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Exponential algebraicity in exponential fields

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Abstract

The exponential algebraic closure operator in an exponential field is always a pregeometry and its dimension function satisfies a weak Schanuel property. It follows that there are at most countably many essential counterexamples to Schanuel's conjecture.

1. *Introduction*

In a field, the notion of algebraicity is captured by the algebraic closure operator, acl. Algebraic closure is a pregeometry, that is, a closure operator of finite character satisfying the Steinitz exchange property

$$a \in \operatorname{acl}(C \cup \{b\}) \smallsetminus \operatorname{acl}(C) \implies b \in \operatorname{acl}(C \cup \{a\})$$

and hence it gives rise to a dimension function, in this case transcendence degree. The analogous closure operator, ecl^F, in an exponential field F was defined by Macintyre [**4**]. In the special case of the real exponential field $\mathbb{R}_{\exp} = \langle \mathbb{R}; +, \cdot, \exp \rangle$, where exp is the usual exponential function $x \mapsto e^x$, Wilkie showed that $\operatorname{ecl}^{\mathbb{R}}$ is a pregeometry. His technique was to define a pregeometry $\operatorname{cl}^{\mathbb{R}}$ by derivations, and, using techniques of o-minimality and real analysis, to construct enough derivations to show that the two closure operators were equal. He later extended the result to the complex exponential field $\mathbb{C}_{\exp}$ [**5**], still using analytic techniques and the major theorem that the real field with exponentiation and restricted analytic functions is o-minimal.

Looking to study $\mathbb{C}_{\exp}$ in another way, Zilber [**6**] constructed an exponential field using the *amalgamation of strong extensions* technique of Hrushovski [**2**], and conjectured that it is isomorphic to $\mathbb{C}_{\exp}$. His exponential field comes with a pregeometry satisfying an important transcendence property, the Schanuel property.

In this paper, I give an algebraic proof of the generalization of Wilkie's result to an arbitrary exponential field.

Theorem 1.1. *For any* (*total or partial*) *exponential field* F, *the closure operator* ecl^F *is a pregeometry, and it always agrees with the pregeometry* cl^F *defined using derivations.*

Furthermore, in every exponential field F, the dimension function associated with ecl^F, which we call *exponential transcendence degree*, etd^F, satisfies a weak form of the Schanuel property.

Theorem 1.2. *Suppose* $C \subseteq F$ *is* ecl^F*-closed. Let* $x_1, \ldots, x_n \in F$. *Then*

$$\delta(\bar{x}/C) := \operatorname{td}(\bar{x}, \exp(\bar{x})/C) - \operatorname{ldim}_{\mathbb{Q}}(\bar{x}/C) \geqslant \operatorname{etd}^F(\bar{x}/C).$$

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For any subsets X and Y of a field F (of characteristic zero), $\mathrm{td}(X/Y)$ means the transcendence degree of the field extension $\mathbb{Q}(X,Y)/\mathbb{Q}(Y)$. Writing $\langle X\rangle_{\mathbb{Q}}$ for the $\mathbb{Q}$-linear span of X, we define $\mathrm{ldim}_{\mathbb{Q}}(X/Y)$ to be the dimension of the quotient $\mathbb{Q}$-vector space $\langle X,Y\rangle_{\mathbb{Q}}/\langle Y\rangle_{\mathbb{Q}}$.

In Hrushovski's constructions, the *predimension function* δ characterizes the dimension function. In this case δ does not directly give information about $\mathrm{ecl}^F(\emptyset)$, but δ and $\mathrm{ecl}^F(\emptyset)$ together determine the dimension function.

THEOREM 1.3. *For any $\bar{x}$ in F, the exponential transcendence degree satisfies:*

$$\mathrm{etd}^F(\bar{x}) = \min\{\delta(\bar{x}\bar{y}/\,\mathrm{ecl}^F(\emptyset)) \mid \bar{y} \subseteq F\}.$$

The full Schanuel property states that $\delta(\bar{x}) \geqslant 0$ for all $\bar{x}$, and under this condition we can replace $\mathrm{ecl}^F(\emptyset)$ by $\emptyset$ in the above theorems. In the complex case this is Schanuel's conjecture, which is considered out of reach. However, we can show the following theorem.

THEOREM 1.4. *There are at most countably many essential counterexamples to Schanuel's conjecture.*

The notion of an essential counterexample must be explained. A counterexample to Schanuel's conjecture is a tuple $\bar{a} = (a_1, \ldots, a_n)$ of complex numbers such that $\delta(\bar{a}) < 0$. If there exists $\bar{a}$ such that $\delta(\bar{a}) < -1$, then for any $b \in \mathbb{C}$, we have $\delta(\bar{a}b) \leqslant \delta(\bar{a}) + 1 < 0$, and so there would be continuum-many counterexamples. However, if $\delta(\bar{a}b) = \delta(\bar{a}) + 1$, then b is not contributing to the counterexample, and so we want to exclude such cases. Note also that the value of $\delta(\bar{a})$ depends only on the $\mathbb{Q}$-linear span of $\bar{a}$. We define an *essential counterexample* to be a counterexample $\bar{a}$ such that $\delta(\bar{a}) \leqslant \delta(\bar{c})$ for any tuple $\bar{c}$ from the $\mathbb{Q}$-span of $\bar{a}$. Thus, every counterexample contains an essential counterexample in its $\mathbb{Q}$-linear span.

To prove these theorems we construct derivations on exponential fields and show that they can be extended to strong extensions of these fields. This seems to be a very non-trivial fact, depending on a theorem of Ax [**1**]. The techniques in this paper can probably be extended to any collection of functions for which a similar result is known. In particular, they should work for fields with formal analogues of the Weierstrass $\wp$-functions, and the exponential maps of other semiabelian varieties, using the analogues of Ax's theorem given in [**3**].

2. *Exponential rings and fields*

In this paper, a *ring* $R = \langle R; +, \cdot\rangle$ is always commutative, with 1. We write $\mathbb{G}_{\mathrm{a}}(R)$ for the additive group $\langle R; +\rangle$ and $\mathbb{G}_{\mathrm{m}}(R)$ for the multiplicative group $\langle R^{\times}; \cdot\rangle$ of units of R.

DEFINITION 2.1. An *exponential ring* (or *E-ring*) is a ring R equipped with a homomorphism $\exp_R$ (also written exp, or $x \mapsto e^x$) from $\mathbb{G}_{\mathrm{a}}(R)$ to $\mathbb{G}_{\mathrm{m}}(R)$.

We adopt the convention that an *E-field* is an E-ring which is a field *of characteristic zero*. Furthermore, an *E-domain* is an E-ring with no zero divisors which is also a $\mathbb{Q}$-algebra.

Note that if R is an E-ring of positive characteristic p (that is, p is the least non-zero natural number such that $\underbrace{1 + \ldots + 1}_{p} = 0$), then $(e^x)^p = e^0 = 1$, for each $x \in R$. In particular, if R is a domain, then p is prime and $0 = (e^x)^p - 1 = (e^x - 1)^p$, and so the exponential map is trivial. This is the reason for the convention that E-domains and E-fields are always of characteristic zero. It will be convenient for defining strong embeddings later to insist that E-domains are $\mathbb{Q}$-algebras.

We will also need the notion of a *partial E-domain*, where the exponential map is defined only on a subgroup of $\mathbb{G}_{\mathrm{a}}(R)$. To have the most useful notion of embedding, we give the formal definition as a two-sorted structure.

DEFINITION 2.2. A *partial E-domain* is a two-sorted structure

$$\langle R, A(R); +_R, \cdot, +_A, (q\cdot)_{q\in\mathbb{Q}}, \alpha, \exp_R\rangle,$$

where $\langle R; +_R, \cdot\rangle$ is a domain, $\langle A(R); +_A, (q\cdot)_{q\in\mathbb{Q}}\rangle$ is a $\mathbb{Q}$-vector space, $\langle A(R); +_A\rangle \xrightarrow{\alpha} \langle R; +_R\rangle$ is an injective homomorphism of additive groups and $\langle A(R); +_A\rangle \xrightarrow{\exp_R} \langle R; \cdot\rangle$ is a homomorphism. We identify $A(R)$ with its image under α, and write $+$ for both $+_A$ and $+_R$.

We take the natural definitions of homomorphisms and embeddings of E-rings and partial E-domains. Thus a homomorphism of E-rings $R \xrightarrow{\varphi} S$ is a ring homomorphism that preserves the exponential map. A homomorphism of partial E-domains is a ring homomorphism such that, for each $x \in A(R)$, we have $\varphi(x) \in A(S)$ and $\exp_S(\varphi(x)) = \varphi(\exp_R(x))$. The two-sorted definition of partial E-domains means that in an embedding $R \hookrightarrow S$, it is possible to have an element $x \in A(S)$ with $x, \exp_S(x) \in R$, but $x \notin A(R)$.

The category of E-rings is defined just by functions and equations, and so there is a notion of a free E-ring. We write $\mathbb{Z}[X]^E$ for the free E-ring on a set of generators X, and call it the E-ring of *exponential polynomials* in indeterminates X. Similarly, for any E-ring R we can consider the free E-ring extension of R on a set of generators X, written $R[X]^E$, and call it the E-ring of exponential polynomials over R (or with coefficients in R). See [**4**] for an explicit construction.

3. *Exponential algebraicity*

Exponential algebraic closure is the analogue in E-domains of the notion of (relative) algebraic closure in pure domains. In a domain R, an element a is algebraic over a subring B if and only if it satisfies a non-trivial polynomial over B. In the E-domain context, we need a slightly more complicated definition.

DEFINITION 3.1. Let R be an E-domain. A *Khovanskii system* (of equations and inequations) consists of, for some $n \in \mathbb{N}$, exponential polynomials $f_1, \ldots, f_n \in R[X_1, \ldots, X_n]^E$, with equations

$$f_i(x_1, \ldots, x_n) = 0 \quad \text{for } i = 1, \ldots, n$$

and the inequation

$$\begin{vmatrix} \dfrac{\partial f_1}{\partial X_1} & \cdots & \dfrac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial X_1} & \cdots & \dfrac{\partial f_n}{\partial X_n} \end{vmatrix} (x_1, \ldots, x_n) \neq 0.$$

In the analytic context of $\mathbb{R}_{\exp}$ or $\mathbb{C}_{\exp}$, the f_i are analytic functions, and the non-vanishing of the Jacobian means that $\bar{x}$ is an *isolated* zero of the system of equations $\bar{f}(\bar{x}) = 0$. However, the notion of a Khovanskii system is purely algebraic, and so we do not need any topology to make sense of it.

DEFINITION 3.2. If B is an E-subring of R, define $a \in \mathrm{ecl}^R(B)$ if and only if there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in R$ and $f_1, \ldots, f_n \in B[X_1, \ldots, X_n]^E$ such that $a = a_1$ and $(a_1, \ldots, a_n)$ is a solution to the Khovanskii system given by the f_i. If $C \subseteq F$ is any subset, let $\hat{C}$ be the E-subring of R generated by C, and define $\mathrm{ecl}^R(C) = \mathrm{ecl}^R(\hat{C})$.

We say that $\mathrm{ecl}^R(C)$ is the *exponential algebraic closure* of C in R. If $a \in \mathrm{ecl}^R(C)$, then we say that a is *exponentially algebraic* over C in R, and otherwise that it is *exponentially transcendental* over C in R. When R is a partial E-domain, the same definition works, but we must be careful only to apply exponential polynomial functions where they are defined.

LEMMA 3.3. *If R is a partial E-domain, then ecl^R is a closure operator with finite character; that is, for any subsets C, B of R, we have the following:*
(1) $C \subseteq \mathrm{ecl}^R(C)$*;*
(2) $B \subseteq C \Rightarrow \mathrm{ecl}^R(B) \subseteq \mathrm{ecl}^R(C)$*;*
(3) $\mathrm{ecl}^R(\mathrm{ecl}^R(C)) = \mathrm{ecl}^R(C)$*;*
(4) $\mathrm{ecl}^R(C) = \bigcup\{\mathrm{ecl}^R(C_0) \mid C_0$ *is a finite subset of* $C\}$.
Furthermore, the closure of any subset is an E-subring of R, and if R is a field, then it is an E-subfield.

Proof. The proof is a straightforward exercise. □

It is also easy to see that if $R \subseteq S$ are E-domains and $C \subseteq R$, then $\mathrm{ecl}^R(C) \subseteq \mathrm{ecl}^S(C) \cap R$. However, unlike in the case of algebraic closure, this inclusion may be strict.

REMARK 3.4. On $\mathbb{R}_{\exp}$ or $\mathbb{C}_{\exp}$, there can only be countably many isolated zeros of a system of equations, and so it follows that there are only countably many exponentially algebraic numbers. It is, of course, a difficult problem to show that any number is even transcendental, and as far as I know there are no real or complex numbers which are known to be exponentially transcendental. It seems likely that the Liouville numbers are all exponentially transcendental, but that may be difficult to prove.

4. *Derivations and differentials*

Derivations play an important role in transcendence theory for pure fields. The analogous notion for exponential fields was first exploited by Wilkie. Here we define exponential derivations and differentials, in analogy with the theory of differentials in commutative algebra.

DEFINITION 4.1. Let R be a partial E-ring and M be an R-module. (There is no exponential structure on M; it is just a module in the usual sense.) A *derivation* from R to M is a map $R \xrightarrow{\partial} M$ such that, for each $a, b \in R$, we have the following:
(1) $\partial(a + b) = \partial a + \partial b$;
(2) $\partial(ab) = a\partial b + b\partial a$.

It is an *exponential derivation* or *E-derivation* if and only if also for each $a \in A(R)$, we have $\partial(\exp(a)) = \exp(a)\partial a$.

Write $\mathrm{Der}(R, M)$ for the set of all derivations from R to M, and $\mathrm{EDer}(R, M)$ for the set of all E-derivations from R to M. For any subset C of R, we write $\mathrm{Der}(R/C, M)$ and $\mathrm{EDer}(R/C, M)$, respectively, for the sets of derivations and E-derivations that vanish on C. It is easy to see that these are R-modules.

We have the *universal derivation* $R \xrightarrow{d} \Omega(R/C)$, where $\Omega(R/C)$ is the R-module generated by symbols $\{dr \mid r \in R\}$, subject only to the relations given by d being a derivation and the relations $dc = 0$ for each $c \in C$. Similarly there is a universal E-derivation, $R \xrightarrow{d} \Xi(R/C)$, where $\Xi(R/C)$ is the quotient of $\Omega(R/C)$ defined by the extra relations of an E-derivation. The universal property is that if $R \xrightarrow{\partial} M$ is any E-derivation vanishing on C, then there is a unique R-linear map ∂^* such that

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[dr, "\partial"'] & \Xi(R/C) \arrow[d, "\partial^*"] \\ & M \end{tikzcd}$$

commutes.

An important special case is when $M = R$. In this case, we write $\mathrm{Der}(R/C)$ for $\mathrm{Der}(R/C, R)$ and $\mathrm{EDer}(R/C)$ for $\mathrm{EDer}(R/C, R)$. When $C = \emptyset$, we also write $\mathrm{Der}(R)$ and $\mathrm{EDer}(R)$.

Unlike in the case of pure fields, it is not easy to see what the derivations on a given E-field are. The reason for this is that a derivation on an E-field F_1 may not extend to an extension E-field $F_2 \supseteq F_1$. This phenomenon also occurs for pure fields, but only in positive characteristic and only in one way, when giving new pth roots.

Example 4.2. Consider the extension of pure fields $\mathbb{F}_p(t) \subseteq \mathbb{F}_p(s)$, where $t = s^p$. On $\mathbb{F}_p(t)$ we have the derivation $\frac{\partial}{\partial t}$; but if ∂ is any derivation on $\mathbb{F}_p(s)$, then $\partial t = \partial(s^p) = ps^{p-1}\partial s = 0$, and so ∂ is not an extension of $\frac{\partial}{\partial t}$.

In pure fields of characteristic zero, if $F_1 \subseteq F_2$, then $\dim \mathrm{Der}(F_2/F_1) = \mathrm{td}(F_2/F_1)$. Furthermore, $a \in \mathrm{acl}(F_1)$ if and only if $\mathrm{td}(F_1(a)/F_1) = 0$ if and only if every derivation on F_2 which vanishes on F_1 also vanishes at a. By analogy, we define a closure operator cl^R on an E-domain R as follows.

Definition 4.3. For R a partial E-domain, $C \subseteq R$ and $a \in R$, define $a \in \mathrm{cl}^R(C)$ if and only if for every $\partial \in \mathrm{EDer}(R/C)$ we have $\partial a = 0$.

By the universal property of $\Xi(R/C)$, we have $a \in \mathrm{cl}^R(C)$ if and only if $da = 0$ in $\Xi(R/C)$.

Lemma 4.4. *The operator* cl^R *is a closure operator satisfying the exchange property. Furthermore, the closure of any subset is an E-subring, and if* R *is a field, then it is an E-subfield.*

Proof. It is immediate that $C \subseteq \mathrm{cl}^R(C)$; that if $C_1 \subseteq C_2$ then $\mathrm{cl}^R(C_1) \subseteq \mathrm{cl}^R(C_2)$; and that $\mathrm{cl}^R(\mathrm{cl}^R(C)) = \mathrm{cl}^R(C)$. It is also immediate that $\mathrm{cl}^R(C)$ is closed under the E-ring operations and under taking multiplicative inverses. For exchange, suppose that $a \in \mathrm{cl}^R(Cb)$ but $b \notin \mathrm{cl}^R(Ca)$. Then there is an E-derivation ∂ that vanishes on C such that $\partial a = 0$ and $\partial b = 1$.

Let $\partial' \in \mathrm{EDer}(R/C)$ and $\partial'' = \partial' - (\partial' b)\partial$. Then $\partial' a = \partial'' a$, but $\partial'' b = 0$ and $a \in \mathrm{cl}^R(Cb)$, and so $\partial'' a = 0$. Hence $\partial' a = 0$, and so $a \in \mathrm{cl}^R(C)$. □

Wilkie explicitly builds finite character into the definition of cl^R to give a pregeometry. In fact this is not necessary, as finite character holds already.

Proposition 4.5. *Suppose R is a partial E-domain, $C \subseteq R$ and $a \in \mathrm{cl}^R(C)$. Then there is a finite subset C_0 of C and a finitely generated partial E-subring R_0 of R such that $a \in \mathrm{cl}^{R_0}(C_0)$. Furthermore, cl^R has finite character, and is a pregeometry.*

Proof. We have $da = 0$ in $\Xi(R/C)$. We use a simple compactness argument. Let L be a formal language with a constant symbol for each finite sum $\sum r_i ds_i$ with the $r_i, s_i \in R$. Let T be the L-theory consisting of all instances of the axioms saying that these symbols represent elements of the R-module $\Xi(R/C)$, that is, the axioms of an R-module, the axioms saying that d is an E-derivation, and the axioms $dc = 0$ for each $c \in C$. Then $T \vdash da = 0$. Hence, by compactness there is a finite subtheory T_0 of T such that $T_0 \vdash da = 0$. Let C_0 be the subset of C consisting of those c such that the axiom $dc = 0$ appears in T_0. Let R_0 be the partial E-subring of R generated by all the $r \in R$ that occur in some axiom of T_0. Then we must have $da = 0$ in $\Xi(R_0/C_0)$, and also in $\Xi(R/C_0)$. Thus $a \in \mathrm{cl}^{R_0}(C_0)$, and $a \in \mathrm{cl}^R(C_0)$, which gives the finite character of cl^R. Lemma 4.4 shows that cl^R satisfies the other axioms of a pregeometry. □

We now begin to relate our closure operators cl^R and ecl^R.

Lemma 4.6. *The quotient $\Xi(R/C)$ can also be characterized as the R-module generated by the symbols $\{dr \mid r \in R\}$ subject to the relations*

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{r})\, dr_i = 0 \tag{*}$$

for each $f \in C[\bar{X}]^E$ and tuple $\bar{r}$ from R such that $f(\bar{r}) = 0$.

Proof. The relation $d(x+y) = dx + dy$ comes from $f = X_1 + X_2 - X_3$, and similarly for the other basic relations axiomatizing E-derivations. Conversely, the relations $(*)$ follow from the axioms of E-derivations by the chain rule. □

Proposition 4.7. *Let R be a partial E-domain and C be a subset of R. Then $\mathrm{ecl}^R(C) \subseteq \mathrm{cl}^R(C)$.*

Proof. Both closures of C are E-subrings of R, and so we may assume that C is an E-subring. Suppose $a_1, \ldots, a_n \in \mathrm{ecl}^R(C)$, as witnessed by being a solution to the Khovanskii system formed by $f_1, \ldots, f_n \in C[X_1, \ldots, X_n]^E$. Suppose $\partial \in \mathrm{Der}(R/C)$, and let J be the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix} (\bar{a}).$$

Then by Lemma 4.6,

$$J\begin{pmatrix}\partial a_1\\ \vdots \\ \partial a_n\end{pmatrix} = 0.$$

Since $\bar{a}$ solves the Khovanskii system, the determinant $|J| \neq 0$, and so J has an inverse with coefficients in the field of fractions of R. Clearing denominators, for some non-zero $r \in R$ the matrix rJ^{-1} has coefficients in R. Then

$$r\begin{pmatrix}\partial a_1\\ \vdots \\ \partial a_n\end{pmatrix} = 0$$

and hence $\partial a_i = 0$ for each i, as R is a domain. So each a_i lies in $\mathrm{cl}^R(C)$. □

It will be useful to have a stronger form of Lemma 4.6 for finitely generated extensions of partial E-fields, where we consider only relations between the chosen generators.

LEMMA 4.8. *Suppose $C \hookrightarrow F$ is an inclusion of partial E-fields, $a_1, \dots, a_n$ is a $\mathbb{Q}$-linear basis for $A(F)$ over $A(C)$ and F is generated as a field by $A(F) \cup \exp(A(F))$. Then $\Xi(F/C)$ is the F-vector space generated by $da_1, \dots, da_n$ subject to the relations*

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{a})\, da_i = 0 \tag{*}$$

for each $f \in C[\bar{X}]^E$ such that $f(\bar{a}) = 0$.

Proof. The differentials de^{a_i} satisfy $de^{a_i} = e^{a_i} da_i$, and so are in the span of the da_i. The field F is algebraic over $C(\bar{a}, e^{\bar{a}})$; thus these differentials span $\Omega(F/C)$, and hence they certainly span $\Xi(F/C)$. We must show that the basic relations axiomatizing E-derivations follow from the relations $(*)$. All of the exponential relations $de^b = e^b db$ follow from those for the a_i by $\mathbb{Q}$-linearity. We are left with the algebraic relations between the elements of F. Suppose

$$p\left(\frac{f_1(\bar{a})}{g_1(\bar{a})}, \dots, \frac{f_m(\bar{a})}{g_m(\bar{a})}\right) = 0 \tag{†}$$

with $p \in C[Y_1, \dots, Y_m]$ and the f_i, g_i exponential polynomials, with $g_i(\bar{a}) \neq 0$. Clearing the denominators, we get an exponential polynomial $h(\bar{X})$ such that (†) is equivalent to $h(\bar{a}) = 0$, and so

$$d\,[p(f_1(\bar{a})/g_1(\bar{a}), \dots, f_m(\bar{a})/g_m(\bar{a}))] = 0 \iff d[h(\bar{a})] = 0,$$

but this is if and only if $\sum_{i=1}^{n} \frac{\partial h}{\partial X_i}(\bar{a})\, da_i = 0$, which is of the form $(*)$. Thus the relations of the form $(*)$ are enough to characterize $\Xi(F/C)$. □

5. *Strong extensions*

We need the following theorem of Ax.

THEOREM 5.1 [**1**, Theorem 3]. *Let F be a field of characteristic 0, let Δ be a set of derivations on F and let $C = \bigcap_{\partial \in \Delta} \ker \partial$ be the field of constants. Suppose $x_1, \dots, x_n, y_1, \dots, y_n \in F$*

satisfy $\partial y_i = y_i \partial x_i$ *for each* $i = 1, \ldots, n$ *and each* $\partial \in \Delta$. *Then*

$$\operatorname{td}(\bar{x}, \bar{y}/C) \geqslant \operatorname{ldim}_{\mathbb{Q}}(\bar{x}/C) + \operatorname{rk}(\partial x_i)_{\partial \in \Delta, i=1,\ldots,n}$$

Corollary 5.2. *Let* F *be an E-field, and suppose* $C \subseteq F$ *is* cl^F*-closed. Let* $x_1, \ldots, x_n \in F$. *Then*

$$\operatorname{td}(\bar{x}, \exp(\bar{x})/C) - \operatorname{ldim}_{\mathbb{Q}}(\bar{x}/C) \geqslant \dim^F(\bar{x}/C),$$

where $\dim^F(\bar{x}/C)$ *is the dimension in the sense of the pregeometry* cl^F.

Proof. Taking $\Delta = \operatorname{EDer}(F/C)$ and $y_i = \exp(x_i)$, all the differential equations $\partial y_i = y_i \partial x_i$ for $\partial \in \Delta$ are satisfied. Also $C = \bigcap_{\partial \in \Delta} \ker \partial$ because C is cl^F-closed. We can find $x_{i_1}, \ldots, x_{i_m}$ among the x_i, where $m = \dim^F(\bar{x}/C)$, and the derivations $\partial_1, \ldots, \partial_m \in \Delta$ such that $\partial_j x_{i_k} = \delta_{jk}$, the Kronecker delta. Thus $\operatorname{rk}(\partial x_i)_{\partial \in \Delta, i=1,\ldots,n} = m$. We apply Ax's theorem. □

Now let R be any partial E-domain. For any tuple $\bar{x}$ and subset B of $A(R)$, we define

$$\delta(\bar{x}/B) = \operatorname{td}(\bar{x}, \exp(\bar{x})/B, \exp(B)) - \operatorname{ldim}_{\mathbb{Q}}(\bar{x}/B)$$

which, following Hrushovski, we call the *predimension* of $\bar{x}$ over B. Note that if $B = \bar{b}$ is finite, then we have the useful addition formula $\delta(\bar{x}/\bar{b}) = \delta(\bar{x}\bar{b}/0) - \delta(\bar{b}/0)$.

Definition 5.3. We say an embedding $R_1 \hookrightarrow R_2$ of partial E-domains is *strong*, and write $R_1 \lhd R_2$, if and only if for every tuple $\bar{x}$ from $A(R_2)$, we have $\delta(\bar{x}/A(R_1)) \geqslant 0$.

More generally, if B is any subset of $A(R)$ for a partial E-domain R, we say that B is strong in R, and write $B \lhd R$, if and only if for every tuple $\bar{x}$ from $A(R)$, we have $\delta(\bar{x}/B) \geqslant 0$.

Not all E-field extensions are strong. For example, $\mathbb{R}_{\exp} \subseteq \mathbb{C}_{\exp}$ is not strong, since $\delta(i/\mathbb{R}) = \operatorname{td}(i, e^i/\mathbb{R}) - \operatorname{ldim}_{\mathbb{Q}}(i/\mathbb{R}) = 0 - 1 = -1$. This example can be generalized to show that any proper algebraic extension, or even one of finite transcendence degree, cannot be strong.

Lemma 5.4. *If* $F_0 \lhd F$ *is a strong extension of total E-fields and* $\operatorname{td}(F/F_0)$ *is finite, then* $F = F_0$.

Proof. Suppose F is a proper strong extension of F_0. Choose a $\mathbb{Q}$-linear basis $\{b_i \mid i \in I\}$ for F over F_0. Then

$$\operatorname{td}(F/F_0) \geqslant \operatorname{td}\left(\left\{b_i, e^{b_i} \mid i \in I\right\}/F_0\right) \geqslant |I|,$$

which means that I is finite; but then $I = \emptyset$ or F is a finite extension of $\mathbb{Q}$, in particular algebraic, and so $\operatorname{td}(F/F_0) = 0$, in which case $I = \emptyset$ anyway. Thus $F = F_0$. □

However, if we allow partial exponential fields, every strong extension can be split up into a chain of strong extensions of finite transcendence degree. To show this we need some basic properties of strong extensions, which are left as a straightforward exercise.

Lemma 5.5. *For ordinals* α, *let* R_α *be partial E-domains. Then the following conditions hold.*

(i) *The identity* $R_1 \hookrightarrow R_1$ *is strong.*

(ii) *If $R_1 \lhd R_2$ and $R_2 \lhd R_3$, then $R_1 \lhd R_3$. (That is, the composite of strong extensions is strong.)*

(iii) *Suppose λ is an ordinal, $(R_\alpha)_{\alpha<\lambda}$ is a λ-chain of strong extensions (that is, for each $\alpha \leqslant \beta < \lambda$ there is a strong extension $f_{\alpha,\beta} : R_\alpha \lhd R_\beta$ and for all $\alpha \leqslant \beta \leqslant \gamma$, we have $f_{\beta,\gamma} \circ f_{\alpha,\beta} = f_{\alpha,\gamma}$ and $f_{\alpha,\alpha}$ is the identity on R_α), and R is the union of the chain. Then $R_\alpha \lhd R$ for each α.*

(iv) *Suppose $(R_\alpha)_{\alpha<\lambda}$ is a λ-chain of strong extensions with union R, and $R_\alpha \lhd S$ for each α. Then $R \lhd S$.*

Proposition 5.6. *Suppose $F_0 \lhd F$ is a strong extension of partial E-fields, and F_0, F are* exponential-graph-generated, *that is, they are generated as fields by the graphs of their exponential maps, $A(F) \cup \exp(A(F))$, and similarly for F_0. Then for some ordinal λ there is a chain $(F_\alpha)_{\alpha\leqslant\lambda}$ of partial E-domains such that, for all ordinals $0 \leqslant \alpha \leqslant \beta \leqslant \lambda$, the following conditions hold.*

(1) *$F = F_\lambda$;*
(2) *F_α is exponential-graph-generated;*
(3) *for limit β, we have $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$;*
(4) *$\mathrm{td}(F_{\beta+1}/F_\beta)$ is finite;*
(5) *$F_\alpha \lhd F_\beta$.*

Proof. Let λ be the initial ordinal of cardinality $|A(F)|$, and list $A(F)$ as $(r_\alpha)_{\alpha<\lambda}$. We inductively construct F_β satisfying (1)–(5) and such that $r_\beta \in F_{\beta+1}$ and $F_\beta \lhd F$.

At a limit stage β, define $A(F_\beta) = \bigcup_{\alpha<\beta} A(F_\alpha)$. Take F_β to be the partial E-subfield of F generated by $A(F_\beta)$, and so (2) and (3) hold. Condition (5) holds by part (iii) of Lemma 5.5, and $F_\beta \lhd F$ by part (iv) of Lemma 5.5.

For a successor $F_{\beta+1}$, if $r_\beta \in A(F_\beta)$, then take $F_{\beta+1} = F_\beta$. Otherwise, by induction $F_\beta \lhd F$, and so, for any finite tuple $\bar{x}$ from $A(F)$, we have $\delta(\bar{x}/F_\beta) \geqslant 0$. Choose a tuple $\bar{x}$ containing r_β such that $\delta(\bar{x}/R_\beta)$ is minimal. Let $A(F_{\beta+1})$ be the $\mathbb{Q}$-subspace of $A(F)$ generated by $A(F_\beta)$ and $\bar{x}$, and take $F_{\beta+1}$ to be the partial E-subfield of F generated by $A(F_\beta)$. By the minimality of $\delta(\bar{x}/F_\beta)$, we have $F_{\beta+1} \lhd F$. For any $\alpha \leqslant \beta$, since $F_\alpha \lhd F$, it follows that $F_\alpha \lhd F_{\beta+1}$. Also $\mathrm{td}(F_{\beta+1}/F_\beta) \leqslant 2\,|\bar{x}|$, which is finite, and so (4) holds. Finally, $\bigcup_{\alpha<\lambda} A(F_\alpha) = A(F)$, and so $F = F_\lambda$. □

6. *Extending derivations*

Let $F_0 \subseteq F$ be an extension of (pure) fields and let $\partial \in \mathrm{Der}(F_0)$. There are spaces of differentials $\Omega(F)$ and $\Omega(F/F_0)$ appropriate for considering all derivations on F and those that vanish on F_0. We construct an intermediate space of differentials appropriate for considering extensions of ∂ to F.

Definition 6.1. Let $\Omega(F/\partial)$ be the quotient of $\Omega(F)$ by the relations $\sum a_i db_i = 0$ for those $a_i, b_i \in F_0$ such that $\sum a_i \partial b_i = 0$.

We naturally have quotient maps

$$\Omega(F) \twoheadrightarrow \Omega(F/\partial) \twoheadrightarrow \Omega(F/F_0).$$

Lemma 6.2. *Let* $\mathrm{Der}(F/\partial) = \{\eta \in \mathrm{Der}(F) \mid (\exists \lambda \in F)\eta\restriction_{F_0} = \lambda\partial\}$. *Then* $\mathrm{Der}(F/\partial)$ *is the dual space of* $\Omega(F/\partial)$.

Proof. Suppose $\eta \in \mathrm{Der}(F/\partial)$. Then, for each relation $\sum a_i \partial b_i = 0$, we have $\sum a_i \eta b_i = \lambda \sum a_i \partial b_i = 0$, and so η factors as

$$F \xrightarrow{d} \Omega(F/\partial) \xrightarrow{\eta^*} F$$

for some F-linear map η^*. Now if $\Omega(F/\partial) \xrightarrow{\eta^*} F$ is any F-linear map, define $\eta = \eta^* \circ d$. Then $\eta \in \mathrm{Der}(F)$ and we must show $\eta \in \mathrm{Der}(F/\partial)$. If $\partial b = 0$ for some $b \in F_0$, then the relation $db = 0$ holds in $\Omega(F/\partial)$ and so $\eta b = 0$. If this holds for all $b \in F_0$, then we are done. Otherwise choose $b_0 \in F_0$ such that $\partial b_0 \neq 0$ and let $\lambda = \eta b_0 / \partial b_0$. Let $b \in F_0$, and write $b' = \partial b$ and $b'_0 = \partial b_0$. Then $b'_0 \partial b - b' \partial b_0 = 0$; thus $b'_0 db - b' db_0 = 0$ in $\Omega(F/\partial)$, and so $b'_0 \eta b - b' \eta b_0 = 0$, that is, $\eta b = \lambda \partial b$. Hence $\eta\restriction_{F_0} = \lambda\partial$ and so $\eta \in \mathrm{Der}(F/\partial)$. □

Theorem 6.3. *Suppose* $F_0 \lhd F$ *is a strong extension of partial E-fields and* F_0 *is exponential-graph-generated. Then every E-derivation on* F_0 *extends to* F.

Proof. Let F' be the partial E-subfield of F generated by the graph of exponentiation of F. Then every E-derivation on F' extends to F, as only the field operations must be respected and the characteristic is zero. Hence we may assume $F = F'$. Now by Proposition 5.6 it is enough to prove the theorem for extensions of exponential-graph-generated partial E-fields $F_1 \lhd F_2$ such that $\mathrm{td}(F_2/F_1)$ is finite. Let ∂ be an E-derivation on F_1. Let $\mathrm{EDer}(F_2/\partial) = \mathrm{Der}(F_2/\partial) \cap \mathrm{EDer}(F_2)$.

Let $a_1, \ldots, a_n$ be a $\mathbb{Q}$-basis for $A(F_2)$ over $A(F_1)$ and let $\omega_i = de^{a_i}/e^{a_i} - da_i \in \Omega(F_2)$. Let $\hat{\omega}_i$ be the image of ω_i in $\Omega(F_2/F_1)$ under the natural quotient map $\Omega(F_2) \twoheadrightarrow \Omega(F_2/F_1)$.

We use the following intermediate step in the proof of Ax's theorem (Theorem 5.1 of this paper). Although this statement is not isolated in Ax's paper, it can be obtained from his proof. It is also the special case of [**3**, Proposition 3.7], where the group S is $\mathbb{G}_{\mathrm{m}}{}^n$.

Fact 6.4. If the differentials $\hat{\omega}_1, \ldots, \hat{\omega}_n$ are F_2-linearly dependent in $\Omega(F_2/F_1)$ then there is a non-zero $\mathbb{Z}$-linear combination $b = \sum_{i=1}^n m_i a_i$ such that b and e^b are both algebraic over F_1.

Hence if the $\hat{\omega}_i$ are F_2-linearly dependent, then, for some such b, we have

$$\begin{aligned}\delta(b/F_1) &= \mathrm{td}(b, e^b / A(F_1) \cup \exp(A(F_1))) - \mathrm{ldim}_{\mathbb{Q}}(b/A(F_1)) \\ &= \mathrm{td}(b, e^b/F_1) - \mathrm{ldim}_{\mathbb{Q}}(b/A(F_1)) \\ &= 0 - 1 < 0,\end{aligned}$$

which contradicts $F_1 \lhd F_2$. Thus, the $\hat{\omega}_i$ are F_2-linearly independent in $\Omega(F_2/F_1)$.

Let Λ be the F_2-subspace of $\Omega(F_2)$ generated by $\omega_1, \ldots, \omega_n$. The space of derivations $\mathrm{Der}(F_2)$ is the dual space of $\Omega(F_2)$, and so we can consider the annihilator of Λ in it. By definition of Λ, we have $\mathrm{EDer}(F_2/F_1) = \mathrm{Der}(F_2/F_1) \cap \mathrm{Ann}(\Lambda)$ and $\mathrm{EDer}(F_2/\partial) = \mathrm{Der}(F_2/\partial) \cap \mathrm{Ann}(\Lambda)$. We have shown that the image of Λ in $\Omega(F_2/F_1)$ has dimension n, and hence the image of Λ in $\Omega(F_2/\partial)$ also has dimension n. The subspaces $\mathrm{Der}(F_2/F_1)$ and $\mathrm{Der}(F_2/\partial)$ of $\mathrm{Der}(F_2)$ are dual to the quotients $\Omega(F_2/F_1)$ and $\Omega(F_2/\partial)$ of $\Omega(F_2)$, and hence $\mathrm{Ann}(\Lambda)$ has codimension n in $\mathrm{Der}(F_2/F_1)$, and also in $\mathrm{Der}(F_2/\partial)$.

If $\partial = 0$, then the result is trivial. Otherwise, $\dim \mathrm{Der}(F_2/\partial) = \dim \mathrm{Der}(F_2/F_1) + 1$, and so $\dim \mathrm{EDer}(F_2/\partial) = \dim \mathrm{EDer}(F_2/F_1) + 1$. Thus there is $\eta \in \mathrm{EDer}(F_2/\partial) \smallsetminus \mathrm{EDer}(F_2/F_1)$. Then $\eta\restriction_{F_1} = \lambda\partial$ for some non-zero λ. Let $\eta' = \lambda^{-1}\eta$. Then η' extends ∂ to F_2. □

If F is an E-field with $\mathrm{EDer}(F) = \{0\}$, then of course this zero derivation extends to any E-field extension. Not all E-field extensions of F are strong, and so the converse to the theorem is false.

7. *Proofs of the main theorems*

Proposition 7.1. *If F is a partial E-field and C is a subset of F, then $\mathrm{cl}^F(C) \subseteq \mathrm{ecl}^F(C)$.*

Proof. Suppose $a \in \mathrm{cl}^F(C)$. We may assume that F is exponential-graph-generated, that C is finite and that $C \subseteq A(F)$. Now a must be algebraic over the graph of exponentiation, and we know that both $\mathrm{cl}^F(C)$ and $\mathrm{ecl}^F(C)$ are relatively algebraically closed in F, and so it is enough to prove the proposition for a in the graph of exponentiation. Also, replacing a by some a' with $\exp(a') = a$ if necessary, we may assume that $a \in A(F)$.

By Proposition 4.5, there is F_1, an exponential-graph-generated partial E-subfield of F such that $A(F_1)$ contains C and a and is finitely generated, and such that $a \in \mathrm{cl}^{F_1}(C)$. Choose such an F_1 with $\mathrm{ldim}_{\mathbb{Q}}(A(F_1)/C)$ minimal.

Let $F_2 = \mathrm{cl}^{F_1}(C)$. We claim that $F_2 = F_1$. Certainly $a \in F_2$, and so if $F_2 \neq F_1$, then by minimality of F_1 we have $a \notin \mathrm{cl}^{F_2}(C)$. Thus there is an E-derivation $\partial \in \mathrm{EDer}(F_2/C)$ that does not extend to F_1. Then, by Theorem 6.3, we have $F_2 \not\triangleleft F_1$, but that contradicts Corollary 5.2 since F_2 is cl^{F_1}-closed in F_1. Hence $F_2 = F_1$.

By Lemma 4.8, we have that $\Xi(F_1/C)$ is generated by $da_1, \ldots, da_n$, subject to the relations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(\bar{a})\, da_i = 0$$

for $f \in C[\bar{X}]^E$ such that $f(\bar{a}) = 0$ in F_1. Since $F_1 = \mathrm{cl}^{F_1}(C)$, we have $\Xi(F_1/C) = 0$. Hence, we can choose $f_1, \ldots, f_n$ such that the matrix $J = \left(\frac{\partial f_j}{\partial X_i}(\bar{a})\right)_{i,j=1}^{n}$ has rank n, that is, it is non-singular. Thus $a \in \mathrm{ecl}^{F_1}(C)$.

Now $F_1 \subseteq F$, and so by the remark after Lemma 3.3, we have $\mathrm{ecl}^{F_1}(C) \subseteq \mathrm{ecl}^F(C)$. Hence $a \in \mathrm{ecl}^F(C)$, as required. □

Together with Proposition 4.7, that completes the proof that $\mathrm{ecl}^F = \mathrm{cl}^F$ for any partial E-field F, and it follows that ecl^F is a pregeometry. Theorem 1.1 is established, and Theorem 1.2 follows from Corollary 5.2.

Proof of Theorem 1.3. Let $C = \mathrm{ecl}^F(\emptyset)$ and choose $\bar{y}$ such that $r := \delta(\bar{x}\bar{y}/C)$ is minimal. Let F_0 be the exponential-graph-generated partial E-field extension of C with $A(F_0)$ generated by $\bar{x}\bar{y}$ over C. Using the notation from the proof of Theorem 6.3, we obtain $\mathrm{EDer}(F_0/C) = \mathrm{Der}(F_0/C) \cap \mathrm{Ann}(\Lambda)$, but $\mathrm{Ann}(\Lambda)$ has codimension $\mathrm{ldim}_{\mathbb{Q}}(\bar{x}\bar{y}/C)$ in $\mathrm{Der}(F_0/C)$ by Fact 6.4, hence

$$\mathrm{ldim}_{F_0} \mathrm{EDer}(F_0/C) = \mathrm{td}(F_0/C) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}\bar{y}/C) = \delta(\bar{x}\bar{y}/C) = r.$$

Since $\bar{y}$ is chosen with minimal δ, we have $F_0 \triangleleft F$, and so, by Theorem 6.3, these E-derivations all extend to F. Hence $\mathrm{etd}^F(\bar{x}) \geqslant r$. Then, by Theorem 1.2, we have $\mathrm{etd}^F(\bar{x}) = r$, as required. □

To prove Theorem 1.4, we give a more general result.

PROPOSITION 7.2. *In any partial E-field F, if $\bar{a}$ is an essential counterexample to the Schanuel property, then $\bar{a}$ is contained in* $\operatorname{ecl}^F(\emptyset)$.

Proof. Let $\bar{a}$ be a tuple from F, write $\langle \bar{a} \rangle_{\mathbb{Q}}$ for its $\mathbb{Q}$-linear span, let $B = \langle \bar{a} \rangle_{\mathbb{Q}} \cap \operatorname{ecl}^F(\emptyset)$ and suppose that $\bar{a} \not\subseteq \operatorname{ecl}^F(\emptyset)$; thus $\langle \bar{a} \rangle_{\mathbb{Q}} \neq B$. Then

$$\operatorname{td}(\bar{a}, \exp(\bar{a})/B, \exp(B)) \geqslant \operatorname{td}(\bar{a}, \exp(\bar{a})/\operatorname{ecl}^F(\emptyset))$$

and $\operatorname{ldim}_{\mathbb{Q}}(\bar{a}/B) = \operatorname{ldim}_{\mathbb{Q}}(\bar{a}/\operatorname{ecl}^F(\emptyset))$. Hence

$$\delta(\bar{a}/B) \geqslant \delta(\bar{a}/\operatorname{ecl}^F(\emptyset)) \geqslant \operatorname{etd}^F(\bar{a}) \geqslant 1.$$

and thus $\delta(B) = \delta(\bar{a}) - \delta(\bar{a}/B) < \delta(\bar{a})$; hence $\bar{a}$ is not an essential counterexample. □

Now, by Remark 3.4, we have that $\operatorname{ecl}^{\mathbb{C}}(\emptyset)$ is countable, which proves Theorem 1.4.

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