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On Modular Homology in the Boolean Algebra, II

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Let R be an associative ring with identity and Ω an n-element set. For $k \leq n$ consider the R-module M_k with k-element subsets of Ω as basis. The *r-step inclusion map* $\partial_r : M_k \to M_{k-r}$ is the linear map defined on this basis through $\partial_r(\Delta) := \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{\binom{k}{r}}$ where the Γ_i are the $(k-r)$-element subsets of Δ. For $m < r$ one obtains chains

$$\mathscr{M} : 0 \overset{\partial_r}{\leftarrow} M_m \overset{\partial_r}{\leftarrow} M_{m+r} \overset{\partial_r}{\leftarrow} M_{m+2r} \overset{\partial_r}{\leftarrow} M_{m+3r} \overset{\partial_r}{\leftarrow} \cdots \overset{\partial_r}{\leftarrow} 0$$

of inclusion maps which have interesting homological properties if R has characteristic $p > 0$. V. B. Mnukhin and J. Siemons (*J. Combin. Theory* **74**, 1996 287–300; *J. Algebra* **179**, 1995, 191–199) introduced the notion of *p-homology* to examine such sequences when $r = 1$ and here we continue this investigation when r is a power of p. We show that any section of $\mathscr{M}$ not containing certain *middle terms* is p-exact and we determine the homology modules for such middle terms. Numerous infinite families of irreducible modules for the symmetric groups arise in this fashion. Among these the *semi-simple inductive systems* discussed by A. Kleshchev (*J. Algebra* **181**, 1996, 584–592) appear and in the special case $p = 5$ we obtain the *Fibonacci representations* of A. J. E. Ryba (*J. Algebra* **170**, 1994, 678–686). There are also applications to permutation groups of order co-prime to p, resulting in *Euler–Poincaré* equations for the number of orbits on subsets of such groups.

1. INTRODUCTION

Let Ω be a set of finite size n and R an associative ring with identity. For $k \leq n$ consider the R-module M_k which has k-element subsets of Ω as basis. So M_k consists of all formal sums $f = \sum_{|\Delta|=k} f_\Delta \Delta$ with $\Delta \subseteq \Omega$ and

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$f_\Delta \in R$. When $r \geq 0$ is an integer the *r-step inclusion map* $\partial_r : M_k \to M_{k-r}$ is the linear map defined through $\partial_r(\Delta) := \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{\binom{k}{r}}$ where Γ_i are the $(k-r)$-element subsets of Δ. So fixing some $m < r$ we obtain a chain of inclusion maps

$$\mathscr{M} : 0 \overset{\partial_r}{\leftarrow} M_m \overset{\partial_r}{\leftarrow} M_{m+r} \overset{\partial_r}{\leftarrow} M_{m+2r} \overset{\partial_r}{\leftarrow} M_{m+3r} \overset{\partial_r}{\leftarrow} \cdots .$$

When R has characteristic $p > 0$ then ∂_r^p is the zero map and one is interested in the homological properties of $\mathscr{M}$. In the papers [10, 11] with Valery Mnukhin we have defined the notion of a *p-homological* and *p-exact* sequence—see the definition in Section 3—and this paper is a continuation of that work.

For the purpose of p-homology it is necessary to investigate certain subchains of $\mathscr{M}$. We assume now that R is a ring of prime characteristic $p > 0$ and that r is a power of p. For integers $0 < i^* < p$ and $0 \leq k^* \equiv m \bmod(r)$ with $k^* + i^* r < pr$ we consider the subsequence

$$\mathscr{M}_{k^*, i^*} : 0 \leftarrow M_{k^*} \leftarrow M_{k^* + i^* r} \leftarrow M_{k^* + pr} \leftarrow M_{k^* + (i^* + p)r} \leftarrow M_{k^* + 2i^* r} \cdots .$$

Here each arrow represents the relevant power of ∂_r and as $\partial_r^p = 0$ we see that $\mathscr{M}_{k^*, i^*}$ is homological. ($\mathscr{M}$ would be p-exact by definition if all subsequences $\mathscr{M}_{k^*, i^*}$ of the kind just described were exact).

Let $(*) : M_{k-ir} \leftarrow M_k \leftarrow M_{k+(p-i)r}$ be any three consecutive terms of $\mathscr{M}_{k^*, i^*}$, thus either $k \equiv k^* \bmod(pr)$ with $i = p - i^*$ or $k \equiv k^* + i^* r \bmod(pr)$ with $i = i^*$. We say that $(*)$ or (k, ir) is a *middle term* for $\mathscr{M}_{k^*, i^*}$ if $n < 2k + (p-i)r < n + pr$. Note that $\mathscr{M}_{k^*, i^*}$ may have no middle terms (take n odd, $p = 2$, and $r = 1$, for example), but if there is one then it is unique and we can speak of *the* middle term for $\mathscr{M}_{k^*, i^*}$.

In Theorem 3.2 we prove that any section of $\mathscr{M}_{k^*, i^*}$ not containing its middle term is exact. This result is already contained in Bier's paper [2] and it is proved there via rank arguments based on Wilson's work [15]. Our proof is more direct. In Section 2 we develop the general calculus for inclusion maps in $\bigoplus_{k \geq 0} M_k$. This leads to an "integration theorem" which allows us to write down a pre-image F with $\partial_r^{p-i}(f) = f$ for any f with $\partial_r^i(F) = 0$ unless (k, ir) is a middle term. If $M_{k-ir} \leftarrow M_k \leftarrow {}_{k+(p-i)r}$ is a middle term denote the kernel of ∂_r^i by $K_{k,ir}^n := \ker \partial_r^i \cap M_k^n$ and so let $H_{k,ir}^n := K_{k,ir}^n / \partial_r^{p-i}(M_{k+(p-i)r}^n)$ be the only non-trivial homology module of $\mathscr{M}_{k^*, i^*}$.

In Section 5 we determine explicit generators for all $H_{k,ir}^n$ and in Section 6 we classify completely the homologies of the 1-step map. In Theorem 6.2 it is shown that $H_{k,i}^n$ is isomorphic to $H_{k,i+1}^{n-1} \oplus H_{k-1,i-1}^{n-1}$ as $\mathrm{Sym}(\Omega \setminus \alpha)$-modules. This is used to construct infinite families of irreducible $\mathrm{Sym}(\Omega)$-

modules for arbitrary primes $p > 2$ occurring as the homology modules of the 1-step inclusion map.

For an arbitrary prime $p > 2$ the family $\{H^n_{k,i} : 0 < i < p,\ k \leq n,\ 2k - i + 1 = n\}$ is an example of a *semi-simple inductive system* as considered by Kleshchev [9] and in the case $p = 5$ we obtain Ryba's *Fibonacci representations* of [13]. In [3] Bier has shown how certain spin modules over $GF(2)$ arise as the homology modules of the 2-step map when R has characteristic 2. These examples illustrate that the notion of p-homology leads to worthwhile results on the modular representation theory of the symmetric groups. For the modules of the r-step maps with $r > 1$ in general not much appears in the literature. Our description of the $H^n_{k,ir}$ in Section 5 is explicit enough to allow a full analysis of these modules which may be presented in a subsequent paper.

Other applications concern permutation groups. Permutations act naturally on $\bigoplus_{k \geq 0} M_k$ and for any group $G \subseteq \mathrm{Sym}(\Omega)$ we can consider the *orbit module* in M_k defined as $M_k^G := \{f \in M_k : f^g = f\ \forall g \in G\}$. Its natural basis are the "orbit sums" $\Sigma_{\Gamma^* \in \Gamma^G} \Gamma^*$ where Γ^G as usual denotes $\{\Gamma^g : g \in G\}$. In particular, the dimension of M_k^G is the number of G-orbits on the k-element subsets of Ω. As $\partial_r(M_k^G) \subseteq (M_{k-r}^G)$ we obtain sequences of the kind

$$\mathscr{M}^G_{k^*,i^*} : 0 \leftarrow M^G_{k^*} \leftarrow M^G_{k^*+i^*r} \leftarrow M^G_{k^*+pr} \leftarrow M^G_{k^*+(i^*+p)r} \leftarrow M^G_{k^*+2pr} \cdots,$$

where each term is a submodule of the corresponding term in $\mathscr{M}_{k^*,i^*}$. Such a sequence is automatically homological but may fail to be exact at terms where $\mathscr{M}_{k^*,i^*}$ is exact. However, in Theorem 4.1 we show that if the order of G is not divisible by p, then any section of $\mathscr{M}^G_{k^*,i^*}$ not containing its middle term is exact.

If (k, ir) is a middle term of $\mathscr{M}^G_{k^*,i^*}$ we let $H^G_{k,ir} := K_{k,ir} \cap M^G_k / \partial_r^{p-i}(M^G_{k+(p-i)r})$ denote the only non-trivial homology module in $\mathscr{M}^G_{k^*,i^*}$. The results in Section 5 give generators for $H^G_{k,ir}$ and in Theorem 4.5 we obtain Euler–Poincaré equations for the number of G-orbits on k-element subsets of Ω when p does not divide the order of G. For $p = 2$ we have made use of such equations before in [10] for groups of odd order.

Modular p^*-homology over rings R of characteristic $p > 0$ for sequences such as $\mathscr{M}$ above can be considered for more general classes of partially ordered sets. Note that it will be necessary to distinguish between the characteristic of the ring and the interval length p^* appearing in the definition of p^*-homology; for the Boolean lattice these happen to coincide. If $(\mathscr{P}, \leq, |*|)$ is a ranked poset and if M_k denotes the R-module with $\{x \in \mathscr{P} : |x| = k\}$ as basis then one can define order maps $\partial : M_k \to M_{k-1}$ analogous to ∂_r and therefore the question of p^*-homology can be studied also for such ranked posets. In [12] we have done this for projective spaces over $GF(q)$ when the coefficient ring R has characteristic $p \neq q$ (for $p = q$ no homologies occur). The results are interesting: One principal difference to the Boolean case is that p^* indeed is different from the characteristic of the coefficient ring. On the other hand, also in projective spaces every chain such as $\mathscr{M}_{k^*, i^*}$ is inexact in at most one position and so gives rise to just one non-trivial homology. We hope that these facts support the suggestion that the modular homology considered here uncovers some deeper properties of partially ordered sets.

2. THE r-STEP INCLUSION MAP

Throughout R will be an associative ring with identity and Ω a finite set of cardinality n. We let 2^{Ω} denote the collection of subsets of Ω and $R2^{\Omega}$ the R-module with 2^{Ω} as a basis.

For an integer k the collection of all k-element subsets of Ω is denoted by $\Omega^{\{k\}}$. Furthermore, we let $R\Omega^{\{k\}} \subset R2^{\Omega}$ denote the submodule with k-element subsets as basis. We will abbreviate $R\Omega^{\{k\}}$ by M_k^n or simply M_k if the context is clear. We identify $\Omega^{\{1\}}$ with Ω and $1 \cdot \varnothing$ with $1 \in R$ so that in particular $M_0 = \{r \cdot \varnothing : r \in R\} = R$. Also, we put $M_k = 0$ whenever $k < 0$ or $k > n$ and refer to R as the *coefficient ring* of M_k.

For $f = \sum f_{\Delta} \Delta \in R2^{\Omega}$ the *support* of f is the union of all Δ for which $f_{\Delta} \neq 0$; we will denote it by $\operatorname{supp} f$. The *support size* of f is $\|f\| := |\operatorname{supp} f|$. Two elements f and h of $R2^{\Omega}$ are said to be *disjoint* if $\operatorname{supp} f$ and $\operatorname{supp} h$ are disjoint sets.

The Boolean operation of set union is easily extended to a product on $R2^{\Omega}$: if $f = \sum f_{\Delta} \Delta$ and $h = \sum h_{\Gamma} \Gamma$ are elements of $R2^{\Omega}$ we define

$$f \cup h \quad \sum_{\Delta, \Gamma} f_{\Delta} h_{\Gamma} (\Delta \cup \Gamma).$$

It is a simple matter to check that $R2^{\Omega}$ with this product is an associative algebra with the empty set as identity.

For an integer $r \geq 0$ the *r-step inclusion map* on 2^{Ω} is the linear map $\partial_r : R2^{\Omega} \to R2^{\Omega}$ given by

$$\partial_r(\Delta) = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{\binom{k}{r}},$$

where Δ is a k-element subset of Ω and where the Γ_i are the $(k - r)$-element subsets of Δ. In particular, ∂_0 is the identity map, $\partial_r(\Delta) = 0$ if $|\Delta| < r$, and $\partial_r(\Delta) = 1 \in R$ if $r = |\Delta|$. Clearly, this map restricts to homomorphisms

$$\partial_r : M_k \to M_{k-r}.$$

Furthermore, it is easily verified that for any positive integer s we have

$$\binom{r+s}{r}\partial_{r+s} = \partial_r \partial_s.$$

Let f be an element of $R2^{\Omega}$. Then for any α in Ω we can write f uniquely as

$$f = \alpha \cup f_\alpha + l$$

in such a way that α does not belong to $\operatorname{supp} f_\alpha \cup \operatorname{supp} l$. Also very important is the following

LEMMA 2.1. *If f and h are disjoint elements in $R2^{\Omega}$ then $\partial_r(f \cup h) = \sum_{j=0}^{r} \partial_j(f) \cup \partial_{r-j}(h)$.*

Proof. The identity is obvious when f and h are two disjoint sets and follows from linearity in general. ∎

We shall use these basic facts without further reference. The next theorem generalizes Theorem 2.1 of [10] where the result was proved for the one-step map.

THEOREM 2.2. *For any coefficient ring with identity the kernel of $\partial_r : M_k \to M_{k-r}$ is generated by elements of support size at most* $\max\{2k - r + 1, k\}$.

Remark. When the coefficient ring is an integral domain of characteristic 0 then the minimum support size of elements in the kernel is $\max\{2k - r + 1, k\}$ exactly. This can be seen from simple rank arguments. However, in non-zero characteristic there may be elements of smaller support size. Corollary 3.4 later gives more information.

Proof. The proof follows closely that of Theorem 2.1 in [10] and proceeds by induction on n. For $n = 1, 2$ the result is easily verified. Therefore suppose that $n > 2$ and that $f \in \ker \partial_r \cap M_k^n$ with $\|f\| > \max\{2k - r + 1, k\}$. Furthermore, we may assume that $r \le k$, for if $r > k$ then the standard basis of M_k^n is contained in the kernel of ∂_r and the result clearly holds.

We pick $\alpha \in \operatorname{supp} f$ and write $f = \alpha \cup f_\alpha + l$ where $\alpha \notin \operatorname{supp} f_\alpha \cup \operatorname{supp} l$. Then by Lemma 2.1

$$0 = \partial_r(f) = \alpha \cup \partial_r(f_\alpha) + \partial_{r-1}(f_\alpha) + \partial_r(l).$$

Therefore, $f_\alpha \in \ker \partial_r \cap M_{k-1}^{n-1}$ and by the inductive hypothesis we may write $f_\alpha = \sum_{i=1}^{s} w_i$ with $w_i \in \ker \partial_r \cap M_{k-1}^{n-1}$ and $\|w_i\| \le \max\{2k - r - 1, k$

$-1\}$. As $\|f\| > 2k - r + 1 = k + 1 + (k - r) \geq k + 1$, we have $\|f\| \geq \|w_i\| + 2$ for $i = 1, \dots, s$. Hence, whenever $1 \leq i \leq s$, we may choose α_i in Ω with $\alpha_i \neq \alpha$ and $\alpha_i \in \operatorname{supp} f \setminus \operatorname{supp} w_i$. Then $\|(\alpha - \alpha_i \cup w_i)\| = 2 + \|w_i\| \leq \max\{2k - r + 1, k + 1\} = \max\{2k - r + 1, k\}$ and $\partial_r((\alpha - \alpha_i) \cup w_i) = 0$ by Lemma 2.1. Furthermore, $f = \sum_{i=1}^{s}(\alpha - \alpha_i) \cup w_i + \sum_{i=1}^{s} \alpha_i \cup w_i + l$ and as $\sum_{i=1}^{s} \alpha_i \cup w_i + l \in \ker \partial_r \cap M_k^{n-1}$ we may invoke the induction hypothesis to complete the proof. ∎

LEMMA 2.3. *Suppose that R is an associative ring with identity and has prime characteristic $p > 0$. Let $r \geq 1$ be a power of p and let f be an element of* $\ker \partial_r \cap M_k$. *If $0 < s < p$ satisfies $2k + sr \leq n$ then there exists F in M_{k+sr} with $\partial_r^s(F) = f$.*

Proof. By the theorem above we may assume that $\|f\| \leq \max\{2k - r + 1, k\}$.

Suppose firstly that $\|f\| \leq k$ and that $k > 2k - r + 1$. Therefore we also have $r > k$ and $r \neq 1$. As $k \leq n - sr - k$ there exists $\Gamma \subseteq \Omega$ with $\Gamma \cap \operatorname{supp} f = \varnothing$ and $\|\Gamma\| = sr + k$. We define $F := (s!)^{-1}\partial_k(\Gamma) \cup f \in M_{k+sr}$ and show by induction on $t \leq s$ that $\partial_r^t(F) = t!(s!)^{-1}\partial_{tr+k}(\Gamma) \cup f$. Taking $t = s$ will then complete the proof in this case. For $t = 0$ the result is certainly true and supposing the result holds for $t \leq s - 1$ we calculate, using Lemma 2.1,

$$\begin{aligned}
\partial_r^{t+1}(F) &= \partial_r\Big(t!(s!)^{-1}\big(\partial_{tr+k}(\Delta) \cup f\big)\Big) \\
&= t!(s!)^{-1}\left(\sum_{j=0}^{r} \partial_{r-j}\big(\partial_{tr+k}(\Gamma)\big) \cup \partial_j(f)\right) \\
&= t!(s!)^{-1}\left(\sum_{j=0}^{k} \binom{(t+1)r + k - j}{tr + k} \partial_{(t+1)r+k-j}(\Gamma) \cup \partial_j(f)\right).
\end{aligned}$$

To deduce that $\binom{(t+1)r+k-j}{tr+k} \equiv 0 \bmod(p)$ if $0 < j \leq k$ we use

Fact 1. (see [1, p. 8]). For a positive integer m and a prime p the largest integer l such that p^l divides $m!$ is $\sum_{n=1}^{\infty}\lfloor m/p^n \rfloor$ (where $\lfloor x \rfloor$ denotes the integral part of the real number x).

And

Fact 2. For any real numbers x and y we have $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$.

Therefore, if $r = p^d$ with $d \geq 1$, then the largest integer l such that p^l divides $\binom{(t+1)r+k-j}{tr+k}$ is

$$\sum_{n=1}^{\infty}\left(\left\lfloor\frac{(t+1)r+k-j}{p^n}\right\rfloor - \left\lfloor\frac{tr+k}{p^n}\right\rfloor - \left\lfloor\frac{r-j}{p^n}\right\rfloor\right)$$
$$\geq (t+1) - t + \sum_{n=1}^{d-1}\left(\left\lfloor\frac{(t+1)r+k-j}{p^n}\right\rfloor - \left\lfloor\frac{tr+k}{p^n}\right\rfloor - \left\lfloor\frac{r-j}{p^n}\right\rfloor\right)$$

which is strictly greater than zero. (Note that each term of the sum is nonnegative by Fact 2.) So we have indeed shown that $\binom{(t+1)r+k-j}{tr+k} \equiv 0 \bmod(p)$ if $0 < j \leq k$.

Furthermore, we can calculate

$$\begin{aligned}\binom{(t+1)r+k}{tr+k} &= \prod_{-tr<l\leq k}\left(\frac{(t+1)r+l}{tr+l}\right)\\ &= \prod_{-tr<l\leq k}\left(1+\frac{r}{tr+l}\right)\\ &\equiv \left(1+\frac{1}{t}\right)\left(1+\frac{1}{t-1}\right)\cdots\left(1+\frac{1}{2}\right)\left(1+\frac{1}{1}\right)\\ &= t+1\end{aligned}$$

which completes the induction and gives the result in this case.

We may now suppose that $\|f\| \leq 2k - r + 1 \leq n - (s+1)r + 1$. Therefore there exists $\Gamma \subseteq \Omega$ with $\Gamma \cap \operatorname{supp} f = \emptyset$ and $\|\Gamma\| = (s+1)r - 1$. We then define $F := (s!)^{-1}\partial_{r-1}(\Gamma) \cup f$ and show inductively that $\partial_r^t(F) = t!(s!)^{-1}\partial_{(t+1)r-1}(\Gamma) \cup f$ whenever $t \leq s$. Taking $t = s$ will then complete the proof in this case. For $t = 0$ the result is certainly true and if we suppose the result holds for $t \leq s - 1$ then

$$\begin{aligned}\partial_r^{t+1}(F) &= \partial_r\left(t!(s!)^{-1}\partial_{(t+1)r-1}(\Gamma) \cup f\right)\\ &= t!(s!)^{-1}\sum_{j=1}^{r}\partial_j\left(\partial_{(t+1)r-1}(\Gamma)\right) \cup \partial_{r-j}(f)\\ &= t!(s!)^{-1}\sum_{j=1}^{r}\binom{(t+1)r+j-1}{j}\partial_{(t+1)r-1+j}(\Gamma) \cup \partial_{r-j}(f).\end{aligned}$$

We note that we are done if $r = 1$. Therefore we suppose that $r > 1$. But then

$$\binom{(t+1)r+j-1}{j} = \frac{((t+1)r+j-1)}{(j-1)} \cdot \frac{((t+1)r+j-2)}{(j-2)} \cdot \cdots \cdot \frac{(t+1)r+1}{1} \cdot \frac{(t+1)r}{j}$$

and we see that all terms in the product will be $\equiv 1 \bmod(p)$ except the last. This will be $\equiv 0 \bmod(p)$ unless $j = r$ when it will be $t+1$. This completes the induction and hence also the proof. ∎

The main result of this section is the next theorem which shows that Corollary 2.3 of [10] can be extended to r-step maps.

THEOREM 2.4 (The Integration Theorem). *Suppose that R is an associative ring with identity of prime characteristic p. Let $r \geq 1$ be a power of p and let f be an element of M_k. Suppose further that $\partial_r^i(f) = 0$ with $0 < i < p$ and that $j \in \{1, \ldots, p-i\}$ satisfies $2k + jr \leq n$. Then there exists F in M_{k+jr} with $\partial_r^j(F) = f$.*

Proof. The proof is by induction on i. For $i = 1$ the result holds by the preceding lemma. Suppose the result holds for $i \leq p-2$ and that $f \in \ker \partial_r^{i+1} \cap M_k$ with $j \in \{1, \ldots, p-(i+1)\}$ satisfying $2k + jr \leq n$. Then $\partial_r^i(\partial_r(f)) = 0$. Also $\partial_r(f) \in M_{k-r}$ and $2(k-r) \leq n - jr - 2r \leq n - (j+1)r$ where $j+1 \in \{2, \ldots, p-i\}$. Therefore by the inductive hypothesis there exists $H \in M_{k+jr}$ with $\partial_r^{j+1}(H) = \partial_r(f)$. But then $\partial_r(\partial_r^j(H) - f) = 0$ and $2k + jr \leq n$. So by the preceding lemma, there exists $J \in M_{k+jr}$ with $\partial_r^j(J) = \partial_r^j(H) - f$. Hence $f = \partial_r^j(H - J)$ and the induction is complete. ∎

3. HOMOLOGICAL SEQUENCES

Throughout this chapter the coefficient ring R has prime characteristic $p > 0$, $r \geq 1$ is a power of p, and Ω is some finite of cardinality n.

We observe that $\partial_r^p : R2^\Omega \to R2^\Omega$ is the zero map. To see this recall the formula $\binom{r+s}{r}\partial_{r+s} = \partial_r \partial_s$ from Section 2. By induction it follows that

$$\partial_r^j = \binom{jr}{r} \cdot \cdots \cdot \binom{2r}{r} \cdot \binom{r}{r} \partial_{jr}.$$

Now notice that $\binom{pr}{r} = p\binom{pr-1}{r-1} \equiv 0 \bmod(p)$.

The results in Section 2 lead us to investigate homology. We recall the definitions: if $\chi : A \to B$ and $\psi : B \to C$ are homomorphisms then the sequence $A \to B \to C$ is *homological* at B if $\ker(\psi) \supseteq \chi(A)$. In this case $H := \ker(\psi)/\chi(A)$ is the *homology module* at B and the sequence is *exact* if $H = 0$, that is, if $\ker(\psi) = \chi(A)$. A longer sequence

$$\mathscr{A}: \cdots \leftarrow A_{j-2} \leftarrow A_{j-1} \leftarrow A_j \leftarrow A_{j+1} \leftarrow A_{j+2} \leftarrow \cdots$$

is *homological* (*exact*) if it has that property at every A_i. In [10, 11] we have introduced the following

DEFINITION. $\mathscr{A}$ is *p-exact* (*p-homological*) if all subsequences of the kind $\mathscr{A}_{k^*, i^*} : \cdots \leftarrow A_{k^*} \leftarrow A_{k^*+i^*} \leftarrow A_{k^*+p} \leftarrow A_{k^*+i^*+p} \leftarrow A_{k^*+2p} \leftarrow A_{k^*+i^*+2p} \leftarrow \cdots$ are exact (homological) for every k^* and $0 < i^* < p$. (The arrows are the natural compositions of the maps in $\mathscr{A}$.)

As is pointed out in Bier's paper [2], this kind of homology was first considered in the works [8] of Mayer in 1947, see also [14].

Now select some $m < r$ and consider the sequence

$$\mathscr{M}: 0 \overset{\partial_r}{\leftarrow} M_m \overset{\partial_r}{\leftarrow} M_{m+r} \overset{\partial_r}{\leftarrow} M_{m+2r} \overset{\partial_r}{\leftarrow} M_{m+3r} \overset{\partial_r}{\leftarrow} \cdots.$$

In order to investigate its p-homological properties we fix integers $0 < i^* < p$ and $0 \le k^* \equiv m \bmod(r)$ with $k^* + i^* r < pr$ to obtain the subsequence

$$\mathscr{M}_{k^*, i^*}: 0 \leftarrow M_{k^*} \leftarrow M_{k^*+i^*r} \leftarrow M_{k^*+pr} \leftarrow M_{k^*+(i^*+p)r} \leftarrow M_{k^*+2pr} \cdots$$

in which each arrow represents the relevant power of ∂_r. Since $\partial_r^p : R2^\Omega \to R2^\Omega$ is the zero map this sequence is homological.

For general parameters $|\Omega| = n$, $0 < i < p$, and k we let $K^n_{k,ir}$ denote $\ker \partial_r^i \cap M^n_k$ and let

$$H^n_{k,ir} := K^n_{k,ir} / \partial_r^{p-i}\left(M^n_{k+(p-i)r}\right)$$

be the corresponding homology module. If $f \in K^n_{k,ir}$ then we denote its coset in $H^n_{k,ir}$ by

$$[f] := f + \partial_r^{p-i}\left(M^n_{k+(p-i)r}\right).$$

As before the superscript n can be dropped if the context is clear. We begin by stating a consequence of the Integration Theorem of Section 2:

LEMMA 3.1. *Suppose that R is an associative ring with identity and has prime characteristic p. Let $r \ge 1$ be a power of p and suppose that $0 < i < p$ satisfies $2k + (p - i)r \le n$. Then $H^n_{k,ir} = 0$.*

To extend this result let now $M_{k-ir} \leftarrow M_k \leftarrow M_{k+(p-i)r}$ be any three consecutive terms of $\mathscr{M}_{k^*,i^*}$. (So either $k \equiv k^* \bmod(pr)$ and $i = p - i^*$ or $k \equiv k^* + i^* r \bmod(pr)$ and $i = i^*$.) We say that (k, ir) is a *middle term* for $\mathscr{M}_{k^*,i^*}$ if $n < k + (k + (p-i)r) < n + pr$, indicating that $M_{k-ir} \leftarrow M_k \leftarrow M_{k+(p-i)r}$ is nearest to the middle of $\mathscr{M}_{k^*,i^*}$. Note that there may be no middle terms for $\mathscr{M}_{k^*,i^*}$ (take n odd, $p = 2$, and $r = 1$, for example). However, if there is a middle term, then it is easy to see that there is at most one so that we can talk of *the* middle term for $\mathscr{M}_{k^*,i^*}$. We extend the use of this term slightly and refer also to $M_{k-ir} \leftarrow M_k \leftarrow M_{k+(p-i)r}$ as the *middle term* of $\mathscr{M}_{k^*,i^*}$. Further, $M_{k-ir} \leftarrow M_k \leftarrow M_{k+(p-i)r}$ will be called a middle term, or a middle term of $\mathscr{M}$, if it is the middle term for some $\mathscr{M}_{k^*,i^*}$.

The following result appears already in Bier's paper [2, Satz 2]. The proof there is based on Wilson's rank formula [15] which yields the p-rank of the incidence matrix of k-subsets versus $(k - ir)$-subsets of Ω.

THEOREM 3.2. *Suppose that R is an associative ring with identity and has prime characteristic $p > 0$. Let $r \geq 1$ be a power of p. Then $H^n_{k,ir} = 0$ unless (k, ir) is a middle term.*

COROLLARY 3.3. *A section of $\mathscr{M}$ containing no middle terms is p-exact.*

COROLLARY 3.4. *If the coefficient ring has prime characteristic $p > 0$ and if (k, ir) is not a middle term then the kernel of $\partial^i_r : M_k \to M_{k-ir}$ is generated by elements of support size at most $k + (p - i)r$.*

Proofs. The corollaries are clear. To prove the theorem we introduce a new linear map $U_r : R2^\Omega \to R2^\Omega$ defined by $U_r(\Delta) = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{\binom{n-k}{r}}$ where Δ is a k-element subset of Ω and where the Γ_i are the $(k + r)$-element subsets of Ω containing Δ. Note that for $0 < i < p$ the matrix representing $U^i_r : M_k \to M_{k+ir}$ is the transpose of the matrix representing $\partial^i_r : M_{k+ir} \to M_k$. In particular, $U^i_r : M_k \to M_{k+ir}$ and $\partial^i_r : M_{k+ir} \to M_k$ have the same rank. Furthermore, the linear map $c : R2^\Omega \to R2^\Omega$ defined by $c(\Delta) = \Omega \setminus \Delta$ is a module isomorphism which satisfies $cU_r c = \partial_r$.

Let $M_a \leftarrow M_b \leftarrow M_{a+pr}$ be consecutive terms of $\mathscr{M}_{k^*,i^*}$ where without loss of generality $b = a + ir$. If $a + b + pr \leq n$ then this sequence will be exact by Lemma 3.1. Hence we may assume that $a + b + pr \geq n + pr$. We consider the sequence of modules $M_{n-(a+pr)} \leftarrow M_{n-b} \leftarrow M_{n-a}$. Since $2n - (a + b) \leq n$, Lemma 3.1 implies that this sequence is exact. But then we calculate

$$\begin{aligned}\dim H_{b,ir} &= \dim K_{b,ir} - \dim \partial_r^{p+i}\big(M_{a+pr}\big)\\ &= \dim M_b - \dim \partial_r^i(M_b) - \dim \partial_r^{p-i}\big(M_{a+pr}\big)\\ &= \dim M_b - \dim U_r^i(M_a) - \dim U_r^{p-i}(M_b)\\ &= \dim M_{n-b} - \dim \partial_r^i(M_{n-a}) - \dim \partial_r^{p-i}(M_{n-b})\\ &= \dim K_{n-b,(p-i)\cdot r} - \dim \partial_r^i(M_{n-a})\\ &= \dim H_{n-b,(p-i)\cdot r}\\ &= 0.\end{aligned}$$

This completes the proof. ∎

4. GROUP ACTIONS AND THE EULER–POINCARÉ EQUATION

We shall show that there is a canonical way to attach submodules of $R2^{\Omega}$ to any permutation group on Ω. These give rise to homological sequences to which we can apply the result of the last section in order to establish exactness.

As before, the coefficient ring R has prime characteristic $p > 0$, $r \geq 1$ is a power of p, Ω is some finite set of cardinality n, and $G \subseteq \mathrm{Sym}(\Omega)$ is a permutation group on Ω.

Let g be a permutation of Ω. Then g acts on 2^{Ω} by $\Gamma \mapsto \Gamma^g := \{\gamma^g : \gamma \in \Gamma\}$ which can be extended linearly to the whole of $R2^{\Omega}$. It is not difficult to see that g commutes with ∂_r and so images and kernels of ∂_r^i are left invariant by permutations. This also implies that permutations act as linear maps on the homology modules $H_{k,ir}$.

We define the *orbit module* of G in M_k as

$$M_k^G := \{f \in M_k : f^g = f, \forall g \in G\}.$$

The natural basis for M_k^G are the "orbit sums" $\sum_{\Gamma^* \in \Gamma^G} \Gamma^*$ where Γ^G as usual denotes $\{\Gamma^g : g \in G\}$. In particular

$$n_k^G := \dim M_k^G$$

is the number of G-orbits on $\Omega^{\{k\}}$. As $\partial_r(M_k^G) \subseteq (M_{k-r}^G)$ we obtain sequences of orbit modules. Therefore select as before some $m < r$ and consider the sequence

$$\mathscr{M}^G : 0 \overset{\partial_r}{\leftarrow} M_m^G \overset{\partial_r}{\leftarrow} M_{m+r}^G \overset{\partial_r}{\leftarrow} M_{m+2r}^G \overset{\partial_r}{\leftarrow} M_{m+3r}^G \overset{\partial_r}{\leftarrow} \cdots$$

which is certainly p-homological. In order to investigate p-exactness we fix integers $0 < i^* < p$ and $0 \le k^* \equiv m \bmod(r)$ with $k^* + i^* r < pr$ to obtain the subsequence

$$\mathscr{M}^G_{k^*, i^*} : 0 \leftarrow M^G_{k^*} \leftarrow M^G_{k^* + i^* r} \leftarrow M^G_{k^* + pr} \leftarrow M^G_{k^* + (i^* + p)r} \leftarrow M^G_{k^* + 2pr} \cdots$$

of $\mathscr{M}^G$ in which arrows are appropriate powers of ∂_r.

For arbitrary parameters $|\Omega| = n$, $0 < i < p$, and k we let $K^G_{k, ir}$ denote $\ker \partial_r^i \cap M^G_k$ and let

$$H^G_{k, ir} := K^G_{k, ir} / \partial_r^{p-i}\left(M^G_{k+(p-i)r}\right)$$

be the corresponding homology module. If $f \in K^G_{k, ir}$ we denote its coset in $H^G_{k, ir}$ by

$$[f] := f + \partial_r^{p-i}\left(M^G_{k+(p-i)r}\right).$$

The dimension of $H^G_{k, ir}$ is the *Betti number*

$$\beta^G_{k, ir} := \dim H^G_{k, ir}.$$

In particular, if G is the identity group then $H^G_{k, ir} = H^n_{k, ir}$ and we put

$$\beta^n_{k, ir} := \dim H^n_{k, ir}.$$

By Theorem 3.2 we have $\beta^n_{k, ir} = 0$ unless (k, ir) is the middle term of $\mathscr{M}_{k^*, i^*}$ in which case we refer to $\beta^n_{k, ir}$ as *the* Betti number of $\mathscr{M}_{k^*, i^*}$. Middle terms for $\mathscr{M}^G_{k^*, i^*}$ and $\mathscr{M}^G$ are defined as before. We now examine $\mathscr{M}^G_{k^*, i^*}$ for exactness.

THEOREM 4.1. *Suppose that R is a ring of prime characteristic $p > 0$. Let $r \ge 1$ be a power of p and G a permutation group on Ω whose order is not divisible by p. Then $H^G_{k, ir} = 0$ unless (k, ir) is a middle term.*

COROLLARY 4.2. *If p does not divide the order of G then any section of $\mathscr{M}^G$ containing no middle terms is p-exact.*

Remark. Theorem 3.2 is the special case of Theorem 4.1 when G is the identity group on Ω. Theorem 4.1 states that all but one of the Betti numbers of $\mathscr{M}^G_{k^*, i^*}$ are trivial. Therefore, if (k, ir) is the middle term of $\mathscr{M}^G_{k^*, i^*}$, we call $\beta^G_{k, ir} = \dim H^G_{k, ir}$ *the* Betti number of $\mathscr{M}^G_{k^*, i^*}$.

Proof. Let $M^G_a \leftarrow M^G_b \leftarrow M^G_{a+pr}$ be consecutive terms of $\mathscr{M}^G_{k^*, i^*}$ where without loss of generality $b = a + ir$. Suppose that $b + a + pr \le n$ or that $b + a + pr \ge n + pr$. If $f \in K^G_{b, ir} \subseteq K_{b, ir}$ then by Theorem 3.2 there exists

$F \in M_{a+pr}$ with $\partial_r^{p-i}(F) = f$. But then $|G|^{-1}\sum_{g \in G} F^g \in M_{a+pr}^G$ and $\partial_r^{p-i}(|G|^{-1}\sum_{g \in G} F^g) = |G|^{-1}\sum_{g \in G} \partial_r^{p-i}(F)^g = f$. This completes the proof. ∎

Before we continue we note that Theorem 4.1 can be used to compute the modular rank of certain *orbit inclusion matrices* of G: For $s \leq t$ let $W_{s,t}^G$ be the matrix whose columns are indexed by G-orbits on $\Omega^{\{t\}}$, rows by G-orbits on $\Omega^{\{s\}}$, with (i, j)-entry, for a fixed t-set Γ in the j^{th} orbit, counting the number of s-element subsets $\Delta \subseteq \Gamma$ belonging to the i^{th} orbit.

It is easy to see that $W_{k-r,k}^G$, viewed as a matrix over R, is the matrix of $\partial_r : M_k^G \to M_{k-r}^G$. The following extends Theorem 4.2 of [10].

COROLLARY 4.3. *If p does not divide the order of G, if r is a power of p, and if k, $0 < i < p$ satisfy $2k - ir \leq n$ then the p-rank of $W_{k-ir,k}^G$ is $n_{k-ir}^G - n_{k-pr}^G + n_{k-(p+i)r}^G - n_{k-2pr}^G \cdots$.*

Proof. $0 \leftarrow \cdots \leftarrow M_{k-pr-ir}^G \leftarrow M_{k-pr}^G \leftarrow M_{k-ir}^G \leftarrow M_k^G$ is exact according to the preceding corollary. ∎

The *Euler–Poincaré Equation* for a homological sequence states that its characteristic (i.e., the alternating sum of the dimensions) is equal to the alternating sum of its Betti numbers, see for instance Chapter IX.4 in [7] or Chapter XX.3 in [6]. As $\mathscr{M}_{k^*,i^*}^G$ has particularly simple homologies when G has order co-prime to p this becomes a strong result. We denote by

$$C_{H_{k,ir}}(G) := \left\{[f] \in H_{k,ir} : [f]^g = [f]\ \forall g \in G\right\}$$

the centralizer of G in $H_{k,ir}$, or in other words, the fixed-module of G on $H_{k,ir}$. We give an alternative characterization of $H_{k,ir}^G$.

PROPOSITION 4.4. *If the coefficient ring has prime characteristic $p > 0$ and if $G \subseteq \mathrm{Sym}(\Omega)$ has order co-prime to p then $H_{k,ir}^G \cong C_{H_{k,ir}}(G)$.*

Proof. First we note that

$$\begin{aligned} C_{H_{k,ir}}(G) &= \left\{[f] \in H_{k,ir} : [f]^g = [f]\ \forall g \in G\right\} \\ &= \left(K_{k,ir}^G + \partial_r^{p-i}\left(M_{k+(p-i)r}\right)\right) / \partial_r^{p-i}\left(M_{k+(p-i)r}\right) \end{aligned}$$

since if $[f] = [f]^g$ for all $g \in G$ then $[f] = [|G|^{-1}\sum_{g \in G} f^g]$ and $|G|^{-1}\sum_{g \in G} f^g$ is fixed by the group. We clearly have $\partial_r^{p-i}(M_{k+(p-i)r}^G) \subseteq \partial_r^{p-i}(M_{k+(p-i)r}) \cap K_{k,ir}^G$. Moreover, if $F \in M_{k+(p-i)r}$ with $\partial_r^{p-i}(F)$ fixed by the group then $\partial_r^{p-i}(F) = \partial_r^{p-i}(|G|^{-1}\Sigma_{g\in G}F^g)$ showing that $\partial_r^{p-i}(M^G_{k+(p-i)r}) = \partial_r^{p-i}(M_{k+(p-i)r}) \cap K^G_{k,ir}$. But then

$$\begin{aligned} H^G_{k,ir} &= K^G_{k,ir}/\partial_r^{p-i}\big(M_{k+(p-i)r}\big) \cap K^G_{k,ir} \\ &\cong \Big(K^G_{k,ir} + \partial_r^{p-i}\big(M_{k+(p-i)r}\big)\Big)/\partial_r^{p-i}\big(M_{k+(p-i)r}\big) \\ &= C_{H_{k,ir}}(G). \quad \blacksquare \end{aligned}$$

As usual, we put the binomial coefficient $\binom{n}{k}$ equal to zero if $k < 0$ or if $k > n$:

THEOREM 4.5 (The Euler–Poincaré Equation). *If the coefficient ring has prime characteristic $p > 0$ and if $r \geq 1$ is a power of p, let (k, ir) with $0 < i < p$ be the middle term of $\mathscr{M}_{k^*,i^*}$ and $\beta^n_{k,ir} = \dim H^n_{k,ir}$ its Betti number.*

Suppose that $G \subseteq \mathrm{Sym}(\Omega)$ has order not divisible by p and let $\beta^G_{k,ir} = \dim H^G_{k,ir}$ be the Betti number of $\mathscr{M}^G_{k^*,i^*}$. Then

$$\begin{aligned} \beta^n_{k,ir} &= \sum_{t\in\mathbf{Z}} \binom{n}{k-prt} - \binom{n}{k-ir-prt} \\ &\geq \beta^G_{k,ir} = \sum_{t\in\mathbf{Z}} n^G_{k-prt} - n^G_{k-ir-prt} \end{aligned}$$

and G induces a fixed-point-free representation of degree $\beta^n_{k,ir} - \beta^G_{k,ir}$ on $H_{k,ir}/C$ where $C \cong H^G_{k,ir}$ is the fixed module of G on $H_{k,ir}$.

Proof. By Theorem 4.1, $\mathscr{M}^G_{k^*,i^*}$ has at most one non-trivial homology and so the Euler–Poincaré formula gives $\beta^G_{k,ir} = \Sigma_{t\in\mathbf{Z}} n^G_{k-prt} - n^G_{k-ir-prt}$ as the n^G_j are the dimensions of the modules in $\mathscr{M}^G_{k^*,i^*}$. The equation for $\beta^n_{k,ir}$ is the special case when $G = 1$ and the inequality follows from Proposition 4.4. Finally, the centralizer of G in $H_{k,ir}/C$ is trivial as p does not divide $|G|$. $\blacksquare$

Remarks. (1) Consider the function $\varphi^n_{k,ir} := \Sigma_{t\in\mathbf{Z}}\binom{n}{k-prt} - \binom{n}{k-ir-prt}$ for general n, k, i, r. It is clearly periodic in k and ir and Theorem 4.5 states that $\beta^n_{k,ir}$ agrees with $\varphi^n_{k,ir}$ when (k, ir) is a middle term while $\beta^n_{k,ir} = 0$ otherwise. Some fascinating observations can be made: For $p = 2$, $r = 1$ we have $\varphi^n_{k,i} = 0$; for $p = 3$ and $r = 1$ we get $\varphi^n_{k,i} \in \{0, 1\}$ while for $p = 5$, $r = 1$ we find that $\varphi^n_{k,i}$ is 0 or the $(n-1)^{\mathrm{st}}$, n^{th}, or $(n-1)^{\mathrm{st}}$ Fibonacci number. See also Remark 2 following Theorem 6.5.

(2) The inequality $\beta^n_{k,ir} \geq \beta^G_{k,ir}$ may not hold for groups of order divisible by p. For instance, when $p = 3$ and G is C_6 acting on six points, we have $\beta^6_{3,1} = 1$ but $\beta^G_{3,1} = 2$.

(3) For middle terms the inequality $\varphi^n_{k,ir} \geq \beta^G_{k,ir}$ gives interesting results about the orbits on subsets of permutation groups of order co-prime to p, in particular if $\varphi^n_{k,ir}$ is small. This was first used in Theorem 6.1 in [10].

(4) Any functional relation for $\varphi^n_{k,ir}$ will give information about $\beta^n_{k,ir}$. For instance, it is clear that $\varphi^n_{k,ir} = \varphi^{n-1}_{k,ir} + \varphi^{n-1}_{k-1,ir}$ as this holds for binomial coefficients. This leads to the corollary below. But there are less obvious relations and some of these will be made more explicit in Section 6.

COROLLARY 4.6. (i) *If* $0 < i < p$ *and* $n + 1 \leq 2k + (p-i)r \leq n + pr - 1$ *then* $\beta^n_{k,ir} = \beta^{n-1}_{k,ir} + \beta^{n-1}_{k-1,ir}$. (ii) *If* $0 < i < p$ *and* $n < 2k + p - i < n + p$, *then* $\beta^n_{k,i} = \beta^{n-1}_{k,i+1} + \beta^{n-1}_{k-1,i-1}$.

Proof. In (i) the conditions on the parameters mean that (k, ir) is a middle term for a set of size n and that (k, ir) and $(k-1, ir)$ are middle terms for a set of size $n-1$. Hence $\beta^n_{k,ir} = \varphi^n_{k,ir}$, $\beta^{n-1}_{k,ir} = \varphi^{n-1}_{k,ir}$, and $\beta^{n-1}_{k-1,ir} = \varphi^{n-1}_{k-1,ir}$. The result follows from $\varphi^n_{k,ir} = \varphi^{n-1}_{k,ir} + \varphi^{n-1}_{k-1,ir}$. Similarly, for (ii) write out the terms of $\varphi^n_{k,i}$ and use the relation for the binomial coefficients. ∎

5. GENERATORS OF THE KERNELS

In this section we construct generators for $K^n_{k,ir}$ for general $0 \leq k \leq n$ and $0 < i < p$. This then also provides generators for the homology modules $H^n_{k,ir}$ and $H^G_{k,ir}$ for groups of order co-prime to p.

If $2k + (p-i)r \leq n$ or $2k + (p-i)r \geq n + pr$ then Theorem 3.2 implies that $K_{k,ir} = \partial_r^{p-i}(M_{k+(p-i)r})$ which provides an efficient set of generators. Therefore we restrict our attention to finding a generating set for $K_{k,ir}$ when $n < 2k + (p-i)r < n + pr$, that is, when (k, ir) is a middle term.

Moreover, if $k < ir$ then $K_{k,ir} = M_k$ and so we can assume that $k \geq ir$. When $ir \leq k$ and $2k - ir + 1 \leq n$ we define

$$\begin{aligned} C_{k,ir} := \Big\{ &(\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2) \cup \cdots \cup (\alpha_t - \beta_t) \cup \Gamma : \\ &\alpha_j, \beta_{j*} \in \Omega, \Gamma \subseteq \Omega, \alpha_j \neq \beta_{j*} \\ &\text{for } 1 \leq j, j* \leq t, t = k - ir + 1, |\Gamma| = ir - 1 \Big\}. \end{aligned}$$

LEMMA 5.1. *Let R be any coefficient ring with identity and let r and i be positive integers. If* $k = ir$ *and* $k + 1 \leq n$ *then* $K_{k,ir} = \langle C_{k,ir} \rangle$.

Proof. Certainly $\ker \partial_r^i \cap M_k$ is spanned by $\{\Gamma - \Delta : \Gamma, \Delta \subseteq \Omega$ and $|\Gamma| = |\Delta| = k\}$. We show that $\Gamma - \Delta \in \langle C_{k,ir} \rangle$ by induction on $|\Gamma \setminus \Delta|$.

For $|\Gamma \setminus \Delta| = 0$ or 1 this certainly holds. Therefore suppose that $|\Gamma \setminus \Delta| \geq 2$ and that

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_j, \gamma_{j+1}, \ldots, \gamma_k\}$$

and

$$\Delta = \{\gamma_1, \gamma_2, \ldots, \gamma_j, \alpha_{j+1}, \ldots, \alpha_k\},$$

where $j = k - |\Gamma \setminus \Delta|$. Then we let $\Theta = \{\gamma_1, \gamma_2, \ldots, \gamma_j, \gamma_{j+1}, \alpha_{j+2}, \ldots, \alpha_k\}$ and note that $\Gamma - \Delta = \Gamma - \Theta + \Theta - \Delta$ where $|\Gamma \setminus \Theta| = |\Gamma \setminus \Delta| - 1$ and $|\Theta \setminus \Delta| = 1$. Invoking the induction hypothesis completes the proof. ∎

LEMMA 5.2. *Let R be a ring of prime characteristic p and $r \geq 1$ a power of p. If $0 < i < p$ and $ir \leq k$ let (k, ir) be a middle term. Then $K_{k,ir}^n = \langle \partial_r^{p-i}(M_{k+(p-i)r}^n), C_{k,ir} \rangle$.*

Proof. We proceed by induction on n. For small values of n the result is easily verified. So suppose that the lemma is true for all values $< n$. By the above lemma we may assume that $k < ir$. Let $f \in K_{k,ir}$ be given. We assume that $\operatorname{supp} f = \Omega$ otherwise we may use induction or Theorem 3.2 to complete the proof. We write $f = \alpha \cup f_\alpha + l$ where $\operatorname{supp} f_\alpha \cup \operatorname{supp} l \subseteq \Omega^* := \Omega \setminus \alpha$. Then $\partial_r^i(f) = 0$ implies that $\partial_r^i(f_\alpha) = 0$, that is, $f_\alpha \in K_{k-1,ir}^{n-1}$. Then either by induction, the above lemma, or Theorem 3.2 we see that $f_\alpha \in \langle \partial_r^{p-i}(M_{k-1+(p-i)r}^{n-1}), C_{k-1,ir}^{n-1} \rangle$. We write $f_\alpha = \partial_r^{p-i}(F) + \sum_j r_j c_j$ with $r_j \in R$, $c_j \in C_{k-1,ir}^{n-1}$, and $F \in M_{k-1+(p-i)r}^{n-1}$. Since $2k - ir < n$ and $\|c_j\| = 2k - ir - 1$ we may select $\alpha_j \in \Omega^* \setminus \operatorname{supp} c_j$. We let $h := f - \sum_j r_j(\alpha - \alpha_j) \cup c_j - \partial_r^{p-i}(\alpha \cup F)$ and note that $h \in K_{k,ir}^{n-1}$, $\sum_j r_j(\alpha - \alpha_j) \cup c_j \in C_{k,ir}$ and $\partial_r^{p-i}(\alpha \cup F) \in \partial_r^{p-i}(M_{k+(p-i)r})$. Therefore by induction or Theorem 3.2 the proof is complete. ∎

We collect the results of this section so far together in the following.

THEOREM 5.3. *Let R be a ring of prime characteristic p, $r \geq 1$ a power of the prime p, and let $0 < i < p$.*

(i) *If (k, ir) is not a middle term then $K_{k,ir} = \partial_r^{p-i}(M_{k+(p-i)r})$, and*

(ii) *If (k, ir) is a middle term then $K_{k,ir} = M_k$ for $k < ir$ and $K_{k,ir} = \langle \partial_r^{p-i}(M_{k+(p-i)r}), C_{k,ir} \rangle$ for $ir \leq k$.*

From this we obtain immediately expressions for the homology modules:

COROLLARY 5.4. *If (k, ir) is a middle term, then*

$$H^n_{k,ir}\begin{cases} = M_k/\partial_r^{p-i}\big(M_{k+(p-i)r}\big) & \text{if } k < ir \\ \cong \langle C_{k,ir}\rangle/\langle C_{k,ir}\rangle \cap \partial_r^{p-i}\big(M_{k+(p-i)r}\big) & \text{if } ir \le k. \end{cases}$$

Further, if $G \subset \mathrm{Sym}(\Omega)$ has order co-prime to p, then

$$H^G_{k,ir}\begin{cases} = M_k^G/M_k^G \cap \partial_r^{p-i}\big(M_{k+(p-i)r}\big) & \text{if } k < ir \\ \cong M_k^G \cap \langle C_{k,ir}\rangle/M_k^G \cap \langle C_{k,ir}\rangle \cap \delta_r^{p-i}\big(M_{k+(p-i)r}\big) & \text{if } ir \le k. \end{cases}$$

Proof. The first part is clear and the second follows from Proposition 4.4. ∎

It is clear that the module generated by $C_{k,ir}$ is of special importance and we will examine it in terms of the standard representation theory of the symmetric groups; as a reference we suggest Chapter 7 of [5].

Suppose now that R is a field of characteristic $p > 0$. Let

$$c = (\alpha_1 - \beta_1) \cup (\alpha_2 - \beta_2) \cup \cdots \cup (\alpha_t - \beta_t) \cup \Gamma$$

be an element in $C_{k,ir}$ with $t = k - ir + 1$ and $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{ir-1}\}$ and define $\Omega^* = \{\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_r, \gamma_1, \ldots, \gamma_{ir-1}\}$. We notice that c corresponds to the polytabloid $\tau \cdot \kappa_\tau$ on Ω^* where

$$\tau = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_t & \gamma_1 & \gamma_2 & \cdots & \gamma_{ir-1} \\ \beta_1 & \beta_2 & \cdots & \beta_t \end{pmatrix}$$

and where $\kappa_\tau = (1 - (\alpha_1, \beta_1)) \cdot (1 - (\alpha_2, \beta_2)) \cdots (1 - (\alpha_t, \beta_t)) \in R\,\mathrm{Sym}(\Omega^*)$ is the signed column stabilizer of τ. So if we let M_k^* denote the R-module with k-element subsets of Ω^* as basis we have obtained

LEMMA 5.5. *$S := \langle M_k^* \cap C_{k,ir}\rangle$ is isomorphic to the Specht module for the partition of Ω^* into 2 parts of size k and $k - ir + 1$. Further, $\langle C_{k,ir}\rangle = S^{\uparrow \mathrm{Sym}(\Omega)}$ is the module induced from S.*

With the use of this lemma and reciprocity arguments one can determine the structure of $\langle C_{k,ir}\rangle$. This is the case in particular when $|\Omega^*|$ is close to $|\Omega|$ and we will use this lemma in the next section to determine the structure of some homology modules in terms of Specht modules.

6. THE HOMOLOGIES OF THE 1-STEP MAP

In this section we restrict our attention to the case $r = 1$ and for simplicity the 1-step map ∂_1 is denoted by ∂. Throughout this section R is a ring of prime characteristic p. In [10] we have shown that in characteristic $p = 2$ all homologies of the 1-step map are trivial. So here throughout $p > 2$.

If $G \subseteq \mathrm{Sym}(\Omega)$ is a permutation group on Ω and $0 < i < p$ then $H^G_{k,i}$ is the homology module relative to Ω as defined in Section 4. If $\alpha \in \Omega$ then we regard the stabilizer G_α of α as a permutation group on $\Omega \setminus \alpha$ and so $H^{G_\alpha}_{k,i}$ denotes the homology module relative to $\Omega \setminus \alpha$. To avoid unpleasant case distinctions we will put $H^{G_\alpha}_{k,i} = H^G_{k,i} = H^n_{k,i} = 0$ when $i = 0$ or $i = p$.

THEOREM 6.1. *Let R be a ring of prime characteristic p, $0 < i < p$, and let G be a permutation group on Ω. Suppose that for some $\alpha \in \Omega$ the size of the orbit α^G is co-prime to p and let N be the normalizer of G_α in* $\mathrm{Sym}(\Omega \setminus \alpha)$.

Then there exists a monomorphism $\Phi : H^G_{k,i} \to H^{G_\alpha}_{k,i+1} \oplus H^{G_\alpha}_{k-1,i-1}$ which commutes with N.

A special case of this theorem is worth mentioning separately. Note that in both theorems we do not require that (k, i) be a middle term:

THEOREM 6.2. *Let R be a ring of prime characteristic p, $0 < i < p$, and let α be an arbitrary element of Ω. Then $H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}$ as* $\mathrm{Sym}(\Omega \setminus \alpha)$*-modules. In particular, if $p > 2$ and $0 \le k \le n$ then $H^n_{k,i} \neq 0$ if and only if (k, i) is a middle term for n.*

Remark. Note that (k, i) is a middle term with respect to Ω if and only if $(k, i + 1)$ and $(k - 1, i + 1)$ are middle terms with respect to $\Omega \setminus \alpha$. Hence Theorem 5.2 and induction on n can be used to give the shortest self-contained proof that $H^n_{k,i}$ is trivial if and only if (k, i) is a middle term.

Proof of Theorem 6.1. Let $f = \alpha \cup f_\alpha + l$ be an element of $K^G_{k,i}$ where $\alpha \notin \mathrm{supp}(f_\alpha) \cup \mathrm{supp}(l)$. Then $0 = \partial^i(f) = \alpha \cup \partial^i(f_\alpha) + i\partial^{i-1}(f_\alpha) + \partial^i(l)$ and so $\partial^i(f_\alpha) = 0$, $l \in K^{G_\alpha}_{k,i+1}$, and $if_\alpha + \partial(l) \in K^{G_\alpha}_{k-1,i-1}$.

Now define the map $\Phi : H^G_{k,i} \to H^{G_\alpha}_{k,i+1} \oplus H^{G_\alpha}_{k-1,i-1}$ by putting

$$\Phi : [f] \mapsto \big([l], [if_\alpha + \partial(l)]\big).$$

To show that this is well defined suppose that $[f] = [h]$ with $h = \alpha \cup h_\alpha + m$ and $\alpha \notin \mathrm{supp}(h_\alpha) \cup \mathrm{supp}(m)$. So there exists some $F = \alpha \cup F_\alpha + L \in M^G_{k+p-i}$ with $\alpha \notin \mathrm{supp}(F_\alpha) \cup \mathrm{supp}(L)$ and $\partial^{p-i}(F) = f - h = \alpha \cup (f_\alpha$

$-h_\alpha) + l - m$. Note that $F_\alpha^g = F_\alpha$ and $L^g = L$ for all $g \in G_\alpha$ and we calculate

$$\begin{aligned}\alpha \cup (f_\alpha - h_\alpha) + l - m \\ &= \partial^{p-i}(\alpha \cup F_\alpha + L) \\ &= \alpha \cup \partial^{p-i}(F_\alpha) + (p-i)\partial^{p-i-1}(F_\alpha) + \partial^{p-i}(L)\end{aligned}$$

implying that $l - m \in \partial^{p-i-1}(M_{k+p-i-1}^{G_\alpha})$ and that $\partial^{p-i}(F_\alpha) = f_\alpha - h_\alpha$. Applying ∂ to the equation gives

$$\begin{aligned}\alpha \cup \partial(f_\alpha - h_\alpha) + f_\alpha - h_\alpha + \partial(l-m) \\ &= \partial^{p-i+1}(\alpha \cup F_\alpha + L) \\ &= \alpha \cup \partial^{p-i+1}(F_\alpha) + (p-i+1)\partial^{p-i}(F_\alpha) + \partial^{p-i+1}(L)\end{aligned}$$

so that $i(f_\alpha - g_\alpha) + \partial(l-m) \in \partial^{p-i+1}(M_{k+p-i}^{G_\alpha})$. Therefore Φ is well defined, clearly linear, and it is a simple matter to check that it commutes with N.

Suppose now that $\Phi([f]) = ([0],[0])$. Then there exists $F \in M_{k+p-i-1}^{G_\alpha}$ with $\partial^{p-i-1}(F) = l$ and there exists $H \in M_{k+p-i}^{G_\alpha}$ with $\partial^{p-i+1}(H) = if_\alpha + \partial(l)$. Then

$$\begin{aligned}\partial^{p-i}(\alpha \cup F) &= \alpha \cup \partial(l) + (p-i)l, \\ \partial^{p-i+1}(\alpha \cup H) &= \alpha \cup \left(if_\alpha + \partial(l)\right) + (p-i+1)\partial^{p-i}(H)\end{aligned}$$

and hence $\partial^{p-i+1}(\alpha \cup H) - (p-i+1)\partial^{p-i}(H) - \partial^{p-i}(\alpha \cup F) = if$. Let $J := \partial(\alpha \cup H) - (p-i+1)H - \alpha \cup F$. Then J is fixed by G_α and we may define

$$J^G := |\alpha^G|^{-1} \sum_{G_\alpha g \in cos(G:G_\alpha)} J^g.$$

Then J^G is fixed by G and $\partial^{p-i}(J^G) = if$. Hence Φ is injective. ∎

Proof of Theorem 6.2. Here we suppose $G = \{1\}$ and let $([l],[m]) \in H_{k,i+1}^{n-1} \oplus H_{k-1,i-1}^{n-1}$. Then $\partial^i(\alpha \cup (i^{-1}(m - \partial(l)) + l)) = 0$ and $\Phi([\alpha \cup i^{-1}(m - \partial(l)) + l]) = ([l],[m])$ showing that Φ is surjective. Alternatively, use Corollary 4.6(ii) to show that $H_{k,i+1}^{n-1} \oplus H_{k-1,i-1}^{n-1}$ has dimension $\beta_{k,i}^n$. As (k,i) is a middle term for n if and only if $(k, i+1)$ and $(k-1, i-1)$ are middle terms for $n-1$ the statement about the non-triviality of $H_{k,i}^n$ is proved by induction on n. This completes the proof. ∎

Theorem 6.2 is useful for investigating the irreducibility of the homology modules which we deal with in the next two results.

THEOREM 6.3. *Let R be a ring of prime characteristic $p > 2$. For $\alpha \in \Omega$ assume that $H^n_{k,i} \cong H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}$ is non-zero and suppose further that $H^{n-1}_{k,i+1}$ and $H^{n-1}_{k-1,i-1}$ are zero or irreducible $R\,\mathrm{Sym}(\Omega \setminus \alpha)$-modules and that they are non-isomorphic if they are both non-zero. Then $H^n_{k,i}$ is an irreducible $R\,\mathrm{Sym}(\Omega)$-module.*

Proof. For a contradiction we will suppose that U is a non-trivial $R\,\mathrm{Sym}(\Omega)$-submodule of $H^n_{k,i}$ and so if Φ is the map of Theorem 6.2 then $\Phi(U)$ is a non-trivial $R\,\mathrm{Sym}(\Omega \setminus \alpha)$-submodule of $H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1}$. Therefore we are done if either $H^{n-1}_{k,i+1}$ or $H^{n-1}_{k-1,i-1}$ is zero. Hence we may assume that $H^{n-1}_{k,i+1}$ and $H^{n-1}_{k-1,i-1}$ are irreducible non-isomorphic $R\,\mathrm{Sym}(\Omega \setminus \alpha)$-modules and further that $n \geq 3$, $k < n$, and $1 < i < p-1$. Therefore $\Phi(U)$ is either $H^{n-1}_{k-1,i-1}$ or $H^{n-1}_{k,i+1}$.

Case 1. $\Phi(U) = H^{n-1}_{k-1,i-1}$. Let $[f]$ be a generator of $H^{n-1}_{k-1,i-1}$ as given in Theorem 5.3. So either $f \in C^{n-1}_{k-1,i-1}$ if $k \geq i$ or f is a $(k-1)$-subset of $\Omega \setminus \alpha$ if $k < i$. As $[f] \in \Phi(U)$ we have $[i^{-1}\alpha \cup f] = \Phi^{-1}([f]) \in U$. But again by Theorem 5.3 we have $\langle [\alpha \cup f]^g : g \in \mathrm{Sym}(\Omega)\rangle = H^n_{k,i}$, a contradiction.

Case 2. $\Phi(U) = H^{n-1}_{k,i+1}$. First assume that $i < k$ and let $f \in C^{n-1}_{k,i+1}$. Then we may write $f = A \cup l$ where $A = \{\alpha_1, \dots, \alpha_i\}$ and $l \in C^{n-i-1}_{k-i,1}$. Hence putting $\bar{f} := -i^{-1}\alpha \cup \partial(f) + f$ we see that $\Phi^{-1}([f]) = [\bar{f}]$, and as $[f] \in H^{n-1}_{k,i+1}$ it follows that $\bar{f}$ belongs to U. Let $\bar{f}^{(\alpha,\alpha_1)}$ be the result of applying the transposition (α, α_1) to $\bar{f}$. Then

$$\begin{aligned}\left[\bar{f}^{(\alpha,\alpha_1)}\right] &= \left[-i^{-1}\alpha_1 \cup \partial\big((A \setminus \alpha_1) \cup \alpha \cup l\big) + (A \setminus \alpha_1) \cup \alpha \cup l\right] \\ &= \left[\alpha \cup \Big(\big((i^{-1}+1)A \setminus \alpha_1 - i^{-1}\partial(A)\big) \cup l\Big) - i^{-1}A \cup l\right].\end{aligned}$$

As $i > 1$ consider the transposition (α, α_2) and compute $[\bar{f}^{(\alpha,\alpha_1)}] - [\bar{f}^{(\alpha,\alpha_2)}] = [(i^{-1}+1)\alpha \cup (\alpha_2 - \alpha_1) \cup A \setminus \{\alpha_1, \alpha_2\} \cup l]$. By Theorem 5.3 the $\mathrm{Sym}(\Omega)$-images of $[\alpha \cup (\alpha_2 - \alpha_1) \cup A \setminus \{\alpha_1, \alpha_2\} \cup l]$ generate $H^n_{k,i}$ which is a contradiction.

Secondly we assume that $2 \leq k \leq i$ and here we let $A := \{\alpha_1, \dots, \alpha_k\}$ so that $[A] \in H^{n-1}_{k,i+1}$. Putting $f := -i^{-1}\alpha \cup \partial(A) + A$, we see that $[f] = \Phi^{-1}([A])$ so that $f \in U$. Further,

$$\begin{aligned}\left[f^{(\alpha,\alpha_1)}\right] &= \left[-i^{-1}\alpha_1 \cup \partial(A \setminus \alpha_1) \cup \alpha + (A \setminus \alpha_1) \cup \alpha\big)\right] \\ &= \left[\alpha \cup \big((i^{-1}+1)A \setminus \alpha_1 - i^{-1}\partial(A)\big) - i^{-1}A\right]\end{aligned}$$

and hence $[f^{(\alpha,\alpha_1)}] - [f^{(\alpha,\alpha_2)}] = [(i^{-1}+1)\alpha \cup (\alpha_2 - \alpha_1) \cup A \setminus \{\alpha_1, \alpha_2\}]$. If $i = k$ then Theorem 5.3 implies as before that the $\mathrm{Sym}(\Omega)$-images of $[\alpha \cup (\alpha_2 - \alpha_1) \cup A \setminus \{\alpha_1, \alpha_2\}]$ generate $H^n_{k,i}$ which is a contradiction.

In any case expressions of the form $\alpha \cup (\alpha_2 - \alpha_1) \cup A \setminus \{\alpha_1, \alpha_2\}$ are differences of two k-element sets and so U has co-dimension as most one in $H^n_{k,i}$. We suppose therefore that $k < i$ and that $1 = \dim(H^n_{k,i}/U) = \dim H^n_{k,i} - \dim H^{n-1}_{k,i+1} = \dim H^{n-1}_{k-1,i-1}$ by Theorem 6.2. As $2 \leq k$ we have $1 \leq i - 1 < i + 1 < p$ so that both $(k-1, i)$ and $(k-2, i-2)$ are middle terms with respect to $n - 2$ and $0 \leq k - 2 < k - 1 \leq n - 2$. Hence by Theorem 6.2 we have $\beta^{n-2}_{k-1,i} > 0$ and $\beta^{n-2}_{k-2,i-2} > 0$ which contradicts $1 = \beta^{n-1}_{k-1,i-1} = \beta^{n-2}_{k-1,i} + \beta^{n-2}_{k-2,i-2}$.

Therefore finally assume that $k = 1$ and as $H^{n-1}_{k-1,i-1} \neq 0$ it follows from Theorem 6.2 that $(0, i-1)$ is a middle term so that $p - i + 1 > n - 1$. As $K^{n-1}_{1,i+1} = M^{n-1}_1$ for $i + 1 > 1$ we have $H^{n-1}_{k,i+1} = M^{n-1}_1 / \partial^{p-(i+1)}(M^{n-1}_{p-i})$. As $H^{n-1}_{k,i+1}$ is irreducible by assumption while M^{n-1}_1 is not, we cannot have $\partial^{p-(i+1)}(M^{n-1}_{p-i}) = 0$. Together with $p - i + 1 > n - 1$ this implies that $n - 1 = p - i$ which means that $H^n_{k,i} = M^n_1 / \partial^{n-1}(M^n_n)$ and this module is know to be irreducible for $p > n$, see [4, p. 18]. This completes the proof. ∎

THEOREM 6.4. *Let R be a ring of prime characteristic $p \neq 2$ and suppose that $0 \leq k \leq n$ and $0 < i < p$ satisfy $2k + p - i = n + p - 1$. Then $H^n_{k,i}$ is irreducible. Furthermore, if (k', i') is another pair of positive integers satisfying the above conditions then $H^n_{k,i} \not\cong H^n_{k',i'}$.*

Proof. We prove this result by induction on n. For small values of n the result is easily verified. Therefore suppose the result holds for $n - 1$. Since $2k + p - (i + 1) = (n - 1) + p - 1$ and $2(k - 1) + p - (i - 1) = (n - 1) + p - 1$ we see inductively that $H^{n-1}_{k,i+1}$ and $H^{n-1}_{k-1,i-1}$ are either zero or irreducible, and that if they are both irreducible then they are non-isomorphic. However, as $p \neq 2$ it is easy to see that $H^{n-1}_{k,i+1}$ and $H^{n-1}_{k-1,i-1}$ cannot both be zero. So Theorem 6.3 implies that $H^n_{k,i}$ is irreducible.

Now suppose that (k', i') is another pair of positive integers satisfying $2k' + p - i' = n + p - 1$ and $0 < i' < p$. We assume for a contradiction that $H^n_{k,i} \cong H^n_{k',i'}$ and so by Theorem 6.2 we have $H^{n-1}_{k,i+1} \oplus H^{n-1}_{k-1,i-1} \cong H^{n-1}_{k',i'+1} \oplus H^{n-1}_{k'-1,i'-1}$. The induction hypothesis then implies that $H^{n-1}_{k-1,i-1} \not\cong H^{n-1}_{k'-1,i'-1}$, hence $H^{n-1}_{k,i+1} \cong H^{n-1}_{k'-1,i'-1}$ and $H^{n-1}_{k-1,i-1} \cong H^{n-1}_{k',i'+1}$.

Moreover, if $H^{n-1}_{k,i+1}$ and $H^{n-1}_{k-1,i-1}$ are both non-zero then the induction hypothesis implies that $k = k' - 1$ and $k - 1 = k'$, giving us a contradiction.

Suppose therefore that $H^{n-1}_{k,i+1}$ is non-zero and $H^{n-1}_{k-1,i-1}$ is zero. But then we see that $i - 1 \equiv i' + 1 \equiv 0 \bmod(p)$ and by the induction hypothesis that $i + 1 = i' - 1$. Hence $2 \equiv i + 1 = i' - 1 \equiv -2 \bmod(p)$, a contradiction. A similar argument works in the case when $H^{n-1}_{k,i+1}$ is zero and $H^{n-1}_{k-1,i-1}$ is non-zero. ∎

Finally we are in a position to identify certain of the homology modules in terms of Specht modules and partitions of Ω:

THEOREM 6.5. *Let R be a ring of prime characteristic $p \neq 2$ and suppose that $i \leq k$ and $0 < i < p$ satisfy $2k + p - i = n + p - 1$. Then $H^n_{k,i}$ is isomorphic to $S^\tau / S^\tau \cap S^{\tau\perp}$ where S^τ is the Specht module corresponding to a partition of Ω into 2 parts of length k and $k - i + 1$.*

Proof. By Corollary 5.4 we have $H^n_{k,i} \cong \langle C_{k,i}\rangle / \langle C_{k,i}\rangle \cap \partial^{p-i}(M_{k+(p-i)r})$ and as $\Omega = \Omega^*$ in Lemma 5.5 we have $S^\tau = \langle C_{k,i}\rangle$. As $S^\tau \cap S^{\tau\perp}$ is the unique maximal submodule of S^τ, the result follows from Theorem 6.4. ∎

Remarks. (1) Provided that $n + 2 \geq p$ there are $(p-1)/2$ distinct pairs of positive integers k and $0 < i < p$ with $2k + p - i = n + p - 1$. So Theorem 6.4 provides $(p-1)/2$ non-isomorphic irreducible $\mathrm{Sym}(\Omega)$-modules and their dimensions are given by the function $\varphi^n_{k,i}$ of Section 4.

(2) When $p = 5$ these modules are precisely the Fibonacci representations of the symmetric groups described in Ryba's paper [13]. Such systems of representations have been generalized in Kleshchev's work [9]. For general prime $p > 2$ the collection $\mathscr{H} := \{H^n_{k,i} : k < n,\ 0 \leq i < p,\ 2k - i + 1 = n\}$ is an example of the *semi-simple inductive systems* discussed in [9]. In fact, $\mathscr{H}$ consists precisely of the modules arising from 2-part partitions which satisfy Kleshchev's condition of Theorem 2.1 in [9]. We conjecture that such semi-simple inductive systems for partitions with more than 2 parts arise also as homologies for suitable posets.

In the remainder of this section we give the complete decomposition of the $H^n_{k,i}$. Let a be an integer satisfying $0 < a < p$. For $0 < i < p$ we define module homomorphisms

$$\rho : H^n_{k,i} \to H^n_{k,i+1}$$

and

$$\partial : H^n_{k,i} \to H^n_{k-1,i-1}$$

by $\rho([f]) := [f]$ and $\partial([f]) := [\partial(f)]$, respectively. It is a simple matter to check that these maps are well-defined. We record some properties of these homomorphisms in the following:

LEMMA 6.6. *If $2k + p - i = n + a$ then*

(a) *$\rho : H^n_{k,i} \to H^n_{k,i+1}$ is surjective if $i \geq a - 1$ and*

(b) *$\partial : H^n_{k,i} \to H^n_{k-1,i-1}$ is surjective if $i \leq p - (a-1)$.*

Proof. (a) Let f be in $K^n_{k,i+1}$. Then $\partial^i(f) \in K^n_{k-i,1}$ and $2(k-i) + p - 1 = n + (a-1) - i \leq n$. By the Integration Theorem (or indeed, by

Lemma 2.3) there exists F in $M^n_{k+p-(i+1)}$ with $\partial^{p-1}(F) = \partial^i(f)$. But then $\partial^i(f - \partial^{p-(i+1)}(F)) = 0$ and $\rho([f + \partial^{p-(i+1)}(F)]) = [f]$.

(b) Let f be in $K^n_{k-1,i-1}$. Then $2(k-1)+1 = n - (p-(a-1)-i) \le n$ and by the Integration Theorem (or indeed, by Lemma 2.3) there exists F in $K^n_{k,i}$ with $\partial(F) = f$. But then $\partial([F]) = [f]$. ∎

We now present two further results which will help us determine the composition factors of the homology modules.

LEMMA 6.7. *If* $2k + p - i = n + a$ *then* $H^n_{k,i} \cong H^n_{k,a}$.

Proof. We notice that $k + p - i + k + p - a \equiv n \bmod(p)$ so that $H^n_{k,i}$ and $H^n_{k,a}$ will have the same dimension. Without loss of generality we may suppose that $i < a$ and then we look at the map $\rho^{a-i} : H^n_{k,i} \to H^n_{k,a}$. If $f \in K^n_{k,a}$ then $\partial^i(f) \in K^n_{k-i,a-i}$ and $2(k-i) + p - (a-i) = n$ so that, by the Integration Theorem, there exists F in M_{k+p-a} with $\partial^{p-(a-i)}(F) = \partial^i(f)$. But then $\partial^i(f - \partial^{p-a}(F)) = 0$ and $\rho^{a-i}([f - \partial^{p-a}(F)]) = [f]$. ∎

LEMMA 6.8. $H^n_{k,i} \cong H^n_{n-k,p-i}$.

Proof. Suppose $2k + p - i = n + a$ and, without loss of generality, that $n - k \ge k$. But then $n - 2k = p - (i + a)$ and we can look at the map $\partial^{p-(i+a)} : H^n_{n-k,p-i} \to H^n_{k,a}$. Suppose that $f \in K^n_{k,a}$. Then $2k + p - (i + a) = n$ and by the Integration Theorem there exists $F \in K^n_{n-k,p-i}$ with $\partial^{p-(i+a)}(F) = f$. But then $\partial^{p-(i+a)}([F]) = [f]$. However, $H^n_{n-k,p-i}$ and $H^n_{k,i}$ have the same dimension and hence applying the previous result completes the proof. ∎

We are now in a position to determine the composition factors of all homology modules. Since $H^n_{k,i} \cong H^n_{n-k,p-i}$ it suffices to consider the case when $2k + p - i = n + a$ and $0 < a < p/2$.

THEOREM 6.9. *Let* $2k + p - i = n + a$ *and* $0 < a < p/2$. *Then the composition factors of* $H^n_{k,i}$ *each have multiplicity one and are given as follows*:

(a) $\{H^n_{k-j,i+a-1-2j} : j = 0, \dots, a-1\}$ *if* $a \le i \le p - a$.

(b) $\{H^n_{k-j,i+a-1-2j} : j = 0, \dots, i-1\}$ *if* $i < a$ *and*

(c) $\{H^n_{k-j,i+a-1-2j} : j = i-(p-a), \dots, a-1\}$ *if* $i > p - a$.

Proof. The proof is by induction on a. Suppose firstly that $i < a$. Then $H^n_{k,i} \cong H^n_{k,a}$ and $2k + p - a = n + i$ with $i < a < p - i$ so that, by induction, the composition factors of $H^n_{k,i}$ are $\{H^n_{k-j,i+a-1-2j} : j = 0, \dots, i-1\}$.

Secondly suppose that $i > p - a$. Then $H^n_{k,i} \cong H^n_{n-k,p-a}$ and $2(n-k) + p - (p-a) = n + p - i < n + a$ with $p - i < p - a < i$. By induction the composition factors of $H^n_{k,i}$ are therefore

$$\left\{H^n_{n-k-j,2p-(a+i)-1-2j} : j = 0, \ldots, p-i-1\right\}$$
$$= \left\{H^n_{k-l,i+a-1-2l} : l = i-(p-a), \ldots, a-1\right\}.$$

Finally suppose that $a \leq i \leq p-a$. By Lemma 6.6 all composition factors of $H^n_{k,i+1}$ and $H^n_{k-1,i-1}$ will be composition factors of $H^n_{k,i}$. Since $2(k-1)+p-(i-1) = 2k+p-(i+1) = n+(a-1)$ and $a-1 \leq i-1 < i+1 \leq p-(a-1)$ we can assume inductively that

$$\left\{H^n_{k-1-j,i-1+a-2-2j} : j = 0, \ldots, a-2\right\}$$
$$\cup\left\{H^n_{k-l,i+1+a-2-2l} : l = 0, \ldots, a-2\right\}$$
$$= \left\{H^n_{k-j,i+a-1-2j} : j = 0, \ldots, a-1\right\}$$

are all composition factors of $H^n_{k,i}$. This set consists precisely of the composition factors of $H^n_{k-i,i-1}$ together with $H^n_{k,i+a-1}$. Furthermore, we notice that $\dim H^n_{k,i} - \dim H^n_{k-1,i-1} = \dim H^n_{k,i+a-1}$ since $k-1+k-(i+a-1) \equiv n \bmod(p)$. Since all the modules in $\{H^n_{k-j,i+a-1-2j} : j = 0, \ldots, a-1\}$ are irreducible and non-isomorphic we are done. ∎

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