^{-i}} = \sum_j \epsilon_j$. Our main result is:

PROPOSITION 5.3.8 *The map $\epsilon_{\lambda,\lambda^{-i}}$ is an FH-isomorphism between $H_T^{\lambda^{-i}}$ and V_i.*

Proof: By Lemma's 5.3.6 and 5.3.5 we have

$$\begin{aligned}\{s\}^i\theta_{\lambda,\lambda^{-i}}\epsilon_{\lambda,\lambda^{-i}} &= \sum_j\sum_{\pi\in T}\{t\}^i\pi\alpha_j \\ &= \{t\}^i\theta_\lambda.\end{aligned}$$

Then as $\theta_{\lambda,\lambda^{-i}}$ and $\epsilon_{\lambda,\lambda^{-i}}$ are FH-homomorphisms and V_i is cyclic we see that $\epsilon_{\lambda,\lambda^{-i}}$ maps $H_{\lambda,\lambda^{-i}}^{\lambda^{-i}}$ onto V_i. To show injectivity note that by Lemma 5.3.7 the homomorphisms ϵ_j are injective and linearly independent and hence their sum is injective.
□

THEOREM 5.3.9 *We have the decomposition*

$$H^{\lambda} \cong_{FH} \bigoplus_{i=1}^{r} H_T^{\lambda^{-i}}.$$

Proof: Clearly we have $H^{\lambda} = \bigoplus_{i=1}^{r} V_i$ and the result now follows from Proposition 5.3.8. □

5.3.2 Branching the standard map

In what follows we are interested in the restriction of the standard map to H^{λ} and $H_T^{\lambda^{-i}}$. Thus we adopt the following convention. We consider ψ_{λ} as a map with domain H^{λ}. Hence if $\{t\}$ is a λ-tabloid we define $\psi_{\lambda} : H^{\lambda} \to H^{\lambda'}$ by

$$\psi_{\lambda} : \{t\}\theta_{\lambda} \mapsto \sum_{g \in G_{\{t\}}} (\{t\}')\theta_{\lambda'} g.$$

We then consider the map $\psi_{\lambda,\lambda^{-i}}$ as a map with domain $H_T^{\lambda^{-i}}$. Hence we define $\psi_{\lambda,\lambda^{-i}} : H_T^{\lambda^{-i}} \to H^{(\lambda^{-i})'}$ by

$$\psi_{\lambda,\lambda^{-i}} : \{s\}\theta_{\lambda,\lambda^{-i}} \mapsto \sum_{h \in H_{\{s\}^i}} (\{s\}')\theta_{(\lambda^{-i})'} h.$$

Suppose that λ has q removable nodes. We write W_i for the subspace of $H^{\lambda'}$ spanned by those $\{t_1\}\theta_{\lambda'}$ with n in the $(q-i+1)$th column of $\{t_1\}$. The reason for the reverse numbering of the W_i is that it allows us write $(\{t\}^i)'\theta_{\lambda'} \in W_i$.

LEMMA 5.3.10 (THE PUSH DOWN LEMMA) *For each i we have $(V_i)\psi_{\lambda} \subseteq \bigoplus_{j \geq i} W_j$.*

Proof: One can see that a $(\{t\}^i)'$-column of length λ'_j contains an element from a $\{t\}^i$-row of length λ_i iff $j \leq r-i+1$. Hence a $(\{t\}^i)'$-column of length λ'_j contains an element from the same $\{t\}^i$-row as n iff $j \leq r-i+1$. Now let $g \in G_{\{t\}^i}$. Then there exists an x in the same $\{t\}^i$-row as n such that $xg = n$. By the above discussion x lies in the $(\{t\}^i)'$-column of length λ'_j for some $j \leq r-i+1$. Hence n lies in the jth column of $(\{t\}^i)'g$ for some $j \leq r-i+1$. Thus $j = r-k+1$ for some k with $k \geq i$. Hence $(\{t\}^i)'g \in W_k$ with $k \geq i$. $\square$

Example 5.3.11 *To illustrate the proof of Lemma 5.3.10 let $n = 12$ and consider $\{t\}^2$ below. We see that the column of length 4 of $(\{t\}^2)'$ contains an element from all rows of $\{t\}^2$. The columns of length 3 contain an element from rows of length λ_1 and λ_2 and finally the column of length 2 only contains an element from rows of length λ_1.*

$$\{t\}^2 = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 12 & \\ \cline{1-3} 11 & & & \\ \cline{1-1} \end{array} \quad , \quad \{t\}^{2\prime} = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 12 & \\ 11 & & & \end{array} \quad .$$

Now define a map $\psi_i : V_i \rightarrow W_i$ by

$$\psi_i : \{t\}^i\theta_\lambda \mapsto \sum_{h \in H_{\{t\}^i}} (\{t\}^i)'\theta_{\lambda'}h.$$

The rest of this section is dedicated to the study of the map ψ_i and how it relates to ψ_λ and $\psi_{\lambda,\lambda^{-i}}$. Thus we fix some notation concerning the tabloid $\{t\}^i$ that we will use throughout the proofs. Let n lie in a row of length $l+1$ in $\{t\}^i$ and let the elements that lie in the same row as n be $x_1, \ldots, x_l$ with $x_j < x_{j+1}$. Define $\sigma_j = (n, x_j)$ with the convention that $\sigma_{l+1} = (n, n) = 1$. Thus the σ_j are a complete set of coset representatives of $H_{\{t\}^i}$ in $G_{\{t\}^i}$. The main combinatorial lemma that we need is the following:

Lemma 5.3.12 *With the notation above let x_j lie in a column of $(\{t\}^i)'$ that has the same length as the column that n lies in. Then there exists $\pi_j \in S_{(\lambda')^*}$ and $h_j \in H_{\{t\}^i}$ such that $(\{t\}^i)'\sigma_j = (\{t\}^i)'\pi_j h_j$.*

Proof: Without loss assume that the column that contains x_j is left of the column that contains n. Let the column that contains x_j consist of the $(p+1)$-many elements $c_1, c_2, \ldots, c_p, x_j$ and let the column that contains n consist of the elements

$d_1, d_2, \ldots, d_p, n$. Further for each k we may assume that c_k and d_k lie in the same row of $\{t\}^i$. Then the left column in $(\{t\}^i)'\sigma_j$ consists of $c_1, c_2, \ldots, c_p, n$ and the right $d_1, d_2, \ldots, d_p, x_j$. Hence let $h_j = \prod_k (c_k, d_k)$. Then $h_j \in H_{\{t\}^i}$ as h_j doesn't move n and each pair c_k, d_k lie in the same $\{t\}^i$-row. Let $\pi_j \in S_{(\lambda')^*}$ be the twist element that swaps the two columns. Thus the left column of $(\{t\}^i)'\pi_j$ consists of $n, d_1, \ldots, d_p$ and the right $x_j, c_1, \ldots, c_p$. Hence the left column of $(\{t\}^i)'\pi_j h_j$ consists of $c_1, c_2, \ldots, c_p, n$ and the right $d_1, d_2, \ldots, d_p, x_j$. Thus $(\{t\}^i)'h_j\pi_j = (\{t\}^i)'\sigma_i$. $\square$

EXAMPLE 5.3.13 *To illustrate the proof of Lemma 5.3.12 consider $\{t\}^2$ in Example 5.3.11. With the notation above we have $\sigma_2 = (10, 12)$, $\pi_2 = (2, 3)$ and $h_2 = (2, 3)(6, 7)$. Hence we have:*

$$\{t\}^2 = \begin{array}{cccc} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 12 & \\ \cline{1-3} 11 & & & \\ \cline{1-1} \end{array} , \qquad (\{t\}^2)' = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 12 & \\ 11 & & & \end{array} ,$$

$$(\{t\}^2)'\sigma_2 = \begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 12 & 10 & \\ 11 & & & \end{array} = (\{t\}^2)'\pi_2 h_2 \qquad .$$

LEMMA 5.3.14 *Let $\{t\}^i\theta_\lambda\psi_\lambda = \sum_j w_j$ where $w_j \in W_j$. Then w_i is a scalar multiple of $\{t\}^i\theta_\lambda\psi_i$.*

Proof: By definition of ψ_λ we have

$$\begin{aligned} \{t\}^i\theta_\lambda\psi_\lambda &= \sum_{g \in G_{\{t\}^i}} (\{t\}^i)'\theta_{\lambda'} g \\ &= \sum_{j=1}^{l+1} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)'\theta_{\lambda'}\sigma_j h \\ &= \sum_{j=1}^{l+1} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)'\sigma_j h \theta_{\lambda'}. \end{aligned}$$

Again let n lie in a column of length p in $(\{t\}^i)'$. Let J be the set of indices such that x_j lies in a column of length p in $(\{t\}^i)'\sigma_j$. Then

$$
\begin{aligned}
w_i &= \sum_{j \in J} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)' \sigma_j h \theta_{\lambda'} && (33)\\
&= \sum_{j \in J} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)' (\pi_j h_j) h \theta_{\lambda'} && (34)\\
&= \sum_{j \in J} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)' \pi_j h \theta_{\lambda'} && (35)\\
&= \sum_{j \in J} \sum_{h \in H_{\{t\}^i}} (\{t\}^i)' \pi_j \theta_{\lambda'} h && (36)\\
&= \sum_{j \in J} \sum_{h \in H_{\{t\}^i}} \sum_{\pi \in S_{(\lambda')^*}} (\{t\}^i)' \pi_j \pi h && (37)\\
&= |J| \sum_{h \in H_{\{t\}^i}} \sum_{\pi \in S_{(\lambda')^*}} (\{t\}^i)' \pi h && (38)\\
&= |J| \sum_{h \in H_{\{t\}^i}} (\{t\}^i)' \theta_{\lambda'} h && (39)\\
&= |J| \{t\}^i \theta_\lambda \psi_i. && (40)
\end{aligned}
$$

Above (34) follows from (33) by using the $\pi_j \in S_{(\lambda')^*}$ and $h_j \in H_{\{t\}^i}$ of Lemma 5.3.12. Secondly (35) follows from (34) by absorbing h_j into the sum over $H_{\{t\}^i}$. Third (36) follows from (35) as $\theta_{\lambda'}$ is an FG-homomorphism. Fourth (37) follows from (36) by definition of the map $\theta_{\lambda'}$. Fifth (38) follows from (37) by absorbing π_j into the sum over the twist group. Sixth (39) follows from (38) by definition of $\theta_{\lambda'}$. Finally (40) follows from (39) by definition of ψ_i. □

Proposition 5.3.15 *We have the equality* $\epsilon_{\lambda,\lambda^{-i}} \circ \psi_i = \psi_{\lambda,\lambda^{-i}} \circ \epsilon_{\lambda'(\lambda')^{-i}}$.

Proof: As all the modules involved are transitive FH-permutation modules it is enough to show that $\epsilon_{\lambda,\lambda^{-i}} \circ \psi_i$ and $\psi_{\lambda,\lambda^{-i}} \circ \epsilon_{\lambda'(\lambda')^{-i}}$ agree on the λ^{-i}-tabloid $\{s\}^i$. To ease notation we let ϵ_μ and $\epsilon_{\mu'}$ denote $\epsilon_{\lambda,\lambda^{-i}}$ and $\epsilon_{\lambda',(\lambda')^{-i}}$ respectively. Similarly

θ_μ and $\theta_{\mu'}$ will denote $\theta_{\lambda,\lambda^{-i}}$ and $\theta_{\lambda',(\lambda')^{-i}}$. Then we have

$$
\begin{aligned}
(\{s\}^i\theta_\mu)\epsilon_\mu \circ \psi_i &= (\{t\}^i\theta_\lambda)\psi_i && (41)\\
&= \sum_{h\in H_{\{t\}^i}} (\{t\}^i)'\theta_{\lambda'}h && (42)\\
&= \sum_{h\in H_{\{t\}^i}} (\{s\}^i)'\theta_{\mu'}\epsilon_{\mu'}h && (43)\\
&= \sum_{h\in H_{\{t\}^i}} (\{s\}^i)'\theta_{\mu'}h\epsilon_{\mu'} && (44)\\
&= \sum_{h\in H_{\{s\}^i}} (\{s\}^i)'\theta_{\mu'}h\epsilon_{\mu'} && (45)\\
&= \{s\}^i\theta_\mu\psi_{\lambda,\lambda^i}\epsilon_{\mu'}. && (46)
\end{aligned}
$$

First (41) follows from the definition of ϵ_μ. Second (42) follows from (41) by definition of ψ_i. Third (43) follows from (42) by definition of $\epsilon_{\mu'}$. Fourth (44) follows from (43) as h commutes with $\epsilon_{\mu'}$. Fifth (45) follows from (44) as $H_{\{t\}^i} = H_{\{s\}^i}$. Sixth (46) follows from (45) by definition of ψ_μ. □

LEMMA 5.3.16 *The map ψ_i is injective iff the map $\psi_{\lambda,\lambda^{-i}}$ is injective.*

Proof: First suppose that $\psi_{\lambda,\lambda^{-i}}$ is injective. By Lemma 5.3.8 we have ϵ_i' is injective. By Proposition 5.3.15 we have $\epsilon_i \circ \psi_i = \psi_{\lambda,\lambda^{-i}} \circ \epsilon_i'$. Hence $\epsilon_i \circ \psi_i$ is injective. By Lemma 5.3.8 the map ϵ_i is injective and surjective. Thus the map ψ_i is injective. Conversely suppose that ψ_i is injective. Then $\epsilon_i \circ \psi_i$ is injective by Lemma 5.3.8. Hence $\psi_{\lambda,\lambda^{-i}} \circ \epsilon_i'$ is injective. Thus $\psi_{\lambda,\lambda'}$ is injective. □

THEOREM 5.3.17 *Suppose that $\psi_{\lambda,\lambda^{-i}}$ is injective for each i. Then the standard map of λ is injective.*

Proof: For a contradiction suppose $0 \neq \sum_k v_k \in ker(\psi_\lambda)$ with $v_i \in V_i$. Let i be the smallest index such that $v_i \neq 0$. Then by Lemma 5.3.14 we see that

$$v\psi_\lambda = v_i\psi_i + \left(\sum_{k>i} v_k\right)\psi_\lambda.$$

By Lemma 5.3.10 we know that $(\sum_{k>i} v_k)\psi_\lambda$ is a sum of tabloids none of which are in W_i. Hence $v_i\psi_i = 0$. Hence ψ_i is not injective and so by Lemma 5.3.16 the map $\psi_{\lambda,\lambda^{-i}}$ is not injective. This is a contradiction as $\psi_{\lambda,\lambda^{-i}}$ is injective. Thus ψ_λ is injective. □

LEMMA 5.3.18 *Let $\lambda = (\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$. Then $ker(\psi_r) \subseteq ker(\psi_\lambda)$.*

Proof: Let $v_r \in ker(\psi_r)$. By Lemma's 5.3.10 and 5.3.14 we have that $v_r\psi = v_r\psi_r$. Then as $v_r \in ker(\psi_r)$ we have $v_r\psi = 0$. □

DEFINITION 5.3.19 *Let $\mathbb{P}$ denote the set of all partitions. Write $\lambda > \mu$ if μ is obtained from λ by removing the removable node in the last part of λ. Write $\lambda \gg \nu$ if there exists a chain of partitions $\lambda = \lambda(1) > \cdots > \lambda(d) > \nu$.*

We can now prove the following result that generalizes the result of Black and List.

THEOREM 5.3.20 *Let $\lambda \gg \nu$ and suppose ψ_λ is injective. Then ψ_ν is injective.*

Proof: First note that as we are removing the last node A_s we have $H_T^{\lambda^{-s}} = H^{\lambda^{-s}}$ since it is easy to see that $S_{(\lambda^{-s})^*} \subseteq S_{\lambda^*}$. Hence $\psi_{\lambda,\lambda^{-i}} = \psi_{\lambda^{-i}}$. By definition of $\gg$ there exists a chain of partitions $\lambda = \lambda(1) > \cdots > \lambda(d) > \nu$ and we induct on i that $\psi_{\lambda(i)}$ is injective. The base step is the hypothesis that ψ_λ is injective. By Lemma 5.3.18 we know that the kernel of the $\lambda(i+1)$-standard map is contained in the kernel of the $\lambda(i)$-standard map. By induction the $\lambda(i)$-standard map is injective and so the $\lambda(i+1)$-standard map is injective. Hence the standard map of $\lambda(d) = \nu$ is injective. □

5.3.3 Partitions with at most two removable nodes

First suppose that λ has a single removable node. That is $\lambda = (b^a)$. With the notation that we have accrued throughout this section we have $H^{(b^a)} = V_1$ and $\lambda^{-1} = (b^{a-1}, b-1)$. Then $S_{\lambda^*} = S_a$ and $S_{(\lambda^{-1})^*} = S_{a-1}$. Hence $S_{\lambda^*} \cap S_{(\lambda^{-1})^*} = S_{(\lambda^{-1})^*} = S_{a-1}$. Hence $H_T^\lambda = H^{\lambda^{-1}}$. Thus by Proposition 5.3.8 we have that $H^\lambda \cong_{FH} H^{\lambda^1}$ and by Theorem 5.3.17 the standard map $\psi_{(b^a)}$ is injective iff $\psi_{(b^{a-1}, b-1)}$ is injective. Now suppose that λ has two removable nodes. If $\psi_{\lambda,\lambda^{-2}} = \psi_{\lambda^{-2}}$ has a kernel then ψ_λ has a kernel by Theorem 5.3.20. If both $\psi_{\lambda,\lambda^{-1}}$ and $\psi_{\lambda^{-2}}$ are injective then by Theorem 5.3.17 the map ψ_λ is injective. Finally consider the case that $\psi_{\lambda^{-2}}$ is injective and but $\psi_{\lambda,\lambda^{-1}}$ is not injective. In this case the main result is Theorem 5.3.22 below.

LEMMA 5.3.21 *The map $\epsilon_{\lambda,\lambda^{-2}}$ is an FH-isomorphism from $ker(\psi_{\lambda,\lambda^{-2}})$ into $ker(\psi_2)$.*

Proof: Let $v \in ker(\psi_{\lambda,\lambda^{-2}})$. Then by Proposition 5.3.15 we have

$$v\epsilon_{\lambda,\lambda^{-2}} \circ \psi_2 = v\psi_{\lambda,\lambda^{-2}} \circ \epsilon_{\lambda',(\lambda')^{-2}} = 0$$

Conversely if $w \in ker(\psi_2)$ then as $\epsilon_{\lambda,\lambda^{-2}}$ is an isomorphism we have there exists $w = v\epsilon^{-1}_{\lambda,\lambda^{-2}} \in J^{\lambda,\lambda^{-2}}$. Hence again by Proposition 5.3.15 we have $w\psi_{\lambda,\lambda^{-2}} = 0$. □

THEOREM 5.3.22 *Suppose that ψ_{λ^2} is injective and $\psi_{\lambda,\lambda^{-1}}$ is not injective. Then $ker(\psi_\lambda)$ is FH-isomorphic to a submodule of $ker(\psi_{\lambda,\lambda^{-1}}) \bigoplus ker(\psi_{\lambda,\lambda^{-1}})$.*

Proof: Let $v = v_1 + v_2 \in ker(\psi_\lambda)$ with $v_i \in V_i$. Then by Lemma's 5.3.10 and 5.3.14 we may write $v\psi_\lambda = v_1\psi_1 + v_1\psi^\sim + v_2\psi_2$ where $v_1\psi_1 \in W_1$ and $v_1\psi^\sim, v_2\psi_2 \in W_2$. Hence $v_1 \in ker(\psi_1)$ and $v_1\psi^\sim = -v_2\psi_2$. As ψ_2 is injective, for $v_1 \in ker(\psi_1)$ there exists a unique v_1' such that $v_1\psi^\sim = -v_2\psi_2$. Define an FH-homomorphism $\phi : V_1 \to V_2$ by $v_1 \mapsto v_1'$. Hence $ker(\psi_\lambda) = ker(\psi_1) \bigoplus im(\phi)$. The result now follows from Lemma 5.3.21. □

Chapter 6

The standard map of partitions with at most four parts

In this chapter we collect together all the techniques we have amassed throughout this thesis and apply them to partitions with at most four parts. We will obtain full results for partitions with at most three parts and partial results for four part partitions.

6.1 Partitions with at most three parts

When a partition has at most three parts it will turn out that that the standard map is the best possible homomorphism as the attractive Theorem 6.1.3 shows. We begin with an elementary lemma;

LEMMA 6.1.1 *The bad three part partitions are those of the shape* $(b+1, b, b)$.

Proof: Suppose that λ is bad. Then there is a hook $A(i,j)$ whose leg is longer than its arm. As λ has three parts the leg must have length two and the arm length one. Thus the bottom two parts of λ must have equal length and the first part one greater. □

Theorem 6.1.2 *A partition with at most three parts has injective standard map iff it is good.*

Proof: By Theorem 5.2.4 we know that if λ is bad then the standard map of λ is not injective. Hence it suffices to show that if λ is good and has at most three parts then the standard map is injective. If λ has two parts then λ is extremely good and the result holds by Theorem 4.2.21. Suppose that λ has three parts. If $\lambda = (a, b, b)$ with $a \geq b + 2$ then λ is extremely good. If all three parts of λ are distinct then λ is extremely good. Thus by Lemma 6.1.1 it remains to show that the partition (a, a, b) with $b \leq a$ has injective standard map. By Theorem 4.1.17 the standard map of (a^3) is injective. Recalling Definition 5.3.19 we see $(a, a, b) \ll (a^3)$. Hence by Corollary 5.3.20 the standard map of (a, a, b) is injective. □

Theorem 6.1.3 *Let λ be a partition with at most three parts. Then the following are equivalent:*

(i) There exists an injective map $H^\lambda \to H^{\lambda'}$.

(ii) The partition λ is good.

(iii) The standard map of λ is injective.

Proof: Part (i) implies (ii) by Theorem 5.2.4. Part (ii) implies (iii) by Theorem 6.1.2. Part (iii) clearly implies (i). □

6.2 Good partitions with four parts

In this section we look at what can be said about partitions with four parts. We stress that we only obtain partial results in this section as we lack two base cases. As with three part partitions we begin by grouping the partitions into three families.

Lemma 6.2.1 *Let λ be a partition with at most four parts that dominates its conjugate. Suppose that λ is good but not extremely good. Then λ has one of the following three shapes:*

(i) $(a^2, (a-2)^2)$ *with* $a \geq 4$.

(ii) $(a^2, a-1, a-2)$ *with* $a \geq 4$.

(iii) (a^3, b) *with* $b < a$ *and* $a \geq 4$.

Proof: If λ has one or four removable nodes then λ is extremely good. Suppose λ has three removable nodes. In this case λ has exactly two parts of equal length. If $\lambda_2 = \lambda_3$ or $\lambda_3 = \lambda_4$ then it is easy to see that λ is good iff λ is extremely good. Thus a good but not extremely good partition with three removable nodes has type *(ii)*. Finally suppose λ has two removable nodes then either three parts have equal length or $\lambda = (a^2, b^2)$. If the first three parts have equal length then λ is of type *(iii)*. If the last three parts have equal length then it is easy to see λ is good iff it is extremely good. Thus suppose $\lambda = (a^2, b^2)$. If λ is good but not extremely good it is easy to see that $a - b = 2$ and λ has type *(i)*. □

Lemma 6.2.2 *Suppose that the standard maps of the partitions* $(4^2, 2^2)$ *and* $(4^2, 3, 2)$ *are injective. Then for* $a \geq 4$ *the standard maps of the partitions* $(a^2, (a-2)^2)$ *and* $(a^2, a-1, a-2)$ *are injective.*

Proof: Induct on a. The base step is the hypothesis. The left-most column of both $(a^2, (a-2)^2)$ and $(a^2, a-1, a-2)$ are good. By induction the standard maps of the partitions $((a-1)^2, (a-3)^2)$ and $((a-1)^2, a-2, a-3)$ are injective. Hence by Theorem 4.1.15 the standard maps of the partitions $(a^2, (a-2)^2)$ and $(a^2, a-1, a-2)$ are injective. □

Proposition 6.2.3 *Suppose that the standard maps of the partitions* $(4^2, 2^2)$ *and* $(4^2, 3, 2)$ *are injective. Then a partition with at most four parts has injective standard map iff it is good.*

Proof: By Theorem 4.1.17 we know that the standard maps of (b^a) are injective for $a \leq 4$ and $b \geq a$. Hence by Theorem 4.2.21 all very good four part partitions are injective. By Hypothesis $(4^2, 2^2)$ and $(4^2, 3, 2)$ are injective. Hence by Lemma 6.2.2

the standard maps of partitions of the shape $(a^2, (a-2)^2)$ and $(a^2, a-1, a-2)$ are injective. Finally by Theorem 4.1.17 the standard map of (a^4) is injective. Then $(a^4) \gg (a^3, b)$. Hence by Corollary 5.3.20 the standard map of (a^3, b) is injective. Hence Lemma 6.2.1 shows that we have accounted for all good four part partitions.

□

6.3 Bad partitions with four parts

In this section we show that studying bad partitions with at least four parts is difficult. We know that the standard maps of these partitions are not injective so it remains to decide if there exists an injective map $H^\lambda \to H^{\lambda'}$. First we classify the bad partitions with four parts into three natural families. We then show that two of these families behave well. That is, the rank of the standard map $\psi_\lambda|$ decides the existence of an injective map $H^\lambda \to H^{\lambda'}$. However we then study the third family of partitions for whom the standard map is not injective but for whom there often exists an injective map $H^\lambda \to H^{\lambda'}$.

LEMMA 6.3.1 *A four part bad partition that dominates its conjugate has one of the following three shapes:*

(i) (a, b^3) *with* $b \geq 3$ *and* $a - b < 3$

(ii) $(a+1, a^2, b)$ *with* $a \geq 2$*,*

(iii) $(a, b+1, b, b)$ *with* $b \geq 3$*.*

Proof: A bad partition with two removable nodes must be of type (i). In a bad partition with three removable nodes either the hook $A(1,2)$ or $A(2,3)$ is bad. If $A(1,2)$ is bad the partition has type (ii) and if $A(1,2)$ is bad it has type (iii). □

PROPOSITION 6.3.2 *Let* λ *be a partition of type* (i) *or* (ii) *as in Lemma 6.3.1. Then there is no injective map* $H^\lambda \to H^{\lambda'}$*.*

Proof: If λ is of type (i) then by Theorem 3.5.2 there is no injective map $H^\lambda \to H^{\lambda'}$. The proof for type (ii) partitions is similar. Again let X and X' denote the bases of H^λ and $H^{\lambda'}$ respectively and then follow the proof of Theorem 3.5.2. □

6.3.1 Composable sequences

In this short subsection we collect together some tools and notation that will make our assault on the remaining type (iii) partitions a little less painful.

DEFINITION 6.3.3 *The sequence of tabloids $\{t\}^1, \{t\}^2, \ldots, \{t\}^r$ is composable if:*

(i) *Every row of $\{t\}^i$ is a subrow of $\{t\}^1$ or $\{t\}^{i+1}$, and*

(ii) *The tabloid $\{t\}^{i+1}$ is obtained from $\{t\}^i$ by moving a subset of size p_i from a set of size n_i into a set of size m_i with $n_i - p_i \geq m_i$.*

THEOREM 6.3.4 *Let $\{t\}^1, \{t\}^2, \ldots, \{t\}^r$ be a composable sequence of tabloids. Suppose that the shape of $\{t\}^i$ is λ^i. Then the map $\phi : M^{\lambda^1} \to M^{\lambda^r}$ given by $\{t\}^1 \mapsto \sum_{g \in G_{\{t\}^1}} \{t\}^r g$ is injective.*

Proof: Let λ^i denote the shape of $\{t\}^i$ and define $\phi_i : M^{\lambda^i} \to M^{\lambda^{i+1}}$ by $\phi_i : \{t\}^i \mapsto \sum_{g \in G_{\{t\}^i}} \{t\}^{i+1} g$. It follows by induction using Theorem 2.3.4 that the map ϕ is a scalar multiple of the maps ϕ_i and by Theorem 3.2.2 each map ϕ_i is injective. □

It will be useful to have a more general version of Proposition 4.1.5. To this end let λ and μ be partitions of n and suppose ν is a partition obtained by removing some rows from μ. Clearly the twist group of ν is a subgroup of that of μ. Define a map $\theta_\nu : M^\mu \to M^\mu$ by $\{s\} \mapsto \sum_{\pi \in S_{\nu^*}} \{s\}\pi$. Denote the image of θ_ν by H^μ_ν. Let $\{t\}$ and $\{s\}$ be λ- and μ-tabloids respectively and suppose $\{s_1\}$ is a ν-tabloid obtained from $\{s\}$ by removing rows. Denote the ith smallest element of the jth column of $\{s_1\}$ by $x_{i,j}$. Then we have:

LEMMA 6.3.5 *With the notation above suppose that for each i the set $\{x_{i,j}\}_j$ is a subset of a row of $\{t\}$. Then the image of the map $\phi : M^\lambda \to M^\mu$ is a submodule of H^μ_ν.*

Proof: The proof mimics that of Proposition 4.1.5. Let the jth smallest element in the ith row of the tail of $\{t\}$ be $x_{i,j}$. Then for $\pi \in S_{\nu^*}$ define $a_{\pi \in G_{\{t\}}}$ by $x_{i,j}a_\pi = x_{i,j\pi^{-1}}$. Then we have $\{s\}\pi = \{s\}a_\pi$. Let A denote the set $\{a_\pi \mid \pi \in S_{\nu^*}\}$. By hypothesis for each π we have $a_\pi \in G_{\{t\}}$. Hence $A \subseteq G_{\{t\}}$. Let σ_i be a complete set of coset representatives of A in $G_{\{t\}}$. Then we have

$$\begin{aligned}\{t\}\phi &= \sum_{g \in G_{\{t\}}} \{s\}g \\ &= \sum_i \sum_{a \in A} \{s\}a\sigma_i.\end{aligned}$$

The element $\sum_{a \in A}\{s\}a$ is in the basis of H_ν^λ which is an FG-module. Hence for each i the element $\sum_{a \in A}\{s\}a\sigma_i$ belongs to H_ν^μ and so $\{t\}\phi$ lies in H_ν^λ. □

6.3.2 The partitions $[2(b+c-1), b+c-1, b^c]$

In this subsection we will be looking at the partitions $\lambda = (2a, a, b^c)$ with $a = b+c-1$ and $c \geq 2$. It will turn out that the case $a \geq b+c-1$ can easily be derived from this special case (cf. Subsection 6.3.3). We shall call the parts $(2a)$, (a) and (b^c) the *antidote*, *head* and *tail* of λ respectively. We shall spend the rest of this subsection proving Theorem 6.3.6:

THEOREM 6.3.6 *Let $\lambda = (2a, a, b^c)$. Suppose that the standard map of (b^c) is injective. Then the standard map of λ is not injective but there exists an injective map $H^\lambda \to H^{\lambda'}$.*

For the λ-tabloid $\{t\}$ write $\{t\} = \{t\}^A \cup \{t\}^H \cup \{t\}^T$, where $\{t\}^A, \{t\}^H$ and $\{t\}^T$ are the rows of $\{t\}$ that correspond to the antidote, head and tail of λ respectively. Call these smaller tabloids the antidote, head and tail of $\{t\}$. For example

$$\{t\} = \begin{array}{llllllllllll} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 & 17 & 18 \\ \cline{1-6} 19 & 20 & 21 & 22 \\ \cline{1-4} 23 & 24 & 25 & 26 \\ \cline{1-4} 27 & 28 & 29 & 30 \\ \cline{1-4} \end{array} ,$$

$$\{t\}^A = \begin{array}{llllllllllll} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline \end{array} ,$$

$$\{t\}^H = \begin{array}{llllll} \hline 13 & 14 & 15 & 16 & 17 & 18 \\ \hline \end{array} , \quad \{t\}^T = \begin{array}{llll} \hline 19 & 20 & 21 & 22 \\ \hline 23 & 24 & 25 & 26 \\ \hline 27 & 28 & 29 & 30 \\ \hline \end{array} .$$

Define the tabloid $\{t\}^\circ$ to be $\{t\} = \{t\}^A \cup \{t\}^H \cup (\{t\}^T)'$. We again call the parts $\{t\}^A, \{t\}^H$ and $(\{t\}^T)'$ the antidote, head and tail of $\{t\}^\circ$. The shape λ° of $\{t\}^\circ$ is $(2a, a, c^b)$. We define the antidote, head and tail of λ° to be the shapes of the tabloids parts $\{t\}^A, \{t\}^H$ and $(\{t\}^T)'$ respectively.

$$\{t\}^\circ = \begin{array}{|l|l|l|l|llllllll} \hline \multicolumn{1}{l}{1} & \multicolumn{1}{l}{2} & \multicolumn{1}{l}{3} & \multicolumn{1}{l}{4} & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline \multicolumn{1}{l}{13} & \multicolumn{1}{l}{14} & \multicolumn{1}{l}{15} & \multicolumn{1}{l}{16} & 17 & 18 \\ \cline{1-6} 19 & 20 & 21 & 22 \\ 23 & 24 & 25 & 26 \\ 27 & 28 & 29 & 30 \end{array} .$$

Define a map $\phi : M^\lambda \to M^{\lambda^\circ}$ by $\phi : \{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^\circ g$ and let $\phi|$ denote its restriction to H^λ.

Lemma 6.3.7 *The map $\phi|$ is injective iff the standard map of (b^c) is injective.*

Proof: Let ν be the partition obtained by removing the tail from λ. Then define the FG-isomorphism $\varphi_1 : M^\lambda \to (M^{\lambda^T} \otimes M^\nu)^G$ by $\{t\} \mapsto (\{t\}^A \cup \{t\}^H, \{t\}^T)$. Then define the FG-isomorphism $\varphi_2 : (M^{(\lambda^T)'} \otimes M^\nu)^G \to M^{\lambda^\circ}$ by $(\{t\}^A \cup \{t\}^H, (\{t\}^T)') \mapsto \{t\}^\circ$. The result now follows from Corollary 2.5.14. $\square$

Let $\{t\}^\wedge$ denote the tabloid obtained by moving the ith and $(b+i)$th largest elements from the antidote of $\{t\}^\circ$ into the ith column of $\{t\}^\circ$. The antidote, head and tail of $\{t\}^\wedge$ are then the parts of $\{t\}^\wedge$ that correspond to the antidote, head and tail of $\{t\}^\circ$. The shape of $\{t\}^\wedge$ is the composition $\lambda^\wedge = [2a - 2b, a, (c+2)^b]$.

$$\{t\}^\wedge = \begin{array}{|c|c|c|c|cccccc} 1 & 2 & 3 & 4 & 9 & 10 & 11 & 12 & & \\ 5 & 6 & 7 & 8 & 13 & 14 & 15 & 16 & 17 & 18 \\ 19 & 20 & 21 & 22 & & & & & & \\ 23 & 24 & 25 & 26 & & & & & & \\ 27 & 28 & 29 & 30 & & & & & & \end{array} .$$

The twist group of $\lambda^\wedge$ contains the twist groups of the antidote, head and tail of $\lambda^\wedge$. Define the map $\theta_{\lambda^\wedge T} : M^{\lambda^\wedge} \to M^{\lambda^\wedge}$ by $\{t\}^\wedge \mapsto \sum_{\pi \in S_{(\lambda^\wedge T)^*}} \{t\}^\wedge \pi$. Then define $H^{\lambda^\wedge}_{\lambda^\wedge T}$ to be the image of $\theta_{\lambda^\wedge T}$. Now define a map $\psi^\sim : M^{\lambda^\circ} \to M^{\lambda^\wedge}$ by $\{t\} \mapsto \sum_{g \in G_{\{t\}^\circ}} \{t\}^\wedge$.

LEMMA 6.3.8 *The image of $\phi \circ \psi^\sim$ lies in $H^{\lambda^\wedge}_{\lambda^\wedge T}$.*

Proof: The antidote of $\{t\}^\circ$ is equal to that of $\{t\}$. The ith column in the tail of $\{t\}^\wedge$ is obtained from that of $\{t\}^\circ$ by adding the ith and $(b+i)$th smallest elements of the antidote of $\{t\}^\circ$. Hence every row or column of $\{t\}^\circ$ is a subrow or column of $\{t\}$ or $\{t\}^\wedge$. Hence the map $\phi \circ \psi^\sim$ is a scalar multiple of the map $M^\lambda \to M^{\lambda^\wedge}$ given by $\{t\} \mapsto \sum_{g \in G_{\{t\}}} \{t\}^\wedge g$. Let $x_{i,j}$ denote the ith smallest element of the jth column of the tail of $\{t\}^\wedge$. For $i = 1, 2$ the subset $\{x_{i,j}\}_j$ is a subset of the antidote of $\{t\}$. When $i > 2$ the subset $\{x_{i,j}\}_j$ is a subset of the $(i-2)$th row of the tail of $\{t\}$. The result now follows from Theorem 6.3.5. □

LEMMA 6.3.9 *The map $\psi^\sim$ is injective.*

To prove Lemma 6.3.9 we produce a composable sequence as follows. For $0 \leq i \leq b$ define the tabloid $\{t\}^i$ by letting $\{t\}^0 = \{t\}^\circ$ and obtaining $\{t\}^{i+1}$ from $\{t\}^i$ by moving the i and $(b+i)$th smallest elements of the antidote of $\{t\}^i$ into the ith column of $\{t\}^i$. Then $\{t\}^\wedge = \{t\}^b$.

$$\{t\}^1 = \begin{array}{cccccccccccc} \hline & 2 & 3 & 4 & & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 13 & 14 & 15 & 16 & 17 & 18 & & & & & & \\ \cline{1-6} \multicolumn{1}{|c|}{19} & \multicolumn{1}{c|}{20} & \multicolumn{1}{c|}{21} & \multicolumn{1}{c|}{22} & & & & & & & & \\ \multicolumn{1}{|c|}{23} & \multicolumn{1}{c|}{24} & \multicolumn{1}{c|}{25} & \multicolumn{1}{c|}{26} & & & & & & & & \\ \multicolumn{1}{|c|}{27} & \multicolumn{1}{c|}{28} & \multicolumn{1}{c|}{29} & \multicolumn{1}{c|}{30} & & & & & & & & \\ \multicolumn{1}{|c|}{1} & & & & & & & & & & & \\ \multicolumn{1}{|c|}{5} & & & & & & & & & & & \end{array} ,$$

$$
\{t\}^2 = \begin{array}{cccccccccccc}
\hline
 & & 3 & 4 & & & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
13 & 14 & 15 & 16 & 17 & 18 & & & & & & \\
\cline{1-6}
\multicolumn{1}{|c|}{19} & \multicolumn{1}{c|}{20} & \multicolumn{1}{c|}{21} & \multicolumn{1}{c|}{22} & & & & & & & & \\
\multicolumn{1}{|c|}{23} & \multicolumn{1}{c|}{24} & \multicolumn{1}{c|}{25} & \multicolumn{1}{c|}{26} & & & & & & & & \\
\multicolumn{1}{|c|}{27} & \multicolumn{1}{c|}{28} & \multicolumn{1}{c|}{29} & \multicolumn{1}{c|}{30} & & & & & & & & \\
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & & & & & & & & & & \\
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & & & & & & & & & & \\
\end{array},
$$

$$
\{t\}^3 = \begin{array}{cccccccccccc}
\hline
 & & & 4 & & & & 8 & 9 & 10 & 11 & 12 \\
\hline
13 & 14 & 15 & 16 & 17 & 18 & & & & & & \\
\cline{1-6}
\multicolumn{1}{|c|}{19} & \multicolumn{1}{c|}{20} & \multicolumn{1}{c|}{21} & \multicolumn{1}{c|}{22} & & & & & & & & \\
\multicolumn{1}{|c|}{23} & \multicolumn{1}{c|}{24} & \multicolumn{1}{c|}{25} & \multicolumn{1}{c|}{26} & & & & & & & & \\
\multicolumn{1}{|c|}{27} & \multicolumn{1}{c|}{28} & \multicolumn{1}{c|}{29} & \multicolumn{1}{c|}{30} & & & & & & & & \\
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & & & & & & & & & \\
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} & & & & & & & & & \\
\end{array},
$$

$$
\{t\}^4 = \begin{array}{cccccccccccc}
\hline
 & & & & & & & & 9 & 10 & 11 & 12 \\
\hline
13 & 14 & 15 & 16 & 17 & 18 & & & & & & \\
\cline{1-6}
\multicolumn{1}{|c|}{19} & \multicolumn{1}{c|}{20} & \multicolumn{1}{c|}{21} & \multicolumn{1}{c|}{22} & & & & & & & & \\
\multicolumn{1}{|c|}{23} & \multicolumn{1}{c|}{24} & \multicolumn{1}{c|}{25} & \multicolumn{1}{c|}{26} & & & & & & & & \\
\multicolumn{1}{|c|}{27} & \multicolumn{1}{c|}{28} & \multicolumn{1}{c|}{29} & \multicolumn{1}{c|}{30} & & & & & & & & \\
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & & & & & & & & \\
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & & & & & & & & \\
\end{array} = \{t\}^\wedge.
$$

LEMMA 6.3.10 *The sequence $\{t\}^i$ is composable.*

Proof: The antidote of $\{t\}^i$ is obtained from that of $\{t\}^1$ by removing the $1, 2, \ldots (i-1)$th and $1+b, 2+b, \ldots, (b+i-1)$th smallest elements. For $k \neq i$ the kth column of the tail of $\{t\}^i$ is equal to that of $\{t\}^{i-1}$. The ith column of the tail of $\{t\}^{i+1}$ is obtained from that of $\{t\}^i$ by adding the ith and $(b+i)$th smallest elements of the antidote of $\{t\}^i$. Hence every row or column of $\{t\}^i$ is a subrow or subcolumn of $\{t\}^1$ or $\{t\}^{i+1}$. Finally we have that the antidote of $\{t\}^i$ has length $2a-2i$ and the $(i+1)$st column of $\{t\}^i$ has length c. Hence we must show $2a-2i \geq c$ for $0 \leq i \leq b$.

To this end we have

$$
\begin{aligned}
2a - 2i &\geq 2a - 2b && (47)\\
&= 2b + 2c - 2 - 2b && (48)\\
&= 2c - 2 \\
&\geq c. && (49)
\end{aligned}
$$

Where (47) holds as $i \leq b$. Second (48) follows from (47) as $a = b + c - 1$. Finally (49) holds as $c \geq 2$. Hence the sequence is composable. $\square$

Proof of Lemma 6.3.9: By Lemma 6.3.10 we have that the sequence of tabloids $(\{t\}^i)_{i=0}^b$ is composable. Hence by Theorem 6.3.4 the map $\psi^\sim$ is injective. $\square$

Notice that the antidote of $\{t\}^\wedge$ has $2(a-b)$ elements. Define a tabloid $\{t\}^\perp$ to be the tabloid obtained from $\{t\}^\wedge$ in the following way. Create $(a-b)$-many columns of size two from the antidote of $\{t\}^\wedge$ by moving the ith and $(a-b+i)$th smallest elements into the ith of these columns. Then move each element of the head of $\{t\}^\wedge$ into a column of its own.

$$
\{t\}^\perp = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
1 & 2 & 3 & 4 & 9 & 10 & 13 & 14 & 15 & 16 & 17 & 18 \\
5 & 6 & 7 & 8 & 11 & 12 \\
19 & 20 & 21 & 22 \\
23 & 24 & 25 & 26 \\
27 & 28 & 29 & 30
\end{array} .
$$

LEMMA 6.3.11 *The shape of* $\{t\}^\perp$ *is* λ'.

Proof: Since $\lambda = (2a, a, b^c)$ we have $\lambda' = [(c+2)^b, 2^{a-b}, 1^a]$. The shape of $\lambda^\wedge$ is the composition $[2(a-b), a, (c+2)^b]$. The construction of $\{t\}^\perp$ from $\{t\}^\wedge$ then changes changes the part $2(a-b)$ into (2^{a-b}) and (a) into (1^a). $\square$

Define the map $\theta : M^{\lambda^\wedge} \to M^{\lambda'}$ by $\{t\}^\wedge \mapsto \sum_{g \in G_{\{t\}^\wedge}} \{t\}^\perp g$. We say that (2^{a-b}), (1^a) and $(c+2)^b$ respectively are the antidote, head and tail of λ'.

LEMMA 6.3.12 *The map θ injects $(H^{\lambda^\wedge}_{\lambda^{\wedge T}})$ into $H^{\lambda'}$.*

Proof: It is clear that the map θ is injective. The twist group of λ' is the direct product of the twist groups of the antidote, head and tail of $\{t\}^\perp$. Let z_i denote the element in the ith column of the head of $\{t\}^\perp$. Then the set $\{z_i\}_i$ is the head of $\{t\}^\wedge$ and so by Theorem 6.3.5 we have that the twist group of the head of λ' fixes the image of θ. Next let $y_{i,j}$ denote the ith smallest element of the jth column of the antidote of $\{t\}^\perp$. The set $\{y_{i,j}\}_{i,j}$ is the antidote of $\{t\}^\wedge$ and so by Theorem 6.3.5 the twist group of the antidote of λ' fixes the image of θ. Finally the tails of $\{t\}^\wedge$ and $\{t\}^\perp$ are the same so let π be an element of the twist group of the tail of λ'. As in the proof of Theorem 6.3.5 let $x_{i,j}$ denote the ith smallest element in the jth column of the tail of $\{t\}^\perp$. Then define $a_\pi \in G$ by $a_\pi : x_{i,j} \mapsto x_{i,(j)\pi}$. Then we have $\{t\}^\wedge\pi = \{t\}^\wedge a_\pi$ and $\{t\}^\perp\pi = \{t\}a_\pi$. Then we have

$$\begin{aligned}\{t\}^\wedge\pi\theta &= \{t\}a_\pi\theta \\ &= \sum_{g\in G_{\{t\}^\wedge}} \{t\}^\perp a_\pi g.\end{aligned}$$

Hence θ maps the space fixed by the twist group of the tail of $\lambda^\wedge$ to the space fixed by the twist group of the tail of λ'. □

Proof of Theorem 6.3.6: Let ψ^* denote the composition $\phi \circ \psi^\sim \circ \theta$. Then $\psi^*| = \phi| \circ \psi^\sim \circ \theta$. By Lemma 6.3.8 the image of $\phi| \circ \psi^\sim$ lies in $(H^{\lambda^\wedge}_{\lambda^{\wedge T}})$. Hence by Lemma 6.3.12 the image of $\psi^*|$ lies in $H^{\lambda'}$. By Lemma's 6.3.7, 6.3.9 and 6.3.12 the maps $\phi|, \psi^\sim$ and $\theta|$ are injective. Hence ψ^* is injective. □

6.3.3 The partitions $(a, b+c-1, b^c)$

In this subsection we generalize slightly the results of Subsection 6.3.2. Thoughout this section we fix $\mu = (a, b+c-1, b^c)$ with $a = 2(b+c-1)+r$ for a positive integer r and $c \geq 2$. Let $\{s\}$ be a μ-tabloid and let $\{t\}$ denote the tabloid obtained from $\{s\}$ by moving the largest r elements from the antidote of $\{s\}$ into the head of $\{s\}$. Then define the map $\epsilon : M^\lambda \to M^{\lambda^\dagger}$ by $\{s\} \mapsto \sum_{g \in G_{\{s\}}} \{t\}g$. An immediate consequence of Theorem 6.3.4 is the following:

LEMMA 6.3.13 *The map ϵ is injective.*

□

THEOREM 6.3.14 *Let $\mu = (a, b+c-1, b^c)$ with $a \geq 2(b+c-1)+r$. Then the standard map of μ is not injective but there exists an injective map $H^\mu \to H^{\mu'}$.*

Proof: As the hook $A(2,3)$ of μ is bad the standard map of μ is not injective. The proof then mimics that of Theorem 6.3.6. Let ψ^* denote the composition $\epsilon \circ \phi \circ \psi^\sim \circ \theta$. Then by Lemma 6.3.13 the map ϵ is injective. Clearly ϵ maps H^μ into H^λ. The proof that the map $(\phi \circ \psi^\sim \circ \theta)|$ is injective follows verbatim from that of Theorem 6.3.6. Now notice that the shape of $\{t\}^\perp$ is μ' instead of λ'. The result now follows. □

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