

ORBITS IN FINITE INCIDENCE STRUCTURES

1. INTRODUCTION

A finite incidence structure (S, X, I) consists of finite sets S and X , the points and blocks of the structure, where the matrix I defines the incidence of a point with a block. In this paper we consider tactical decompositions of a structure into point classes and block classes. The interest in tactical decompositions stems from the fact that they are generalizations of the orbit partitions under the action of a collineation group. Therefore the results in this paper apply in particular to groups of automorphisms of a finite structure.

Let P_S be an arbitrary partition of S and P_X an arbitrary partition of X . In Theorem 2.3 we construct a refinement (P_S^*, P_X^*) which is a tactical decomposition of the structure. We show that (P_S^*, P_X^*) is unique with respect to the minimality of the number of classes in P_S^* and P_X^* . The algorithm that produces this tactical decomposition only uses some elementary linear algebra and does not involve automorphisms of the incidence structure. This suggests that tactical decompositions might be of particular interest for the investigation of rigid structures admitting only the identity collineation.

In Section 3 we consider structures in which the point cardinality is equal to the rank of I . Here we prove, that for a tactical decomposition the block partition entirely determines the point classes. Our result (Theorem 3.1) contains a number of well-known theorems. If the structure under consideration is a 2-design, for instance, we obtain a generalization and an independent proof of the Dembowski–Parker–Hughes theorem. In the case of the subset lattice of a finite set the theorem improves a now classical result by Livingstone and Wagner. A similar orbit theorem is obtained for the subspace lattice of finite affine and projective geometries. In a forthcoming paper [13] I have proved that certain general classes of finite graphs have maximal incidence rank. Here we obtain yet another orbit theorem.

In Section 4, finally, we investigate the relationship between the point action and the block action of an automorphism of the structure. Here we show that again the block permutation completely determines the action on points if the rank of I is the cardinality of S . In Theorem 4.2 we give a formula for the number of point orbits of given length in terms of the block permutation.

2. TACTICAL DECOMPOSITIONS

Let S and X be finite sets and I a $(0, 1)$ -incidence matrix where the rows of I are indexed by S and the columns by X . We call the elements of S points and

the elements of X blocks. We say ‘ s is incident with x ’, and denote this by sIx , in the case that the (s, x) -entry of I is 1. The triple (S, X, I) is called a finite incidence structure.

For a field R of characteristic zero we define the space RS as the vector space over R whose basis vectors are the points in S . In a similar way we define the block space RX . Two linear maps arise naturally from the incidence relation: $\partial^- : RX \rightarrow RS$, given by $\partial^-(x) = \sum_{sIx} s$ for all x in X and $\partial^+ : RS \rightarrow RX$, given by $\partial^+(s) = \sum_{sIx} x$ for all $s \in S$. The matrices of ∂^- and ∂^+ (for the bases S and X) are, of course, the incidence matrix I and its transpose I' , respectively.

Let P_S be a partition of S into disjoint classes $P_{S,1}, \dots, P_{S,i}, \dots$ and define the vector p_i in RS as the sum over all points in the class $P_{S,i}$. As P_S is a partition, the p_i 's are linearly independent. Hence they form a basis of the subspace $RP_S = \langle p_i | i = 1, \dots \rangle$ of RS which corresponds to the partition P_S . Vice versa, we call a subspace of RS a partition space, provided it has a basis p_1, \dots, p_i, \dots such that the p_i 's have the above form for some partition of S . It is crucial for our argument and easily verifiable that partitions and partition spaces are in one-one correspondence so that P_S is uniquely determined by RP_S . The partial order relation between two partitions P_S and P'_S of the point set is given by $P_S \leq P'_S$ if and only if every class of P'_S is a subclass of some class in P_S . We say P'_S is finer than or equal to P_S in this case. Clearly $P_S \leq P'_S$ is equivalent to $RP_S \subseteq RP'_S$.

For a partition P_X of the block set into disjoint classes $P_{X,1}, \dots, P_{X,j}, \dots$ the vectors $q_j = \sum_{x \in P_{X,j}} x$ and the partition space $RP_X = \langle q_j | j = 1, \dots \rangle \subseteq RX$ are defined in the same fashion. In the following we consider pairs (P_S, P_X) of partitions of S and X respectively.

DEFINITION. A pair (P_S, P_X) is a *tactical decomposition* of the incidence structure (S, X, I) provided the following two dual conditions are satisfied:

- T^- : For any given point s and any j , the number of blocks in $P_{X,j}$ incident with s only depends on the class $P_{S,i}$ that contains s .
- T^+ : For any given block x and any i , the number of points in $P_{S,i}$ incident with x only depends on the class $P_{X,j}$ that contains x .

Tactical decompositions (for which also the term *generalized orbits* is in common use) were first investigated by Dembowski [5] and are characterized by the following invariance conditions:

PROPOSITION 2.1. *The pair (P_S, P_X) satisfies T^- if and only if $\partial^-(RP_X)$ is a subspace of RP_S . Similarly, (P_S, P_X) satisfies T^+ if and only if $\partial^+(RP_S)$ is contained in RP_X .*

Proof. Let $q = \sum_{x \in P_{X,j}} x$ be a basis vector of RP_X and $\partial^-(q) = \sum_{x \in P_{X,j}} \sum_{sIx} s = v$. We arrange this sum into $v = \sum_{s \in S} v_s \cdot s = \sum_i \sum_{s \in P_{S,i}} v_s \cdot s$, ($v_s \in R$), using the fact that P_S is a partition of S . We observe that v is contained in RP_S if and

only if $v_s = v_{s'}$, whenever s and s' belong to the same class. This, on the other hand, is equivalent to the condition T^- . The second part of the proof is just the dual of the first. \square

We will use the proposition to calculate tactical decompositions. First, however, we need the concept of the point partition generated by a subset W of RS .

Let $v = \sum_{s \in S} v_s \cdot s$ be any vector in W . We write v uniquely as $\sum_i r_i \cdot \sum_k s_{ik}$ in such a way that each point in S occurs exactly once among the s_{ik} and $r_i \neq r_{i'}$, if $i \neq i'$. Thus v produces a partition $P_S(v)$ of S with classes $\{s_{1,k} | k = 1, \dots\}, \dots, \{s_{i,k} | k = 1, \dots\}$ and so on. If we interpret v as an R -valued function on S , the classes of $P_S(v)$ are precisely the level surfaces of v , i.e. the point sets where v has constant value r_i (compare also to chapter 10 in Wielandt [14]).

Now let $P_S(W)$ be the coarsest partition of S for which $P_S(v) \leq P_S(W)$ for all $v \in W$. Let $L(W)$ denote the partition space $RP_S(W)$. The following is an immediate consequence of these definitions.

PROPOSITION 2.2. *Let $W \subseteq RS$ and RP_S some partition space containing W . Then $W \subseteq L(W) \subseteq RP_S$ and $W = L(W)$ if and only if W is a partition space itself.*

Subspaces U of RX define partitions of X and partition spaces $L(U)$ in the same way. We can now state the main result of this section.

THEOREM 2.3. *Let (S, X, I) be a finite incidence structure and let P_S and P_X be partitions of the points and blocks, respectively. Then there is a unique tactical decomposition (P_S^*, P_X^*) with minimal class numbers satisfying $P_S \leq P_S^* \leq P'_S$ and $P_X \leq P_X^* \leq P'_X$ for any tactical decomposition (P'_S, P'_X) with $P_X \leq P'_S$ and $P_X \leq P'_X$.*

Proof. The tactical decomposition (P_S^*, P_X^*) can be constructed in the following fashion. Since $P_S \leq P'_S$ and $P_X \leq P'_X$, we obtain $RP_S \subseteq \partial^-(RP'_X) + RP_S \subseteq RP'_S$ and $RP_X \subseteq \partial^+(RP_S) + RP_X \subseteq RP'_X$ by Proposition 2.1, as (P'_S, P'_X) is a tactical decomposition. By Proposition 2.2, we also have $RP_S \subseteq L(\partial^-(RP'_X) + RP_S) \subseteq RP'_S$ and $RP_X \subseteq L(\partial^+(RP_S) + RP_X) \subseteq RP'_X$. We now define P_S^1 by $RP_S^1 = L(\partial^-(RP'_X) + RP_S)$ and P_X^1 by $RP_X^1 = L(\partial^+(RP_S) + RP_X)$. Thus we obtain $RP_S \subseteq RP_S^1 \subseteq L(\partial^-(RP_X^1) + RP_S^1) \subseteq RP'_S$ and $RP_X \subseteq RP_X^1 \subseteq L(\partial^+(RP_S^1) + RP_X^1) \subseteq RP'_X$ as above. Inductively, let P_S^i and P_X^i be given by $RP_S^i = L(\partial^-(RP_X^{i-1}) + RP_S^{i-1})$ and $RP_X^i = L(\partial^+(RP_S^{i-1}) + RP_X^{i-1})$, respectively. In this way, we obtain two ascending chains $RP_S \subseteq RP_S^1 \subseteq \dots \subseteq RP_S^i \subseteq \dots \subseteq RP'_S$ and $RP_X \subseteq RP_X^1 \subseteq \dots \subseteq RP_X^i \subseteq \dots \subseteq RP'_X$, which become stationary as all dimensions are finite. Thus, for $RP_S^k = RP_S^{k+1}$ and $RP_X^k = RP_X^{k+1}$ we conclude $\partial^-(RP_X^k) \subseteq RP_S^k$ and $\partial^+(RP_S^k) \subseteq RP_X^k$. Hence, according to Proposition 2.1, $P_S^* = P_S^k$ and $P_X^* = P_X^k$ are the partitions of a tactical decomposi-

tion. Uniqueness now easily follows from the fact that $P_S^* \leq P'_S$ and $P_X^* \leq P'_X$ hold for an arbitrary tactical decomposition finer than or equal to (P_S, P_X) . This completes the proof. \square

3. ORBIT THEOREMS

For any incidence structure, the orbits on points and blocks of an automorphism group G form a tactical decomposition. We denote point orbits by $S(G)$ and block orbits by $X(G)$, where G is an arbitrary group of automorphisms of the structure. Clearly the partitions $P_S = \{S\}$ and $P_X = \{X\}$ (consisting of one class each) satisfy $P_S \leq S(G)$ and $P_X \leq X(G)$. According to Theorem 2.3, the tactical refinement (P_S^*, P_X^*) of (P_S, P_X) has the property $P_S^* \leq S(G)$ and $P_X^* \leq X(G)$ for any group of automorphisms. I wish to emphasize at this point that this minimal tactical decomposition is obtained quite independently of automorphisms. Therefore (P_S^*, P_X^*) and, in particular, their class numbers are invariants related to the symmetries of an incidence structure which so far have not been examined. The class number $|P_S^*|$, for instance, is a lower bound for the number of point orbits of all automorphisms. In general, however, one cannot conclude that this bound is attained by any automorphism group: in many instances tactical decompositions cannot be associated to automorphisms. This fact has already been pointed out in Dembowski's paper [5] in the case of projective planes.

In this section we consider a given tactical decomposition of a finite structure and study the relationship between its point and orbit partitions. In many geometrical situations the number of point orbits of an automorphism group is known to be less than or equal to the number of block orbits. The following theorem shows that much more can be proved.

THEOREM 3.1. *Let (S, X, I) be a finite incidence structure and assume that the rank of I is equal to the number of points in S . If (P_S, P_X) is a tactical decomposition, then the block partition P_X uniquely determines the point partition through the relation $\partial^-(RP_X) = RP_S$. Moreover, the number of classes in P_S is given by $|P_S| = |P_X| - \dim((\text{kern } \partial^-) \cap RP_X)$.*

Proof. By the assumption on the rank of I , the map $\partial^+ : RS \rightarrow RX$ is injective. Since ∂^- and ∂^+ are adjoint to each other in the standard inner products, $\partial^- \partial^+$ is injective, and hence is a bijection on RS . If (P_S, P_X) is a tactical decomposition, by Proposition 2.1, the map

$$\partial^- \partial^+ : RP_S \xrightarrow{\partial^+} RP_X \xrightarrow{\partial^-} RP_S$$

is also a bijection. This implies $\partial^-(RP_X) = RP_S$.

Hence the point partition is determined by the block partition. Since the class numbers are the dimensions of RP_S and RP_X , we get $|P_S| = |P_X| - \dim((\text{kern } \partial^-) \cap RP_X)$. This completes the proof. \square

In case that the rank of I is not maximal, the point partition is, in general, only partially determined by $L(\partial^-(RP_X)) \subseteq RP_S$. The argument here can easily be modified to show that $|P_S| \leq |P_X| - \dim((\text{kern } \partial^-) \cap RP_X) + |S| - \text{rank } I$. (This is the content of Theorem 2.1 in Block [2].) The following list, however, shows that the condition on the maximality of the rank of I is quite general.

1. Let S be the class of k -element subsets and X the class of l -element subsets of a set of size n , where $k \leq l$ and $k + l \leq n$. If incidence is defined by set inclusion, the rank of I is $|S|$. In this case Theorem 3.1 generalizes the theorem of Livingstone and Wagner. For reference compare [3], [9] and [13].

2. The incidence matrix of a 2-design has point rank. This is the essence of Fisher's inequality. Here the second part of Theorem 3.1 contains the Dembowski-Parker-Hughes theorem in [5], [7] and [10].

3. Affine and projective spaces: Consider the d -dimensional affine and projective spaces $AG(d, q)$ and $PG(d, q)$ over the field of q elements. Let S be the collection of e -dimensional and X the collection of f -dimensional subspaces where the incidence relation is inclusion. Kantor [8] has shown that I has maximal rank, and hence $\text{rank } I = |S|$ if $0 \leq e < f \leq d - e - 1$.

4. Graphs: Let S be the vertices and X the edges of a finite undirected graph without loop edges. The incidence matrix of such a graph has point rank if and only if each component contains some cycle of odd length. This result is proved in a forthcoming paper [13]. Here Theorem 3.1 is an orbit theorem for graphs.

The point partition of a tactical decomposition does not, in general, determine the block partition. A structure, for instance, may have two automorphism groups which are both point transitive, but which have non-identical orbits on blocks. Some information about P_X , however, can be ascertained from the fact that $L(\partial^+(RP_S)) \subseteq RP_X$. Thus, if P_X^* is the partition belonging to $L(\partial^+(RP_S))$, then $P_X^* \leq P_X$, regardless of the rank of I .

4. AUTOMORPHISMS

An automorphism of a structure consists of a permutation g_S of S and a permutation g_X of X such that incidence is preserved: sIx if and only if $g_S(s)I g_X(x)$. These permutations extend to linear maps on RS and RX . One verifies easily that $g = (g_S, g_X)$ is an automorphism if and only if $\partial^+ g_S = g_X \partial^+$ or, equivalently, $\partial^- g_X = g_S \partial^-$. For convenience the corresponding permutation matrices will also be denoted by g_S and g_X .

THEOREM 4.1. *Let (S, X, I) be a finite incidence structure with $\text{rank } I = |S|$ and let (g_S, g_X) be an automorphism of the structure. Then g_X uniquely determines g_S . If g_X is an arbitrary permutation of X then g_X is the block permuta-*

tion of an automorphism if and only if $\partial^- g_X \partial$ is a permutation of S independent of the choice of ∂ as a right inverse of ∂^- .

Proof. Since $\text{rank } \partial^- = \text{rank } I = |S|$ by assumption, ∂^- has a right inverse. If (g_S, g_X) is an automorphism, $\partial^- g_X = g_S \partial^-$ implies $\partial^- g_X \partial = g_S$ for any right inverse ∂ of ∂^- . Hence g_X determines g_S . If g_X is an arbitrary permutation for which $\partial^- g_X \partial = g_S$ is a permutation independent of ∂ , we have to show that $\partial^- g_X = g_S \partial^-$. Let v be the difference between two right inverses of ∂^- . Then $\partial^- v = 0$. By assumption we have $\partial^- g_X v = 0$. Taking $v = (\text{id} - \partial \partial^-)$, we obtain $0 = \partial^- g_X (\text{id} - \partial \partial^-) = \partial^- g_X - \partial^- g_X \partial \partial^- = \partial^- g_X - g_S \partial^-$. Hence $\partial^- g_X = g_S \partial^-$ and the theorem is proved. \square

The tactical decomposition generated by an automorphism $g = (g_S, g_X)$ are the orbits of g on points and blocks. If $RS(g)$ and $RX(g)$ denote the corresponding partition spaces, then $RS(g) = \langle \sum_i g_S^i(s) \mid s \in S \rangle$ and $RX(g) = \langle \sum_j g_X^j(x) \mid x \in X \rangle$. One verifies quite easily that these spaces are characterized by $RS(g) = \{v \mid v \in RS, g_S(v) = v\} = \text{kern}(g_S - 1)$ and $RX(g) = \{w \mid w \in RX, g_X(w) = w\} = \text{kern}(g_X - 1)$. If rank I is equal to $|S|$, Theorem 3.1 implies $\partial^-(RX(g)) = RS(g)$. This relation determines the point orbits of g . The total number of orbits on points $n(g_S)$ is given by $n(g_S) = n(g_X) - \dim(\text{kern}(g_X - 1) \cap \text{kern } \partial^-)$. Independently of Theorem 3.1 we calculate the number of orbits of given length.

THEOREM 4.2. *Let $g = (g_S, g_X)$ be an automorphism of the finite structure (S, X, I) where the rank of I is $|S|$. Suppose $M = I^- \cdot I$ for some right inverse I^- of I and let μ be the Möbius function. Then the number $n_i(g_S)$ of g_S -orbits of length i is given by*

$$n_i(g_S) = \frac{1}{i} \sum_{k \mid i} \mu(k) \text{trace}(M \cdot g_X^{i/k}).$$

The total orbit numbers $n(g_S)$ and $n(g_X)$ are related through $n(g_X) - n(g_S) = \dim(\text{kern}(g_X - 1) \cap \text{kern } \partial^-) = \sum_{i \leq |X|} \sum_{k \mid i} \mu(k) / i \text{ trace}((1 - M) \cdot g_X^{i/k})$.

Proof. The trace of a permutation matrix G is the number $n_1(G)$ of symbols fixed by G . The symbols invariant under G^2 either are points fixed by G or belong to 2-cycles of G . Hence $\text{trace}(G^2) = n_1(G) + 2n_2(G)$ and by induction one proves easily that $\text{trace}(G^k) = \sum_{i \mid k} i \cdot n_i(G)$. Using Möbius inversion, the cycle numbers therefore are given by

$$n_i(G) = \frac{1}{i} \sum_{k \mid i} \mu(k) \text{trace}(G^{i/k}).$$

From $I g_X = g_S I$ and $g_S = I g_X I^-$ we conclude $g_S^i = I g_X^i I^-$. Therefore,

$$\begin{aligned} \text{trace}(g_S^i) &= \sum_{r,s,t} (I)_{rs} (g_X^i)_{st} (I^-)_{tr} = \sum_{r,s,t} (I^-)_{tr} (I)_{rs} (g_X^i)_{st} \\ &= \text{trace}((I^- \cdot I) g_X^i) = \text{trace}(M \cdot g_X^i). \end{aligned}$$

Hence we obtain $n_i(g_S) = (1/i)\sum_{k|i} \mu(k)\text{trace}(M \cdot g_X^{i/k})$ by the above formula. This proves the first part of Theorem 4.2.

The total orbit number of a permutation G is given by

$$n(G) = \sum_i n_i(G) = \sum_i \sum_{k|i} \frac{\mu(k)}{i} \text{trace}(G^{i/k}).$$

Therefore,

$$\begin{aligned} n(g_X) - n(g_S) &= \sum_i n_i(g_X) - n_i(g_S) \\ &= \sum_{i \leq |X|} \sum_{k|i} \frac{\mu(k)}{i} (\text{trace}(g_X^{i/k}) - \text{trace}(g_S^{i/k})) \\ &= \sum_{i \leq |X|} \sum_{k|i} \frac{\mu(k)}{i} (\text{trace}(g_X^{i/k}) - \text{trace}(M \cdot g_X^{i/k})) \\ &= \sum_{i \leq |X|} \sum_{k|i} \frac{\mu(k)}{i} \text{trace}((1 - M) \cdot g_X^{i/k}). \end{aligned}$$

The equation $n(g_X) - n(g_S) = \dim(\text{kern}(g_X - 1) \cap \text{kern } \partial^-)$ follows from Theorem 3.1 as we have pointed out above. □

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