Permutation groups on unordered sets I

By

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I. Introduction. Let $G$ be a permutation group on a finite or infinite set $S$. Consider the system $X_k$ of all $k$-element subsets of $S$ and the natural action of $G$ on $X_k$. The numbers $n_k$ of $G$-orbits on $X_k$ form a non-decreasing sequence for $k \leq \frac{1}{2} \cdot |S|$, but little else is known apart from this fact. See [1, 3].

In this note we examine the growth of $n_k$ (if these numbers are finite) in terms of the groups induced by $G$ on subsets of $S$. If $G$ is $(k-1)$-fold homogeneous on $S$ and $l \geq k$, a rough estimate for the growth rate is $(\binom{n_{k+1}}{k}) \leq (\binom{k}{l-1}) \cdot n_l$. Much sharper results are obtained if the action induced on subsets is rich.

The notation used is standard. The setwise and pointwise stabilizers of a subset $Y$ of $S$ are denoted by $G(r)$ and $G(y)$ respectively. The group $G(Y) = G(Y)/G(y)$ always is considered as a permutation group on $Y$. The orbits of $G$ on $X_k$ are denoted by $X_k(G)$ and $n_k = |X_k(G)|$.

II. Arrangements. Let $H$ be a group acting on a set $Y$ of finite size $l$ and let $x (\neq Y)$ be a subset of $Y$. We allow $x$ to be empty. An arrangement is a collection \{x; y_1, y_2, \ldots, y_t\} such that a) all $y_i$ have size $k = |x| + 1$ and contain $x$, b) $Y = \cup y_i$ and c) for $i \neq j$, $y_i$ and $y_j$ belong to different $H$-orbits. The set $x$ is called the centre of the arrangement. Clearly $t = l - k + 1$. A second arrangement $A' = \{x'; y'_1, y'_2, \ldots, y'_t\}$ is isomorphic to $A = \{x; y_1, y_2, \ldots, y_t\}$ if there is some $h$ in $H$ such that $A^h = A'$. Notice that two arrangements are isomorphic if and only if their centres belong to the same $H$-orbit. The total number of non-isomorphic arrangements with centre size $k - 1$ is denoted by $m(H, k)$. Clearly $m(H, k) \leq (\binom{k-1}{l-1})$ and equality holds if and only if $H$ is the identity on $Y$. We determine the structure of groups for which arrangements exist and determine the numbers $m(H, k)$ for some small values of $k$.

Theorem 2.1. Let $H \neq 1$ be a permutation group on a set $Y$ of size $l$ and let $k \leq l$. Suppose that $x = \{x, \beta, \ldots\}$ is the centre of an arrangement with $|x| = k - 1$. Then

i) $k > 1$. (In fact $m(H, 1) = 0$ if $H \neq 1$ and $m(H, 1) = 1$ if $H = 1$.)

ii) If $k = 2$, then $H$ is an elementary abelian $2$-group and $m(H, 2)$ is the number of $H$-orbits on the points of $Y$ that have length $|H|$.

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iii) If \( k = 3 \), then \( |H_{(x)}| \leq 2 \). If \( |H_{(x)}| = 2 \), then \( H = \text{Sym}(2) \) and \( m(H, 3) = |O| - 1 \) or \( H = \text{Sym}(3) \) and \( m(H, 3) = 1 \).

iv) If \( k = 3 \) and \( |H_{(x)}| = 1 \), then \( |H_{a}| \) and \( |H_{b}| \) are at most 2. Let \( O_{a} \) and \( O_{b} \) be the orbits of \( x \) and \( y \) respectively. Then the graph on \( O_{a} \cup O_{b} \) with edge set \( x^{H} \) has the following connected components: type 1 for \( |H_{a}| = |H_{b}| = 1 \) and \( O_{a} \neq O_{b} \), type 2 for \( |H_{a}| = |H_{b}| = 2 \) and \( O_{a} \neq O_{b} \), type 3 for \( |H_{a}| = 2 \) and \( O_{a} \neq O_{b} \), type 4 for \( |H_{a}| = 1 \) and \( O_{a} = O_{b} \), or type 5 for \( |H_{a}| = 2 \) and \( O_{a} = O_{b} \).

**Proof.** First we note that \( H_{(x)} \) acts as the identity on \( Y - x \) if \( x \) is a centre of an arrangement. This in particular proves the statement i). If \( k = 2 \), let \( O \) be the orbit of \( x \). If \( h \neq 1 \) is in \( H \), then also \( \beta = \alpha h \) is a centre and \( \beta \in \{ \alpha, \beta \} \cap \{ \alpha, \beta \}^{h} \) implies that these two sets are the same. Therefore \( \beta^{h} = \alpha \), \( h^2 = 1 \) and \( H \) is an elementary abelian 2-group of order \( |H| = |O| \). Vice versa, if \( H \) is an elementary abelian 2-group and if \( y \) belongs to an orbit of length \( |H| \), then \( y \) is the centre of an arrangement. For if \( y \in \{ y, \delta \} \cap \{ y, \delta \}^{h} \) for some \( h \) in \( H \), then either \( \gamma^{h} = \gamma \) and \( h = 1 \) or \( \gamma^{h} = \delta \) and \( \gamma = \delta^{h} \). In both cases \( \{ y, \delta \} \) is fixed by \( h \) and so \( y \) is a centre. This proves ii).

Now we assume that \( x = \{ \alpha, \beta \} \) is a centre of size \( k - 1 = 2 \). By the initial remark, \( |H_{(x)}| \) has size at most 2. Consider the case \( |H_{(x)}| = 2 \). Let \( O \) be the orbit containing \( x \) and \( y \). If \( O = x \), \( H = \text{Sym}(2) \). If \( O \neq x \), then any \( H \)-image is a centre again and as there is a transposition \( (\alpha, \beta)(\ldots)(\ldots) \), the images must intersect \( x \) in a point. Counting these images we obtain \( |x^{H}| = \frac{1}{2} \cdot |H| = (|O| - 2) \cdot 2 + 1 \), or \( |O| \cdot (4 - |H_{a}|) = 6 \). Therefore \( |O| = 3 \), \( |H_{a}| = 2 \) and \( H \) is the symmetric group on \( O \). As \( H \) is generated by transpositions fixing all points in \( Y - x \), \( H \) acts as the identity on \( Y - x \) and the only centres are the three isomorphic pairs in \( O \). Therefore \( m(H, 3) = 1 \) which proves iii).

Secondly consider the case \( |H_{(x)}| = 1 \). Suppose that \( k \) in \( H_{x} \) displaces \( \beta \) i.e. \( k: \gamma \to \beta \to \delta \). As \( \{ \alpha, \beta, \gamma \} \) and \( \{ \alpha, \beta, \gamma \}^{h} \) both contain \( x \) we conclude that \( \gamma = \delta \). Therefore \( |H_{a}| \leq 2 \) and similarly \( |H_{b}| \leq 2 \). Consider the graph on the vertices \( O_{a} \cup O_{b} \) with edge set \( x^{H} \). If \( O_{a} \neq O_{b} \), it is bipartite with respective degrees \( d_{a} = |H_{a}| \) and \( d_{b} = |H_{b}| \). This results in the components of type 1-3. If \( O_{a} = O_{b} \), the degree is \( d_{a} = 2 \cdot |H_{a}| = 2 \) or \( 4 \). If \( h = (\alpha, \beta, \gamma, \ldots, \delta) \ldots (\ldots) \) maps \( x \) onto \( \beta \), then \( \{ \alpha, \beta, \delta \} \) and \( \{ \alpha, \beta, \delta \}^{h} \) both contain \( x \). Therefore \( \gamma = \delta \) and \( h \) has order 3. If \( |H_{a}| = 1 \), the edges \( x, \{ \alpha, \gamma \} \) and \( \{ \gamma, \beta \} \) form a component of the graph. This is type 4. If \( |H_{a}| = 2 \), there is some \( k = (\alpha)(\beta, \xi) \ldots \) in \( H_{x} \) with \( \xi \neq \gamma \) and \( \xi \) must be displaced by \( h = (\alpha, \beta, \gamma)(\xi, \theta, \eta) \ldots \) From this one concludes that \( k = (\alpha)(\beta, \xi)(\gamma, \theta)(\eta) \ldots \) The resulting images of \( x \) form a component of type 5. This completes the proof.
We suppose now that for any subset \( Y \) of \( Y \) some group \( H \), acting on \( Y \) is given. Denote this collection of groups by \( \mathcal{G} = \{ H \} \). Let \( x \) be a given set of size \( k-1 \) and \( \mathcal{Y} = \{ x; y | x \subset y \text{ and } y \subseteq Y \text{ has size } k \} \). We say that \( \mathcal{Y} \) is a flag arrangement for \( \mathcal{G} \), if the following is true: Whenever \( A = \{ x; y_1, y_2, \ldots, y_i \} \subseteq \mathcal{Y} \), then \( A \) is an arrangement in \( Y_i = y_1 \cup y_2 \cup \ldots \cup y_i \) for the group \( H_i \). Two flag arrangements with centres \( x \) and \( x' \) are isomorphic if \( x^h = x' \) for some \( h \in H \), the group on \( Y \). Let \( m(\mathcal{G}, k) \) be the number non-isomorphic flag arrangements for \( \mathcal{G} \).

III. The growth of the sequence \( n_k \). Let \( G \) be a permutation group on a finite or infinite set \( S \). If \( X_i(G) = \{ O_1, \ldots, O_j, \ldots \} \) are the orbits on \( l \)-element subsets we define \( m_i(l, k) = m(\mathcal{G}, k) \) where \( \mathcal{G} \) is the collection of groups \( G^{x_i} \) induced by \( G \) on the subsets \( Y_i \subseteq Y \) for some fixed \( Y \) in \( O_j \). It is clear that the definition does not depend upon the choice of \( Y \) in \( O_j \).

**Theorem 3.1.** Suppose that \( G \) acts \((k - 1)\)-fold homogeneously on a set \( S \) with a finite number of orbits on \( X_k \) for some \( k \). If \( l \geq k \) let \( t = l - k + 1 \). Then

\[
\binom{n_k}{t} \leq \sum_{i=1}^{t} m_i(l, k).
\]

**Proof.** Let \( O_1, \ldots, O_{n_k} \) be all orbits of \( G \) on \( X_k \) and select some set \( x \) of size \( k - 1 \). For any \( t \) distinct orbits \( O_1, \ldots, O_t \), we select \( y_i \) in \( O_i \) for \( i = 1, \ldots, t \) such that \( x \subset y_i \). This is possible because \( G \) is \( k-1 \) homogeneous. Then \( \mathcal{Y} = \{ x; y_1, \ldots, y_i \} \) is a flag arrangement for \( \mathcal{G} = \{ G^{y_i} | Y_i \subseteq Y \} \) where \( Y = y_1 \cup y_2 \cup \ldots \cup y_i \). This is a consequence of the fact that the \( y_i \) belong to distinct \( G \)-orbits on \( X_k \). We label the collection \( Q_1, \ldots, Q_t \) by \( j \) if \( Y \) belongs to \( O_j \). (Of course the label is not necessarily uniquely determined). In all we require \( \binom{n_k}{t} \) labels where a label may be used several times.

Suppose therefore that also the sequence \( Q_1, Q_2, \ldots, Q_t \) obtains the label \( j \). Then there are \( y'_i \supset x, y'_i \subset Q_i \) for \( i = 1, \ldots, t \) such that \( Y' = y'_1 \cup y'_2 \cup \ldots \cup y'_t \) belongs to the same orbit as \( Y \). Let therefore \( g \) in \( G \) be such that \( Y'' = Y \). Then \( \{ x; y_1, \ldots, y_i \} \) and \( \{ x''; y'_1, \ldots, y'_t \} \) are flag arrangements for \( \mathcal{G} \). However, they are not isomorphic as \( \{ Q_1, \ldots, Q_t \} \neq \{ Q'_1, \ldots, Q'_t \} \). Therefore a label \( j \) may be used at most \( m_j(l, k) \) times. This gives the required inequality.

We note several consequences of the theorem:

**Corollary 3.2.** Let \( G \) be a transitive permutation group on a set \( S \) with a finite number \( n_2 \) of orbits on \( X_2 \). For a given \( l \geq 3 \) let \( n_{i,1} \) be the number of orbits \( O \) for which \( G^Y = 1, Y \in O \) and let \( n_{i,2} \) be the number of orbits \( O' \) for which \( G^Y \) is an elementary abelian 2-group, \( Y \in O' \). Then \( \binom{n_2}{l} \leq l \cdot n_{i,1} + l/2 \cdot n_{i,2} \).

**Corollary 3.3.** Suppose that \( G \) acts doubly homogeneously on a set \( S \) with a finite number \( n_3 \) of orbits on \( X_3 \). Let \( n_{i,j} \) be the number of orbits \( O \) for which \( |G^Y| = j, Y \in O \) and \( j = 1, 2, 3, \) or \( 6 \). Then \( n_3(n_3 - 1) \leq 12 \cdot n_{4,1} + 6 \cdot n_{4,2} + 2 \cdot(n_{4,3} + n_{4,6}) \).

We also note the following theorem which gives a bound for \( n_2 \) if the action induced on subsets is sufficiently rich:
Corollary 3.4. Let $G$ be transitive on a finite or infinite set $S$. Suppose there is a value $l$ such that the following holds: Whenever $Y \subseteq S$ has size $l$ and $s \in Y$ then there is a subset $Y'$, $s \in Y' \subseteq Y$ with the following properties a) $G^{Y'} \neq 1$ and b) if $G^{Y'}$ is an elementary abelian 2-group, then the orbit of $s$ under $G^{Y'}$ has length different from $|G^{Y'}|$. Then $n_2 < l - 1$.

Proof of 3.2. If $G^Y = 1$ on $Y$ then $m_i(1, 2) \leq 1$ for the orbit containing $Y$ and if $G^Y$ is an elementary abelian 2-group on $Y$, then $m_i(l, k) \leq l/2$ for the orbit containing $Y$ by theorem 2.1. The conclusion now follows from theorem 3.1.

Proof of 3.3. Using theorem 2.1 we get the bounds $m_i(4, 3) \leq 6$ if $G^Y = 1$, $m_i(4, 3) \leq 3$ if $|G^Y| = 2$ and $m_i(4, 3) \leq 1$ if $|G^Y| = 3$ or 6. In all other cases $m_i(4, 3) = 0$. The conclusion now follows from theorem 3.1.

Proof of 3.4. The hypothesis together with theorem 2.1 implies that no element of $Y$ is the center of a flag arrangement. Therefore $m_i(l, 2) = 0$ for all orbits and so $n_2 < l - 1$ by theorem 3.1.

A simple but useful fact on orbits on $X_k$ and $X_l$ in general is

**Theorem 3.5.** Let $G$ be a permutation group on a finite or infinite set with finite numbers $n_k$ and $n_l$ of orbits on $X_k$ and $X_l$ for some $k < l$. Let $E = O_1 \cup O_2 \cup \ldots \cup O_s$ be a union of distinct orbits of $G$ on $X_l$ and let $r_i$ denote the number of orbits of $G^{X_k}$ on the $k$-element subsets of $Y_i \in O_i$. Suppose the following holds about $E$: If $Q_1$ and $Q_2$ are any given $G$-orbits on $X_k$, then there exist $x_1, y_1, \ldots, y_t, x_2$ such that $x_1 \in y_1$, $|y_i \cap y_{i+1}| \geq k$ for $i = 1, \ldots, t - 1$, $y_t \supset x_2$ with $x_1 \in O_1$, $x_2 \in O_2$ and $y_i \in E$. Then

$$n_k \leq \sum_{i=1}^s \binom{t}{2} + 1.$$ 

Proof. We consider the graph whose vertices are the orbits $X_k(G)$. Two distinct orbits $Q$ and $Q'$ are linked by an edge $e$ if there are $x \in Q$ and $x' \in Q'$ such that $x \cup x' \subseteq y \in E$. We label this edge by $j$ if $y$ belongs to $O_j$. The condition on $E$ implies that this graph is connected. Therefore the total number of edges is at least $n_k - 1$. On the other hand, a label $j$ may be used at most $\binom{t}{2}$ times. This yields the inequality.

We conclude with the following inequalities obtained from a theorem on orbits in graphs [4].

**Theorem 3.6.** Let $G$ be a permutation group on a finite set $S$. Suppose that $X_2$ is a disjoint union $E_1 \cup E_2 \cup \ldots \cup E_r$ where each $E_i$ is a union of $G$-orbits on $X_2$.

a) If each graph $(S, E_i)$, $(i = 1, \ldots, r)$, is connected then $n_1 \leq r^{-1} \cdot n_2 + 1$.

b) If every connected component of $(S, E_i)$ contains a circular path of odd length for all $i = 1, \ldots, r$, then $n_1 \leq r^{-1} \cdot n_2$.

Proof. Let $G_i$ be the graph with vertices $S$ and edge set $E_i$. Then $G$ is a group of automorphisms of $G_i$ and we denote the number of orbits of $G$ on $E_i$ by $|E_i(G)|$. By theorems 3.1 and 3.2 in [4] we have $n_1 \leq |E_1(G)| + 1$ and as $n_2 = \sum |E_i(G)|$ the assertion a) follows. If all connected components of $G_i$ contain a cycle of odd length, then
$n_1 \leq |E_1(G)|$ as a consequence of theorem 2.1 and the proof of theorem 3.1 in [4]. This yields b).

References


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