SOME TOPICS IN THE REPRESENTATION THEORY OF THE SYMMETRIC AND GENERAL LINEAR GROUPS

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Thesis submitted to the University of London

for the degree of Doctor of Philosophy

and to Imperial College

for the Diploma of Imperial College

Abstract

The ordinary irreducible representations of the group $\mathfrak{S}_n$ are known to be indexed by the set of partitions $\{\lambda \vdash n\}$, and given a partition λ, the corresponding representation is denoted S^λ. For a given prime p, an open problem in representation theory is to determine the composition factors of S^λ reduced modulo p.

One method used to obtain a first approximation to the multiplicity of a potential composition factor is the application of the LLT algorithm, a recursive algorithm on n which gives the decomposition numbers for the Hecke algebra $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$. Given a p-modular irreducible representation of $\mathfrak{S}_n$, D^μ say, the LLT algorithm will provide a lower bound for the multiplicity of D^μ as a composition factor of any S^λ. We will use the LLT algorithm to explicitly find these bounds in the cases where λ has at most four parts. This information is then reconsidered in terms of $\mathfrak{S}_n$-modules.

Similar results hold for the representations of the general linear groups $\mathrm{GL}_n(q)$; in particular when $p \nmid q$, the ordinary irreducible unipotent representations of $\mathrm{GL}_n(q)$ are indexed by the set $\{\lambda \vdash n\}$ and when reduced modulo p, a first approximation to the multiplicity of their composition factors is given by application of the LLT algorithm. We will consider how the results obtained by using the LLT algorithm relate to the theory of irreducible $\mathrm{GL}_n(q)$-modules.

In addition, we will look at some other interesting results which can be proved in the representation theory of $\mathfrak{S}_n$ and $\mathrm{GL}_n(q)$.

Acknowledgements

Firstly, I would like to thank my supervisor, Gordon James, for introducing me to the subject covered in my thesis, and for his most productive methods of encouragement. I am grateful also to Andrew Mathas for his help with GAP, without which this thesis would have been a lot shorter, and contained many fewer tables, and to Matt Fayers for his discussions of Specht modules.

Warmest thanks to my family, in particular to my father, for his constant support, and to Hilary, who has regularly bought me dinner. I also owe many thanks to other mathematicians at Imperial, especially Alain and Jason for their patience, and Gursh for his most interesting conversations regarding computers. Also, to all my friends, particularly those of Trevelyan road, who made me feel comparatively sane, and K, who tried to actually keep me sane.

This research was sponsored by an EPSRC grant.

Dedication

This thesis is dedicated to the memory of Avril Friel.

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1 Introduction

We wish to consider the representation theory of the following four related algebras.

- The Iwahori–Hecke algebra $\mathcal{H}_{F,q}(\mathfrak{S}_n)$
- The Iwahori–Hecke algebra $\mathcal{H}_0 = \mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$ where q is a root of unity in $\mathbb{C}$
- The symmetric group algebra $F\mathfrak{S}_n \cong \mathcal{H}_{F,1}(\mathfrak{S}_n)$
- The general linear group algebra $K\mathrm{GL}_n(q)$ where char $K = p$ and $p \nmid q$

In particular, we note that both $\mathcal{H}_0$ and $F\mathfrak{S}_n$ are just specific Hecke algebras. The Hecke algebras also occur naturally in the representation theory of the general linear groups; a sketched proof of this result is given by Mathas [26] in the introduction to his recent book. Another point of interest is the relationship between the representation theory of the symmetric and general linear groups. It is well known that there is a close connection between the representations of $\mathfrak{S}_n$ over a field F and the representations of $\mathrm{GL}_n(F)$ over the same field F. However in 1982, James [18] proved that there are also many striking analogues between the representations of $F\mathfrak{S}_n$ and $K\mathrm{GL}_n(q)$ in non–defining characteristic, that is, when the characteristic of K does not divide q. In fact, by setting $q = 1$ in the statement of many of the theorems proved for the representation theory of the general linear groups, we recover a true result concerning that of the symmetric groups. It should be stressed that although a result may hold, the corresponding proof may be quite different. Furthermore, although many of the important theorems regarding the symmetric groups do have an analogue in the general linear groups, there are others that do not, or possibly the resulting analogue is more complicated than expected.

Another major connection between the above algebras concerns the study of the irreducible modules. For each Hecke algebra $\mathcal{H}_{F,q}(\mathfrak{S}_n)$, the ordinary irreducible modules are indexed by the set of partitions of n. They are known as the Specht modules. Furthermore, in each case, the irreducible modules over a field of positive characteristic are always indexed by a subset of the set of partitions of n. Thus for fixed n, the decomposition matrices of the Hecke algebras all tend to have a very similar structure. In particular, it may be shown that if the partitions are arranged according to a certain partial order, then each matrix in question is lower unitriangular. An open problem remains to determine the decomposition matrices; in general, the composition factors of a Specht module over a field of positive characteristic are not known. An exception to this statement is given by the algebra $\mathcal{H}_0$, since an iterative algorithm, described by Lascoux, Leclerc and Thibon [23] in 1996, will compute the decomposition matrices for $\mathcal{H}_0$. It will be seen that the decomposition matrix for $\mathcal{H}_0$ will provide information about the decomposition matrix of an arbitrary Hecke algebra.

In the general linear groups, the unipotent ordinary irreducible modules are again known as the Specht modules, and are again indexed by the set of partitions of n. Furthermore, over any field of non–defining characteristic, the unipotent irreducible modules are also indexed by the set of partitions of n. The full decomposition matrices for the general linear groups can be constructed from just the unipotent part; we will therefore consider only this unipotent part. Again, the structure of these matrices are very similar to that of the decomposition matrices of the Hecke algebras; they can also be shown to be lower unitriangular, and information about them is provided by the decomposition matrix of $\mathcal{H}_0$.

We shall proceed as follows. Chapter 2 introduces the Iwahori–Hecke algebra, and the concept of decomposition matrices. We will briefly discuss the irreducible modules, before moving on the the algebra $\mathcal{H}_0$. As noted above, there exists a recursive algorithm, known as the LLT algorithm, which will compute the decomposition matrices for $\mathcal{H}_0$. A computer package such as GAP [S$^+$95] will produce specific entries. However, the recursive nature of the algorithm means that it is impractical to apply for large values of n. Nevertheless, for certain 'simple' types of partition, it is possible to adapt the LLT algorithm to produce explicit results. In Chapter 2, we ask, and answer, three questions concerning $\mathcal{H}_0$-modules.

- Given a partition λ with at most 2 parts, which Specht modules corresponding to partitions with at most 3 parts contain D^λ as a composition factor, and with what multiplicity does it occur?

- Given a partition μ with at most 3 parts, what are the composition factors of the Specht module S^μ?

- Given a partition λ with at most 3 parts, which Specht modules corresponding to partitions with at most 4 parts contain D^λ as a composition factor, and with what multiplicity does it occur?

From this information, we may deduce the multiplicity of a composition factor D^λ in a Specht module S^μ for any partition μ with at most 4 parts.

In Chapter 3, we consider the symmetric group algebra $F\mathfrak{S}_n$. We will explain some of the theory concerning the irreducible modules; in particular we give an explicit construction of the Specht modules. We then look at the homomorphisms between the Specht modules and certain permutation modules. Throughout, the main reference will be [17]. Having discussed the relationship between the decomposition matrices of the group algebra $F\mathfrak{S}_n$ and the decomposition matrices of $\mathcal{H}_0$, we can make use of the information previously obtained by the LLT algorithm to derive results about the decomposition matrices of $F\mathfrak{S}_n$. We then attempt, with varying degrees of success, to reprove these results without consideration of the LLT algorithm.

Chapter 4 may be seen as an analogue of Chapter 3, concerning the general linear group algebra $K\mathrm{GL}_n(q)$ in non–defining characteristic. Again, we look in some detail at the general theory; the main reference will be [18]. It will be seen that the situation here is more complicated; in particular, we cannot generally describe an explicit basis for a Specht module as a vector space. The relationship between the decomposition matrices of $K\mathrm{GL}_n(q)$ and $\mathcal{H}_0$ is again considered. Together with the results of Chapter 2, this leads to information regarding the decomposition matrices of $K\mathrm{GL}_n(q)$, which we will then try to reprove.

Chapters 5 and 6 consider the Specht modules, that is, the ordinary irreducible modules of the symmetric and general linear group algebras. It is intended that Chapters 2, 3 and 4 should be self–contained; however Chapters 5 and 6 contain results which have been referenced, but not fully explained.

Chapter 5 concerns the Specht modules of the symmetric group algebras. In Chapter 3, we described these Specht modules. A problem that remains open is to fully classify which such ordinary irreducible modules remain irreducible modulo a prime p. We will demonstrate that some previously unclassified Specht modules are indeed reducible, giving strength to a conjecture by James and Mathas [21]. The second part of Chapter 5 considers homomorphisms between Specht modules. In particular, we produce an analogue of a result by Donkin [8] which involved the multiplicity of certain composition factors.

Chapter 6 considers the Specht modules of the general linear groups. We begin by looking at homomorphisms between Specht modules; the intention is to produce an analogue of the results discussed in Chapter 5. Due to the lack of a semistandard homomorphism theorem [17] for $K\mathrm{GL}_n(q)$ Specht modules, the situation here is by no means as simple. Nevertheless, although the conjecture remains unproved, the evidence provided seems to indicate that a very similar result may indeed be true. Finally, we look at an interesting property of the elements of the Specht module, which may have applications in determining a basis of the Specht modules.

2 The LLT algorithm

2.1 The Hecke algebra

Definition 2.1.1. Let F be a field and let $q \in F \setminus \{0\}$. The Iwahori–Hecke algebra $\mathcal{H} = \mathcal{H}_{F,q}(\mathfrak{S}_n)$ of $\mathfrak{S}_n$ is the unital associative F-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i - q)(T_i + 1) &= 0, && \text{for } i = 1, 2, \ldots, n-1, \\
T_i T_j &= T_j T_i, && \text{for } 1 \leq i < j-1 \leq n-2, \\
T_{i+1} T_i T_{i+1} &= T_i T_{i+1} T_i, && \text{for } i = 1, 2, \ldots, n-2.
\end{aligned}$$

Note that if $q = 1$ then $\mathcal{H} \cong F\mathfrak{S}_n$.

Definition 2.1.2. Define e to be the least postive integer such that

$$1 + q + q^2 + \ldots + q^{e-1} = 0$$

and let $e = \infty$ if no such integer exists.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ be partitions of n. (Write $\lambda, \mu \vdash n$.) Say that λ is e-regular if no e parts of λ are the same; else say that λ is e-singular. We define a partial order $\trianglerighteq$ on the set $\{\lambda \mid \lambda \vdash n\}$ by $\lambda \trianglerighteq \mu$ if and only if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all j, and say that $\lambda \vartriangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$.

Then we can define $\mathcal{H}$-modules S^λ (known as the Specht module) and D^λ. The following results hold.

1. Suppose $\mathcal{H} = \mathcal{H}_{\mathbb{C}(q),q}(\mathfrak{S}_n)$. Then $\{S^\lambda \mid \lambda \vdash n\}$ form the complete set of non-isomorphic irreducible $\mathcal{H}$-modules.

2. $D^\lambda \neq 0$ if and only if λ is e-regular and $\{D^\lambda \mid \lambda \vdash n \text{ and } \lambda\ e\text{-regular}\}$ form the complete set of non-isomorphic irreducible $\mathcal{H}$-modules.

Using this, we may now define the decomposition matrix of $\mathcal{H}$. For $\lambda, \mu \vdash n$ and λ e-regular, set $d_{\mu\lambda} = [S^\mu : D^\lambda]$ to be the composition multiplicity of D^λ in S^μ. The matrix $D = (d_{\mu\lambda})$ where $\lambda, \mu \vdash n$ and λ is e-regular is called the decomposition matrix of $\mathcal{H}$. Then these further facts hold.

1. $d_{\lambda\lambda} = 1$ for all $\lambda \vdash n$.

2. $d_{\mu\lambda} = 0$ unless $\lambda \trianglerighteq \mu$.

Hence by arranging the rows and columns of D in a suitable order, we get that D is lower unitriangular.

2.2 The LLT algorithm

In general, we have no way of calculating the decomposition matrices for $\mathcal{H}$. However, in 1996, a paper by Lascoux, Leclerc and Thibon [23] gave an iterative and computational method which they claimed could solve the decomposition matrices for $\mathcal{H}_0 = \mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$. This claim, known as the LLT conjecture, has since been proved [1], and a suitable computer program (for example the GAP share package Specht [S$^+$95]) will simply return the appropriate values. However, this gives us no extra information beyond the matrix itself, and the LLT algorithm (as it is now known) is purely combinatorial. The work in this chapter discusses some decomposition numbers corresponding to Specht modules indexed by simple partitions.

In addition, the LLT algorithm produces decomposition matrices not just for $\mathcal{H}_0$ but for other algebras, including

- The q-Schur Algebra $\mathcal{S}_{\mathbb{C},q}$.

- $FGL_n(q)$ where char F is 'sufficiently large'. (More later.)

- A first approximation to arbitrary such Schur and Hecke algebras, and hence to $F\mathfrak{S}_n$ for char $F = p > 0$.

Definition 2.2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition. Then the λ-diagram $[\lambda]$ is the set of nodes $\{(i,j) \mid 1 \leq j \leq \lambda_i \text{ and } i \geq 1\}$. We form the conjugate partition λ' by reflecting the diagram $[\lambda]$ in the line $i = j$.

The e-residue diagram of λ, is the diagram obtained by replacing each node by the number $(j - i) \bmod e$ and the e-ladder diagram of λ is the diagram obtained by replacing each node by the number $i - j + (j - 1)e$.

Example 2.2.2. Let $\lambda = (4,2,1)$ and $e = 3$. Then the diagram, e-residue diagram and e-ladder diagram of λ are respectively given by:

$$\begin{array}{cccc} * & * & * & * \\ * & * & & \\ * & & & \end{array} \qquad \begin{array}{cccc} 0 & 1 & 2 & 0 \\ 2 & 0 & & \\ 1 & & & \end{array} \qquad \begin{array}{cccc} 0 & 2 & 4 & 6. \\ 1 & 3 & & \\ 2 & & & \end{array}$$

Definition 2.2.3. A node $\gamma \notin [\lambda]$ is said to be addable if $[\lambda \cup \gamma]$ is the diagram of a partition. A node $\gamma \in [\lambda]$ is said to be removable if $[\lambda \setminus \gamma]$ is the diagram of a partition.

Example 2.2.4. Let $\lambda = (5,3,3,1)$. Then the removable nodes are those indicated by an underscore in the first diagram and the addable nodes those indicated by a dot in the second:

$$\begin{array}{ccccc} * & * & * & * & \underline{*} \\ * & * & * & & \\ * & * & \underline{*} & & \\ \underline{*} & & & & \end{array} \qquad \begin{array}{cccccc} * & * & * & * & * & \cdot \\ * & * & * & \cdot & & \\ * & * & * & & & \\ * & \cdot & & & & \\ \cdot & & & & & \end{array}$$

Definition 2.2.5. Let λ, μ be partitions and suppose the e-residue diagram of μ is formed by adding d nodes, all of residue i, to the e-residue diagram of λ. Write $\lambda \xrightarrow{d:i} \mu$. Define

$$F_i^d(\lambda) = \sum_{\lambda \xrightarrow{d:i} \mu} v^{N_i(\lambda,\mu)} \mu$$

where $N_i(\lambda, \mu)$ is given by

$$N_i(\lambda,\mu) = \sum_{\gamma \in [\mu]\setminus[\lambda]} \#\{\gamma' \mid \gamma' \text{ an addable } i\text{-node of } \mu \text{ above } \gamma\}$$
$$-\#\{\gamma' \mid \gamma' \text{ a removable } i\text{-node of } \lambda \text{ above } \gamma\}.$$

The term v may be regarded simply as a parameter.

Definition 2.2.6. The LLT algorithm can now be implemented. The algorithm works by calculating the 'crystallized decomposition matrix' of $\mathcal{H}_0$. This is a lower unitriangular matrix with the same structure as the decomposition matrix of $\mathcal{H}_0$, but whose lower triangular entries are elements of $v\mathbb{N}[v]$, where 0 is included as a natural number. The decomposition matrix of $\mathcal{H}_0$ is then obtained by setting $v = 1$. If $\lambda \vdash n$ is e-regular, define $B(\lambda)$ to be the column of the crystallized decomposition matrix indexed by λ. Since the LLT algorithm is recursive, we assume that we know

$B(\mu)$ where $\mu \vdash m$ and $m < n$ or $m = n$ and $\lambda \rhd \mu$. This is reasonable, since if $n = 1$ the crystallized decomposition matrix is simply the identity matrix and if μ dominates no other e-regular partition of n then the properties of the algorithm mean that $B(\mu)$ can be found.

To find $B(\lambda)$ for given λ, we operate the LLT algorithm as follows.

1. Write down the e-residue diagram of λ. Construct the partition τ by removing those nodes in $[\lambda]$ with maximal ladder number. Suppose there are d such nodes and that they have (common) e-residue i.

2. By induction, we know $B(\tau)$. Set $C_\lambda = F_i^d B(\tau)$ with $F_i^d B(\tau)$ defined in the obvious manner. Then C_λ is of the form

$$\begin{aligned} C_\lambda &= \sum_{\lambda \trianglerighteq \nu} c_{\nu\lambda}(v)\nu \\ &= B(\lambda) - \sum_{\lambda \rhd \nu} \alpha_{\nu\lambda}(v) B(\nu) \end{aligned}$$

   where $\alpha_{\nu\lambda}(v) \in \mathbb{N}[v + v^{-1}]$ and $c_{\nu\lambda}(v) \in \mathbb{N}[v, v^{-1}]$.

3. Find the most dominant partition, ν_0, such that $c_{\nu_0\lambda}(v)$ does not belong to $v\mathbb{N}[v]$. If no such partition exists then $B(\lambda) = C_\lambda$ and we are done. Otherwise, $\alpha_{\nu_0\lambda}(v)$ is the unique polynomial in $v + v^{-1}$ such that the coefficient of v^i in $\alpha_{\nu_0\lambda}(v)$ is equal to the coefficient of v^i in $c_{\nu_0\lambda}(v)$ for all $i \leq 0$.

   Replace C_λ with the element $C_\lambda - \alpha_{\nu_0\lambda}(v)B(\nu_0)$ and repeat step 3 until all the coefficients $c_{\nu\lambda}(v)$ belong to $v\mathbb{N}[v]$ for $\lambda \rhd \nu$.

Example 2.2.7. Consider $\lambda = (3, 2)$ and $e = 2$. We wish to find $B(3, 2)$.

1. Find τ. The e-ladder diagram of λ is $\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & \end{matrix}$. Hence τ is given by $\begin{matrix} * & * \\ * & \end{matrix}$ where we have removed 2 nodes, both of e-residue 0.

2. Looking up $\tau = (2, 1)$ we find that $B(\tau) = (2, 1)$. We calculate $C_\lambda = F_0^2 B(\tau)$:

$$\begin{matrix} 0 & 1 \\ 1 & \end{matrix} \xrightarrow{2:0} \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & \\ & & \end{matrix} + v \begin{matrix} 0 & 1 & 0 \\ 1 & & \\ 0 & & \end{matrix} + v^2 \begin{matrix} 0 & 1 \\ 1 & 0 \\ 0 & \end{matrix}.$$

3. Since this is of the correct form,

$$B(3, 2) = (3, 2) + v(3, 1^2) + v^2(2^2, 1).$$

Example 2.2.8. Consider $\lambda = (5)$ and $e = 2$. We wish to find $B(5)$.

1. Find τ. The e-ladder diagram of λ is $\begin{array}{ccccc}0&1&2&3&4\end{array}$. Hence τ is given by $\begin{array}{cccc}*&*&*&*\end{array}$ where we have removed 1 node, of e-residue 0.

2. Looking up $\tau = (4)$ we find that $B(\tau) = (4) + v(3,1) + v(2,1^2) + v^2(1^4)$. We calculate $C_\lambda = F_0^1 B(\tau)$:

$$\begin{array}{cccc}0&1&0&1\end{array} + v\,\begin{array}{ccc}0&1&0\\1&&\end{array} + v\,\begin{array}{cc}0&1\\1&\\0&\end{array} + v^2\,\begin{array}{c}0\\1\\0\\1\end{array} \xrightarrow{1:0} \begin{array}{ccccc}0&1&0&1&0\end{array}$$

$$+ \begin{array}{ccc}0&1&0\\1&0&\end{array} + v\,\begin{array}{ccc}0&1&0\\1&&\\0&&\end{array} + v\,\begin{array}{ccc}0&1&0\\1&&\\0&&\end{array} + v^2\,\begin{array}{cc}0&1\\1&0\\0&\end{array} + v^2\,\begin{array}{c}0\\1\\0\\1\\0\end{array}.$$

3. This is not of the correct form. The partition $\nu = (3,2)$ has coefficient $1 \notin v\mathbb{N}[v]$. Hence $\alpha_{\nu\lambda}(v) = 1$. We subtract $B(3,2)$ once from C_λ to get

$$(5) + (3,2) + 2v(3,1^2) + v^2(2^2,1) + v^2(1^5) - [(3,2) + v(3,1^2) + v^2(2^2,1)]$$

$$= (5) + v(3,1^2) + v^2(1^5).$$

This is of the correct form, hence

$$B(5) = (5) + v(3,1^2) + v^2(1^5).$$

Remark 2.2.9. There are two very useful facts which are not immediately apparent from this description. These are as follows.

- For given $\mu \vdash n$ and $d, i \in \mathbb{N}$ suppose that $F_i^d B(\mu)$ is of the form

$$\lambda + v \sum_{\lambda \rhd \nu} c_{\nu\lambda}(v)\nu$$

for polynomials $c_{\nu\lambda}(v) \in \mathbb{N}[v]$. Then $B(\lambda) = F_i^d B(\mu)$.

- The partial order on the set of partitions of n gives that if ν has exactly s parts and μ exactly t parts with $s < t$ then $\mu \ntrianglerighteq \nu$. Suppose we only consider some dominant partitions, that is, those partitions with at most s

parts. Given a partition λ, we would like to find out which such dominant Specht modules contain D^λ as a composition factor. The format of the LLT algorithm means that we can achieve this by simply ignoring, at every single step, those partitions with more than s parts.

2.3 Partitions with at most 3 parts

Throughout the rest of this chapter, we will be working with the Hecke algebra $\mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_n)$, where q is a primitive e^{th} root of unity in $\mathbb{C}$.

Definition 2.3.1. Let $B'_s(\lambda)$ be the part of the column of the crystallized decomposition matrix indexed by λ which corresponds to only the rows containing partitions with at most s parts. Clearly if λ itself has more than s parts then $B'_s(\lambda) = 0$. For the rest of Section 2.3, we consider $B'_3(\lambda)$, denoted simply $B'(\lambda)$.

Example 2.3.2. Let $n = 5$ and $e = 3$. Then the columns of the crystallized decomposition matrix of $\mathcal{H}$ are given by:

$$\begin{array}{lllll} B(5) &=& (5)+v(2^2,1) &=& B'(5). \\ B(4,1) &=& (4,1)+v(3,2) &=& B'(4,1). \\ B(3,2) &=& (3,2)+v(1^5). & & B'(3,2) = (3,2). \\ B(2^2,1) &=& (2^2,1)+v(2,1^3). & & B'(2^2,1) = (2^2,1). \end{array}$$

Setting $v = 1$ we get that $D^{(5)}$ is a composition factor of $S^{(5)}$ and $S^{(2^2,1)}$, each with multiplicity 1. Similarly for other cases.

Theorem 2.3.3. *Let $2 \le e < \infty$ and suppose $\lambda = (\lambda_1, \lambda_2)$ is a one or two part e-regular partion of $(\lambda_1 + \lambda_2) = n$. Suppose that*

$$\begin{aligned} \lambda_1 - \lambda_2 &= ae + i - 1 \\ \lambda_2 &= be + j - 1 \end{aligned}$$

where $0 \le i, j < e$. Define

$$\begin{aligned} m_2 &:= be - 1 \\ m_1 - m_2 &:= ae - 1 \end{aligned}$$

so that $\lambda = (m_1 + i + j, m_2 + j)$. Also if $(i + j) > e$, let

$$I = (i + j) - e.$$

Then $B'(\lambda)$ is given by the following sums of partitions.

$\underline{i = 0, j = 0:}$

(m_1, m_2)

$\underline{i = 0, j > 0:}$

$a = 1, b = 0:$

$(m_1 + j, m_2 + j)$

$a \geq 2, b = 0:$

$(m_1 + j, m_2 + j) + v(m_1 + j - e, m_2 + e, j)$

$a \geq 1, b \geq 1:$

$(m_1 + j, m_2 + j) + v(m_1 + j, m_2, j) + v^2(m_1, m_2 + j, j)$

$\underline{i > 0, j = 0:}$

$a = 0, b = 1:$

$(m_1 + i, m_2)$

$a = 0, b \geq 2:$

$(m_1 + i, m_2) + v(m_1, m_2 - e + i, e)$

$a \geq 1, b \geq 1:$

$(m_1 + i, m_2) + v(m_1, m_2 + i) + v^2(m_1, m_2, i)$

$\underline{i > 0, j > 0, i + j < e:}$

$a = 0, b = 0:$

$(m_1 + i + j, m_2 + j)$

$a = 1, b = 0:$

$(m_1 + i + j, m_2 + j) + v(m_1 + j, m_2 + i + j)$

$a \geq 2, b = 0:$

$(m_1 + i + j, m_2 + j) + v(m_1 + j, m_2 + i + j) + v(m_1 + i + j - e, m_2 + e, j) +$
$v^2(m_1 + j - e, m_2 + e, i + j)$

$a = 0, b = 1:$

$(m_1 + i + j, m_2 + j) + v(m_1 + i + j, m_2, j)$

$a = 0, b \geq 2:$

$(m_1 + i + j, m_2 + j) + v(m_1 + i + j, m_2, j) + v(m_1 + j, m_2 + i + j - e, e) +$
$v^2(m_1, m_2 + i + j - e, e + j)$

$a \geq 1, b \geq 1:$

$(m_1 + i + j, m_2 + j) + v(m_1 + j, m_2 + i + j) + v(m_1 + i + j, m_2, j) +$
$v^2(m_1 + j, m_2, i + j) + v^2(m_1, m_2 + i + j, j) + v^3(m_1, m_2 + j, i + j)$

$\underline{i + j = e:}$

$a = 0, b = 0:$

(m_1+e, m_2+j)

$a=1, b=0$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)$

$a\geq 2, b=0$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)+v(m_1, m_2+e, j)$

$a=0, b=1$:

$(m_1+e, m_2+j)+v(m_1+e, m_2, j)$

$a=1, b=1$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)+v(m_1+e, m_2, j)+v^2(m_1, m_2+j, e)$

$a\geq 2, b=1$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)+v(m_1+e, m_2, j)+v(m_1, m_2+e, j)+$
$v^2(m_1, m_2+j, e)$

$a=0, b\geq 2$:

$(m_1+e, m_2+j)+v(m_1+e, m_2, j)+v(m_1+j, m_2, e)$

$a=1, b\geq 2$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)+v(m_1+e, m_2, j)+v(m_1+j, m_2, e)+$
$v^2(m_1, m_2+j, e)$

$a\geq 2, b\geq 2$:

$(m_1+e, m_2+j)+v(m_1+j, m_2+e)+v(m_1+e, m_2, j)+v(m_1+j, m_2, e)+$
$v(m_1, m_2+e, j)+v^2(m_1, m_2+j, e)$

$\underline{i+j>e:}$

$a=0, b=0$:

$(m_1+e+I, m_2+j)+v(m_1+e, m_2+j, I)$

$a=1, b=0$:

$(m_1+e+I, m_2+j)+v(m_1+e, m_2+j, I)+v(m_1+j, m_2+e+I)+$
$v^2(m_1+j, m_2+e, I)$

$a\geq 2, b=0$:

$(m_1+e+I, m_2+j)+v(m_1+e, m_2+j, I)+v(m_1+j, m_2+e+I)+$
$v^2(m_1+j, m_2+e, I)+v(m_1+I, m_2+e, j)+v^2(m_1, m_2+e+I, j)$

$a=0, b=1$:

$(m_1+e+I, m_2+j)+v(m_1+e, m_2+j, I)+v(m_1+e+I, m_2, j)+$
$v^2(m_1+e, m_2+I, j)$

$a=1, b=1$:

$(m_1+e+I, m_2+j)+v(m_1+e, m_2+j, I)+v(m_1+j, m_2+e+I)+$
$v^2(m_1+j, m_2+e, I)+v(m_1+e+I, m_2, j)+v^2(m_1+e, m_2+I, j)+$
$v^2(m_1+I, m_2+j, e)+v^3(m_1, m_2+j, e+I)$

$a\geq 2, b=1$:

$(m_1+e+I, m_2+j) + v(m_1+e, m_2+j, I) + v(m_1+j, m_2+e+I)+$
$v^2(m_1+j, m_2+e, I) + v(m_1+e+I, m_2, j) + v^2(m_1+e, m_2+I, j)+$
$v^2(m_1+I, m_2+j, e) + v^3(m_1, m_2+j, e+I) + v^2(m_1+I, m_2+j, e)+$
$v^3(m_1, m_2+j, e+I)$

$a = 0, b \geq 2$:

$(m_1+e+I, m_2+j) + v(m_1+e, m_2+j, I) + v(m_1+e+I, m_2, j)+$
$v^2(m_1+e, m_2+I, j) + v(m_1+j, m_2+I, e) + v^2(m_1+j, m_2, e+I)$

$a = 1, b \geq 2$:

$(m_1+e+I, m_2+j) + v(m_1+e, m_2+j, I) + v(m_1+j, m_2+e+I)+$
$v^2(m_1+e, m_2+I, j) + v(m_1+e+I, m_2, j) + v^2(m_1+e, m_2+I, j)+$
$v(m_1+j, m_2+I, e) + v^2(m_1+j, m_2, e+I) + v^2(m_1+I, m_2+j, e)+$
$v^3(m_1, m_2+j, e+I)$

$a \geq 2, b \geq 2$:

$(m_1+e+I, m_2+j) + v(m_1+e, m_2+j, I) + v(m_1+j, m_2+e+I)+$
$v^2(m_1+e, m_2+I, j) + v(m_1+e+I, m_2, j) + v^2(m_1+e, m_2+I, j)+$
$v(m_1+j, m_2+I, e) + v^2(m_1+j, m_2, e+I) + v(m_1+I, m_2+e, j)+$
$v^2(m_1, m_2+e+I, j) + v^2(m_1+I, m_2+j, e) + v^3(m_1, m_2+j, e+I)$

Proof to follow.

Corollary 2.3.4. *We recover the decomposition matrix by setting $v = 1$. If λ is a partition with at most 2 parts and μ a partition with at most 3 parts then $[S^\mu : D^\lambda]$ is either 0 or 1 for any irreducible D^λ.*

Remark 2.3.5. If $e = 2$ then the case $a = 0$ does not occur and the only cases for (i, j) are $(i = 0, j = 0), (i = 1, j = 0), (i = 0, j = 1)$ and $(i + j = e)$. In these situations, the theorem still holds.

Example 2.3.6. Suppose $e = 3$ and $\lambda = (8, 6)$. Using the definitions above, we get that $(i, j) = (0, 1)$, $(m_1, m_2) = (7, 5)$ and $(a, b) = (1, 2)$. If we look up this case above, we find that

$$\begin{aligned} B'(\lambda) &= (m_1+j, m_2+j) + v(m_1+j, m_2, j) + v^2(m_1, m_2+j, j) \\ &= (8,6) + v(8,5,1) + v^2(7,6,1) \end{aligned}$$

and that $D^{(8,6)}$ is a composition factor of $S^{(8,6)}$, $S^{(8,5,1)}$ and $S^{(7,6,1)}$, each with multiplicity 1.

Figure 2.1 shows several cases for $e = 3$. In the example above, we would simply look up the box with leading entry $(8, 6)$. Its entries are the Specht modules corresponding to partitions with at most 3 parts which contain $D^{(8,6)}$ as a composition factor.

Figure 2.1: $e = 3$

	a	**0**		**1**			**2**		
b	**i** **j**	**1**	**2**	**0**	**1**	**2**	**0**	**1**	**2**
0	**1**	(0)	(1)	(2)	$(3) + v(2,1)$	$(4) + v(2,2)$	$(5)+ v(2,2,1)$	$(6) + v(3,2,1)+ v(5,1) + v^2(2,2,2)$	$(7) + v(4,2,1)+ v(5,2)$
	2	$(1,1)$	$(2,1)+ v(1,1,1)$	$(3,1)$	$(4,1) + v(3,2)$	$(5,1) + v(4,1,1)+ v(3,3) + v^2(3,2,1)$	$(6,1)+ v(3,2,2)$	$(7,1) + v(4,2,2)+ v(6,2)$	$(8,1) + v(7,1,1)+ v(6,3) + v^2(6,2,1)+ v(5,2,2) + v^2(4,3,2)$
1	**0**	$(2,2)$	$(3,2)$	$(4,2)$	$(5,2) + v(4,3)+ v^2(4,2,1)$	$(6,2) + v(4,4)+ v^2(4,2,2)$	$(7,2)$	$(8,2) + v(7,3)+ v^2(7,2,1)$	$(9,2) + v(7,4)+ v^2(7,2,2)$
	1	$(3,3)+ v(3,2,1)$	$(4,3)+ v(4,2,1)$	$(5,3)+ v(5,2,1)+ v^2(4,3,1)$	$(6,3) + v(6,2,1)+ v(5,4) + v^2(5,2,2)+ v^2(4,4,1)+ v^3(4,3,2)$	$(7,3) + v(7,2,1)+ v(5,5) + v^2(4,3,3)$	$(8,3)+ v(8,2,1)+ v^2(7,3,1)$	$(9,3) + v(9,2,1)+ v(8,4) + v^2(8,2,2)+ v^2(7,4,1)+ v^3(7,3,2)$	$(10,3) + v(10,2,1)+ v(8,5) + v(7,5,1)+ v^2(7,3,3)$
	2	$(4,4)+ v(4,2,2)$	$(5,4)+ v(4,4,1)+ v(5,2,2)+ v^2(4,3,2)$	$(6,4)+ v(6,2,2)+ v^2(4,4,2)$	$(7,4) + v(7,2,2)+ v(6,5)+ v^2(4,4,3)$	$(8,4) + v(7,4,1)+ v(6,6) + v^2(6,5,1)+ v(8,2,2) + v^2(7,3,2)+ v^2(5,4,3) + v^3(4,4,4)$	$(9,4)+ v(9,2,2)+ v^2(7,4,2)$	$(10,4) + v(10,2,2)+ v(9,5) + v(7,5,2)+ v^2(7,4,3)$	$(11,4) + v(10,4,1)+ v(9,6) + v^2(9,5,1)+ v(11,2,2) + v^2(10,3,2)+ v(8,5,2) + v^2(7,6,2)+ v^2(8,4,3) + v^3(7,4,4)$
2	**0**	$(5,5)+ v(4,3,3)$	$(6,5)+ v(4,4,3)$	$(7,5)$	$(8,5) + v(7,6)+ v^2(7,5,1)$	$(9,5) + v(7,7)+ v^2(7,5,2)$	$(10,5)$	$(11,5) + v(10,6)+ v^2(10,5,1)$	$(12,5) + v(10,7)+ v^2(10,5,2)$
	1	$(6,6)+ v(6,5,1)+ v(5,4,3)+ v^2(4,4,4)$	$(7,6)+ v(7,5,1)+ v(5,5,3)$	$(8,6)+ v(8,5,1)+ v^2(7,6,1)$	$(9,6) + v(9,5,1)+ v(8,7) + v^2(8,5,2)+ v^2(7,7,1)+ v^3(7,6,2)$	$(10,6) + v(10,5,1)+ v(8,8) + v(8,5,3)+ v^2(7,6,3)$	$(11,6)+ v(11,5,1)+ v^2(10,6,1)$	$(12,6) + v(12,5,1)+ v(11,7) + v^2(11,5,2)+ v^2(10,7,1)+ v^3(10,6,2)$	$(13,6) + v(13,5,1)+ v(11,8) + v(11,5,3)+ v(10,8,1) + v^2(10,6,3)$
	2	$(7,7)+ v(7,5,2)+ v(6,5,3)$	$(8,7)+ v(7,7,1)+ v(8,5,2)+ v(6,6,3)+ v^2(6,5,4)$	$(9,7)+ v(9,5,2)+ v^2(7,7,2)$	$(10,7) + v(10,5,2)+ v(9,8) + v(9,5,3)+ v^2(7,7,3)$	$(11,7) + v(10,7,1)+ v(9,9) + v^2(9,8,1)+ v(11,5,2) + v^2(10,6,2)+ v(9,6,3) + v^2(9,5,4)+ v^2(8,7,3) + v^3(7,7,4)$	$(12,7)+ v(12,5,2)+ v^2(10,7,2)$	$(13,7) + v(13,5,2)+ v(12,8) + v(12,5,3)+ v(10,8,2) + v^2(10,7,3)$	$(14,7) + v(13,7,1)+ v(12,9) + v^2(12,8,1)+ v(14,5,2) + v^2(13,6,2)+ v(12,6,3) + v^2(12,5,4)+ v(11,8,2) + v^2(10,9,2)+ v^2(11,7,3) + v^3(10,7,4)$

Theorem 2.3.7 (The Column Addition Theorem). *[26] Let $\lambda, \mu \vdash n$ where $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_t)$. Suppose*

$$\lambda_0 = \underbrace{(\lambda_1 + k, \ldots, \lambda_s + k, k, \ldots, k)}_{l}$$

$$\mu_0 = \underbrace{(\mu_1 + k, \ldots, \lambda_t + k, k, \ldots, k)}_{l}$$

are such that $\lambda_0, \mu_0 \vdash n + lk$ and λ_0 is e-regular. Then

$$[S^\mu : D^\lambda] = [S^{\mu_0} : D^{\lambda_0}].$$

Corollary 2.3.8. *Given any e-regular partition λ, we may read off the multiplicity $[S^\mu : D^\lambda]$ for any Specht module S^μ such that μ has at most 3 parts.*

Proof. Consider an e-regular partition λ. If λ has at most 2 parts, then Corollary 2.3.8 follows from Theorem 2.3.3, and if λ has more than 3 parts then it cannot be a composition factor of any Specht module S^μ where μ has at most 3 parts. Consider $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ having exactly 3 parts. From Theorem 2.3.3, we know which Specht modules corresponding to partitions with at most 2 parts contain D^λ as a composition factor. But $S^{(\mu_1,\mu_2,\mu_3)}$ contains a composition factor $D^{(\lambda_1,\lambda_2,\lambda_3)}$ if and only if $\mu_3 \geq \lambda_3$ and $S^{(\mu_1-\lambda_3,\mu_2-\lambda_3,\mu_3-\lambda_3)}$ contains a composition factor $D^{(\lambda_1-\lambda_3,\lambda_2-\lambda_3)}$, which information is known. □

Now let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition with at most 3 parts. Write

$$\lambda = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}$$

where α_i is the e-residue of the last node on line i, with the convention that if $\lambda_2 = 0$ then $\alpha_2 = e - 2$ and if $\lambda_3 = 0$ then $\alpha_3 = e - 3$, that is, if we add a node to line i, that node will have residue $\alpha_i + 1$. Note α_i is only defined up to modulus e; no distinction will be made between α_i and $\alpha_i + e$. Let $\equiv$ denote equivalence modulo e. Otherwise the notation is that defined in Theorem 2.3.3.

Proof of Theorem 2.3.3. The proof uses some general results (see [26]); it is possible avoid these and proceed using only the LLT algorithm, but this is even more time consuming. For the remaining cases, the proof is by induction.

General Results:

1. $d_{\mu\lambda} \neq 0$ only if λ and μ have the same e-core.

2. Carter's criterion: Set $h(i,j)$ to be the hook length of node (i,j), that is $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$ where λ' denotes the partion conjugate to λ. Over a field of characteristic 0, set $\nu_e(k) = 0$ if $e \mid k$ and -1 otherwise. Then if $\nu_e(h(a,c)) = \nu_e(h(b,c))$ for all nodes (a,c) and (b,c) in $[\lambda]$ we have that S^λ is irreducible.

$\underline{i=0,\ j=0\ (1)}$: Recall that $m_2 \equiv e-1$ and $m_1 \equiv e-2$. This partition has e-core $(2e-2, e-1)$. If the Specht module corresponding to a partition μ has $D^{(m_1,m_2)}$ as a composition factor, then μ has e-core $(2e-2, e-1)$. Suppose $\mu = (\mu_1, \mu_2, \mu_3)$ has such an e-core. Form the partition $\nu = (\nu_1, \nu_2, \nu_3)$ by removing all possible horizontal e-rim hooks, that is hooks of leg length 0, from μ so that $\nu_i \equiv \mu_i$ for all i and ν_3, $\nu_2 - \nu_3$, $\nu_1 - \nu_2 < e$.

Now suppose that $\nu_2 - \nu_3 < e-1$. Since μ and hence ν has e-core $(2e-2, e-1)$, we have that $\nu_2 \geq e-1$ and hence we can remove a rim hook beginning at the node $(2, \nu_2)$ and working downwards. But this produces a partition whose second part is less than ν_3 where $\nu_3 \leq e-1$ and which cannot have the correct e-core. Thus $\nu_2 - \nu_3 = e-1$. A similar argument shows that $\nu_1 - \nu_2 = e-1$. Hence $\nu = (\nu_3 + 2e - 2, \nu_3 + e - 1, \nu_3)$ (where $\nu_3 < e$) is an e-core, and for it to have the correct form, we must have that $\nu_3 = 0$. But ν was formed by removing horizontal hooks from μ. Thus the only partitions with at most 3 parts and with such an e-core are of the form $(re-2, se-1, te)$ for some integers r, s, t and by Carter's criterion, these are irreducible.

Hence $B'(m_1, m_2) = (m_1, m_2)$.

Now assume that $i > 0$ and $j > 0$ unless otherwise stated.

$\underline{j=0,\ a,b \geq 1\ (2)}$: Proof is by induction on i for $1 \leq i < e$.

$i = 1$: By (1), $B'(m_1, m_2) = (m_1, m_2)$. Then $F^1_{e-2} B'(m_1, m_2)$ is given by

$$\begin{pmatrix} e-3 & e-3 & e-3 \\ m_1 & m_2 & 0 \end{pmatrix} \xrightarrow{1,e-2} \begin{pmatrix} e-2 & e-3 & e-3 \\ m_1+1 & m_2 & 0 \end{pmatrix} + v \begin{pmatrix} e-3 & e-2 & e-3 \\ m_1 & m_2+1 & 0 \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-3 & e-2 \\ m_1 & m_2 & 1 \end{pmatrix}$$

which, since it is of the correct form, is equal to $B'(m_1+1, m_2)$.

$2 \leq i < e$: Assume true for $i-1$. Then $F^1_{i-3} B'(m_1+i-1, m_2)$ is given by

$$\begin{pmatrix} i-4 & e-3 & e-3 \\ m_1+i-1 & m_2 & 0 \end{pmatrix} + v \begin{pmatrix} e-3 & i-4 & e-3 \\ m_1 & m_2+i-1 & 0 \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-3 & i-4 \\ m_1 & m_2 & i-1 \end{pmatrix}$$
$$\xrightarrow{1,i-3} \begin{pmatrix} i-3 & e-3 & e-3 \\ m_1+i & m_2 & 0 \end{pmatrix} + v \begin{pmatrix} e-3 & i-3 & e-3 \\ m_1 & m_2+i & 0 \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-3 & i-3 \\ m_1 & m_2 & i \end{pmatrix}$$

since $e-3 \not\equiv i-3, i-4$ and $m_1 - m_2, m_2 \geq e-1$.

$\underline{i=0,\ a,b \geq 1\ (3)}$: Proof is by induction on j for $1 \leq j < e$.

$j = 1$: By (1), $B'(m_1, m_2) = (m_1, m_2)$. Then $F^2_{e-2} B'(m_1, m_2)$ is given by

$$\begin{pmatrix} e-3 & e-3 & e-3 \\ m_1 & m_2 & 0 \end{pmatrix} \xrightarrow{2,e-2} \begin{pmatrix} e-2 & e-2 & e-3 \\ m_1+1 & m_2+1 & 0 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & e-2 \\ m_1+1 & m_2 & 1 \end{pmatrix} + v^2\begin{pmatrix} e-3 & e-2 & e-2 \\ m_1 & m_2+1 & 1 \end{pmatrix}$$

$2 \le j < e$: Assume true for $j-1$. Then $F^2_{j-3}B'(m_1+j-1, m_2+j-1)$ is given by

$$\begin{pmatrix} j-4 & j-4 & e-3 \\ m_1+j-1 & m_2+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} j-4 & e-3 & j-4 \\ m_1+j-1 & m_2 & j-1 \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-4 & j-4 \\ m_1 & m_2+j-1 & j-1 \end{pmatrix} \xrightarrow{2,j-3} \begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & j-3 \\ m_1 & m_2+j & j \end{pmatrix}$$

since $e-3 \not\equiv j-3, j-4$ and $m_1-m_2, m_2 \ge e-1$.

<u>$i+j<e,\ a,b \ge 1$ (4)</u>: If $j = e-1$, this case does not occur. Hence fix j with $1 \le j < e-1$. Proof is by induction on i for $1 \le i < e-j$.

$i=1$: By (3), $F^1_{j-2}B'(m_1+j, m_2+j)$ is given by

$$\begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & j-3 \\ m_1 & m_2+j & j \end{pmatrix} \xrightarrow{1,j-2} \begin{pmatrix} j-2 & j-3 & e-3 \\ m_1+j+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & j-2 & e-3 \\ m_1+j & m_2+j+1 & 0 \end{pmatrix} + v\begin{pmatrix} j-2 & e-3 & j-3 \\ m_1+j+1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & j-2 \\ m_1+j & m_2 & j+1 \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-2 & j-3 \\ m_1 & m_2+j+1 & j \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & j-2 \\ m_1 & m_2+j & j+1 \end{pmatrix}$$

$2 \le i < e-j$: Assume true for $i-1$. Then $F^1_{i+j-3}B'(m_1+i+j-1, m_2+j)$ is given by

$$\begin{pmatrix} i+j-4 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-4 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} i+j-4 & e-3 & j-3 \\ m_1+i+j-1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & i+j-4 \\ m_1+j & m_2 & i+j-1 \end{pmatrix} + v^2\begin{pmatrix} e-3 & i+j-4 & j-3 \\ m_1 & m_2+i+j-1 & j \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & i+j-4 \\ m_1 & m_2+j & i+j-1 \end{pmatrix} \xrightarrow{1,i+j-3} \begin{pmatrix} i+j-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-3 & e-3 \\ m_1+j & m_2+i+j & 0 \end{pmatrix} + v\begin{pmatrix} i+j-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & i+j-3 \\ m_1+j & m_2 & i+j \end{pmatrix} + v^2\begin{pmatrix} e-3 & i+j-3 & j-3 \\ m_1 & m_2+i+j & j \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & i+j-3 \\ m_1 & m_2+j & i+j \end{pmatrix}$$

since $i+j-4 \not\equiv j-3, e-3$ and $i+j-3 \not\equiv j-3, e-3$ and $a,b \ge 1$.

<u>$i+j=e,\ a,b=1$ (5)</u>: Note that $m_1 = m_1-m_2 = e-1$. Fixing j, proof comes from (4) and (3).

By (3), unless $j=e-1$, $F^1_{e-3}B'(m_1+e-1, m_2+j)$ is given by

$$\begin{pmatrix} e-4 & e+j-3 & e-3 \\ m_1+e-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e+j-3 & e-4 & e-3 \\ m_1+j & m_2+e-1 & 0 \end{pmatrix} + v\begin{pmatrix} e-4 & e-3 & e+j-3 \\ m_1+e-1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & e-4 \\ m_1+j & m_2 & e-1 \end{pmatrix} + v^2\begin{pmatrix} e-3 & e-4 & j-3 \\ m_1 & m_2+e-1 & j \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & e-4 \\ m_1 & m_2+j & e-1 \end{pmatrix}$$

$$\xrightarrow{1,e-3} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+i+j & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1 & m_2+j & i+j \end{pmatrix} \text{ where } i+j=e.$$

If $j=e-1$ then $i=1$, and by (3) $F^1_{j-2}B'(m_1+j, m_2+j)$ is given by

$$\begin{pmatrix} j-3 & j-3 & e-3 \\ m_2+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & j-3 \\ m_1 & m_2+j & j \end{pmatrix}$$
$$\xrightarrow{1,j-2} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+j+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+j+1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+j+1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1 & m_2+j & j+1 \end{pmatrix} \text{ where } 1+j=e.$$

Note $m_2=e-1$ and $m_1=m_2+e-1$.

$\underline{i+j>e,\ a,b=1\ (6)}$: Note that $m_2=m_1-m_2=e-1$. Case $j=1$ does not occur, so fix j with $2\le j<e$ and use induction on I, for $1\le I<j$.

$I=1$ $(i+j=e+1)$: By (5), $F^1_{e-2}B'(m_1+e, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j-1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1 & m_2+j & i+j-1 \end{pmatrix}$$
$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & e-2 \\ m_1+e & m_2+j & 1 \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & e-2 & e+j-3 \\ m_1+j & m_2+e+1 & 0 \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & e-2 \\ m_1+j & m_2+e & 1 \end{pmatrix} +$$
$$v\begin{pmatrix} e-2 & e-3 & j-3 \\ m_1+e+1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & e-2 & j-3 \\ m_1+e & m_2+1 & j \end{pmatrix} +$$
$$v^2\begin{pmatrix} e-2 & e+j-3 & e-3 \\ m_1+1 & m_2+j & e \end{pmatrix} + v^3\begin{pmatrix} e-3 & e+j-3 & e-2 \\ m_1 & m_2+j & e+1 \end{pmatrix}$$

$2\le I<j$: Assume true for $I-1$. Then $F^1_{I-3}B'(m_1+e+I-1, m_2+j)$ is given by

$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-4 & e-3 \\ m_1+j & m_2+e+I-1 & 0 \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & I-4 \\ m_1+j & m_2+e & I-1 \end{pmatrix} +$$
$$v\begin{pmatrix} I-4 & e-3 & j-3 \\ m_1+e+I-1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & I-4 & j-3 \\ m_1+e & m_2+I-1 & j \end{pmatrix} +$$
$$v^2\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+I-1 & m_2+j & e \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1 & m_2+j & e+I-1 \end{pmatrix}$$
$$\xrightarrow{1,e+I-3} \begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-3 & e-3 \\ m_1+j & m_2+e+I & 0 \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & I-3 \\ m_1+j & m_2+e & I \end{pmatrix} +$$
$$v\begin{pmatrix} I-3 & e-3 & j-3 \\ m_1+e+I & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & I-3 & j-3 \\ m_1+e & m_2+I & j \end{pmatrix} +$$
$$v^2\begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+I & m_2+j & e \end{pmatrix} + v^3\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1 & m_2+j & e+I \end{pmatrix}$$

since $I-4\not\equiv j-3, e-3$ and $I-3\not\equiv j-3, e-3$ and $a,b=1$.

The proof for the case $i+j\ge e$ where

1. $a=1$ and $b\ge 2$

2. $a \geq 2$ and $b = 1$

3. $a, b \geq 2$

is then very similar to the proofs of (5) and (6).

$\underline{i + j \leq e,\ a, b = 0\ (7)}$: In this case, $m_1 + i + j < e - 1$. Hence $(m_1 + i + j, m_2 + j)$ is an e-core and so $B'(m_1 + i + j, m_2 + j)$ (and indeed $B(m_1 + i + j, m_2 + j)) = (m_1 + i + j, m_2 + j)$.

$\underline{i + j > e,\ a, b = 0\ (8)}$: Note that $m_2 = m_1 - m_2 = -1$. Case $j = 1$ does not occur, so fix j with $2 \leq j < e$ and use induction on I for $1 \leq I < j$.
$I = 1$ $(i + j = e + 1)$: By (7), $F^1_{e-2}B'(m_1 + e, m_2 + j)$ is given by
$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} \xrightarrow{1,e-2} \begin{pmatrix} e-2 & j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & e-2 \\ m_1+e & m_2+j & 1 \end{pmatrix}$$
$2 \leq I < j$: Assume true for $I - 1$. Then $F^1_{I-3}B'(m_1 + e + I - 1, m_2 + j)$ is given by
$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix}$$
$$\xrightarrow{1,I-3} \begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix}$$
since $I - 4 \not\equiv j - 3, e - 3$ and $I - 3 \not\equiv j - 3, e - 3$.

$\underline{i = 0,\ a = 1,\ b = 0\ (9)}$: Note that $m_2 = -1$, $m_1 - m_2 = e - 1$. We use induction on j for $1 \leq j < e$.
$j = 1$: By (7), $F^1_{e-2}B'(m_1 + e, m_2 + j)$ is given by
$$\begin{pmatrix} e-3 & e-2 & e-3 \\ (m_1-e)+e & m_2+1 & 0 \end{pmatrix} \xrightarrow{1,e-2} \begin{pmatrix} e-2 & e-2 & e-3 \\ m_1+1 & m_2+1 & 0 \end{pmatrix} \text{ (Note } m_2 + j = 0.)$$
$2 \leq j < e$: Assume true for $j - 1$. Then $F^2_{j-3}B'(m_1 + j - 1, m_2 + j - 1)$ is given by
$$\begin{pmatrix} j-4 & j-4 & e-3 \\ m_1+j-1 & m_2+j-1 & 0 \end{pmatrix} \xrightarrow{2,j-3} \begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix}$$
since $e - 3 \not\equiv j - 4, j - 3$.

$\underline{i + j \leq e,\ a = 1,\ b = 0\ (10)}$: Note that $m_2 = -1$, $m_1 - m_2 = e - 1$. Fix j with $1 \leq j < e$ and use induction on i, for $1 \leq i \leq e - j$.
$i = 1$ $(i + j = e + 1)$: By (9), $F^1_{j-2}B'(m_1 + j, m_2 + j)$ is given by
$$\begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} \xrightarrow{1,e+j-2} \begin{pmatrix} j-2 & j-3 & e-3 \\ m_1+j+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & j-2 & e-3 \\ m_1+j & m_2+j+1 & 0 \end{pmatrix}$$
$2 \leq i \leq e - j$: Assume true for $i - 1$. Then $F^1_{i+j-3}B'(m_1 + i + j - 1, m_2 + j)$ is given by
$$\begin{pmatrix} i+j-4 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-4 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix}$$

$$\xrightarrow{1,i+j-3} \begin{pmatrix} i+j-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-3 & e-3 \\ m_1+j & m_2+i+j & 0 \end{pmatrix}$$

since $i+j-4 \not\equiv j-3, e-3$ and $i+j-3 \not\equiv j-3, e-3$.

$\underline{i+j > e,\ a=1,\ b=0\ (11)}$: Note that $m_2 = -1$, $m_1 - m_2 = e-1$. Case $j=1$ does not occur, so fix j with $2 \leq j < e$ and use induction on I, for $1 \leq I < j$.

$I = 1$ $(i+j = e+1)$: By (10), $F^1_{e-2}B'(m_1+e, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix}$$

$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & e-2 \\ m_1+e & m_2+j & 1 \end{pmatrix} +$$

$$v\begin{pmatrix} j-3 & e-2 & e-3 \\ m_1+j & m_2+e+1 & 0 \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & e-2 \\ m_1+j & m_2+e & 1 \end{pmatrix}$$

$2 \leq I < j$: Assume true for $I-1$. Then $F^1_{I-3}B'(m_1+e+I-1, m_2+j)$ is given by

$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix} +$$

$$v\begin{pmatrix} j-3 & I-4 & e-3 \\ m_1+j & m_2+e+I-1 & 0 \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & I-4 \\ m_1+j & m_2+e & I-1 \end{pmatrix}$$

$$\xrightarrow{1,I-3} \begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix} +$$

$$v\begin{pmatrix} j-3 & I-3 & e-3 \\ m_1+j & m_2+e+I & 0 \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & I-3 \\ m_1+j & m_2+e & I \end{pmatrix}$$

since $I-4 \not\equiv j-3, e-3$ and $I-3 \not\equiv j-3, e-3$.

$\underline{i=0,\ a=2,\ b=0\ (12)}$: Note that $m_2 = -1$, $m_1 - m_2 = 2e-1$. Use induction on j for $1 \leq j < e$.

$j = 1$: By (11), $F^1_{e-2}B'(m_1, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & e-2 & e-3 \\ m_1 & 0 & 0 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & e-3 \\ m_1+1-e & e-1 & 0 \end{pmatrix}$$

$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & e-2 & e-3 \\ m_1+1 & 0 & 0 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & e-2 \\ m_1+1-e & e-1 & 1 \end{pmatrix}$$

(Note $m_2 + j = 0$ and $m_1 - e + 1 = e - 1$.)

$2 \leq j < e$: Assume true for $j-1$. Then $F^2_{j-3}B'(m_1+j-1, m_2+j-1)$ is given by

$$\begin{pmatrix} j-4 & j-4 & e-3 \\ m_1+j-1 & m_2+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} j-4 & e-3 & j-4 \\ m_1+j-e-1 & m_2+e & j-1 \end{pmatrix}$$

$$\xrightarrow{2,j-3} \begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j-e & m_2+e & j \end{pmatrix}$$

since $j-4 \not\equiv e-3$.

$\underline{i+j < e,\ a=2,\ b=0\ (13)}$: Note that $m_2 = -1$, $m_1 - m_2 = 2e-1$. Case $j = e-1$ does not occur, so fix j with $1 \leq j < e-1$ and use induction on i for $1 \leq i < e-j$.

$i = 1$: By (12), $F^1_{j-2}B'(m_1+j, m_2+j)$ is given by

$$\begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j-e & m_2+e & j \end{pmatrix}$$

$$\xrightarrow{1,j-2} \begin{pmatrix} j-2 & j-3 & e-3 \\ m_1+j+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & j-2 & e-3 \\ m_1+j & m_2+j+1 & 0 \end{pmatrix} +$$

$$v\begin{pmatrix} j-2 & e-3 & j-3 \\ m_1+j-e+1 & m_2+e & j \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & j-2 \\ m_1+j-e & m_2+e & j+1 \end{pmatrix}$$

$2 \le i < j$: Assume true for $i-1$. Then $F^1_{i+j-3}B'(m_1+i+j-1, m_2+j)$ is given by

$$\begin{pmatrix} i+j-4 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-4 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} i+j-4 & e-3 & j-3 \\ m_1+i+j-e-1 & m_2+e & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & i+j-4 \\ m_1+j-e & m_2+e & i+j-1 \end{pmatrix}$$
$$\xrightarrow{1,i+j-3} \begin{pmatrix} i+j-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & i+j-3 & e-3 \\ m_1+j & m_2+i+j & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} i+j-3 & e-3 & j-3 \\ m_1+i+j-e & m_2+e & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & i+j-3 \\ m_1+j-e & m_2+e & i+j \end{pmatrix}$$

since $i+j-4 \not\equiv j-3, e-3$ and $i+j-3 \not\equiv j-3, e-3$.

$i+j=e,\ a=2,\ b=0$ (14): Note that $m_2=-1$, $m_1-m_2=2e-1$. Fixing j, proof comes from (12) and (13).

By (13), if $j<e-1$, $F^1_{e-3}B'(m_1+e-1, m_2+j)$ is given by

$$\begin{pmatrix} e-4 & j-3 & e-3 \\ m_1+e-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-4 & e-3 \\ m_1+j & m_2+e-1 & 0 \end{pmatrix}$$
$$+v\begin{pmatrix} e-4 & e-3 & j-3 \\ m_1-1 & m_2+e & j \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & e-4 \\ m_1+j-e & m_2+e & e-1 \end{pmatrix}$$
$$\xrightarrow{1,e-3} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+i+j & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j-e & m_2+e & j \end{pmatrix}$$

since $j-3 \not\equiv e-3, e-4$.

If $j=e-1$ then $i=1$ and by (12) $F^1_{e-3}B'(m_1+j, m_2+j)$ is given by

$$\begin{pmatrix} e-4 & e-4 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-4 & e-3 & e-4 \\ m_1+j-e & m_2+e & j \end{pmatrix}$$
$$\xrightarrow{1,e-3} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+j+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+j+1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+j-e+1 & m_2+e & j \end{pmatrix} \quad \text{(Note } m_2+e=j.\text{)}$$

$i+j>e,\ a=2,\ b=0$ (15): Note that $m_2=-1$, $m_1-m_2=2e-1$. Case $j=1$ does not occur, so fix j with $2 \le j < e$ and use induction on I, for $1 \le I < j$.

$I=1$ ($i+j=e+1$): By (14), $F^1_{e-2}B'(m_1+e, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2+i+j-1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1 & m_2+i+j-1 & j \end{pmatrix} \xrightarrow{1,e-2} \begin{pmatrix} e-2 & j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & j-3 & e-2 \\ m_1+e & m_2+j & 1 \end{pmatrix} + v\begin{pmatrix} j-3 & e-2 & e-3 \\ m_1+j & m_2+e+1 & 0 \end{pmatrix} +$$
$$v^2\begin{pmatrix} j-3 & e-3 & e-2 \\ m_1+j & m_2+e & 1 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & j-3 \\ m_1+1 & m_2+e & j \end{pmatrix} +$$
$$v^2\begin{pmatrix} e-3 & e-2 & j-3 \\ m_1 & m_2+e+1 & j \end{pmatrix}$$

$2 \le I < j$: Assume true for $I-1$. Then $F^1_{I-3}B'(m_1+e+I-1, m_2+j)$ is given by

$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-4 & e-3 \\ m_1+j & m_2+e+I-1 & 0 \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & I-4 \\ m_1+j & m_2+e & I-1 \end{pmatrix} +$$

$$v\begin{pmatrix} I-4 & e-3 & j-3 \\ m_1+I-1 & m_2+e & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & I-4 & j-3 \\ m_1 & m_2+e+I-1 & j \end{pmatrix}$$
$$\xrightarrow{1,I-3}\begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-3 & e-3 \\ m_1+j & m_2+e+I & 0 \end{pmatrix} + v^2\begin{pmatrix} j-3 & e-3 & I-3 \\ m_1+j & m_2+e & I \end{pmatrix} +$$
$$v\begin{pmatrix} I-3 & e-3 & j-3 \\ m_1+I & m_2+e & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & I-3 & j-3 \\ m_1 & m_2+e+I & j \end{pmatrix}$$

since $I-4 \not\equiv j-3, e-3$ and $I-3 \not\equiv j-3, e-3$.

$\underline{a>2,\ b=0\ (16)}$: We only need to prove the case when $a=3, i=0, j=1$. The proof for all other cases of $a=3, b=0$ then follows exactly from $(11-14)$ above, and the case $a>3$ follows by an obvious induction.

By (14), $F^1_{e-2}B'(m_1+j-1, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j-e & m_2+j+e-1 & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j-e-1 & m_2+e & j \end{pmatrix}$$
$$\xrightarrow{1,e-2}\begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j & m_2+j & 0 \end{pmatrix} + \begin{pmatrix} j-3 & j-3 & e-3 \\ m_1+j-e & m_2+j+e & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j-e & m_2+e & j \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & j-3 \\ m_1+j-e & m_2+e & j \end{pmatrix} +$$
$$v^2\begin{pmatrix} e-3 & j-3 & j-3 \\ m_1-e & m_2+e+j & j \end{pmatrix}$$

The coefficient of $(m_1+j-e, m_2+j+e, 0)$ is equal to 1 so this equation is not of the correct form. However, by (3) $B'(m_1+j-e, m_2+j+e, 0)$ is given by

$$(m_1+j-e, m_2+j+e) + v(m_1+j-e, m_2+e, j) + v^2(m_1-e, m_2+e+j, j).$$

Hence

$$B'(m_1+j, m_2+j) = (m_1+j, m_2+j) + v(m_1+j-e, m_2+e, j).$$

$\underline{j=0,\ a=0,\ b=1\ (17)}$: Note that $m_2 = e-1$, $m_1-m_2=-1$. Also, if $e=2$ then the situation $a=0$ does not occur. Hence for the remainder of the proof of Theorem 2.3.3, we may assume $e>2$. We use induction on i for $1 \le i < e$.

$i=1$: By (8), $F^1_{e-3}B'(m_1+i, m_2-1)$ is given by

$$\begin{pmatrix} i-3 & e-4 & e-3 \\ (m_1-e)+e+i & m_2-1 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-4 & i-3 \\ m_1 & m_2-1 & i \end{pmatrix} \xrightarrow{1,e-3}\begin{pmatrix} i-3 & e-3 & e-3 \\ m_1+i & m_2 & 0 \end{pmatrix}$$

(Note $m_1 = m_2-1$)

$2 \le i < e$: Assume true for $i-1$. Then $F^1_{i-3}B'(m_1+i-1, m_2)$ is given by

$$\begin{pmatrix} i-4 & e-3 & e-3 \\ m_1+i-1 & m_2 & 0 \end{pmatrix} \xrightarrow{1,i-3}\begin{pmatrix} i-3 & e-3 & e-3 \\ m_1+i & m_2 & 0 \end{pmatrix}$$

since $i-4, i-3 \not\equiv e-3$.

$\underline{i+j \le e,\ a=0,\ b=1\ (18)}$: Note that $m_2 = e-1$, $m_1 - m_2 = -1$. Fix i with $1 \le j < e$ and use induction on j, for $1 \le j \le e-i$.

$j=1$: By (17), $F^1_{e-2}B'(m_1+i+j, m_2+j-1)$ is given by

$$\begin{pmatrix} i+j-3 & e-3 & e-3 \\ m_1+i+j & m_2 & 0 \end{pmatrix} \xrightarrow{1,e-2} \begin{pmatrix} i+j-3 & e-2 & e-3 \\ m_1+i+j & m_2+1 & 0 \end{pmatrix}$$
$$+v\begin{pmatrix} i+j-3 & e-3 & e-2 \\ m_1+i+j & m_2 & 1 \end{pmatrix}$$

$2 \le j \le e-j$: Assume true for $j-1$. Then $F^1_{j-3}B'(m_1+i+j, m_2+j-1)$ is given by

$$\begin{pmatrix} i+j-3 & j-4 & e-3 \\ m_1+i+j & m_2+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} i+j-3 & e-3 & j-4 \\ m_1+i+j & m_2 & j-1 \end{pmatrix}$$
$$\xrightarrow{1,j-3} \begin{pmatrix} i+j-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} i+j-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix}$$

since $j-4 \not\equiv i+j-3, e-3$ and $j-3 \not\equiv i+j-3, e-3$.

$\underline{i+j > e,\ a=0,\ b=1\ (19)}$: Note that $m_2 = e-1$, $m_1 - m_2 = -1$. Case $j=1$ does not occur, so fix j with $2 \le j < e$ and use induction on I, for $1 \le I < j$.

$I = 1 (i+j = e+1)$: By (18), $F^1_{e-2}B'(m_1+e, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+e & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+e & m_2 & j \end{pmatrix}$$
$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & e+j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & 1 \end{pmatrix} +$$
$$v\begin{pmatrix} e-2 & e-3 & j-3 \\ m_1+e+1 & m_2 & j \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-2 & j-3 \\ m_1+e & m_2+1 & j \end{pmatrix}$$

$2 \le I < j$: Assume true for $I-1$. Then $F^1_{I-3}B'(m_1+e+I-1, m_2+j)$ is given by

$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix} +$$
$$v\begin{pmatrix} I-4 & e-3 & j-3 \\ m_1+e+I-1 & m_2 & j \end{pmatrix} + v^2 \begin{pmatrix} e-3 & I-4 & j-3 \\ m_1+e & m_2+1 & j \end{pmatrix}$$
$$\xrightarrow{1,I-3} \begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix} +$$
$$v\begin{pmatrix} I-3 & e-3 & j-3 \\ m_1+e+I & m_2 & j \end{pmatrix} + v^2 \begin{pmatrix} e-3 & I-3 & j-3 \\ m_1+e & m_2+I & j \end{pmatrix}$$

since $I-4 \not\equiv j-3, e-3$ and $I-3 \not\equiv j-3, e-3$.

$\underline{j=0,\ a=0,\ b=2\ (20)}$: Note that $m_2 = 2e-1$, $m_1 - m_2 = -1$. Use induction on i for $1 \le i < e$.

$i=1$: By (19), $F^1_{e-3}B'(m_1+1, m_2-1)$ is given by

$$\begin{pmatrix} e-2 & e-4 & e-3 \\ m_1+1 & m_2-1 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-4 & e-2 \\ m_1 & m_2-1 & 1 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & e-4 \\ m_1+1 & m_2-e & e-1 \end{pmatrix}$$
$$+ v^2 \begin{pmatrix} e-3 & e-2 & e-4 \\ m_1 & m_2-e+1 & e-1 \end{pmatrix}$$
$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & e-3 & e-3 \\ m_1+1 & m_2 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-2 & e-3 \\ m_1 & m_2-e+1 & e \end{pmatrix}$$

$2 \le i < e$: Assume true for $i-1$. Then $F^1_{i-3}B'(m_1+j-1, m_2)$ is given by

$$\begin{pmatrix} i-4 & e-3 & e-3 \\ m_1+i-1 & m_2 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & i-4 & e-3 \\ m_1 & m_2-e+i-1 & e \end{pmatrix}$$

$$\xrightarrow{1,i-3} \begin{pmatrix} i-3 & e-3 & e-3 \\ m_1+i & m_1 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & i-3 & e-3 \\ m_1 & m_2-e+i & e \end{pmatrix}$$

since $i-3, i-4 \not\equiv e-3$.

$i+j<e,\ a=0,\ b=2$ (21): Note that $m_2 = 2e-1$, $m_1 - m_2 = -1$. Case $i = e-1$ does not occur, so fix i with $1 \le i < e-1$ and use induction on j, for $1 \le j < e-i$.

$j=1$: By (18), $F^1_{j-2}B'(m_1+i+j, m_2)$ is given by

$$\begin{pmatrix} i-2 & e-3 & e-3 \\ m_1+i+j & m_2 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & i-2 & e-3 \\ m_1 & m_2-e+i+j & e \end{pmatrix}$$

$$\xrightarrow{1,e-2} \begin{pmatrix} i-2 & e-2 & e-3 \\ m_1+i+j & m_2+1 & 0 \end{pmatrix} + v\begin{pmatrix} i-2 & e-3 & e-2 \\ m_1+i+1 & m_2 & j \end{pmatrix} +$$
$$v\begin{pmatrix} e-2 & i-2 & e-3 \\ m_1+1 & m_2-e+i+j & e \end{pmatrix} + v^2\begin{pmatrix} e-3 & i-2 & e-2 \\ m_1 & m_2-e+i+j & e+1 \end{pmatrix}$$

$2 \le j < e-i$: Assume true for $j-1$. Then $F^1_{j-3}B'(m_1+i+j, m_2+j-1)$ is given by

$$\begin{pmatrix} i+j-3 & j-4 & e-3 \\ m_1+i+j & m_2+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} i+j-3 & e-3 & j-4 \\ m_1+i+j & m_2 & j-1 \end{pmatrix} +$$
$$v\begin{pmatrix} j-4 & i+j-3 & e-3 \\ m_1+j-1 & m_2-e+i+j & e \end{pmatrix} + v^2\begin{pmatrix} e-3 & i+j-3 & j-4 \\ m_1 & m_2-e+i+j & e+j-1 \end{pmatrix}$$
$$\xrightarrow{1,j-3} \begin{pmatrix} i+j-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} i+j-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & i+j-3 & e-3 \\ m_1+j & m_2-e+i+j & e \end{pmatrix} + v^2\begin{pmatrix} e-3 & i+j-3 & j-3 \\ m_1 & m_2-e+i+j & e+j \end{pmatrix}$$

since $j-4 \not\equiv i+j-3, e-3$ and $j-3 \not\equiv i+j-3, e-3$.

$i+j=e,\ a=0,\ b=2$ (22): Note that $m_2 = 2e-1$, $m_1 - m_2 = -1$. Using induction on j:

$j=1$: By (20), $F^1_{e-3}B'(m_1+i+j-1, m_2+j)$ is given by

$$\begin{pmatrix} e-4 & e-2 & e-3 \\ m_1+i+j-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-4 & e-3 & e-2 \\ m_1+i+j-1 & m_2 & j \end{pmatrix} +$$
$$v\begin{pmatrix} e-2 & e-4 & e-3 \\ m_1+j & m_2-e+i+j-1 & e \end{pmatrix} + v^2\begin{pmatrix} e-3 & e-4 & e-2 \\ m_1 & m_2-e+i+j-1 & e+j \end{pmatrix}$$
$$\xrightarrow{1,e-3} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2-e+i+j & e \end{pmatrix} \text{ (Note } e>2.)$$

$2 \le j < e$: Assume true for $j-1$. Then $F^1_{j-3}B'(m_1+i+j, m_2+j-1)$ is given by

$$\begin{pmatrix} e-3 & j-4 & e-3 \\ m_1+i+j & m_2+j-1 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-3 & j-4 \\ m_1+i+j & m_2 & j-1 \end{pmatrix} +$$
$$v\begin{pmatrix} j-4 & e-3 & e-3 \\ m_1+j-1 & m_2-e+i+j & e \end{pmatrix} \xrightarrow{1,j-3} \begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+i+j & m_2+j & 0 \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+i+j & m_2 & j \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2-e+i+j & e \end{pmatrix}$$

since $j-4, j-3 \not\equiv e-3$.

$i+j>e,\ a=0,\ b=2$ (23): Note that $m_2 = 2e-1$, $m_1 - m_2 = -1$. Case $j=1$ does not occur, so fix j with $2 \le j < e$ and use induction on I, for $1 \le I < j$.

$I = 1$ ($i+j = e+1$): By (22), $F^1_{e-2}B'(m_1+e, m_2+j)$ is given by

$$\begin{pmatrix} e-3 & j-3 & e-3 \\ m_1+e & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-3 & j-3 \\ m_1+e & m_2 & j \end{pmatrix} + v\begin{pmatrix} j-3 & e-3 & e-3 \\ m_1+j & m_2 & j \end{pmatrix}$$
$$\xrightarrow{1,e-2} \begin{pmatrix} e-2 & j-3 & e-3 \\ m_1+e+1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & e-2 \\ m_1+e & m_2+j & 1 \end{pmatrix} +$$
$$v\begin{pmatrix} e-2 & e-3 & j-3 \\ m_1+e+1 & m_2 & j \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-2 & j-3 \\ m_1+e & m_2+1 & j \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & e-2 & e-3 \\ m_1+j & m_2+1 & e \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & e-2 \\ m_1+j & m_2 & e+1 \end{pmatrix}$$

$2 \leq I < j$: Assume true for $I-1$. Then $F^1_{I-3}B'(m_1+e+I-1, m_2+j)$ is given by

$$\begin{pmatrix} I-4 & j-3 & e-3 \\ m_1+e+I-1 & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-4 \\ m_1+e & m_2+j & I-1 \end{pmatrix} +$$
$$v\begin{pmatrix} I-4 & e-3 & j-3 \\ m_1+e+I-1 & m_2 & j \end{pmatrix} + v^2\begin{pmatrix} e-3 & I-4 & j-3 \\ m_1+e & m_2+I-1 & j \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-4 & e-3 \\ m_1+j & m_2+I-1 & e \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & I-4 \\ m_1+j & m_2 & e+I-1 \end{pmatrix}$$
$$\xrightarrow{1,I-3} \begin{pmatrix} I-3 & j-3 & e-3 \\ m_1+e+I & m_2+j & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & j-3 & I-3 \\ m_1+e & m_2+j & I \end{pmatrix} +$$
$$v\begin{pmatrix} I-3 & e-3 & j-3 \\ m_1+e+I & m_2 & j \end{pmatrix} + v^2 \begin{pmatrix} e-3 & I-3 & j-3 \\ m_1+e & m_2+I & j \end{pmatrix} +$$
$$v\begin{pmatrix} j-3 & I-3 & e-3 \\ m_1+j & m_2+I & e \end{pmatrix} + v^2 \begin{pmatrix} j-3 & e-3 & I- \\ m_1+j & m_2 & e+I \end{pmatrix}$$

since $I-4 \not\equiv j-3, e-3$ and $I-3 \not\equiv j-3, e-3$.

$\underline{a=0,\ b>2\ (24)}$: We only need to prove the case when $b=3$, $j=0$, $i=1$. The proof for all other cases of $a=0$, $b=3$ then follow exactly from $(20-23)$ above, and the case $b>3$ follows by an obvious induction.

By (23), $F^1_{e-3}B'(m_1+1, m_2-1)$ is given by

$$\begin{pmatrix} e-2 & e-4 & e-3 \\ m_1+1 & m_2-1 & 0 \end{pmatrix} + v\begin{pmatrix} e-3 & e-4 & e-2 \\ m_1 & m_2-1 & 1 \end{pmatrix} + v\begin{pmatrix} e-2 & e-3 & e-4 \\ m_1+1 & m_2-e & e-1 \end{pmatrix} +$$
$$v^2 \begin{pmatrix} e-3 & e-2 & e-4 \\ m_1 & m_2-e+1 & e-1 \end{pmatrix} + v\begin{pmatrix} e-4 & e-2 & e-3 \\ m_1-1 & m_2-e+1 & e \end{pmatrix} +$$
$$v^2 \begin{pmatrix} e-4 & e-3 & e-2 \\ m_1-1 & m_2-e & e+1 \end{pmatrix} \xrightarrow{1,e-3} \begin{pmatrix} e-2 & e-3 & e-3 \\ m_1+1 & m_2 & 0 \end{pmatrix} +$$
$$\begin{pmatrix} e-2 & e-3 & e-3 \\ m_1+1 & m_2-e & e \end{pmatrix} + v\begin{pmatrix} e-3 & e-2 & e-3 \\ m_1 & m_2-e+1 & e \end{pmatrix} +$$
$$v\begin{pmatrix} e-3 & e-2 & e-3 \\ m_1 & m_2-e+1 & e \end{pmatrix} + v^2 \begin{pmatrix} e-3 & e-3 & e-2 \\ m_1 & m_2-e & e+1 \end{pmatrix}$$

This, however, is not of the correct form because one coefficient, that of (m_1+1, m_2-e, e), is equal to 1. However by (2)

$$B'(m_1-e+1, m_2-2e) = (m_1-e+1, m_2-2e) + v(m_1-e, m_2-2e+1) \\ + v^2(m_1-e, m_2-2e, 1).$$

Hence

$$B'(m_1+1, m_2-e, e) = (m_1+1, m_2-e, e) + v(m_1, m_2-e+1, e)$$

$$+ v^2(m_1, m_2 - e, e + 1)$$

and so

$$B'(m_1 + j, m_2 + j) = (m_1 + j, m_2 + j) + v(m_1 + j - e, m_2 + e, j).$$

□

Remark 2.3.9. It is convenient to represent these results pictorially. Since we obtain the decomposition matrix of $\mathcal{H}$ by setting $v = 1$, we shall not record the power of v concerned in the equation; no useful information is lost this way.

Suppose we lay the pairs (λ_1, λ_2) where $(\lambda_1 \geq \lambda_2)$ in a grid with λ_2 increasing downward and $\lambda_1 - \lambda_2$ increasing across the page (see Figure 2.1 for an example). Then we can partition them geometrically into the sets in Figure 2.2. The 9 large squares denote the regions $a \in \{0, 1, \geq 2\}$, $b \in \{0, 1, \geq 2\}$ and, with i increasing across the square and j downward, are then further partitioned into the cases

$\{i = 0,\ j = 0\}, \{i > 0,\ j = 0\}, \{i = 0,\ j > 0\}, \{i, j > 0,\ i + j < e\}$,

$\{i, j > 0,\ i + j = e\}, \{i, j > 0,\ i + j > e\}$

in the obvious manner.

Now label each occurring arrangement of the numbers m_1, m_2, i and j as follows:

$\underline{i = 0,\ j = 0:}$

O1. (m_1, m_2)

$\underline{i = 0,\ j > 0:}$

O1. $(m_1 + j, m_2 + j)$

A1. $(m_1 + j - e, m_2 + e, j)$

A2. $(m_1, m_2 + j, j)$

A3. $(m_1 + j, m_2, j)$

$\underline{i > 0,\ j = 0:}$

O1. $(m_1 + i, m_2)$

O2. $(m_1, m_2 + i)$

B1. $(m_1, m_2 + i - e, e)$

B2. (m_1, m_2, i)

$\underline{i > 0,\ j > 0,\ i + j < e:}$

O1. $(m_1 + i + j, m_2 + j)$

O2. $(m_1 + j, m_2 + i + j)$
C1. $(m_1 + i + j, m_2, j)$
C2. $(m_1, m_2 + j, i + j)$
C3. $(m_1 + j, m_2, i + j)$
C4. $(m_1, m_2 + i + j, j)$
C5. $(m_1 + i + j - e, m_2 + e, j)$
C6. $(m_1 + j - e, m_2 + e, i + j)$
C7. $(m_1, m_2 + i + j - e, e + j)$
C8. $(m_1 + j, m_2 + i + j - e, e)$

$\underline{i + j = e}$

O1. $(m_1 + e, m_2 + j)$
O2. $(m_1 + j, m_2 + e)$
D1. $(m_1 + e, m_2, j)$
D2. $(m_1, m_2 + j, e)$
D3. $(m_1 + j, m_2, e)$
D4. $(m_1, m_2 + e, j)$

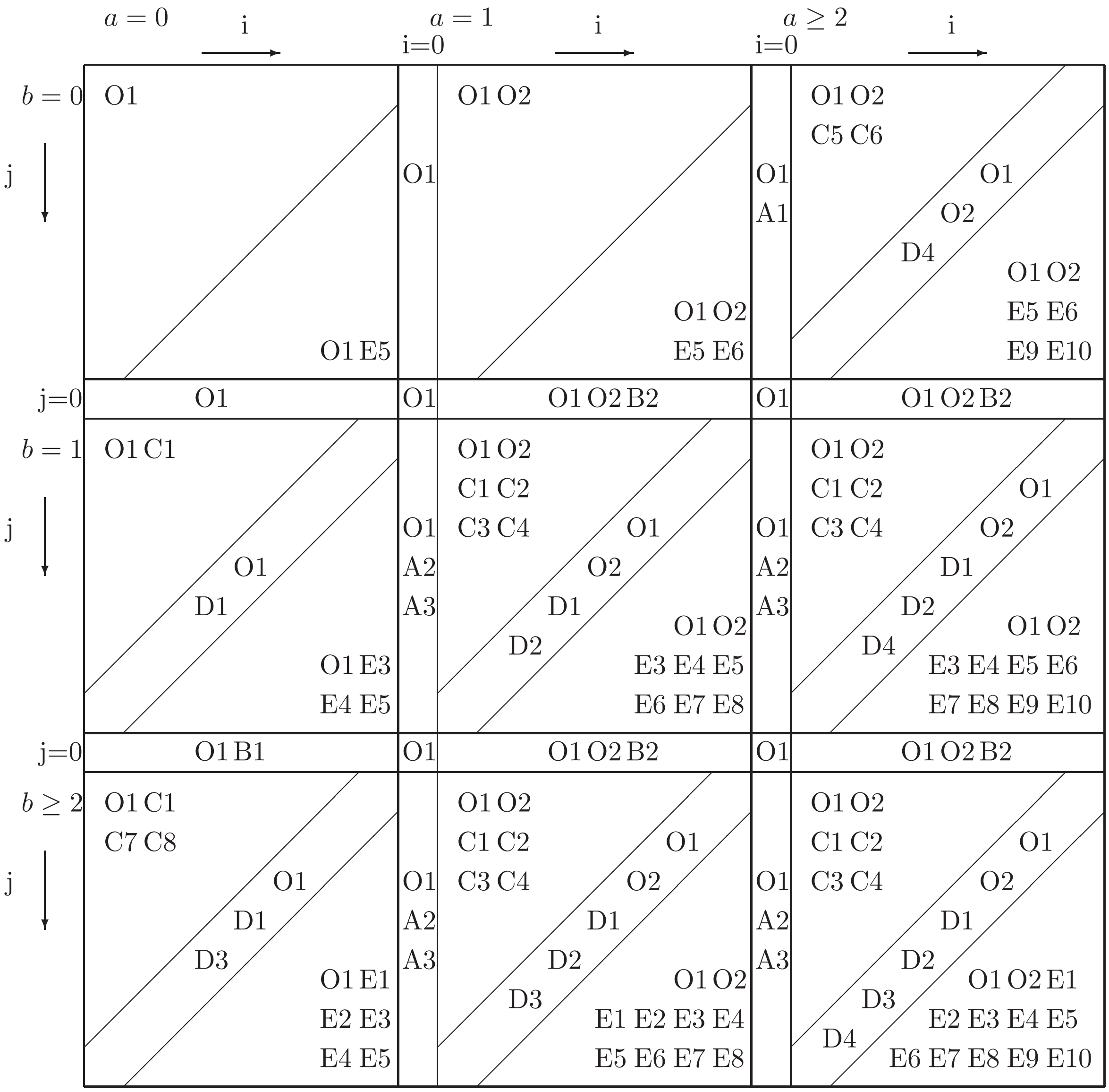
$\underline{i + j > e}$

O1. $(m_1 + e + I, m_2 + j)$
O2. $(m_1 + j, m_2 + e + I)$
E1. $(m_1 + j, m_2 + I, e)$
E2. $(m_1 + j, m_2, e + I)$
E3. $(m_1 + e + I, m_2, j)$
E4. $(m_1 + e, m_2 + I, j)$
E5. $(m_1 + e, m_2 + j, I)$
E6. $(m_1 + j, m_2 + e, I)$
E7. $(m_1 + I, m_2 + j, e)$
E8. $(m_1, m_2 + j, e + I)$
E9. $(m_1 + I, m_2 + e, j)$
E10. $(m_1, m_2 + e + I, j)$

Note that the entries O1 and O2 occur repeatedly. In fact, O1 represents λ itself, that is (m_1+i+j, m_2+j). However, if $j = 0$, this may be written (m_1+i, m_2), and likewise in other cases. Similarly O2 represents the partition $(m_1 + j, m_2 + i + j)$. Naturally other repetitions also happen, but we chose to differentiate between situations like

$$\text{C1.} \quad (m_1 + i + j, m_2, j) \qquad\qquad i, j > 0 \quad i + j < e$$

Figure 2.2: A representation of Theorem 2.3.3



D1.	$(m_1 + e, m_2, j)$	$i, j > 0$	$i + j = e$
E3.	$(m_1 + e + I, m_2, j)$	$i, j > 0$	$i + j > e$

We may then represent Theorem 2.3.3 (without the powers of v) as in Figure 2.2.

Example 2.3.10. Consider the previous example, $e = 3$ and $\lambda = (8, 6)$. Here $a = 1, b = 2, i = 0, j = 1$. Hence, looking at the vertical box down the left hand side of the square $\{a = 1, b \geq 2\}$, we see that it contains 3 entries, $O1, A2, A3$. Looking these up above, we find that, as before, they correspond to $(m_1 + j, m_2 + j), (m_1, m_2 + j, j), (m_1 + j, m_2, j)$. Note that from observation of $O1, A2, A3$ we can deduce that the only one or two part partion μ such that D^λ is a composition factor of S^μ is λ itself. If another such partition existed, it would be labelled $O2$.

Remark 2.3.11. Theorem 2.3.3 shows some points which are not clear with the usual LLT algorithm. Firstly, setting $v = 1$, it is easy to see that if μ is a partition with at most 3 parts then $[S^\mu : D^\lambda]$ is either 0 or 1 for all λ. Secondly, no matter how large n is, no D^λ is a composition factor of more than 12 such Specht modules (although this upper bound is frequently reached). In addition, the information from Theorem 2.3.3 is used to deduce Theorem 2.3.12.

Theorem 2.3.12. *Let $e > 3$. (See Remark 2.3.14.) Consider the Specht module $S^{(\mu_1,\mu_2,\mu_3)}$. Define $|x|$ such that $|x| \equiv x \mod e$ and $0 \leq |x| < e$. Let*

$$\begin{aligned} l &= |\mu_1 + 2| \\ k &= |\mu_2 + 1| \\ s &= |\mu_3| \end{aligned}$$

and let the following shorthand denote the given inequalities:

- $\alpha := \mu_2 - \mu_3 < e - 1$
- $\beta := \mu_2 - \mu_3 > e - 1$
- $\gamma := \mu_1 - \mu_2 < e - 1$
- $\delta := \mu_1 - \mu_2 > e - 1$

Label the partitions with at most 3 parts which correspond to the irreducible modules as follows.

1. (μ_1, μ_2, μ_3)
2. $(\mu_1 - l + k, \mu_2 + l - k, \mu_3)$
3. $(\mu_1 - l + k + e, \mu_2 + l - k - e, \mu_3)$
4. $(\mu_1 - l + k, \mu_2 - k + s, \mu_3 + l - s)$
5. $(\mu_1 - l + k + e, \mu_2 - k + s - e, \mu_3 + l - s)$
6. $(\mu_1 - l + s, \mu_2, \mu_3 + l - s)$
7. $(\mu_1, \mu_2 - k + s, \mu_3 + k - s)$
8. $(\mu_1 + e, \mu_2 - k + s - e, \mu_3 + k - s)$
9. $(\mu_1 - l + s, \mu_2 + l - k, \mu_3 + k - s)$
10. $(\mu_1 - l + s + e, \mu_2 + l - k - e, \mu_3 + k - s)$
11. $(\mu_1 - l + k, \mu_2 - k + s + e, \mu_3 + l - s - e)$
12. $(\mu_1 - l + k + e, \mu_2 - k + s, \mu_3 + l - s - e)$
13. $(\mu_1 - l + k + 2e, \mu_2 - k + s - e, \mu_3 + l - s - e)$

14. $(\mu_1 - l + s + e, \mu_2, \mu_3 + l - s - e)$
15. $(\mu_1, \mu_2 - k + s + e, \mu_3 + k - s - e)$
16. $(\mu_1 + e, \mu_2 - k + s, \mu_3 + k - s - e)$
17. $(\mu_1 - l + s, \mu_2 + e + l - k, \mu_3 + k - s - e)$
18. $(\mu_1 - l + s + e, \mu_2 + l - k, \mu_3 + k - s - e)$
19. $(\mu_1 - l + s + 2e, \mu_2 + l - k - e, \mu_3 + k - s - e)$
20. $(\mu_1 - l + k + e, \mu_2 - k + s + e, \mu_3 + l - s - 2e)$
21. $(\mu_1 - l + s + 2e, \mu_2, \mu_3 + l - s - 2e)$
22. $(\mu_1 - l + s + e, \mu_2 + l - k + e, \mu_3 + k - s - 2e)$
23. $(\mu_1 + e, \mu_2, \mu_3 - e)$
24. $(\mu_1 - l + k, \mu_2 + l - k + e, \mu_3 - e)$
25. $(\mu_1 - l + k + e, \mu_2 + l - k, \mu_3 - e)$

The composition factors of $S^{(\mu_1,\mu_2,\mu_3)}$ are given by the following tables. All the partitions listed in $1-25$ above are distinct, and each composition factor occurs with multiplicity 1.

$A : \mu_3 = 0$

		1	*2*	*3*
$l = k$		$\star$		
$k > l = 0$	$\mu_2 < e - 1$	$\star$		
	$\mu_2 > e - 1$	$\star$	$\star$	
$k > l > 0$		$\star$	$\star$	
$l > k$	$\mu_2 < e - 1$	$\star$		
	$\mu_2 \geq e - 1$	$\star$		$\star$

$B : 1 \leq \mu_3 < e$

			1	2	3	4	5	6	7	8	9	10
$l = k = s$			$\star$									
$l = k > s$			$\star$									
$s > l = k$			$\star$					$\star$	$\star$	$\star$		
$l > k = s$			$\star$		$\star$							
$k = s > l$			$\star$	$\star$				$\star$				
$k > l = s$	α		$\star$									
	β		$\star$	$\star$								
$l = s > k$	α	γ	$\star$							$\star$		
	α	δ	$\star$						$\star$	$\star$		
	β	γ	$\star$		$\star$							
	β	δ	$\star$		$\star$				$\star$			
$k > l > s$			$\star$	$\star$								
$l > s > k$	α		$\star$		$\star$				$\star$	$\star$		$\star$
	β		$\star$		$\star$				$\star$			$\star$
$s > k > l$	α		$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	
	β		$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$	
$l > k > s$	α		$\star$									
	β		$\star$		$\star$							
$k > s > l$	α		$\star$					$\star$				
	β		$\star$	$\star$		$\star$		$\star$				
$s > l > k$	α	γ	$\star$					$\star$				$\star$
	α	δ	$\star$					$\star$	$\star$	$\star$	$\star$	$\star$
	β	γ	$\star$		$\star$		$\star$	$\star$				
	β	δ	$\star$		$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	

$C : \mu_3 = e$

			1	2	3	11	12	14	15	16	18	19	23	24	25
$l = k = 0$			$\star$												
$l = k > 0$			$\star$					$\star$	$\star$	$\star$					
$l > k = 0$			$\star$		$\star$			$\star$							$\star$
$k > l = 0$	α	γ	$\star$							$\star$			$\star$	$\star$	
	α	δ	$\star$						$\star$	$\star$			$\star$		
	β	γ	$\star$	$\star$									$\star$	$\star$	
	β	δ	$\star$	$\star$					$\star$				$\star$		
$l > k > 0$	α	γ	$\star$					$\star$				$\star$			
	α	δ	$\star$					$\star$	$\star$	$\star$	$\star$	$\star$			
	β	γ	$\star$		$\star$		$\star$	$\star$							$\star$
	β	δ	$\star$		$\star$		$\star$	$\star$	$\star$	$\star$	$\star$				$\star$
$k > l > 0$	α	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		$\star$	$\star$	
	α	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		$\star$		
	β	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$		$\star$	$\star$	
	β	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$		$\star$		

$D : e < \mu_3 < 2e$

			1	2	3	4	5	6	7	8	9	10	11
$l = k = s$			$\star$										
$l = k > s$			$\star$										
$s > l = k$			$\star$					$\star$	$\star$	$\star$			
$l > k = s$			$\star$		$\star$								
$k = s > l$			$\star$	$\star$				$\star$					
$k > l = s$	α	γ	$\star$										
	α	δ	$\star$										
	β	γ	$\star$	$\star$									
	β	δ	$\star$	$\star$									
$s = l > k$	α	γ	$\star$							$\star$			
	α	δ	$\star$						$\star$	$\star$			
	β	γ	$\star$		$\star$								
	β	δ	$\star$		$\star$				$\star$				
$l > k > s$	α	γ	$\star$										
	α	δ	$\star$										
	β	γ	$\star$		$\star$								
	β	δ	$\star$		$\star$								
$k > s > l$	α	γ	$\star$					$\star$					$\star$
	α	δ	$\star$					$\star$					
	β	γ	$\star$	$\star$		$\star$		$\star$					$\star$
	β	δ	$\star$	$\star$		$\star$		$\star$					
$s > l > k$	α	γ	$\star$					$\star$				$\star$	
	α	δ	$\star$					$\star$	$\star$	$\star$	$\star$	$\star$	
	β	γ	$\star$		$\star$		$\star$	$\star$					
	β	δ	$\star$		$\star$		$\star$	$\star$	$\star$	$\star$	$\star$		
$k > l > s$	α	γ	$\star$	$\star$									$\star$
	α	δ	$\star$	$\star$									$\star$
	β	γ	$\star$	$\star$									$\star$
	β	δ	$\star$	$\star$									$\star$
$l > s > k$	α	γ	$\star$		$\star$				$\star$	$\star$		$\star$	
	α	δ	$\star$		$\star$				$\star$	$\star$		$\star$	
	β	γ	$\star$		$\star$				$\star$			$\star$	
	β	δ	$\star$		$\star$				$\star$			$\star$	
$s > k > l$	α	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		
	α	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		
	β	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$		
	β	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$		

	12	13	14	15	16	17	18	19	23	24	25
			$\star$	$\star$	$\star$						
			$\star$								$\star$
										$\star$	
					$\star$				$\star$	$\star$	
				$\star$	$\star$				$\star$		
									$\star$	$\star$	
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									$\star$		$\star$
									$\star$		
			$\star$					$\star$			
			$\star$	$\star$	$\star$		$\star$	$\star$			
	$\star$		$\star$								$\star$
	$\star$		$\star$	$\star$	$\star$		$\star$				$\star$
			$\star$				$\star$				
			$\star$	$\star$	$\star$	$\star$	$\star$				
			$\star$							$\star$	
			$\star$	$\star$	$\star$	$\star$				$\star$	
	$\star$		$\star$								
			$\star$								
	$\star$		$\star$								$\star$
			$\star$								$\star$
	$\star$		$\star$	$\star$	$\star$		$\star$		$\star$	$\star$	
	$\star$		$\star$	$\star$	$\star$		$\star$		$\star$		
	$\star$		$\star$	$\star$			$\star$		$\star$	$\star$	
	$\star$		$\star$	$\star$			$\star$		$\star$		
	$\star$	$\star$	$\star$				$\star$		$\star$		$\star$
	$\star$	$\star$	$\star$				$\star$		$\star$		
	$\star$	$\star$	$\star$				$\star$		$\star$		$\star$
	$\star$	$\star$	$\star$				$\star$		$\star$		
						$\star$			$\star$	$\star$	
						$\star$			$\star$		
						$\star$			$\star$	$\star$	
						$\star$			$\star$		

$\underline{E : \mu_3 \geq 2e}$

			1	2	3	4	5	6	7	8	9	10	11	12	13
$l = k = s$			$\star$												
$l = k > s$			$\star$												
$s > l = k$			$\star$					$\star$	$\star$	$\star$					
$l > k = s$			$\star$		$\star$										
$k = s > l$			$\star$	$\star$				$\star$							
$k > l = s$	α	γ	$\star$												
	α	δ	$\star$												
	β	γ	$\star$	$\star$											
	β	δ	$\star$	$\star$											
$s = l > k$	α	γ	$\star$							$\star$					
	α	δ	$\star$						$\star$	$\star$					
	β	γ	$\star$		$\star$										
	β	δ	$\star$		$\star$				$\star$						
$l > k > s$	α	γ	$\star$												
	α	δ	$\star$												
	β	γ	$\star$		$\star$									$\star$	
	β	δ	$\star$		$\star$									$\star$	
$k > s > l$	α	γ	$\star$					$\star$					$\star$		
	α	δ	$\star$					$\star$							
	β	γ	$\star$	$\star$		$\star$		$\star$					$\star$		
	β	δ	$\star$	$\star$		$\star$		$\star$							
$s > l > k$	α	γ	$\star$					$\star$				$\star$		$\star$	
	α	δ	$\star$					$\star$	$\star$	$\star$	$\star$	$\star$			
	β	γ	$\star$		$\star$		$\star$	$\star$						$\star$	
	β	δ	$\star$		$\star$		$\star$	$\star$	$\star$	$\star$	$\star$				
$k > l > s$	α	γ	$\star$	$\star$									$\star$	$\star$	
	α	δ	$\star$	$\star$									$\star$	$\star$	
	β	γ	$\star$	$\star$									$\star$	$\star$	
	β	δ	$\star$	$\star$									$\star$	$\star$	
$l > s > k$	α	γ	$\star$		$\star$				$\star$	$\star$		$\star$		$\star$	$\star$
	α	δ	$\star$		$\star$				$\star$	$\star$		$\star$		$\star$	$\star$
	β	γ	$\star$		$\star$				$\star$			$\star$		$\star$	$\star$
	β	δ	$\star$		$\star$				$\star$			$\star$		$\star$	$\star$
$s > k > l$	α	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$				
	α	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$				
	β	γ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$				
	β	δ	$\star$	$\star$		$\star$	$\star$	$\star$	$\star$		$\star$				

	14	15	16	17	18	19	20	21	22	23	24	25
	$\star$	$\star$	$\star$									
	$\star$											$\star$
											$\star$	
			$\star$							$\star$	$\star$	
		$\star$	$\star$							$\star$		
										$\star$	$\star$	
		$\star$								$\star$		
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	$\star$					$\star$	$\star$	$\star$				
	$\star$	$\star$	$\star$		$\star$	$\star$		$\star$				
	$\star$						$\star$	$\star$				$\star$
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	$\star$				$\star$							
	$\star$	$\star$	$\star$	$\star$	$\star$							
	$\star$										$\star$	
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	$\star$											$\star$
	$\star$											$\star$
	$\star$	$\star$	$\star$		$\star$				$\star$	$\star$	$\star$	
	$\star$	$\star$	$\star$		$\star$				$\star$	$\star$		
	$\star$	$\star$			$\star$				$\star$	$\star$	$\star$	
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				$\star$						$\star$	$\star$	
				$\star$						$\star$		
				$\star$						$\star$	$\star$	
				$\star$						$\star$		

The proof uses the following lemma, in which, for a Specht module S^μ, we note the composition factors of S^μ which correspond to partitions with at most 2 parts. (As previously noted, the multiplicity of each such composition factor is 1.)

Lemma 2.3.13.
Suppose $\mu_3 = 0$. Then

1. *If $l > k$ and $\mu_2 < e-1$ then $S^{(\mu_1,\mu_2)}$ has a composition factor $D^{(\mu_1,\mu_2)}$.*

2. *If $l > k$ and $\mu_2 \geq e-1$ then $S^{(\mu_1,\mu_2)}$ has composition factors $D^{(\mu_1,\mu_2)}$ and $D^{(\mu_1+k-l+e,\mu_2-k+l-e)}$.*

3. *If* $k > l = 0$ *and* $\mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2)}$ *has composition factors* $D^{(\mu_1,\mu_2)}$ *and* $D^{(\mu_1+k-l,\mu_2-k+l)}$.

4. *If* $k > l = 0$ *and* $\mu_2 < e-1$ *then* $S^{(\mu_1,\mu_2)}$ *has a composition factors* $D^{(\mu_1,\mu_2)}$.

5. *If* $k > l > 0$ *then* $S^{(\mu_1,\mu_2)}$ *has composition factors* $D^{(\mu_1,\mu_2)}$ *and* $D^{(\mu_1+k-l,\mu_2-k+l)}$.

Suppose $0 < \mu_3 < e$ *and* $l = k = 0$. *Then*

1. $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1,\mu_2+\mu_3)}$, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ *and* $D^{(\mu_1+\mu_3,\mu_2)}$.

Suppose $0 < \mu_3 < e$ *and* $l = 0$ *and* $k \neq 0$. *Then*

1. *If* $s = k$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* $D^{(\mu_1+\mu_3,\mu_2)}$.

2. *If* $s > k$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1+\mu_3,\mu_2)}$, $D^{(\mu_1+k,\mu_2+\mu_3-k)}$ *and* $D^{(\mu_1+k+e,\mu_2+\mu_3-e-k)}$.

3. *If* $k > s$ *and* $\mu_2 < e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* $D^{(\mu_1+\mu_3,\mu_2)}$.

4. *If* $k > s$ *and* $\mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1+k,\mu_2+\mu_3-k)}$ *and* $D^{(\mu_1+\mu_3,\mu_2)}$.

Suppose $0 < \mu_3 < e$ *and* $l \neq 0$ *and* $k = 0$. *Then*

1. *If* $s = l$ *and* $\mu_2 = e-1$ *and* $\mu_1 - \mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1,\mu_2+\mu_3)}$ *and* $D^{(\mu_1+e,\mu_2+\mu_3-e)}$.

2. *If* $s = l$ *and* $\mu_2 = e-1$ *and* $\mu_1 - \mu_2 < e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* $D^{(\mu_1+e,\mu_2+\mu_3-e)}$.

3. *If* $s = l$ *and* $\mu_2 \neq e-1$ *and* $\mu_1 - \mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* $D^{(\mu_1,\mu_2+\mu_3)}$.

4. *If* $s > l$ *and* $\mu_2 = e-1$ *and* $\mu_1 - \mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1,\mu_2+\mu_3)}$, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$, $D^{(\mu_1+\mu_3-l,\mu_2+l)}$ *and* $D^{(\mu_1+\mu_3+e-l,\mu_2-e+l)}$.

5. *If* $s > l$ *and* $\mu_2 = e-1$ *and* $\mu_1 - \mu_2 < e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* $D^{(\mu_1+\mu_3+e-l,\mu_2-e+l)}$.

6. *If* $s > l$ *and* $\mu_2 \neq e-1$ *and* $\mu_1 - \mu_2 > e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1,\mu_2+\mu_3)}$, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ *and* $D^{(\mu_1+\mu_3-l,\mu_2+l)}$.

7. *If* $l > s$ *and* $\mu_2 = e-1$ *then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has composition factors* $D^{(\mu_1,\mu_2+\mu_3)}$, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ *and* $D^{(\mu_1+\mu_3+e-l,\mu_2-e+l)}$.

8. *If $l > s$ and $\mu_2 \neq e-1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has composition factors $D^{(\mu_1,\mu_2+\mu_3)}$ and $D^{(\mu_1+\mu_3+e-l,\mu_2-e+l)}$.*

Suppose $\mu_3 = e$. Then

1. *If $k > l > 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3,\mu_2)}$.*

2. *If $k > l > 0$ and $\mu_1 - \mu_2 < e-1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ also has a composition factor $D^{(\mu_1-l+k,\mu_2+\mu_3-k+l)}$.*

3. *If $l > k > 0$ and $\mu_2 > 2e-1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3-l+k,\mu_2-k+l)}$.*

4. *If $k > l = 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3,\mu_2)}$.*

5. *If $k > l = 0$ and $\mu_1 - \mu_2 < e-1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ also has a composition factor $D^{(\mu_1+k,\mu_2+\mu_3-k)}$.*

6. *If $l > k = 0$ and then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3-l,\mu_2+l)}$.*

Suppose $e < \mu_3 < 2e-1$. Then

1. *If $l = 0$ and $k > s$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3,\mu_2)}$.*

2. *If $l = 0$ and $k > s$ and $\mu_1 - \mu_2 < e-1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ also has a composition factor $D^{(\mu_1+k,\mu_2+\mu_3-k)}$.*

3. *If $k = 0$ and $l > s$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+\mu_3-l,\mu_2+l)}$.*

Moreover, only the above Specht modules contain composition factors corresponding to 1 or 2 part partitions and these contain only the 1 or 2 part composition factors listed above.

Proof. The proof follows by consideration of Figure 2.3, which uses the notation of Remark 2.3.9. We will describe what happens whenever $\mu_3 > e$; the other cases follow similarly.

Consider $S^{(\mu_1,\mu_2,\mu_3)}$ where $\mu_3 > e$. We wish to determine which partitions $\lambda = (\lambda_1, \lambda_2)$ are such that D^λ is a composition factor of S^μ. This involves looking at the following cases.

C7. $(m_1, m_2 + i + j - e, e + j)$
E2. $(m_1 + j, m_2, e + I)$
E8. $(m_1, m_2 + j, e + I)$

Observe that

1. If $\lambda = (m_1 + i + j, m_2 + j)$ satisfies $a = 0$ and $b \geq 2$ and $i + j < e$ then D^λ is a composition factor of $S^{(m_1, m_2+i+j-e, e+j)}$.

   Moreover, λ is a partition such that $a = 0$ and $b \geq 2$ and $i + j < e$ if and only if $\mu = (m_1, m_2 + i + j - e, e + j)$ is a partition such that $l = 0$ and $k > s$ and $\mu_1 - \mu_2 < e - 1$.

   Hence if $\mu = (\mu_1, \mu_2, \mu_3)$ satisfies $\mu_3 > e$ and $l = 0$ and $k > s$ and $\mu_1 - \mu_2 < e - 1$ then S^μ has a composition factor $D^{(\mu_1+k, \mu_2+\mu_3-k)}$.

2. If $\lambda = (m_1 + e + I, m_2 + j)$ satisfies $b \geq 2$ and $i + j > e$ then D^λ is a composition factor of $S^{(m_1+j, m_2, e+I)}$.

   Moreover λ is a partition such that $b \geq 2$ and $i + j > e$ if and only if $\mu = (m_1 + j, m_2, e + I)$ is a partition such that $k = 0$ and $l > s$.

   Hence if $\mu = (\mu_1, \mu_2, \mu_3)$ satisfies $\mu_3 > e$ and $k = 0$ and $l > s$ and then S^μ has a composition factor $D^{(\mu_1+\mu_3-l, \mu_2+l)}$.

3. If $\lambda = (m_1 + e + I, m_2 + j)$ satisfies $a, b \geq 1$ and $i + j > e$ then $D^{(m_1+i+j, m_2+j)}$ is a composition factor of $S^{(m_1, m_2+j, e+I)}$.

   Moreover $\lambda = (m_1 + i + j, m_2 + j)$ is a partition such that $a, b \geq 1$ and $i + j > e$ if and only if $\mu = (m_1 + j, m_2, e + I)$ is a partition such that $l = 0$ and $k > s$ and$\mu_1 - \mu_2 < e - 1$.

   Hence if $\mu = (\mu_1, \mu_2, \mu_3)$ satisfies $\mu_3 > e$ and $l = 0$ and $k > s$ and then S^μ has a composition factor $D^{(\mu_1+\mu_3, \mu_2)}$.

$\square$

The proof of Theorem 2.3.12 can then be deduced from Lemma 2.3.13 and Theorem 2.3.7 (the column addition theorem).

Remark 2.3.14. If $e = 3$, Theorem 2.3.12 holds in all cases except whenever $\lambda = (m, m, m)$. Then S^λ does not contain a composition factor D^λ $(= 0)$, but all other entries are consistent. If $e = 2$ then Theorem 2.3.12 will give all the composition factors of any S^λ, but may also produce some 2-singular (zero) composition factors. We now consider this situation in more detail. Note that the composition factors of Specht modules corresponding to partitions with at most 3 parts when $e = 2$ were classified by James and Mathas in 1995 [21]. We shall reproduce the results of [[21], Corollary 8.21].

Figure 2.3:

					(μ_1, μ_2)	$(\mu_1 - l + k, \mu_2 + l - k)$	$(\mu_1 - l + k + e, \mu_2 + l - k - e)$	$(\mu_1, \mu_2 + \mu_3)$	$(\mu_1 + e, \mu_2 + \mu_3 - e)$	$(\mu_1 + \mu_3, \mu_2)$	$(\mu_1 + k, \mu_2 + \mu_3 - k)$	$(\mu_1 + k + e, \mu_2 + \mu_3 - k - e)$	$(\mu_1 + \mu_3 - l, \mu_2 + l)$	$(\mu_1 + \mu_3 - l + e, \mu_2 + l - e)$	$(\mu_1 - l + k, \mu_2 + \mu_3 + l - k)$	$(\mu_1 + \mu_3 - l + k, \mu_2 + l - k)$
$\mu_3 = 0$	$l > k$		$\mu_2 < e - 1$		O1											
	$l > k$		$\mu_2 \geq e - 1$		O1		O2									
	$k > l = 0$		$\mu_2 < e - 1$		O1											
	$k > l = 0$		$\mu_2 \geq e - 1$		O1	O2										
	$k > l > 0$				O1	O2										
$0 < \mu_3 < e$	$l = 0$	$k = 0$						D1	D4	B2						
		$s > k > 0$								C2	E4	E10				
		$k = s > 0$								A2						
		$k > s$	$\mu_2 < e - 1$							E5						
			$\mu_2 > e - 1$							E5	C4					
	$k = 0$	$s > l > 0$	$\mu_2 = e - 1$	$\mu_1 - \mu_2 > e - 1$				E3	E9				C3	C6		
				$\mu_1 - \mu_2 < e - 1$										C6		
			$\mu_2 \neq e - 1$	$\mu_1 - \mu_2 > e - 1$				E3	E9				C3			
		$l = s > 0$	$\mu_2 = e - 1$	$\mu_1 - \mu_2 > e - 1$				A3	A1							
				$\mu_1 - \mu_2 < e - 1$					A1							
			$\mu_2 \neq e - 1$	$\mu_1 - \mu_2 > e - 1$				A3								
		$l > s$	$\mu_2 = e - 1$					C1	C5					E6		
			$\mu_2 \neq e - 1$					C1						E6		
$\mu_3 = e$	$k > l > 0$			$\mu_1 - \mu_2 > e - 1$						E7						
				$\mu_1 - \mu_2 < e - 1$						E7					C8	
	$l > k > 0$		$\mu_2 > 2e - 1$													E1
	$l = 0$	$k > 0$		$\mu_1 - \mu_2 > e - 1$						D2						
				$\mu_1 - \mu_2 < e - 1$						D2	B1					
	$k = 0$	$l > 0$											D3			
$e < \mu_3 < 2e - 1$	$l = 0$	$k > s$		$\mu_1 - \mu_2 > e - 1$						E8						
				$\mu_1 - \mu_2 < e - 1$						E8	C7					
	$k = 0$	$l > s$											E2			

Suppose that $e = 2$. Then Figure 2.2 can be simplified to the following.

		$a = 1$		$a \geq 2$	
		$i = 0$	$i = 1$	$i = 0$	$i = 1$
$b = 0$	$j = 1$	O1	O1 O2	O1 A1	O1 O2 D4
$b = 1$	$j = 0$	O1	O1 O2 B2	O1	O1 O2 B2
	$j = 1$	O1 A2 A3	O1 O2 D1 D2	O1 A2 A3	O1 O2 D1 D2 D4
$b \geq 2$	$j = 0$	O1	O1 O2 B2	O1	O1 O2 B2
	$j = 1$	O1 A2 A3	O1 O2 D1 D2 D3	O1 A2 A3	O1 O2 D1 D2 D3 D4

We now use this to table produce an analogue of Lemma 2.3.13. Note that all the composition factors must be 2-regular, and let $\equiv$ denote equivalence mod 2.

Lemma 2.3.15.
Suppose $\mu_3 = 0$. Then

1. *If $\mu_1 > \mu_2$ then $S^{(\mu_1,\mu_2)}$ has a composition factor $D^{(\mu_1,\mu_2)}$.*
2. *If $\mu_1 \equiv \mu_2$ and $\mu_2 \neq 0$ then $S^{(\mu_1,\mu_2)}$ has a composition factor $D^{(\mu_1+1,\mu_2-1)}$.*

Suppose $\mu_3 = 1$. Then

1. *If $\mu_1 \equiv 0$ and $\mu_2 \equiv 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has composition factors $D^{(\mu_1+1,\mu_2)}$ and $D^{(\mu_1+2,\mu_2-1)}$.*
2. *If $\mu_1 \equiv 0$ and $\mu_2 \equiv 1$ and $\mu_1 - \mu_2 > 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ also has a composition factor $D^{(\mu_1,\mu_2+1)}$.*
3. *If $\mu_1 \equiv \mu_2 \equiv 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+1,\mu_2)}$.*
4. *If $\mu_1 \equiv \mu_2 \equiv 1$ and $\mu_1 - \mu_2 \neq 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1,\mu_2+1)}$.*
5. *If $\mu_1 \equiv \mu_2 \equiv 1$ and $\mu_2 = 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+2,\mu_2-1)}$.*

Suppose $\mu_3 = 2$. Then

1. *If $\mu_1 \equiv \mu_2 \equiv 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+2,\mu_2)}$.*
2. *If $\mu_1 \equiv \mu_2 \equiv 1$ and $\mu_1 > \mu_2$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+1,\mu_2+1)}$.*

Moreover, only the above Specht modules contain composition factors corresponding to 1 or 2 part partitions and these contain only the 1 or 2 part composition factors listed above.

Using Lemma 2.3.15 and the column addition theorem, we may deduce the following analogue of Theorem 2.3.12. We will use the more convenient notation of [21].

Theorem 2.3.16. *Let* $e = 2$ *and suppose* $(\mu_1, \mu_2, \mu_3) \equiv (\alpha_1, \alpha_2, \alpha_3) \mod 2$. *Label the partitions* $(\lambda_1, \lambda_2, \lambda_3)$ *corresponding to the irreducible modules as follows.*

1. (μ_1, μ_2)
2. $(\mu_1 + 1, \mu_2 - 1)$
3. (μ_1, μ_2, μ_3)
4. $(\mu_1 + 1, \mu_2 - 1, \mu_3)$
5. $(\mu_1, \mu_2 + 1, \mu_3 - 1)$
6. $(\mu_1 + 1, \mu_2, \mu_3 - 1)$
7. $(\mu_1 + 2, \mu_2 - 1, \mu_3 - 1)$
8. $(\mu_1 + 1, \mu_2 + 1, \mu_3 - 2)$
9. $(\mu_1 + 2, \mu_2, \mu_3 - 2)$

The composition factors of $S^{(\mu_1,\mu_2,\mu_3)}$ *are given by the following tables. All the partitions listed in* $1 - 9$ *above are distinct, and each composition factor occurs with multiplicity 1.*

$A : \mu_2 = 0$

Then $S^{(\mu_1)} \cong D^{(\mu_1)}$.

$B : \mu_2 \geq 1, \mu_3 = 0$

(α_1, α_2)		*1*	*2*
$(0, 1)$ *or* $(1, 0)$		$\star$	
$(0, 0)$ *or* $(1, 1)$	$\mu_1 > \mu_2$	$\star$	$\star$
	$\mu_1 = \mu_2$		$\star$

$C : \mu_3 = 1$

$(\alpha_1, \alpha_2, \alpha_3)$			*3*	*4*	*5*	*6*	*7*
$(1,1,1)$	$\mu_1 > \mu_2$	$\mu_2 > \mu_3$	$\star$	$\star$	$\star$		
		$\mu_2 = \mu_3$			$\star$		$\star$
	$\mu_1 = \mu_2$	$\mu_2 > \mu_3$		$\star$			
		$\mu_2 = \mu_3$					$\star$
$(0,0,1)$	$\mu_1 > \mu_2$	$\mu_2 > \mu_3 + 1$	$\star$	$\star$		$\star$	
		$\mu_2 = \mu_3 + 1$	$\star$			$\star$	
	$\mu_1 = \mu_2$	$\mu_2 > \mu_3 + 1$		$\star$		$\star$	
		$\mu_2 = \mu_3 + 1$				$\star$	
$(0,1,1)$	$\mu_1 > \mu_2 + 1$	$\mu_2 > \mu_3$	$\star$		$\star$	$\star$	$\star$
		$\mu_2 = \mu_3$			$\star$	$\star$	
	$\mu_1 = \mu_2 + 1$	$\mu_2 > \mu_3$	$\star$			$\star$	$\star$
		$\mu_2 = \mu_3$				$\star$	
$(1,0,1)$			$\star$				

$D : \mu_3 \geq 2$

(α_1, α_2)			*3*	*4*	*5*	*6*	*7*	*8*	*9*
$(0,0,0)$ *or* $(1,1,1)$	$\mu_1 > \mu_2$	$\mu_2 > \mu_3$	$\star$	$\star$	$\star$				$\star$
		$\mu_2 = \mu_3$			$\star$				$\star$
	$\mu_1 = \mu_2$	$\mu_2 > \mu_3$		$\star$					$\star$
		$\mu_2 = \mu_3$							$\star$
$(0,0,1)$ *or* $(1,1,0)$	$\mu_1 > \mu_2$	$\mu_2 > \mu_3 + 1$	$\star$	$\star$		$\star$		$\star$	
		$\mu_2 = \mu_3 + 1$	$\star$			$\star$		$\star$	
	$\mu_1 = \mu_2$	$\mu_2 > \mu_3 + 1$		$\star$		$\star$			
		$\mu_2 = \mu_3 + 1$				$\star$			
$(1,0,0)$ *or* $(0,1,1)$	$\mu_1 > \mu_2 + 1$	$\mu_2 > \mu_3$	$\star$		$\star$	$\star$	$\star$		
		$\mu_2 = \mu_3$			$\star$	$\star$			
	$\mu_1 = \mu_2 + 1$	$\mu_2 > \mu_3$	$\star$			$\star$	$\star$		
		$\mu_2 = \mu_3$				$\star$			
$(0,1,0)$ *or* $(1,0,1)$			$\star$						

Note that either Lemmas 2.3.13 and 2.3.15 or Theorems 2.3.12 and 2.3.16 can be used to write a very simple computer program which will determine the composition factors of a Specht module corresponding to any 1,2 or 3 part partition.

2.4 Partitions with at most 4 parts

In this section, we wish to consider $B_4'(\lambda)$. The methods used will be very similar to those discussed in the Section 2.3.

Definition 2.4.1. Let $e \geq 2$ and suppose $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a one, two or three part e-regular partion of $\lambda_1 + \lambda_2 + \lambda_3 = n$. Suppose that

$$\lambda_1 - \lambda_2 = ae + i - 1$$
$$\lambda_2 - \lambda_3 = be + j - 1$$

$$\lambda_3 = ce + k - 1$$

where $0 \leq i, j, k < e$. Define

$$m_3 := ce - 1$$
$$m_2 - m_3 := be - 1$$
$$m_1 - m_2 := ae - 1$$

so that

$$\lambda = (m_1 + i + j + k, m_2 + j + k, m_3 + k).$$

If $e < (i + j + k) < 2e$, let

$$I = (i + j + k) - e.$$

If $e < (j + k)$, let

$$J = (j + k) - e.$$

If $2e < (i + j + k)$, let

$$H = (i + j + k) - 2e$$

so that $1 \leq I, J, H < e$.

We can imagine the entries arranged in a 3-dimensional array, with a, i increasing horiontally across the plane, b, j increasing downwards in the plane and c, k increasing vertically upwards. The 'cube' representing all values of $0 \leq i, j, k < e$ is then divided into the following regions, where we assume that $i, j, k > 0$ unless otherwise stated.

A. $i = j = k = 0$

B. $j = k = 0$

C. $i = k = 0$

D. $k = 0, i + j < e$

E. $k = 0, i + j = e$

F. $k = 0, i + j > e$

G. $i = j = 0$

H. $j = 0, i + k < e$

I. $j = 0, i + k = e$

J. $j = 0, i + k > e$

K. $i = 0, j + k < e$

L. $i = 0, j + k = e$

M. $i = 0, j + k > e$

N. $i + j + k < e$

O. $i + j + k = e$

P. $i + j + k > e, i + j < e,$ $j + k < e$

Q. $i + j = e, j + k < e$

R. $i + j > e, j + k < e$

S. $i + j < e, j + k = e$

T. $i + j = j + k = e$

U. $i + j > e, j + k = e$

V. $i + j < e, j + k > e$

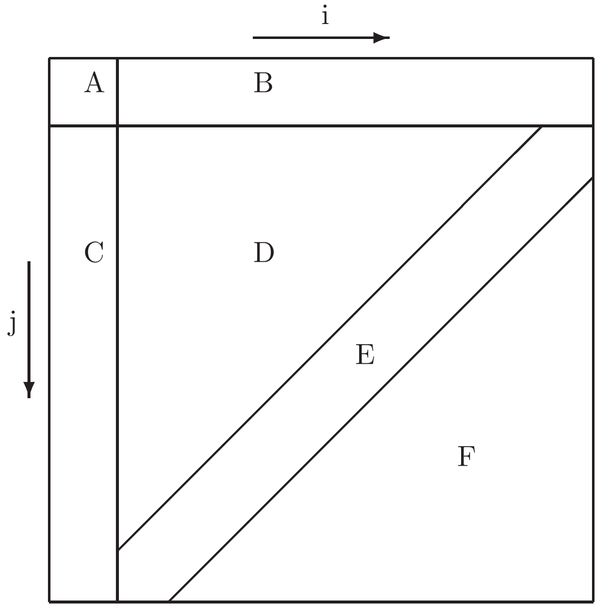
W. $i + j = e, j + k > e$

X. $i + j, j + k > e,$ $i + j + k < 2e$

Y. $i + j + k = 2e$

Z. $i + j + k > 2e$

Figure 2.4: $k = 0$



If $e = 2$ or $e = 3$ some of these cases will not exist. Otherwise they may be drawn as in Figures 2.4 and 2.5.

Conjecture 2.4.2. *Suppose λ is a partition of n with at most 3 parts. Then the Specht modules S^μ which are such that μ has at most 4 parts and which contain D^λ as a composition factor are given by the tables in Appendix A.2.*

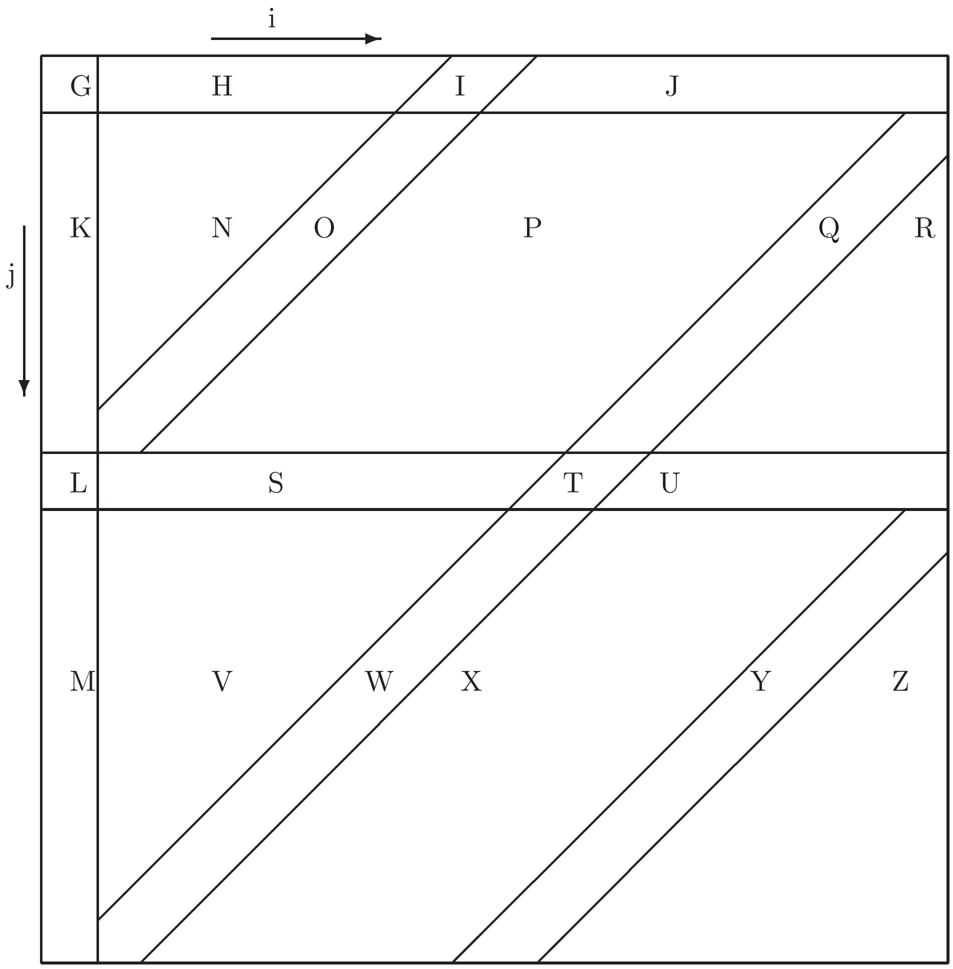
Remark 2.4.3. It is clear by looking at the proof of the comparatively compact Theorem 2.3.3 that writing out a proof for Conjecture 2.4.2 would be extremely tedious, and furthermore would be unlikely to convince the reader who is not prepared to study around 500 cases of proof by induction. Hence, although the results have been checked rigorously, it is presented as a conjecture. Following is a sketch of the proof in some cases when $a = b = c = 1$.

Rough proof of some cases of Conjecture 2.4.2. Suppose $a = b = c = 1$. Note that case A, $(3e-3, 2e-2, e-1)$ is an e-core and so $B(\lambda) = \lambda$. We will now look at case B, proceeding as before by induction on i for $1 \leq i < e$. However, for reasons of space, we do not make the inductive arguments, simply presenting the results.

A	B
(m_1, m_2, m_3)	$(m_1 + i, m_2, m_3)$
	$v(m_1, m_2 + i, m_3)$
	$v^2(m_1, m_2, m_3 + i)$
	$v^3(m_1, m_2, m_3, i)$

We can then move about the cube in the following manner.

Figure 2.5: $k > 0$



A → C: Use induction on j for $1 \leq j < e$.

C → D: Fix j and use induction on i for $1 \leq i < e - j$.

D → E.

E → F: Fix j and use induction on I for $1 \leq I < j$.

A	C	D	E
(m_1, m_2, m_3)	(m_1+j, m_2+j, m_3)	(m_1+i+j, m_2+j, m_3)	(m_1+e, m_2+j, m_3)
		$v(m_1+j, m_2+i+j, m_3)$	$v(m_1+j, m_2+e, m_3)$
	$v(m_1+j, m_2, m_3+j)$	$v(m_1+i+j, m_2, m_3+j)$	$v(m_1+e, m_2, m_3+j)$
		$v^2(m_1+j, m_2, m_3+i+j)$	
	$v^2(m_1+j, m_2, m_3, j)$	$v^2(m_1+i+j, m_2, m_3, j)$	$v^2(m_1+e, m_2, m_3, j)$
		$v^3(m_1+j, m_2, m_3, i+j)$	
	$v^2(m_1, m_2+j, m_3+j)$	$v^2(m_1, m_2+i+j, m_3+j)$	
		$v^3(m_1, m_2+j, m_3+i+j)$	$v^2(m_1, m_2+j, m_3+e)$
	$v^3(m_1, m_2+j, m_3, j)$	$v^3(m_1, m_2+i+j, m_3, j)$	
		$v^4(m_1, m_2+j, m_3, i+j)$	
	$v^4(m_1, m_2, m_3+j, j)$	$v^4(m_1, m_2, m_3+i+j, j)$	
		$v^5(m_1, m_2, m_3+j, i+j)$	$v^3(m_1, m_2, m_3+j, e)$

F	
(m_1+e+I, m_2+j, m_3)	$v^2(m_1+e+I, m_2, m_3, j)$
$v(m_1+e, m_2+j, m_3+I)$	$v^3(m_1+e, m_2+I, m_3, j)$
$v^2(m_1+e, m_2+j, m_3, I)$	$v^4(m_1+e, m_2, m_3+I, j)$
$v(m_1+j, m_2+e+I, m_3)$	$v^2(m_1+I, m_2+j, m_3+e)$
$v^2(m_1+j, m_2+e, m_3+I)$	$v^3(m_1, m_2+j, m_3+e+I)$
$v^3(m_1+j, m_2+e, m_3, I)$	$v^4(m_1, m_2+j, m_3+e, I)$
$v(m_1+e+I, m_2, m_3+j)$	$v^3(m_1+I, m_2, m_3+j, e)$
$v^2(m_1+e, m_2+I, m_3+j)$	$v^4(m_1, m_2+I, m_3+j, e)$
$v^3(m_1+e, m_2, m_3+j, I)$	$v^5(m_1, m_2, m_3+j, e+I)$

A → G: Use induction on k for $1 \leq k < e$.

G → H: Fix k and use induction on i for $1 \leq i < e - k$.

H → I.

I → J: Fix k and use induction on I for $1 \leq I < k$.

G	H	I
(m_1+k, m_2+k, m_3+k)	(m_1+i+k, m_2+k, m_3+k)	(m_1+e, m_2+k, m_3+k)
	$v(m_1+k, m_2+i+k, m_3+k)$	$v(m_1+k, m_2+e, m_3+k)$
	$v^2(m_1+k, m_2+k, m_3+i+k)$	$v^2(m_1+k, m_2+k, m_3+e)$
$v(m_1+k, m_2+k, m_3, k)$	$v(m_1+i+k, m_2+k, m_3, k)$	$v(m_1+e, m_2+k, m_3, k)$
	$v^2(m_1+k, m_2+i+k, m_3, k)$	$v^2(m_1+k, m_2+e, m_3, k)$
	$v^3(m_1+k, m_2+k, m_3, i+k)$	
$v^2(m_1+k, m_2, m_3+k, k)$	$v^2(m_1+i+k, m_2, m_3+k, k)$	$v^2(m_1+e, m_2, m_3+k, k)$
	$v^3(m_1+k, m_2, m_3+i+k, k)$	
	$v^4(m_1+k, m_2, m_3+k, i+k)$	$v^3(m_1+k, m_2, m_3+k, e)$
$v^3(m_1, m_2+k, m_3+k, k)$	$v^3(m_1, m_2+i+k, m_3+k, k)$	
	$v^4(m_1, m_2+k, m_3+i+k, k)$	$v^3(m_1, m_2+k, m_3+e, k)$
	$v^5(m_1, m_2+k, m_3+k, i+k)$	$v^4(m_1, m_2+k, m_3+k, e)$

J	
(m_1+e+I, m_2+k, m_3+k)	$v^3(m_1+k, m_2+e, m_3+I, k)$
$v(m_1+e, m_2+k, m_3+k, I)$	$v^2(m_1+e+I, m_2, m_3+k, k)$
$v(m_1+k, m_2+e+I, m_3+k)$	$v^3(m_1+e, m_2+I, m_3+k, k)$
$v^2(m_1+k, m_2+e, m_3+k, I)$	$v^3(m_1+k, m_2+I, m_3+k, e)$
$v^2(m_1+k, m_2+k, m_3+e+I)$	$v^4(m_1+k, m_2, m_3+k, e+I)$
$v^3(m_1+k, m_2+k, m_3+e, I)$	$v^3(m_1+I, m_2+k, m_3+e, k)$
$v(m_1+e+I, m_2+k, m_3, k)$	$v^4(m_1, m_2+k, m_3+e+I, k)$
$v^2(m_1+e, m_2+k, m_3+I, k)$	$v^4(m_1+I, m_2+k, m_3+k, e)$
$v^2(m_1+k, m_2+e+I, m_3, k)$	$v^5(m_1, m_2+k, m_3+k, e+I)$

G $\to$ K: Fix k and use induction on j for $1 \leq j < e-k$.

K $\to$ L.

L $\to$ M: Fix k and use induction on J for $1 \leq J < k$.

G	K	L
(m_1+k, m_2+k, m_3+k)	$(m_1+j+k, m_2+j+k, m_3+k)$	(m_1+e, m_2+e, m_3+k)
	$v(m_1+j+k, m_2+k, m_3+j+k)$	$v(m_1+e, m_2+k, m_3+e)$
	$v^2(m_1+k, m_2+j+k, m_3+j+k)$	$v^2(m_1+k, m_2+e, m_3+e)$
$v(m_1+k, m_1+k, m_3, k)$	$v(m_1+j+k, m_2+j+k, m_3, k)$	$v(m_1+e, m_2+e, m_3, k)$
	$v^2(m_1+j+k, m_2+k, m_3, j+k)$	
	$v^3(m_1+k, m_2+j+k, m_3, j+k)$	
$v^2(m_1+k, m_2, m_3+k, k)$	$v^2(m_1+j+k, m_2, m_3+j+k, k)$	
	$v^3(m_1+j+k, m_2, m_3+k, j+k)$	$v^2(m_1+e, m_2, m_3+k, e)$
	$v^4(m_1+k, m_2, m_3+j+k, j+k)$	
$v^3(m_1, m_2+k, m_3+k, k)$	$v^3(m_1, m_2+j+k, m_3+j+k, k)$	
	$v^4(m_1, m_2+j+k, m_3+k, j+k)$	
	$v^5(m_1, m_2+k, m_3+j+k, j+k)$	$v^3(m_1, m_2+k, m_3+e, e)$

M	
$(m_1+e+J, m_2+e+J, m_3+k)$	$v(m_1+e+J, m_2+e+J, m_3, k)$
$v(m_1+e+J, m_2+e, m_3+k, J)$	$v^2(m_1+e+J, m_2+e, m_3+J, k)$
$v^2(m_1+e, m_2+e+J, m_3+k, J)$	$v^3(m_1+e, m_2+e+J, m_3+J, k)$
$v(m_1+e+J, m_2+k, m_3+e+J)$	$v^2(m_1+e+J, m_2+J, m_3+k, e)$
$v^2(m_1+e+J, m_2+k, m_3+e, J)$	$v^3(m_1+e+J, m_2, m_3+k, e+J)$
$v^3(m_1+e, m_2+k, m_3+e+J, J)$	$v^4(m_1+e, m_2+J, m_3+k, e+J)$
$v^2(m_1+k, m_2+e+J, m_3+e+J)$	$v^3(m_1+J, m_2+k, m_3+e+J, e)$
$v^3(m_1+k, m_2+e+J, m_3+e, J)$	$v^4(m_1+J, m_2+k, m_3+e, e+J)$
$v^4(m_1+k, m_2+e, m_3+e+J, J)$	$v^5(m_1, m_2+k, m_3+e+J, e+J)$

K $\to$ N: Fix j and k such that $j+k < e$ and use induction on i for $1 \leq i < e-j-k$.

N $\to$ O.

O $\to$ P: Fix j and k such that $j+k < e$ and use induction on I for $1 \leq I < k$.

P$\to$ Q.

Q $\to$ R: Fix j and k such that $j+k < e$ and use induction on I for $k < I < j+k$.

N	O
$(m_1+i+j+k, m_2+j+k, m_3+k)$	(m_1+e, m_2+j+k, m_3+k)
$v(m_1+j+k, m_2+i+j+k, m_3+k)$	$v(m_1+j+k, m_2+e, m_3+k)$
$v(m_1+i+k+k, m_2+k, m_3+j+k)$	$v(m_1+e, m_2+k, m_3+j+k)$
$v^2(m_1+j+k, m_2+k, m_3+i+j+k)$	$v^2(m_1+j+k, m_2+k, m_3+e)$
$v^2(m_1+k, m_2+i+j+k, m_3+j+k)$	$v^2(m_1+k, m_2+e, m_3+j+k)$
$v^3(m_1+k, m_2+j+k, m_3+i+j+k)$	$v^3(m_1+k, m_2+j+k, m_3+e)$
$v(m_1+i+j+k, m_2+j+k, m_3, k)$	$v(m_1+e, m_2+j+k, m_3, k)$
$v^2(m_1+j+k, m_2+i+j+k, m_3, k)$	$v^2(m_1+j+k, m_2+e, m_3, k)$
$v^2(m_1+i+j+k, m_2+k, m_3, j+k)$	$v^2(m_1+e, m_2+k, m_3, j+k)$
$v^3(m_1+j+k, m_2+k, m_3, i+j+k)$	
$v^3(m_1+k, m_2+i+j+k, m_3, j+k)$	$v^3(m_1+k, m_2+e, m_3, j+k)$
$v^4(m_1+k, m_2+j+k, m_3, i+j+k)$	
$v^2(m_1+i+j+k, m_2, m_3+j+k, k)$	$v^2(m_1+e, m_2, m_3+j+k, k)$
$v^3(m_1+j+k, m_2, m_3+i+j+k, k)$	
$v^3(m_1+i+j+k, m_2, m_3+k, j+k)$	$v^3(m_1+e, m_2, m_3+k, j+k)$
$v^4(m_1+j+k, m_2, m_3+k, i+j+k)$	$v^3(m_1+j+k, m_2, m_3+k, e)$
$v^4(m_1+k, m_2, m_3+i+j+k, j+k)$	
$v^5(m_1+k, m_2, m_3+j+k, i+j+k)$	$v^4(m_1+k, m_2, m_3+j+k, e)$
$v^3(m_1, m_2+i+j+k, m_3+j+k, k)$	
$v^4(m_1, m_2+j+k, m_3+i+j+k, k)$	$v^3(m_1, m_2+j+k, m_3+e, k)$
$v^4(m_1, m_2+i+j+k, m_3+k, j+k)$	
$v^5(m_1, m_2+j+k, m_3+k, i+j+k)$	$v^4(m_1, m_2+j+k, m_3+k, e)$
$v^5(m_1, m_2+k, m_3+i+j+k, j+k)$	$v^4(m_1, m_2+k, m_3+e, j+k)$
$v^6(m_1, m_2+k, m_3+j+k, i+j+k)$	$v^5(m_1, m_2+k, m_3+j+k, e)$

P	
$(m_1+e+I, m_2+j+k, m_3+k)$	$v^3(m_1+k, m_2+e+I, m_3, j+k)$
$v(m_1+e, m_2+j+k, m_3+k, I)$	$v^4(m_1+k, m_2+e, m_3+I, j+k)$
$v(m_1+j+k, m_2+e+I, m_3+k)$	$v^2(m_1+e+I, m_2, m_3+j+k, k)$
$v^2(m_1+j+k, m_2+e, m_3+k, I)$	$v^3(m_1+e, m_2+I, m_3+j+k, k)$
$v(m_1+e+I, m_2+k, m_3+j+k)$	$v^3(m_1+e+I, m_2, m_3+k, j+k)$
$v^2(m_1+e, m_2+k, m_3+j+k, I)$	$v^4(m_1+e, m_2+I, m_3+k, j+k)$
$v^2(m_1+j+k, m_2+k, m_3+e+I)$	$v^3(m_1+j+k, m_2+I, m_3+k, e)$
$v^3(m_1+j+k, m_2+k, m_3+e, I)$	$v^4(m_1+j+k, m_2, m_3+k, e+I)$
$v^2(m_1+k, m_2+e+I, m_3+j+k)$	$v^4(m_1+k, m_2+I, m_3+j+k, e)$
$v^3(m_1+k, m_2+e, m_3+j+k, I)$	$v^5(m_1+k, m_2, m_3+j+k, e+I)$
$v^3(m_1+k, m_2+j+k, m_3+e+I)$	$v^3(m_1+I, m_2+j+k, m_3+e, k)$
$v^4(m_1+k, m_2+j+k, m_3+e, I)$	$v^4(m_1, m_2+j+k, m_3+e+I, k)$
$v(m_1+e+I, m_2+j+k, m_3, k)$	$v^4(m_1+I, m_2+j+k, m_3+k, e)$
$v^2(m_1+e, m_2+j+k, m_3+I, k)$	$v^5(m_1, m_2+j+k, m_3+k, e+I)$
$v^2(m_1+j+k, m_2+e+I, m_3, k)$	$v^4(m_1+I, m_2+k, m_3+e, j+k)$
$v^3(m_1+j+k, m_2+e, m_3+I, k)$	$v^5(m_1, m_2+k, m_3+e+I, j+k)$
$v^2(m_1+e+I, m_2+k, m_3, j+k)$	$v^5(m_1+I, m_2+k, m_3+j+k, e)$
$v^3(m_1+e, m_2+k, m_3+I, j+k)$	$v^6(m_1, m_2+k, m_3+j+k, e+I)$

The case Q is more interesting because it is necessary to subtract off terms. Hence the first column of the following table gives the approximation, C_λ, while the second gives those terms which are subtracted. The third column gives $B_4'(\lambda)$ as required.

Q

$\mathbf{C}_\lambda$	$\mathbf{B'_4(m_1+e, m_2+j+k, m_3+k, k)}$	$\mathbf{B'_4}(\lambda)$
$(m_1+e+k, m_2+j+k, m_3+k)$		$(m_1+e+k, m_2+j+k, m_3+k)$
$(m_1+e, m_2+j+k, m_3+k, k)$	$(m_1+e, m_2+j+k, m_3+k, k)$	
$v(m_1+j+k, m_2+e+k, m_3+k)$		$v(m_1+j+k, m_2+e+k, m_3+k)$
$v(m_1+j+k, m_2+e, m_3+k, k)$	$v(m_1+j+k, m_2+e, m_3+k, k)$	
$v(m_1+e+k, m_2+k, m_3+j+k)$		$v(m_1+e+k, m_2+k, m_3+j+k)$
$v(m_1+e, m_2+k, m_3+j+k, k)$	$v(m_1+e, m_2+k, m_3+j+k, k)$	
$v^2(m_1+j+k, m_2+k, m_3+e, k)$	$v^2(m_1+j+k, m_2+k, m_3+e, k)$	
$v^2(m_1+k, m_2+e, m_3+j+k, k)$	$v^2(m_1+k, m_2+e, m_3+j+k, k)$	
$v^2(m_1+k, m_2+j+k, m_3+e+k)$		$v^2(m_1+k, m_2+j+k, m_3+e+k)$
$v^3(m_1+k, m_2+j+k, m_3+e, k)$	$v^3(m_1+k, m_2+j+k, m_3+e, k)$	
$v(m_1+e+k, m_2+j+k, m_3, k)$		$v(m_1+e+k, m_2+j+k, m_3, k)$
$v^2(m_1+e, m_2+j+k, m_3+k, k)$		$v^2(m_1+e, m_2+j+k, m_3+k, k)$
$v^2(m_1+j+k, m_2+e+k, m_3, k)$		$v^2(m_1+j+k, m_2+e+k, m_3, k)$
$v^3(m_1+j+k, m_2+e, m_3+k, k)$		$v^3(m_1+j+k, m_2+e, m_3+k, k)$
$v^2(m_1+e+k, m_2+k, m_3, j+k)$		$v^2(m_1+e+k, m_2+k, m_3, j+k)$
$v^2(m_1+e, m_2+k, m_3+k, j+k)$	$v^2(m_1+e, m_2+k, m_3+k, j+k)$	
$v^3(m_1+k, m_2+e, m_3+k, j+k)$	$v^3(m_1+k, m_2+e, m_3+k, j+k)$	
$v^2(m_1+e+k, m_2, m_3+j+k, k)$		$v^2(m_1+e+k, m_2, m_3+j+k, k)$
$v^3(m_1+e, m_2+k, m_3+j+k, k)$		$v^3(m_1+e, m_2+k, m_3+j+k, k)$
$v^3(m_1+e+k, m_2, m_3+k, j+k)$		$v^3(m_1+e+k, m_2, m_3+k, j+k)$
$v^4(m_1+e, m_2+k, m_3+k, j+k)$		$v^4(m_1+e, m_2+k, m_3+k, j+k)$
$v^3(m_1+j+k, m_2+k, m_3+k, e)$	$v^3(m_1+j+k, m_2+k, m_3+k, e)$	
$v^3(m_1+k, m_2+k, m_3+j+k, e)$		$v^3(m_1+k, m_2+k, m_3+j+k, e)$
$v^4(m_1+k, m_2, m_3+j+k, e+k)$		$v^4(m_1+k, m_2, m_3+j+k, e+k)$
$v^3(m_1+k, m_2+j+k, m_3+e, k)$		$v^3(m_1+k, m_2+j+k, m_3+e, k)$
$v^4(m_1, m_2+j+k, m_3+e+k, k)$		$v^4(m_1, m_2+j+k, m_3+e+k, k)$
$v^4(m_1+k, m_2+j+k, m_3+k, e)$	$v^4(m_1+k, m_2+j+k, m_3+k, e)$	
$v^4(m_1+k, m_2+k, m_3+e, j+k)$	$v^4(m_1+k, m_2+k, m_3+e, j+k)$	
$v^5(m_1+k, m_2+k, m_3+j+k, e)$	$v^5(m_1+k, m_2+k, m_3+j+k, e)$	
$v^5(m_1, m_2+k, m_3+j+k, e+k)$		$v^5(m_1, m_2+k, m_3+j+k, e+k)$

Above, we needed to find $B'_4(m_1+e, m_2+j+k, m_3+k, k)$. By the column addition theorem it is sufficient to consider $\nu = (m_1+e-k, m_2+j, m_3)$. By rearranging this into the form $(m'_1+i'+j'+k', m'_2+j'+k', m'_3+k')$ (where the primed letters correspond in an obvious way to the usual notation) we get that

$$\begin{aligned} k' &= 0 & m'_1 &= m_1 \\ j' &= j & m'_2 &= m_2 \\ i' &= e-j-k & m'_3 &= m_3 \end{aligned}$$

In other words, this is just situation D, with $a' = b' = c' = 1$ as considered before. Thus $B'_4(m'_1+i'+j', m'_2+j', m'_3)$ is given by

$(m_1'+i'+j', m_2'+j', m_3')$	that is	(m_1+e-k, m_2+j, m_3)
$v(m_1'+j', m_2'+i'+j', m_3')$		$v(m_1+j, m_2+e-k, m_3)$
$v(m_1'+i'+j', m_2', m_3'+j')$		$v(m_1+e-k, m_2, m_3+j)$
$v^2(m_1'+j', m_2', m_3'+i'+j')$		$v^2(m_1+j, m_2, m_3+e-k)$
$v^2(m_1'+i'+j', m_2', m_3', j')$		$v^2(m_1+e-k, m_2, m_3, j)$
$v^3(m_1'+j', m_2', m_3', i'+j')$		$v^3(m_1+j, m_2, m_3, e-k)$
$v^2(m_1', m_2'+i'+j', m_3'+j')$		$v^2(m_1, m_2+e-k, m_3+j)$
$v^3(m_1', m_2'+j', m_3'+i'+j')$		$v^3(m_1, m_2+j, m_3+e-k)$
$v^3(m_1', m_2'+i'+j', m_3', j')$		$v^3(m_1, m_2+e-k, m_3, j)$
$v^4(m_1', m_2'+j', m_3', i'+j')$		$v^4(m_1, m_2+j, m_3, e-k)$
$v^4(m_1', m_2', m_3'+i'+j', j')$		$v^4(m_1, m_2, m_3+e-k, j)$
$v^5(m_1', m_2', m_3'+j', i'+j')$		$v^5(m_2, m_2, m_3+j, e-k)$

By adding k columns of length 4, we find that $B_4'(m_1+e, m_2+j+k, m_3+k, k)$ is given by

$(m_1+e, m_2+j+k, m_3+k, k)$	$v^2(m_1+k, m_2+e, m_3+j+k, k)$
$v(m_1+j+k, m_2+e, m_3+k, k)$	$v^3(m_1+k, m_2+j+k, m_3+e, k)$
$v(m_1+e, m_2+k, m_3+j+k, k)$	$v^3(m_1+k, m_2+e, m_3+k, j+k)$
$v^2(m_1+j+k, m_2+k, m_3+e, k)$	$v^4(m_1+k, m_2+j+k, m_3+k, e)$
$v^2(m_1+e, m_2+k, m_3+k, j+k)$	$v^4(m_1+k, m_2+k, m_3+e, j+k)$
$v^3(m_1+j+k, m_2+k, m_3+k, e)$	$v^5(m_2+k, m_2+k, m_3+j+k, e)$

Case R is then straightforward and has not been written out.

L $\to$ S: Fix j and use induction on I for $1 \le I < e-j$.

S $\to$ T.

T $\to$ U: Fix j and use induction on I for $e-j < I < e-1$.

L	S	T
(m_1+e, m_2+e, m_3+k)	(m_1+e+I, m_2+e, m_3+k)	(m_1+e+k, m_2+e, m_3+k)
	$v(m_1+e, m_2+e+I, m_3+k)$	$v(m_1+e, m_2+e+k, m_3+k)$
	$v^2(m_1+e, m_2+e, m_3+k, I)$	$v(m_1+e, m_2+e, m_3+k, k)$
$v(m_1+e, m_2+k, m_3+e)$	$v(m_1+e+I, m_2+k, m_3+e)$	$v(m_1+e+k, m_2+k, m_3+e)$
	$v^2(m_1+e, m_2+k, m_3+e+I)$	
	$v^3(m_1+e, m_2+k, m_3+e, I)$	$v^2(m_1+e, m_2+k, m_3+e, k)$
$v^2(m_1+k, m_2+e, m_3+e)$	$v^2(m_1+k, m_2+e+I, m_3+e)$	
	$v^3(m_1+k, m_2+e, m_3+e+I)$	$v^2(m_1+k, m_2+e, m_3+e+k)$
	$v^4(m_1+k, m_2+e, m_3+e, I)$	$v^3(m_1+k, m_2+e, m_3+e, k)$
$v(m_1+e, m_2+e, m_3, k)$	$v(m_1+e+I, m_2+e, m_3, k)$	$v(m_1+e+k, m_2+e, m_3, k)$
	$v^2(m_1+e, m_2+e+I, m_3, k)$	$v^2(m_1+e, m_2+e+k, m_3, k)$
	$v^3(m_1+e, m_2+e, m_3+I, k)$	$v^3(m_1+e, m_2+e, m_3+k, k)$
$v^2(m_1+e, m_2, m_3+k, e)$	$v^2(m_1+e+I, m_2, m_3+k, e)$	$v^2(m_1+e+k, m_2, m_3+k, e)$
	$v^3(m_1+e, m_2+I, m_3+k, e)$	$v^3(m_1+e, m_2+k, m_3+k, e)$
	$v^4(m_1+e, m_2, m_3+k, e+I)$	
$v^3(m_1, m_2+k, m_3+e, e)$	$v^3(m_1+I, m_2+k, m_3+e, e)$	$v^3(m_1+k, m_2+k, m_3+e, e)$
	$v^4(m_1, m_2+k, m_3+e+I, e)$	
	$v^5(m_1, m_2+k, m_3+e, e+I)$	$v^4(m_1, m_2+k, m_3+e, e+k)$

Note that (m_1+e, m_2+e, m_3+k, k) occurs with multiplicity $v+v^3$. All other composition factors so far have occured with multiplicity 1; indeed in Appendix A.2 they are arranged in length–lexicographic order. Case U is straightforward and has not been written out.

M $\to$ V: Fix j and k with $j+k>e$ and use induction on I for $j+k-e+1 \leq I < k$.
V $\to$ W.
W $\to$ X: Fix j and k with $j+k>e$ and use induction on I for $k<I<e$.
X $\to$ Y.
Y$\to$ Z: Fix j and k with $j+k>e$ and use induction on H for $1 \leq H < j+k-e$.

There is little point in writing out these cases; they are all straightforward except for case Y, when the first approximation gives $(m_1+e+J, m_2+e, m_3+e, k)$ with multiplicity 1. Hence it is necessary to consider $B_4'(m_1+j, m_2+e-k, m_3+e-k)$. This turns out to be an example of case H, where $a'=b'=c'=1$. This method used is then similar to that discussed in case Q.

$\square$

Remark 2.4.4. The case $a=b=c=1$ was self contained; the only partitions whose coefficient did not lie in $v\mathbb{N}[v]$ were such that the corresponding a', b', c' satisfied $a'=b'=c'=1$. This is not usually true.

Corollary 2.4.5. *Although it would be preferable to be able to describe the composition factors of a Specht module corresponding to a partition with at most 4 parts, extracting this information by 'inverting' Conjecture 2.4.2 is far from trivial. However, we can give two easy corollaries:*

1. *If* $e>2$*, the maximum multiplicity of a composition factor* D^λ *in* S^μ *is 4, and this multiplicity is achieved. If* $e=2$ *then the maximum multiplicity is 2 [21].*
2. *Let* $e>2$*. Then for any integer d, there exists a Specht module* $S^\mu = S^{(\mu_1,\mu_2,\mu_3,\mu_4)}$ *such that* S^μ *has at least d distinct composition factors. (Note that if* $e=2$ *this statement does not hold. James and Mathas [21] have shown that in this case* S^μ *has at most 18 composition factors, corresponding to 16 different irreducible modules.)*

Proof.

1. Let $e>2$ and consider case Y where $a=0$ and $b,c>2$. Then in the crystallized decomposition matrix

$$[S^{(m_1+e+J, m_2+k, m_3+e, e)} : D^{(m_1+2e, m_2+e+J, m_3+k)}]$$

$$= 3v^2 + v^4.$$

Thus the composition factor occurs with multiplicity 4. Inspection of all other cases shows that no composition factor occurs with higher multiplicity.

2. Consider $S^{((d+3)e-1,(d+3)e-1,de,de)}$ and look at case H for $a = c = 0$. If $\lambda = (m_1 + i + j + k, m_2 + j + k, m_3 + k)$ satisfies

$$\begin{aligned} a &= c = 0, \\ j &= 0, \\ i + k &< e \\ b &= 2x + y + 3 \end{aligned}$$

(all terms as defined in Appendix A.1) then it is a composition factor of $S^{(m_1-(z+1)e+i+k,m_2-(z+1)e+k,m_3+(z+1)e+k,(z+1)e)}$ for $0 \leq z < x$. Hence for $0 \leq r < d$ we have that

$$D^{((2d-2r+3)e-1,(2d-2r+3)e-1)}$$

is a composition factor of

$$S^{((d-r+3)e-1,(d-r+3)e-1,(d-r)e,(d-r)e)}.$$

Hence

$$S^{((d+3)e-1,(d+3)e-1,de,de)}$$

has composition factors

$$D^{((2d-r+3)e-1,(2d-r+3)e-1,re,re)}$$

for $0 \leq r < d$.

□

Remark 2.4.6. Consider the case of $B'_s(\lambda)$ for $s > 4$. Observation of e-cores and Carter's criterion as before will give that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{s-1})$ where $\lambda_i - \lambda_{i-1} \equiv e - 1$ for $i = 2, \ldots, s-1$ and $\lambda_{s-1} \equiv e - 1$ then $B'_s(\lambda) = \lambda$. Theoretically, we could use this jumping off point to in order to find $B'_s(\lambda')$ for general λ'. In practice, the sheer volume of cases for $s = 4$, and the fact that the number of cases appears to be increasing exponentially means that this is not feasible.

3 The symmetric groups

3.1 Adjustment matrices

Let $\mathcal{H} = \mathcal{H}_{F,q}(\mathfrak{S}_n)$ where char $F = p > 0$. Define e to be the least positive integer such that $1 + q + q^2 + \ldots + q^{e-1} = 0$ with $e = \infty$ if no such integer exists. In the latter case, the decomposition matrix is simply the identity matrix. Hence, take e to be finite and consider $\mathcal{H}_0 = \mathcal{H}_{\mathbb{C},\omega}(\mathfrak{S}_n)$ where ω is a primitive e^{th} root of unity in $\mathbb{C}$. Using the LLT algorithm, we can compute the decomposition matrix of $\mathcal{H}_0$. Let D_0 be the decomposition matrix of $\mathcal{H}_0$ and D the decomposition matrix of $\mathcal{H}$ (which has the same structure), where we assume that the columns and rows are arranged according to the partial order. It can be shown that

$$D = D_0 A$$

where A is a suitably sized lower unitriangular matrix whose entries are non-negative integers [26]. Then A is known as the adjustment matrix. For any e-regular partition μ, suppose $B(\mu)$ is the column of D indexed by μ and $B_0(\mu)$ is the column of D_0 indexed by μ. Then each e-regular partition λ satisfies

$$B(\lambda) = B_0(\lambda) + \sum_{\lambda \rhd \nu} a_{\nu\lambda} B_0(\nu)$$

for coefficients $a_{\nu\lambda} \in \mathbb{N}$.

Example 3.1.1. Let $p = e = 2$ and $n = 7$. Then the equation $D_0 A = D$ is represented by the matrices in Figure 3.1.

In general the adjustment matrices are not known. However if part of the decomposition matrix of $\mathcal{H}$ is known, we can use it to find part of the adjustment matrix, and hence obtain a closer approximation to D. In particular, the composition factors of any Specht module corresponding to a one or two part partition are always known [17].

Example 3.1.2. Let $p = e = 3$ and $n = 13$ and consider the parts of D and D_0 corresponding to one or two part partitions. The entries here are known and

Figure 3.1: $n = 7$ and $p = e = 2$.

$$
\begin{array}{l}
(7)\\ (6,1)\\ (5,2)\\ (4,3)\\ (4,2,1)\\ (5,1^2)\\ (3^2,1)\\ (3,2^2)\\ (4,1^3)\\ (3,2,1^2)\\ (2^3,1)\\ (3,1^4)\\ (2^2,1^3)\\ (2,1^5)\\ (1^7)
\end{array}
\begin{pmatrix}
1 & & & & \\
 & 1 & & & \\
 & & 1 & & \\
 & & & 1 & \\
 & & 1 & & 1\\
1 & & 1 & & \\
 & & & & 1\\
 & & & & 1\\
 & 1 & & 1 & \\
 & & 1 & & 1\\
 & & & 1 & \\
1 & & 1 & & \\
 & & 1 & & \\
 & 1 & & & \\
1 & & & &
\end{pmatrix}
\begin{pmatrix}
1 & & & & \\
 & 1 & & & \\
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & & & 1
\end{pmatrix}
=
\begin{pmatrix}
1 & & & & \\
 & 1 & & & \\
 & & 1 & & \\
 & 1 & & 1 & \\
1 & & 1 & & 1\\
1 & & 1 & & \\
1 & & & & 1\\
1 & & & & 1\\
 & 2 & & 1 & \\
1 & & 1 & & 1\\
 & 1 & & 1 & \\
1 & & 1 & & \\
 & & 1 & & \\
 & 1 & & & \\
1 & & & &
\end{pmatrix}
$$

hence we can find the top of the adjustment matrix A as in Figure 3.2. The first approximation to part of D is found by the LLT algorithm, and is given by matrix D_0 in Figure 3.3. The second approximation is obtained by multiplying D_0 by the part of the adjustment matrix that we discovered in Figure 3.2 to get the matrix D_1 in Figure 3.4.

For most of this chapter, we shall ignore the adjustment matrices and use only the following fact:

$$[S^\mu : D^\lambda]_{\mathcal{H}} \geq [S^\mu : D^\lambda]_{\mathcal{H}_0}$$

or, to use a weaker statement:

Corollary 3.1.3. *If the $\mathcal{H}_0$-module S^μ has a composition factor D^λ then the $\mathcal{H}$-module S^μ has a composition factor D^λ.*

3.2 The symmetric groups

In particular, we consider $q = 1$. Then $\mathcal{H} \cong F\mathfrak{S}_n$, the symmetric group algebra, and $e = \text{char } F = p$. For the rest of this chapter, let F be a field of prime characteristic e[1] and let $q = 1$. Unless otherwise stated, we will be working over a field of characteristic e; and we will write $a = 0$ to mean that it is zero as an element of this field. By previous work we know the structure of the decomposition matrices for $F\mathfrak{S}_n$ and have a lower bound of 1 for many of their entries.

[1]We keep the notation e rather than use p to represent the characteristic of the field both for consistency with the previous section and to tie in with future work on $GL_n(q)$.

Figure 3.2: $n = 13$ and $p = e = 3$.

$$D_0 = \begin{array}{l} (13) \\ (12,1) \\ (11,2) \\ (10,3) \\ (9,4) \\ (8,5) \\ (7,6) \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ 1 & & 1 & & & & & \\ & & 1 & 1 & & & & \\ & & & & 1 & & & \\ & & & 1 & & 1 & & \\ & & & & & 1 & 1 & \\ \vdots & & & & & & & \ddots \end{pmatrix} \qquad D = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ 1 & & 1 & & & & & \\ 1 & & 1 & 1 & & & & \\ & 1 & & & 1 & & & \\ 1 & & & 1 & & 1 & & \\ & & & & & 1 & 1 & \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ 1 & & & 1 & & & & \\ & 1 & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

Figure 3.3: $n = 13$ and $p = e = 3$.

$$D_0 = \begin{array}{l} (13) \\ (12,1) \\ (11,2) \\ (10,3) \\ (9,4) \\ (8,5) \\ (7,6) \\ (11,1^2) \\ (10,2,1) \\ (9,3,1) \\ (8,4,1) \\ (7,5,1) \\ (6^2,1) \\ (9,2^2) \\ (8,3,2) \\ (7,4,2) \\ (6,5,2) \\ (7,3^3) \\ (6,4,3) \\ (5^2,3) \\ (5,4^2) \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ 1 & & 1 & & & & & \\ & & 1 & 1 & & & & \\ & & & & 1 & & & \\ & & & 1 & & 1 & & \\ & & & & & 1 & 1 & \\ & & & & & & & \ddots \\ & & & & & & & \\ 1 & & 1 & 1 & & & & \\ & & & & & & & \\ & & & 1 & & 1 & 1 & \\ & & & & & & & \\ & 1 & & & 1 & & & \\ & & & & & & & \\ & & & & 1 & & & \\ & & & & & & & \\ & & & 1 & & & & \\ & & & & & & & \\ & & & & & & 1 & \\ & & & & & & & \\ \vdots & & & & & & & \ddots \end{pmatrix}$$

Figure 3.4: $n = 13$ and $p = e = 3$.

$$
D_1 = \begin{array}{l}
(13) \\
(12,1) \\
(11,2) \\
(10,3) \\
(9,4) \\
(8,5) \\
(7,6) \\
(11,1^2) \\
(10,2,1) \\
(9,3,1) \\
(8,4,1) \\
(7,5,1) \\
(6^2,1) \\
(9,2^2) \\
(8,3,2) \\
(7,4,2) \\
(6,5,2) \\
(7,3^3) \\
(6,4,3) \\
(5^2,3) \\
(5,4^2) \\

\end{array}
\begin{pmatrix}
1 & & & & & & & \\
 & 1 & & & & & & \\
1 & & 1 & & & & & \\
1 & & 1 & 1 & & & & \\
 & 1 & & & 1 & & & \\
1 & & & 1 & & 1 & & \\
 & & & & & 1 & 1 & \\
 & & & & & & & \ddots \\
 & & & & & & & \\
2 & & 1 & 1 & & & & \\
 & & & & & & & \\
1 & & & 1 & & 1 & 1 & \\
 & & & & & & & \\
 & 2 & & & 1 & & & \\
 & & & & & & & \\
 & 1 & & & 1 & & & \\
 & & & & & & & \\
1 & & & 1 & & & & \\
 & & & & & & & \\
 & & & & & & 1 & \\
 & & & & & & & \\
\vdots & & & & & & & \ddots
\end{pmatrix}
$$

The main reference for this section is [17].

Definition 3.2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition of n. (Write $\lambda \models n$.) Recall that the diagram of λ is given by

$$[\lambda] = \{(i, j) \mid 1 \leq i \text{ and } 1 \leq j \leq \lambda_i\}.$$

Then a λ-tableau is defined to be a bijection from $[\lambda]$ to $\{1, \ldots, n\}$ (or occasionally some other specified set of size n), which will be regarded as '$[\lambda]$ with the nodes replaced by the elements of $\{1, \ldots, n\}$'. A row equivalence relation $\sim_{\text{row}}$ is defined on the set of λ-tableaux in an obvious way, and column equivalence $\sim_{\text{col}}$ is defined similarly. An equivalence class of λ-tableaux under $\sim_{\text{row}}$ is called a λ-tabloid, and we write $\{t\}$ to be the tabloid containing t, denoted with lines between the rows. Hence if

$$t = \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{array}$$

we write the corresponding tabloid as

$$\{t\} = \begin{array}{c} \hline 12 \\ \hline 34 \\ \hline 56 \\ \hline \end{array} = \begin{array}{c} \hline 21 \\ \hline 34 \\ \hline 65 \\ \hline \end{array}.$$

The set of tableaux and the set of tabloids are acted upon in a natural way by $\mathfrak{S}_n$, and we define M^λ to be the $F\mathfrak{S}_n$-module whose basis elements, as a vector space, are given by the set of λ-tabloids, with the natural action

$$\{t\}\pi = \{t\pi\}$$

for $\{t\}$ a λ-tabloid and $\pi \in \mathfrak{S}_n$. For a given tableau t, define its column stabiliser C_t by

$$C_t = \{\pi \in \mathfrak{S}_n \mid t\pi \sim_{\text{col}} t\}.$$

We define a bilinear form $\langle\ ,\ \rangle$ on M^λ by setting

$$\langle \{t_i\}, \{t_j\} \rangle = \begin{cases} 1 & \{t_i\} = \{t_j\} \\ 0 & \text{otherwise} \end{cases}$$

and extending bilinearly.

Definition 3.2.2. For a tableau t, define the corresponding polytabloid e_t by

$$e_t = \sum_{\pi \in C_t} \epsilon(\pi)\{t\pi\}.$$

Now define the Specht module S^λ to be the $F\mathfrak{S}_n$-module spanned by the set of polytabloids. In particular, S^λ is cyclic as an $F\mathfrak{S}_n$-module, and is generated by e_t for any tableau t. Define

$$D^\lambda = \frac{S^\lambda}{S^\lambda \cap S^{\lambda^\perp}}.$$

Note that any maps that we consider in this chapter will be $F\mathfrak{S}_n$-homomorphisms.

Remarks 3.2.3. Some useful facts about the above modules are given by the following statements.

- Submodule Theorem [11]:

  Suppose $M \subseteq M^\lambda$ is an F$\mathfrak{S}_n$-module. Then $M \supseteq S^\lambda$ or $M \subseteq S^{\lambda^\perp}$.

- Quotient Theorem [17, 1.4]:

  $M^\lambda/S^{\lambda^\perp} \cong (S^\lambda)^*$ where, for a module M, M^* represents the dual of M.

- M^μ has a composition factor D^λ only if $\lambda \trianglerighteq \mu$ [17, Lemma 12.1] and if $\lambda \ntrianglerighteq \mu$ then $\mathrm{Hom}(S^\lambda, M^\mu) = 0$ [17, Lemma 13.15].

Corollary 3.2.4. *Let $\lambda, \mu \vdash n$. Then*

1. *Suppose we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\ker\Theta \not\supseteq S^\lambda$ and $\mathrm{Im}\,\Theta \subseteq S^\mu$. Then every composition factor of S^λ is a composition factor of S^μ.*

2. *Suppose λ is e-regular and we can find a map $0 \neq \Theta : S^\lambda \to M^\mu$ such that $\mathrm{Im}\,\Theta \subseteq S^\mu$. Then D^λ is a composition factor of S^μ.*

Proof.

1. Suppose there exists $\Theta : M^\lambda \to M^\mu$ such that $\ker\Theta \not\supseteq S^\lambda$ and $\mathrm{Im}\,\Theta \subseteq S^\mu$. Then $\ker\Theta \subseteq S^{\lambda^\perp}$. Since every composition factor of S^λ is a composition factor of $M^\lambda/S^{\lambda^\perp}$, we therefore have that every composition factor of S^λ is a composition factor of $M/\ker\Theta$, and so every composition factor of S^λ is a composition factor of $\mathrm{Im}\,\Theta$.

   Thus every composition factor of S^λ is a composition factor of S^μ.

2. Suppose that λ is e-regular and there exists $0 \neq \Theta : S^\lambda \to M^\mu$ such that $\operatorname{Im}\Theta \subseteq S^\mu$. Then $\ker\Theta \subseteq S^\lambda \cap S^{\lambda^\perp}$. Every composition factor of $S^\lambda/\ker\Theta$ is a composition factor of $\operatorname{Im}\Theta$ and hence every composition factor of $S^\lambda/\ker\Theta$ is a composition factor of S^μ. But $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda^\perp})$ is a composition factor of $S^\lambda/\ker\Theta$.

   Thus D^λ is a composition factor of S^μ.

$\square$

Definition 3.2.5. We now extend our definition of tableaux, allowing them to have repeated entries. Let $\lambda, \mu \vdash n$. Say that a λ-tableau T has type μ if for every i, the number i occurs μ_i times in T. Set

$$\mathcal{T}(\lambda,\mu) = \{T \mid T \text{ is a } \lambda\text{-tableau of type } \mu\}.$$

Say that T is semistandard if its entries are increasing (non–decreasing) along its rows and strictly increasing down its columns. The relations $\sim_{\text{row}}$ and $\sim_{\text{col}}$ are defined in the obvious way on $\mathcal{T}(\lambda,\mu)$.

Let t be a fixed λ-tableau (of type (1^n)). Given a λ-tableau S of type μ, we define a μ-tabloid $d_t(S)$ by putting $i \in \{1,\dots,n\}$ in row j, where i and j occupy corresponding positions in t and S.

Definition 3.2.6. Now, for each $T \in \mathcal{T}(\lambda,\mu)$, we define a homomorphism

$$\Theta_T : M^\lambda \longrightarrow M^\mu$$

by stipulating that

$$\{t\} \longmapsto \sum_{S \sim_{\text{row}} T} d_t(S)$$

and extending to be a homomorphism. (Note that Θ_T is independent of the choice of t.)

Example 3.2.7. Let $\lambda = (4,2)$ and $\mu = (2^3)$ and let

$$T = \begin{matrix} 1\,1\,2\,2 \\ 3\,3 \end{matrix}\ .$$

Then Θ_T is defined by

$$\Theta_T : \frac{\overline{1\,2\,3\,4}}{\underline{5\,6}} \longrightarrow \frac{\overline{1\,2}}{\frac{3\,4}{\underline{5\,6}}} + \frac{\overline{1\,3}}{\frac{2\,4}{\underline{5\,6}}} + \frac{\overline{1\,4}}{\frac{2\,3}{\underline{5\,6}}} + \frac{\overline{2\,3}}{\frac{1\,4}{\underline{5\,6}}} + \frac{\overline{2\,4}}{\frac{1\,3}{\underline{5\,6}}} + \frac{\overline{3\,4}}{\frac{1\,2}{\underline{5\,6}}}.$$

A particular set of such homomorphisms gives us a new and useful definition of the Specht module. Let d be a positive integer, and choose t such that $0 \leq t < \lambda_{d+1}$. Define $\nu = (\nu_1, \nu_2, \ldots) \models n$ by

$$\nu_i = \begin{cases} \lambda_i + \lambda_{i+1} - t & i = d \\ t & i = d+1 \\ \lambda_i & \text{otherwise} \end{cases}.$$

Let T be the λ-tableau with all entries in row i equal to i, except for $i = d+1$, when there are $\lambda_{d+1} - t$ entries equal to d and t entries equal to $d+1$. Then T is a λ-tableau of type ν, and we write $\psi_{d,t}$ for the homomorphism $\Theta_T : M^\lambda \to M^\nu$. We then have the following.

Theorem 3.2.8 ([17], Corollary 17.18). *For $\lambda \vdash n$*

$$S^\lambda = \bigcap_{d \geq 1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker(\psi_{d,t}).$$

Now let $\hat{\Theta}_T : S^\lambda \to M^\mu$ be the restriction of Θ_T to S^λ and let $\mathcal{T}_0(\lambda, \mu)$ denote the set of semistandard λ-tableaux of type μ. Then unless $e = 2$ and λ is 2-singular we have the following, originally due to Carter and Lusztig [3].

Theorem 3.2.9 ([17], Theorem 13.13). *If λ is a partition of n and μ a composition of n, then the set $\{\hat{\Theta}_T \mid T \in \mathcal{T}_0(\lambda, \mu)\}$ is a basis for $\mathrm{Hom}_{k\mathfrak{S}_n}(S^\lambda, M^\mu)$.*

Furthermore, unless $e = 2$ and λ is e-singular, every homomorphism from S^λ to M^μ can be extended to a homomorphism from M^λ to M^μ.

We call this basis the set of semistandard homomorphisms. Note that if T_1 and T_2 are row equivalent then $\Theta_{T_1} = \Theta_{T_2}$.

We now consider some notation. Suppose that $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2, \mu_3)$, and that T is a λ-tableau of type μ such that T contains a_i of the number i in row 1 and b_i of the number i in row 2, for $i = 1, 2, 3$. Write the map Θ_T as

$$\begin{matrix} 1^{a_1}\, 2^{a_2}\, 3^{a_3} \\ 1^{b_1}\, 2^{b_2}\, 3^{b_3} \end{matrix}.$$

Similar notation will also be used to denote the tableau T. Note that the order of the elements in each row does not affect the homomorphism, but does affect the tableau. Hence if

$$T = \begin{array}{l} 1\,1\,1\,2\,2\,1\,3\,3 \\ 1\,2\,2\,2\,3\,3\,1 \end{array}$$

we would write

$$T = \begin{array}{l} 1^3\,2^2\,1\,3^2 \\ 1\,2^3\,3^2\,1 \end{array}$$

and

$$\Theta_T = \begin{array}{l} 1^3\,2^2\,1\,3^2 \\ 1\,2^3\,3^2\,1 \end{array} = \begin{array}{l} 1^4\,2^2\,3^2 \\ 1^2\,2^3\,3^2 \end{array}.$$

Lemma 3.2.10. *Let T be a λ-tableau of type μ and fix a λ-tableau t. Let s be a μ-tableau so that $\{s\} = d_t(S)$ for some (unique) λ-tableau S of type μ. Then the coefficient of $\{s\}$ in $e_t\Theta_T$ is given by*

$$\sum_{\pi \in C_t} \Big\{ \epsilon(\pi) \mid d_t(S)\pi = d_t(R) \text{ where } R \sim_{row} T \Big\}.$$

Proof. Suppose we have the conditions of Lemma 3.2.10. Note that

$$e_t(\Theta_T) = \{t\}\Theta_T \left(\sum_{\pi \in C_t} \epsilon(\pi)\pi \right)$$

where

$$\{t\}\Theta_T = \sum_{R \sim_{\text{row}} T} d_t(R).$$

□

We observe that

$$\pi \in C_t \text{ implies that } d_t(S)\pi = d_t(R) \text{ where } R \sim_{\text{col}} S.$$

Remark 3.2.11. Suppose there exists $0 \neq \hat{\Theta} : S^\lambda \longrightarrow M^\mu$ such that $\operatorname{Im} \hat{\Theta} \subseteq S^\mu$. Then $S^\mu = \bigcap_{d,t} \ker \psi_{d,t}$ where $\operatorname{Im} \psi_{d,t} \subseteq M^\nu$, say. This gives the diagram

$$S^\lambda \xrightarrow{\hat{\Theta}} M^\mu \xrightarrow{\psi_{d,t}} M^\nu.$$

Consider the map $\hat{\Theta}\psi_{d,t} \in \mathrm{Hom}(S^\lambda, M^\nu)$. It can be uniquely expressed as a sum of semistandard homomorphisms, and $\mathrm{Im}\,\hat{\Theta} \subseteq \ker \psi_{d,t}$ if and only if $\hat{\Theta}\psi_{d,t} = 0$, that is, each of the coefficients in this sum is equal to 0.

Example 3.2.12. Suppose $e = 2$. Then there exists

$$0 \neq \hat{\Theta} : S^{(4,2)} \longrightarrow M^{(2^3)}$$

such that $\mathrm{Im}\,\hat{\Theta} \subseteq S^{(2^3)}$.

Proof. Set $\hat{\Theta} = \begin{array}{l} 1^2\,2^1\,3^1 \\ 2^1\,3^1 \end{array}$. Then

$$\begin{aligned}
\hat{\Theta}\psi_{2,0} &= 4 \begin{array}{l} 1^2\,2^2 \\ 2^2 \end{array} \\
&= 0 \quad \mathrm{mod}\ 2. \\
\hat{\Theta}\psi_{2,1} &= 2 \begin{array}{l} 1^2\,2^1\,3^1 \\ 2^2 \end{array} + 2 \begin{array}{l} 1^2\,2^2 \\ 2^1\,3^1 \end{array} \\
&= 0 \quad \mathrm{mod}\ 2. \\
\hat{\Theta}\psi_{1,0} &= 3 \begin{array}{l} 1^3\,3^1 \\ 1^1\,3^1 \end{array} \\
&= -2(3) \begin{array}{l} 1^4 \\ 3^2 \end{array} \\
&= 0 \quad \mathrm{mod}\ 2. \\
\hat{\Theta}\psi_{1,1} &= 3 \begin{array}{l} 1^3\,3^1 \\ 2^1\,3^1 \end{array} + \begin{array}{l} 1^2\,2^1\,3^1 \\ 1^1\,3^1 \end{array} \\
&= 3 \begin{array}{l} 1^3\,3^1 \\ 2^1\,3^1 \end{array} - \begin{array}{l} 1^3\,3^1 \\ 2^1\,3^1 \end{array} - 2 \begin{array}{l} 1^3\,2^1 \\ 3^2 \end{array} \\
&= 0 \quad \mathrm{mod}\ 2.
\end{aligned}$$

So $\mathrm{Im}\,\hat{\Theta} \subseteq S^{(2^3)}$. □

Thus the main problem is finding the coefficient of each semistandard homomorphism comprising $\hat{\Theta}\psi_{d,t}$.

Lemma 3.2.13. *Suppose* $\Theta : M^{(\lambda_1,\lambda_2)} \to M^{(\mu_1,\mu_2,\mu_3)}$ *is given by*

$$\Theta = \begin{array}{l} 1^{\mu_1-\lambda_2+j+k}\,2^{\mu_2-j}\,3^{\mu_3-k} \\ 1^{\lambda_2-j-k}\,2^j 3^k \end{array}.$$

Then

$$\Theta\psi_{2,t} = \sum_{s=0}^{t} \binom{\mu_2+\mu_3-t-j-k+s}{\mu_2-j}\binom{j+k-s}{j} \begin{array}{l} 1^{\mu_1-\lambda_2+j+k}\,2^{\mu_2+\mu_3-t-j-k+s}\,3^{t-s} \\ 1^{\lambda_2-j-k}\,2^{j+k-s}\,3^{s} \end{array}$$

$$\Theta\psi_{1,t} = \sum_{s=0}^{t} \binom{\mu_1+\mu_2-t-\lambda_2+s+k}{\mu_1-\lambda_2+j+k}\binom{\lambda_2-k-s}{\lambda_2-j-k} \begin{array}{l} 1^{\mu_1+\mu_2-t-\lambda_2+s+k}\,2^{t-s}\,3^{\mu_3-k} \\ 1^{\lambda_2-k-s}\,2^{s}\,3^{k} \end{array}$$

Note that, for the given limits, the binomial coefficient is zero if any of the exponents of the map are negative.

Proof. The map $\psi_{d,t}$ can be thought of as changing $\mu_{d+1}-t$ entries of $(d+1)$ into d. This gives the coefficient of each map above. $\square$

Lemma 3.2.14. *Suppose $\lambda = (\lambda_1, \lambda_2) \vdash n$ and $\nu = (\nu_1, \nu_2, \nu_3) \models n$ where $\nu_1 \geq \lambda_2$. Now suppose that r and s are such that the following are semistandard λ-tableaux of type ν.*

$$T_s = \begin{array}{l} 1^{\nu_1}\,2^{\nu_2-\lambda_2+s}\,3^{\nu_3-s} \\ 2^{\lambda_2-s}\,3^{s} \end{array}$$

$$T_r = \begin{array}{l} 1^{\nu_1}\,2^{\nu_2-\lambda_2+r}\,3^{\nu_3-r} \\ 2^{\lambda_2-r}\,3^{r} \end{array}$$

Now fix a tableau t as in Definition 3.2.5. Claim that the ν-tabloid $d_t(T_s)$ occurs with multiplicity 1 in the image of the polytabloid e_t under the map Θ_{T_s}, and does not occur in the image of e_t under the map Θ_{T_r} for any $r \neq s$.

Proof. The proof is by application of Lemma 3.2.10; the first claim is straightforward. Suppose $r \neq s$. If $d_t(T_s)$ occurs in the image of e_t under Θ_{T_r} then there is a λ-tableau of type ν which is row equivalent to T_r and column equivalent to T_s. However, no such tableau exists. $\square$

Corollary 3.2.15. *Suppose λ and ν satisfy the conditions of Lemma 3.2.14 and that T is the tableau*

$$\begin{array}{l} 1^{\nu_1-\lambda_2+a+b}\,2^{\nu_2-a}\,3^{\nu_3-b} \\ 1^{\lambda_2-a-b}2^{a}\,3^{b} \end{array}$$

where all exponents are non–negative. Then

$$\hat{\Theta}_T = \sum_{s=\lambda_2-\nu_2}^{\nu_3} (-1)^{\lambda_2-a-b}\binom{s}{s-b}\binom{\lambda_2-s}{\lambda_2-s-a} \begin{array}{l} 1^{\nu_1}\,2^{\nu_2-\lambda_2+s}\,3^{\nu_3-s}. \\ 2^{\lambda_2-s}\,3^{s} \end{array}.$$

Note that the limits are such that the binomial coefficient is zero if any the exponents of any map are negative.

Proof. Define T_s as in Lemma 3.2.14 and recall that $\{\hat{\Theta}_{T_s} \mid T_s \text{ is a semistandard tableau}\}$ form a basis of $\mathrm{Hom}(S^\lambda, M^\nu)$. Fix a λ-tableau t. Then

$$e_t\hat{\Theta}_T = e_t\left(\sum_s a_s\hat{\Theta}_{T_s}\right)$$

for some coefficients a_s. For each value of s, we use Lemma 3.2.10 to determine the coefficient of $d_t(T_s)$ on both sides of the equation; by Lemma 3.2.14 this coefficient, on the right hand side, is simply a_s, and this enables us to determine a_s for all s.

□

Lemma 3.2.16. *Let $\lambda = (\lambda_1, \lambda_2) \vdash n$ and $\nu = (\nu_1, \nu_2, \nu_3) \models n$. Now suppose $r > s$ and r and s are such that the following are semistandard λ-tableaux of type ν.*

$$T_s = \begin{array}{l} 1^{\nu_1}\, 2^{\nu_2-\lambda_2+s}\, 3^{\nu_3-s} \\ 2^{\lambda_2-s}\, 3^s \end{array}$$

$$T_r = \begin{array}{l} 1^{\nu_1}\, 2^{\nu_2-\lambda_2+r}\, 3^{\nu_3-r} \\ 2^{\lambda_2-r}\, 3^r \end{array}$$

Fix a λ-tableau t. Then the tabloid $d_t(T_s)$ does not occur in the image of e_t under the map Θ_{T_r}.

Proof. We apply Lemma 3.2.10, noting that there are no λ-tableaux of type ν which are row equivalent to T_r and column equivalent to T_s. □

Corollary 3.2.17. *Let λ and ν be as defined in Lemma 3.2.16 and $\hat{\Theta}_T \in \mathrm{Hom}(S^\lambda, M^\nu)$ be given by*

$$\hat{\Theta}_T = \begin{array}{l} 1^{\nu_1-r}\, 2^{\lambda_1-\nu_1+r} \\ 1^r\, 2^{\nu_1+\nu_2-\lambda_1-r}\, 3^{\nu_3} \end{array}$$

where we assume all exponents are non-negative. Then

$$\hat{\Theta}_T = (-1)^r \binom{\lambda_2-\nu_3}{r} \begin{array}{l} 1^{\nu_1}\, 2^{\lambda_1-\nu_1} \\ 2^{\lambda_2-\nu_3}\, 3^{\nu_3} \end{array}.$$

Proof. Define the tableaux T_s as in Lemma 3.2.14, where all T_s are semistandard. Then $0 \le s \le \nu_3$. Fix a λ-tableau t. Then

$$e_t\hat{\Theta}_T = e_t\left(\sum_s a_s\hat{\Theta}_{T_s}\right)$$

for some coefficients a_s. Suppose $0 \le s < \nu_3$ and that $a_{s'} = 0$ for $s' < s$. We consider the coefficient of $d_t(T_s)$ on both sides of the equation using Lemma 3.2.10; it does not appear on the left hand side, and it occurs on the right hand side with coefficient

a_s. Hence $a_s = 0$ for $s < \nu_3$. The coefficient of ν_3 is then found by considering the coefficient of $d_t(T_{\nu_3})$. $\square$

Lemma 3.2.18. *Suppose $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ and $\nu = (\nu_1, \nu_2, \dots) \models n$ are such that $\nu_i > \lambda_1$ for some i. Then* $\mathrm{Hom}(S^\nu, M^\lambda) = 0$.

Proof. If μ is a partition formed by reordering the parts of ν then $S^\mu \cong S^\nu$ [17], and by Remarks 3.2.3 we have that $\mathrm{Hom}(S^\mu, M^\lambda) = 0$. $\square$

Corollary 3.2.19. *Suppose $\hat{\Theta} : S^\lambda \to M^\mu$. Then if $\mu_1 + \mu_2 - t > \lambda_1$ we have that* $\mathrm{Im}\,\theta \subseteq \ker \psi_{1,t}$ *and if $\mu_2 + \mu_3 - t > \lambda_1$ we have that* $\mathrm{Im}\,\theta \subseteq \ker \psi_{2,t}$.

We end this section with some binomial identities which will prove useful later.

Lemma 3.2.20. *[17] Suppose $1 \le b < e$ where e is prime and $e \mid (A-b+1)$. Then*

$$e \mid \binom{A}{b}, \binom{A-1}{b-1}, \dots, \binom{A-b+1}{1}.$$

Lemma 3.2.21. *Suppose that $A = -2$ (mod e) and $0 \le r < e$. Then*

$$\binom{A+r}{A} = \begin{cases} 1 & r = 0 \\ -1 & r = 1 \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. If $r = 0$ or $r = 1$, the proof is trivial. Let $2 \le r < e$. Then

$$\binom{A+r}{A} = \frac{(A+r)(A+r-1)\dots(A+1)}{r!} = 0 \mod e.$$

$\square$

Lemma 3.2.22. *Suppose $A \ge 0$. Then the following binomial identities hold.*

1. $$\sum_{a=0}^{A} (-1)^a \binom{A}{a} \binom{B+a}{C} = (-1)^A \binom{B}{C-A}$$
2. $$\sum_{a=0}^{A} (-1)^a \binom{A}{a} \binom{B-a}{C} = \binom{B-A}{C-A}$$
3. $$\sum_{a=0}^{A} \binom{A}{a} \binom{B}{C-a} = \binom{B+A}{C}$$

where $B \ge C + A \ge C \ge 0$.

Proof. The proofs of (1) and (3) are by induction on A, the case $A = 0$ being trivially true. Hence for these proofs, assume the statement holds for $A - 1$.

1.
$$\begin{aligned}
\sum_{a=0}^{A} (-1)^a \binom{A}{a}\binom{B+a}{C} &= \sum_{a=0}^{A} (-1)^a \left[\binom{A-1}{a-1} + \binom{A-1}{a}\right]\binom{B+a}{C} \\
&= \sum_{a=0}^{A} (-1)^a \binom{A-1}{a-1}\binom{B+a}{C} \\
&\quad + \sum_{a=0}^{A} (-1)^a \binom{A-1}{a}\binom{B+a}{C} \\
&= \sum_{a'=0}^{A-1} (-1)^{a'+1} \binom{A-1}{a'}\binom{B+a'+1}{C} \\
&\quad + \sum_{a=0}^{A-1} (-1)^a \binom{A-1}{a}\binom{B+a}{C} \\
&= -(-1)^{A-1}\binom{B+1}{C-A+1} + (-1)^{A-1}\binom{B}{C-A+1}
\end{aligned}$$

by the inductive hypothesis

$$\begin{aligned}
&= (-1)^A \left[\binom{B+1}{C-A+1} - \binom{B}{C-A+1}\right] \\
&= (-1)^A \binom{B}{C-A}
\end{aligned}$$

2.
$$\begin{aligned}
\sum_{a=0}^{A} (-1)^a \binom{A}{a}\binom{B-a}{C} &= \sum_{a'=0}^{A} (-1)^{A-a}\binom{A}{a'}\binom{B-A+a'}{C} \\
&= (-1)^A \sum_{a'=0}^{A} (-1)^{a'}\binom{A}{a'}\binom{B-A+a'}{C} \\
&= \binom{B-A}{C-A} \text{ by (1)}
\end{aligned}$$

3.
$$\begin{aligned}
\sum_{a=0}^{A} \binom{A}{a}\binom{B}{C-a} &= \sum_{a=0}^{A} \binom{A-1}{a-1}\binom{B}{C-a} + \sum_{a=0}^{A} \binom{A-1}{a}\binom{B}{C-a} \\
&= \sum_{a'=0}^{A-1} \binom{A-1}{a-1}\binom{B}{C-a-1} + \sum_{a=0}^{A-1} \binom{A-1}{a}\binom{B}{C-a} \\
&= \binom{A+B-1}{C-1} + \binom{A+B-1}{C}
\end{aligned}$$

by the inductive hypothesis

$$= \binom{A+B}{C}$$

$\square$

3.3 Accounting for composition factors

In this section, we fix $e \geq 3$. We will consider the composition factors that we have discovered by the use of the LLT algorithm. The notation will therefore follow that of Chapter 2, and we will reprove some of the results determined therein without reference to the LLT algorithm. Specifically, we will begin by considering some of the results obtained in Lemma 2.3.13 and Corollary 3.1.3.

We first look at two part partitions. If $\mu = (\mu_1, \mu_2)$ then in fact the composition factors of S^μ are known [17]. However, motivated by the LLT algorithm, we will reprove some of these results. The method employed will also be used when we consider three part partitions.

Proposition 3.3.1. *Suppose $0 < a - b < e$ and $e \mid (n - a - b + 1)$ where $n - a \geq a$. Then $S^{(n-a,a)}$ has a composition factor $D^{(n-b,b)}$.*

Proof. Suppose the conditions above hold. Let

$$T = \begin{array}{l} 2^a\, 1^{n-a-b} \\ 1^b \end{array}$$

and let $\Theta = \Theta_T : M^{(n-b,b)} \longrightarrow M^{(n-a,a)}$. Let s be a fixed $(n-b,b)$ tableau. Then, by Lemma 3.2.10, the coefficient of $d_s(T)$ in $e_s\Theta$ is 1, and hence $S^{(n-b,b)} \not\subseteq \ker\Theta$.

Now let $0 \leq t < a$. Then

$$\Theta\psi_{1,t} = \binom{n-b-t}{a-t} \begin{array}{l} 1^{n-b-t}\, 2^t \\ 1^b \end{array} .$$

Now by Lemma 3.2.20 we know that if $a - t < e$ then $e \mid \binom{n-b-t}{a-t}$; that is $\Theta\psi_{1,t} = 0$ and $\operatorname{Im}\Theta \subseteq \ker\psi_{1,t}$. Hence if $a < e$ then there exists a map $\Theta : M^{(n-b,b)} \longrightarrow M^{(n-a,a)}$ such that $\operatorname{Im}\Theta \subseteq S^{(n-a,a)}$ and $S^{(n-b,b)} \not\subseteq \ker\Theta$. By Remarks 3.2.4, every composition factor of $S^{(n-b,b)}$ is a composition factor of $S^{(n-a,a)}$; in particular $D^{(n-b,b)}$ is a composition factor of $S^{(n-a,a)}$.

Now suppose $a \geq e$ and let $\hat{\Theta}$ denote the restriction of Θ to $S^{(n-b,b)}$. By the above remarks, we have that $\hat{\Theta}\psi_{1,t} = 0$ if $a-t < e$. But if $a-t \geq e$ then $(n-b,b) \ntrianglerighteq (n-t,t)$

so that $\mathrm{Hom}(S^{(n-b,b)}, M^{(n-t,t)}) = 0$ and again $\hat{\Theta}\psi_{1,t} = 0$.

Hence $\mathrm{Im}\,\hat{\Theta} \subseteq S^{(n-a,a)}$ and by Remarks 3.2.4 we have that $S^{(n-a,a)}$ has a composition factor $D^{(n-b,b)}$ as required. $\square$

Now fix $\mu = (\mu_1, \mu_2, \mu_3)$ to be a partition with exactly 3 parts. The notation used will be that of Theorem 2.3.12 so that

$$l = |\mu_1 + 2|$$
$$k = |\mu_2 + 1|$$
$$s = |\mu_3|$$

where $0 \leq l, k, s < e$. We shall now consider some of the composition factors of the Specht module $S^{(\mu_1,\mu_2,\mu_3)}$; that is, those composition factors corresponding to a one or two part partition $\lambda = (\lambda_1, \lambda_2)$. Our usual strategy will be to demonstrate a map $\hat{\Theta} : S^\lambda \longrightarrow M^\mu$ such that $\mathrm{Im}\,\hat{\Theta} \subseteq S^\mu$; by Remarks 3.2.4, this shows that D^λ is a composition factor of S^μ. Note that $\mathrm{Im}\,\hat{\Theta} \subseteq S^\mu$ if and only if

$$\hat{\Theta}\psi_{1,t} = 0 \text{ for } 0 \leq t < \mu_2 \text{ and}$$
$$\hat{\Theta}\psi_{2,t} = 0 \text{ for } 0 \leq t < \mu_3.$$

We shall use the following maps from M^λ into M^μ.

Definition 3.3.2. Define $\Theta_k : M^\lambda \rightarrow M^\mu$ by

$$\Theta_1^{(j')} = \begin{matrix} 1^{\mu_1-\lambda_2+j'}\, 2^{\mu_2-j'}\, 3^{\mu_3} \\ 1^{\lambda_2-j'} 2^{j'} \end{matrix}$$

$$\Theta_2 = \begin{matrix} 1^{\mu_1}\, 2^{\lambda_1-\mu_1} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{matrix}$$

$$\Theta_3 = \sum_{i=0}^{\mu_3} (-1)^i \frac{i!(\mu_2-i)!}{(\mu_2-\mu_3)!} \begin{matrix} 1^{\mu_1}\, 2^{\mu_2-\lambda_2+i}\, 3^{\mu_3-i} \\ 2^{\lambda_2-i}\, 3^i \end{matrix}$$

$$\Theta_4 = \sum_{i=0}^{\mu_3} (-1)^i \frac{i!(\mu_2-i)!}{(\mu_2-e)!} \begin{matrix} 1^{\mu_1}\, 2^{\mu_2-\lambda_2+i}\, 3^{\mu_3-i} \\ 2^{\lambda_2-i}\, 3^i \end{matrix}$$

where all exponents are assumed to be non–negative; and we shall also consider the restrictions to the Specht module, which will be denoted $\hat{\Theta}_k$.

We begin by looking at $\mu = (\mu_1, \mu_2, \mu_3)$ where $0 < \mu_3 < e$.

Proposition 3.3.3. *Suppose $\mu_3 < e$ and $k = 0$. Then*

- *If $l > s > 0$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3 - l +$*

$e, \mu_2 + l - e)$.

- *If $s > l > 0$ and $\mu_2 = e - 1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3 - l + e, \mu_2 + l - e)$.*

- *If $s > l > 0$ and $\mu_1 - \mu_2 > e - 1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3 - l, \mu_2 + l)$.*

- *If $l = s > 0$ and $\mu_2 = e - 1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$.*

- *If $l = s > 0$ and $\mu_1 - \mu_2 > e - 1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1, \mu_2 + \mu_3)$.*

Proof. Assume that λ, μ satisfy one of the conditions of Proposition 3.3.3 and note that $\lambda_2 \leq \mu_1$. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \hat{\Theta}_1^{(0)} = \begin{array}{l} 1^{\mu_1-\lambda_2}\, 2^{\mu_2}\, 3^{\mu_3} \\ 1^{\lambda_2} \end{array} .$$

Fix a λ-tableau s and let $S = \begin{array}{l} 2^{\mu_2}\, 3^{\mu_3}\, 1^{\mu_1-\lambda_2} \\ 1^{\lambda_2} \end{array}$. Then the coefficient of the tabloid $d_s(S)$ in $e_s\hat{\Theta}$ is 1 by Lemma 3.2.10, and so $\hat{\Theta}$ is not zero. Now fix t with $0 \leq t < \mu_3$. By Lemma 3.2.13,

$$\hat{\Theta}\psi_{2,t} = \binom{\mu_2 + \mu_3 - t}{\mu_3 - t} \begin{array}{l} 1^{\mu_1-\lambda_2}\, 2^{\mu_2+\mu_3-t}\, 3^{t} \\ 1^{\lambda_2} \end{array} .$$

Now $\mu_3 - t < e$ and $e \mid \mu_2 + 1$. By Lemma 3.2.20, we have that $e \mid \binom{\mu_2 + \mu_3 - t}{\mu_3 - t}$, and so $\hat{\Theta}\psi_{2,t} = 0$. Now fix t with $0 \leq t < \mu_2$. Then

$$\hat{\Theta}\psi_{1,t} = \binom{\mu_1 + \mu_2 - \lambda_2 - t}{\mu_2 - t} \begin{array}{l} 1^{\mu_1+\mu_2-\lambda_2-t}\, 2^{t}\, 3^{\mu_3} \\ 1^{\lambda_2} \end{array} \quad \text{for } 0 \leq t < \mu_2.$$

If $\mu_2 < e$ then $e \mid \mu_1 - \lambda_2 + 1$ and so $e \mid \binom{\mu_1 + \mu_2 - \lambda_2 - t}{\mu_2 - t}$ for $0 \leq t < \mu_2$. Thus $\hat{\Theta}\psi_{1,t} = 0$. In this case, we can extend $\hat{\Theta}$ to a map $\Theta : M^\lambda \to M^\mu$, which is such that $\operatorname{Im}\Theta \subseteq S^\mu$.

If $\mu_2 \geq e$ then $e \mid \binom{\mu_1+\mu_2-\lambda_2-t}{\mu_2-t}$ for $\mu_2 - e < t < \mu_2$. But if $0 \leq t \leq \mu_2 - e$ then, assuming the conditions of Proposition 3.3.3, $\mu_1+\mu_2-t > \lambda_1$ and by Corollary 3.2.19, we have that $\hat{\Theta}\psi_{1,t} = 0$.

Thus if $\mu_2 < e$ we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\operatorname{Im}\Theta \subseteq S^\mu$ and $\ker\Theta \not\supseteq S^\lambda$, and otherwise there exists $0 \neq \hat{\Theta} : S^\lambda \to M^\mu$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$. ☐

Proposition 3.3.4. *Suppose that $\mu_3 < e$ and and $k = 0$ and that either*

- *$l = 0$ or*

- *$s > l > 0$ and $\mu_1 - \mu_2 > e - 1$ or*

- *$l > s$.*

Then S^μ has a composition factor D^λ where $\lambda = (\mu_1, \mu_2 + \mu_3)$.

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.4. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \hat{\Theta}_2 = \begin{matrix} 1^{\mu_1} & \\ 2^{\mu_2}\,3^{\mu_3} \end{matrix} .$$

Note that for the given values of λ and μ, $\hat{\Theta}$ is semistandard, and so non–zero. Fix t such that $0 \leq t < \mu_3$. Then

$$\hat{\Theta}\psi_{2,t} = \binom{\mu_2+\mu_3-t}{\mu_3-t} \begin{matrix} 1^{\mu_1} & \\ 2^{\mu_2+\mu_3-t}\,3^t \end{matrix} .$$

Since $\mu_3 - t < e$ and $e \mid \mu_2 + 1$, we have that $e \mid \binom{\mu_2+\mu_3-t}{\mu_3-t}$, and so $\hat{\Theta}\psi_{2,t} = 0$. Also for $0 \leq t < \mu_2$ it is true that $\mu_1 + \mu_2 - t > \lambda_1$ so that $\hat{\Theta}\psi_{1,t} = 0$.

Thus $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$ as required.

☐

Proposition 3.3.5. *Suppose that $\mu_3 < e$ and $l = 0$ and $s \geq k$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3, \mu_2)$.*

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.5. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \hat{\Theta}_1^{(k)} = \begin{matrix} 1^{\mu_1-\lambda_2+k}\,2^{\mu_2-k}\,3^{\mu_3} \\ 1^{\lambda_2-k}2^k \end{matrix}$$

noting that, since $s \geq k$, the map exists. Fix a λ-tableau s and let

$$S = \begin{array}{l} 3^{\mu_3}\, 2^{\mu_2-k}\, 1^{\mu_1-\lambda_2+k} \\ 2^k\, 1^{\lambda_2-k} \end{array}.$$

Then the coefficient of the tabloid $d_s(S)$ in $e_s\hat{\Theta}$ is 1 by Lemma 3.2.10, and so $\hat{\Theta}$ is not zero. Now fix t with $0 \leq t < \mu_3$. Then

$$\hat{\Theta}\psi_{2,t} = \binom{\mu_2+\mu_3-k-t}{\mu_3-t} \begin{array}{l} 1^{\mu_1-\lambda_2+k}\, 2^{\mu_2+\mu_3-k-t}\, 3^t \\ 1^{\lambda_2-k}\, 2^k \end{array}.$$

Then $\mu_3 < e$ and $e \mid (\mu_2 - k + 1)$. Hence $e \mid \binom{\mu_2+\mu_3-k-t}{\mu_3-t}$ and so $\hat{\Theta}\psi_{2,t} = 0$. Now fix t with $0 \leq t < \mu_2$. We have that

$$\hat{\Theta}\psi_{1,t} = \sum_{r=0}^{t} \binom{\lambda_2-r}{k-r} \binom{\mu_1-t+r}{\lambda_2-t-k+r} \begin{array}{l} 1^{\mu_1-t+r}\, 2^{t-r}\, 3^{\mu_3} \\ 1^{\lambda_2-r}\, 2^r \end{array}.$$

Now if $r \neq k$ then $e \mid \binom{\lambda_2-r}{k-r}$; and if $r = k$ and $\mu_2 - t < e$ then $e \mid \binom{\mu_1-t+r}{\mu_2-t-k+r}$. If $\mu_2 - t \geq e$ then $\mu_1 + \mu_2 - t > \lambda_1$. Hence in all cases $\hat{\Theta}\psi_{1,t} = 0$.

Then if $\mu_2 < e$ we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\operatorname{Im}\Theta \subseteq S^\mu$ and $\ker\theta \not\supseteq S^\lambda$, and otherwise there exists $0 \neq \hat{\Theta} : S^\lambda \to M^\mu$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$. $\square$

Proposition 3.3.6. *Suppose $\mu_3 < e$ and $l = 0$ and that either*

- *$s > k > 0$ or*

- *$k > s$ and $\mu_2 > e - 1$.*

Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + k, \mu_2 + \mu_3 - k)$.

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.6. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \hat{\Theta}_2 = \begin{array}{l} 1^{\mu_1}\, 2^{\lambda_1-\mu_1} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{array}$$

noting that such a map exists and is semistandard. Fix t with $0 \leq t < \mu_3$. Then

$$\hat{\Theta}\psi_{2,t} = \binom{\lambda_2-t}{\mu_3-t} \begin{array}{l} 1^{\mu_1}\, 2^{\lambda_1-\mu_1} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{array}.$$

Then $\mu_3 - t < e$ and $e \mid \lambda_2 - \mu_3 + 1$ and so $e \mid \binom{\lambda_2-t}{\mu_3-t}$ and $\hat{\Theta}\psi_{2,t} = 0$. Now fix t

with $0 \le t < \mu_2$. Then

$$\begin{aligned}
\hat{\Theta}\psi_{1,t} &= \sum_{r=0}^{t} \binom{\mu_1+\mu_2-t-r}{\mu_1} \begin{matrix} 1^{\mu_1+\mu_2-t-r}\, 2^{t-\lambda_2+\mu_3+r} \\ 1^r\, 2^{\lambda_2-\mu_3-r}\, 3^{\mu_3} \end{matrix} \\
&= \sum_{r=0}^{\lambda_2-\mu_3} (-1)^r \binom{\lambda_2-\mu_3}{r} \binom{\mu_1+\mu_2-t-r}{\mu_1} \begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t-\lambda_2+\mu_3} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{matrix}
\end{aligned}$$

by Corollary 3.2.17

$$= \binom{\mu_1+\mu_2-t-\lambda_2+\mu_3}{\mu_1-\lambda_2+\mu_3} \begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t-\lambda_2+\mu_3} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{matrix}$$

by Lemma 3.2.22

$$= \binom{\lambda_1-t}{\mu_2-t} \begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t-\lambda_2+\mu_3} \\ 2^{\lambda_2-\mu_3}\, 3^{\mu_3} \end{matrix}$$

If $\mu_2 - t < e$ then since $e \mid (\lambda_1 - \mu_2 + 1)$ we have that $e \mid \binom{\lambda_1-t}{\mu_2-t}$; and if $\mu_2 - t \ge e$ then $\mu_1 + \mu_2 - t > \lambda_1$. Hence $\hat{\Theta}\psi_{1,t} = 0$. $\square$

Proposition 3.3.7. *Suppose that $\mu_3 < e$ and $l = 0$ and $k > s$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3, \mu_2)$.*

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.7. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\begin{aligned}
\hat{\Theta} = \hat{\Theta}_3 &= \sum_{i=0}^{\mu_3} (-1)^i \frac{i!(\mu_2-i)!}{(\mu_2-\mu_3)!} \begin{matrix} 1^{\mu_1}\, 2^{\lambda_1-\mu_1+i}\, 3^{\mu_3-i} \\ 2^{\lambda_2-i}\, 3^i \end{matrix} \\
&= \sum_{i=0}^{\mu_3} \alpha_i T_i
\end{aligned}$$

where

$$\begin{aligned}
\alpha_i &= (-1)^i i!(\mu_2-\mu_3+1)(\mu_2-\mu_3+2)\dots(\mu_2-i) \\
T_i &= \begin{matrix} 1^{\mu_1}\, 2^{\lambda_1-\mu_1+i}\, 3^{\mu_3-i} \\ 2^{\lambda_2-i}\, 3^i \end{matrix}
\end{aligned}$$

for $0 \le i \le \mu_3$.

Note that for the given values of λ and μ, each map T_i exists and is semistandard. Also $\hat{\Theta} \ne 0$ since, in particular, if $i = \mu_3$ then $\alpha_i = (-1)^{\mu_3}(\mu_3)!$ and since $\mu_3 < e$ we have that $e \nmid \alpha_i$. Fix t with $0 \leq t < \mu_3$. Then

$$\hat{\Theta}\psi_{2,t} = \sum_{r=0}^{t} \sum_{i=r}^{\mu_3-t+r} \alpha_i \binom{\mu_2 - r}{i-r} \binom{\mu_3 - t + r}{i} \begin{array}{l} 1^{\mu_1}\, 2^{\mu_3-t+r}\, 3^{t-r}. \\ 2^{\mu_2-r}\, 3^{r} \end{array}$$

So the coefficient of a semistandard homomorphism $\begin{array}{l} 1^{\mu_1}\, 2^{\mu_3-t+r}\, 3^{t-r} \\ 2^{\mu_2-r}\, 3^{r} \end{array}$ is given by

$$(\ddagger) = \sum_{i=r}^{\mu_3-t+r} (-1)^i \frac{i!\,(\mu_2-i)!}{(\mu_2-\mu_3)!} \binom{\mu_2 - r}{i-r} \binom{\mu_3 - t + r}{i}.$$

Claim that $(\ddagger)$ is zero over $\mathbb{Q}$, hence certainly zero over a field of characteristic e. Note that all of the terms in the binomial equations are non–negative. Working over $\mathbb{Q}$, where $*$ is variable, but is independent of i:

$$\begin{aligned}
(\ddagger) &= \sum_{i=r}^{\mu_3-t+r} (-1)^i \frac{i!(\mu_2-i)!}{(\mu_2-\mu_3)!} \frac{(\mu_2-r)!}{(i-r)!(\mu_2-i)!} \frac{(\mu_3-t+r)!}{i!(\mu_3-t+r-i)!} \\
&= * \sum_{i=r}^{\mu_3-t+r} (-1)^i \frac{1}{(i-r)!(\mu_3-t+r-i)!} \\
&= * \sum_{i=0}^{\mu_3-t} (-1)^i \frac{1}{i!(\mu_3-t-i)!} \\
&= * \sum_{i=0}^{\mu_3-t} (-1)^i \binom{\mu_3-t}{i} \\
&= 0 \text{ since } \mu_3 - t > 0.
\end{aligned}$$

Hence $\hat{\Theta}\psi_{2,t} = 0$. Now fix t with $0 \leq t < \mu_2$. Then

$$(\ddagger\ddagger) = \hat{\Theta}\psi_{1,t} = \sum_{i=0}^{\mu_3} \sum_{\substack{r=max\{0,\\ \mu_2-t-i\}}}^{\substack{min\{\mu_2-t,\\ \mu_2-i\}}} \alpha_i \binom{\mu_1+\mu_2-t-r}{\mu_1} \begin{array}{l} 1^{\mu_1+\mu_2-t-r}\, 2^{t-\mu_2+r+i}\, 3^{\mu_3-i} \\ 1^{r}\, 2^{\lambda_2-r-i}\, 3^{i} \end{array}.$$

However, $\mu_1 \equiv -2 \mod e$ and $r \leq \mu_2 - t \leq \lambda_1 - \mu_1 < e$. Hence

$$\binom{\mu_1+\mu_2-t-r}{\mu_1} \equiv \begin{cases} 1 & r = \mu_2 - t \\ -1 & r = \mu_2 - t - 1 \\ 0 & \text{otherwise} \end{cases}$$

by Lemma 3.2.21. We consider two separate cases.

<u>Case 1: $t \geq \mu_3$</u>

$$(\ddagger\ddagger) = \sum_{i=0}^{\mu_3} \alpha_i \begin{matrix} 1^{\mu_1}\, 2^{i}\, 3^{\mu_3-i} \\ 1^{\mu_2-t}\, 2^{t-i}\, 3^{i} \end{matrix} \quad - \sum_{i=1}^{\mu_3} \alpha_i \begin{matrix} 1^{\mu_1+1}\, 2^{i-1}\, 3^{\mu_3-i} \\ 1^{\mu_2-t-1}\, 2^{t-i+1}\, 3^{i} \end{matrix}$$

But then by Lemma 3.2.15

$$\begin{aligned} \begin{matrix} 1^{\mu_1}\, 2^{i}\, 3^{\mu_3-i} \\ 1^{\mu_2-t}\, 2^{t-i}\, 3^{i} \end{matrix} &= \sum_{s=i}^{\mu_2-t+i} (-1)^{\mu_2-t} \binom{s}{i} \binom{\mu_2-s}{t-i} \begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t+s-\mu_2}\, 3^{\mu_3-s} \\ 2^{\mu_2-s}\, 3^{s} \end{matrix}. \\ \begin{matrix} 1^{\mu_1+1}\, 2^{i-1}\, 3^{\mu_3-i} \\ 1^{\mu_2-t-1}\, 2^{t-i+1}\, 3^{i} \end{matrix} &= \sum_{s=i}^{\mu_2-t+i-1} (-1)^{\mu_2-t} \binom{s}{i} \binom{\mu_2-s}{t-i+1} \begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t+s-\mu_2}\, 3^{\mu_3-s} \\ 2^{\mu_2-s}\, 3^{s} \end{matrix}. \end{aligned}$$

Then if we express $(\ddagger\ddagger)$ as a sum of semistandard homomorphisms, the coefficient of the homomorphism

$$\begin{matrix} 1^{\mu_1+\mu_2-t}\, 2^{t+s-\mu_2}\, 3^{\mu_3-s} \\ 2^{\mu_2-s}\, 3^{s} \end{matrix}$$

(where all exponents are non–negative) is given by

$$\begin{aligned} (\dagger) &= \sum_{i=0}^{s} (-1)^{\mu_2-t} \alpha_i \binom{s}{i} \binom{\mu_2-s}{t-i} - \sum_{i=0}^{s} (-1)^{\mu_2-t-1} \alpha_i \binom{s}{i} \binom{\mu_2-s}{t-i+1} \\ &= \pm \sum_{i=0}^{s} \alpha_i \binom{s}{i} \binom{\mu_2-s+1}{t-i+1}. \end{aligned}$$

Claim that $(\dagger)$ is zero over $\mathbb{Q}$, hence certainly zero over a field of characteristic e. Working over $\mathbb{Q}$ and noting all binomial coefficients are non–negative:

$$\begin{aligned} (\dagger) &= \sum_{i=0}^{s} (-1)^i \frac{i!(\mu_2-i)!}{\mu_2-\mu_3} \frac{s!}{i!(s-i)!} \frac{(\mu_2-s+1)!}{(t-i+1)!(\mu_2-s-t+i)!} \\ &= * \sum_{i=0}^{s} (-1)^i \frac{(\mu_2-i)!}{(s-i)!(t-i+1)!(\mu_2-s-t+i)!} \\ &= * \sum_{i=0}^{s} (-1)^i \binom{\mu_2-i}{t-i+1} \binom{\mu_2-t}{s-i} \\ &= * \sum_{i=0}^{s} (-1)^i \binom{\mu_2-s+i}{t-s+i+1} \binom{\mu_2-t}{i} \\ &= * \sum_{i=0}^{\mu_2-t} (-1)^i \binom{\mu_2-t}{i} \binom{\mu_2-s+i}{\mu_2-t-1} \quad \text{as } s \geq \mu_2-t \\ &= * \binom{\mu_2-s}{-1} \text{ by Lemma 3.2.22} \\ &= 0 \text{ since } \mu_2-s \geq 0. \end{aligned}$$

Case 2: $t < \mu_3$

$$(\ddagger\ddagger) = \sum_{i=0}^{t} \alpha_i \begin{matrix} 1^{\mu_1}\, 2^i\, 3^{\mu_3-i} \\ 1^{\mu_2-t}\, 2^{t-i}\, 3^i \end{matrix} - \sum_{i=1}^{t+1} \alpha_i \begin{matrix} 1^{\mu_1+1}\, 2^{i-1}\, 3^{\mu_3-i} \\ 1^{\mu_2-t-1}\, 2^{t-i+1}\, 3^i \end{matrix}.$$

We can then express $(\ddagger\ddagger)$ as a sum of semistandard homomorphisms. Except for a change in the limits, the coefficient of each homomorphism can be shown to be zero exactly as in Case 1.

Thus $\hat{\Theta}\psi_{1,t} = 0$. □

Proposition 3.3.8. *Suppose $\mu_3 < e$ and $l = k = 0$ and $\mu_2 = e - 1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$*

Proof. Suppose $\mu_3 < e$ and $l = k = 0$ and $\mu_2 = e - 1$. Then by the proof of Proposition 3.3.5, there exists a map

$$\Theta : M^{(\mu_1+\mu_3,\mu_2)} \to M^\mu$$

such that

$$\ker \Theta \not\supseteq S^{(\mu_1+\mu_3,\mu_2)} \text{ and } \operatorname{Im} \Theta \subseteq S^\mu.$$

So by Corollary 3.2.4, every composition factor of $S^{(\mu_1+\mu_3,\mu_2)}$ is a composition factor of S^μ. In particular, by Proposition 3.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3,\mu_2)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^μ. □

Proposition 3.3.9. *Suppose $\mu_3 < e$ and $k = 0$ and $s > l > 0$ and $\mu_2 = e - 1$ and $\mu_1 - \mu_2 > e-1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$*

Proof. Suppose $\mu_3 < e$ and $k = 0$ and $s > l > 0$ and $\mu_2 = e - 1$ and $\mu_1 - \mu_2 > e - 1$. Then by the proof of Proposition 3.3.3, there exists a map

$$\Theta : M^{(\mu_1+\mu_3-l,\mu_2+l)} \to M^\mu$$

such that

$$\ker \Theta \not\supseteq S^{(\mu_1+\mu_3-l,\mu_2+l)} \text{ and } \operatorname{Im} \Theta \subseteq S^\mu.$$

So by Corollary 3.2.4, every composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$ is a composition factor of S^μ. In particular, by Proposition 3.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^μ. □

Proposition 3.3.10. *Suppose $\mu_3 < e$ and $k = 0$ and $l > s$ and $\mu_2 = e - 1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$.*

Proof. Suppose $\mu_3 < e$ and $k = 0$ and $l > s$ and $\mu_2 = e - 1$. Then by the proof of Proposition 3.3.3, there exists a map

$$\Theta : M^{(\mu_1+\mu_3-l,\mu_2+l)} \longrightarrow M^{\mu}$$

such that

$$\ker\Theta \not\supseteq S^{(\mu_1+\mu_3-l,\mu_2+l)} \text{ and } \operatorname{Im}\Theta \subseteq S^{\mu}.$$

So by Corollary 3.2.4, every composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$ is a composition factor of S^{μ}. In particular, by Proposition 3.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^{μ}. □

Remarks 3.3.11. In fact, given that $e < \mu_3$, we have almost reproved all the cases in Lemma 2.3.13; the remainder were the following.

- If $0 < \mu_3 < e$ and $l = k = 0$ and $\mu_2 > e - 1$ then S^{μ} has a composition factor $D^{(\mu_1+e,\mu_2+\mu_3-e)}$.

- If $0 < \mu_3 < e$ and $l = 0$ and $s > k > 0$ then S^{μ} has a composition factor $D^{(\mu_1+k+e,\mu_2+\mu_3-k-e)}$.

- If $0 < \mu_3 < e$ and $k = 0$ and $s > l > 0$ and $\mu_2 > e - 1$ and $\mu_2 - \mu_3 > e - 1$ then S^{μ} has a composition factor $D^{(\mu_1+e,\mu_2+\mu_3-e)}$.

In each case above, we can give an example where S^{λ} contains a composition factor D^{μ}, but there is no non–zero map $\hat{\Theta} : S^{\lambda} \to M^{\mu}$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^{\mu}$.

Example 3.3.12. Suppose $e = 5$. Then $D^{(18,5)}$ is a composition factor of $S^{(13,9,1)}$. Suppose $\hat{\Theta} : S^{(18,5)} \longrightarrow M^{(13,9,1)}$ is such that $\operatorname{Im}\hat{\Theta} \subseteq S^{(13,9,1)}$. Then $\hat{\Theta}$ is of the form

$$a_0 \begin{array}{l} 1^{13}\,2^4\,3 \\ 2^5 \end{array} + a_1 \begin{array}{l} 1^{13}\,2^5 \\ 2^4\,3^1 \end{array}$$

for some coefficients a_0, a_1. Then

$$\begin{aligned} \hat{\Theta}\psi_{1,8} &= 14a_0 \begin{array}{l} 1^{14}\,2^3\,3 \\ 2^5 \end{array} + a_0 \begin{array}{l} 1^{13}\,2^4\,3 \\ 1\,2^4 \end{array} + 14a_1 \begin{array}{l} 1^{14}\,2^4 \\ 2^4\,3 \end{array} + a_1 \begin{array}{l} 1^{14}\,2^5 \\ 1\,2^3\,3 \end{array} \\ &= 14a_0 \begin{array}{l} 1^{14}\,2^3\,3 \\ 2^5 \end{array} - 5a_0 \begin{array}{l} 1^{14}\,2^3\,3 \\ 2^5 \end{array} - 4a_0 \begin{array}{l} 1^{14}\,2^4 \\ 2^4\,3 \end{array} + 14a_1 \begin{array}{l} 1^{14}\,2^4 \\ 2^4\,3 \end{array} - 4a_1 \begin{array}{l} 1^{14}\,2^5 \\ 2^4\,3 \end{array} \end{aligned}$$

by Corollary 3.2.15

$$\equiv -a_0 \begin{array}{l} 1^{14}\,2^3\,3 \\ 2^5 \end{array} + a_0 \begin{array}{l} 1^{13}\,2^4\,3 \\ 1\,2^4 \end{array} \mod 5.$$

Hence $a_0 = 0$. So

$$\begin{aligned} \hat{\Theta}\psi_{1,4} &\equiv \binom{18}{5} a_1 \begin{array}{l} 1^{18} \\ 2^4\,3 \end{array} + 14a_1 \begin{array}{l} 1^{14}\,2^4 \\ 1^4\,3 \end{array} \mod 5 \\ &\equiv 3a_1 \begin{array}{l} 1^{18} \\ 2^4\,3 \end{array} + 4a_1 \begin{array}{l} 1^{18} \\ 2^4\,3 \end{array} \mod 5 \end{aligned}$$

by Corollary 3.2.17. Thus $a_1 = a_0 = 0$ and hence $\hat{\Theta} = 0$.

Similarly we can show that

If $e = 5$ then $S^{(13,10,3)}$ has a composition factor $D^{(19,7)}$ but there is no non–zero map from $S^{(19,7)}$ into $S^{(13,10,3)}$.

If $e = 3$ then $S^{(8,5,2)}$ has a composition factor $D^{(11,4)}$ but there is no non–zero map from $S^{(11,4)}$ into $S^{(8,5,2)}$.

We now consider $\mu_3 \geq e$.

Proposition 3.3.13. *Suppose that $e \leq \mu_3 < 2e$ and $l = 0$ and $k > s$ and $\mu_1 - \mu_2 < e$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + k, \mu_2 + \mu_3 - k)$.*

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.13. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \hat{\Theta}_2 = \begin{array}{l} 1^{\mu_1}\,2^{\lambda_1-\mu_1} \\ 2^{\lambda_2-\mu_3}\,3^{\mu_3} \end{array}$$

noting that such a map exists and is semistandard. Fix t with $0 \leq t < \mu_3$. Then

$$\hat{\Theta}\psi_{2,t} = \binom{\lambda_2 - t}{\mu_3 - t} \begin{array}{l} 1^{\mu_1}\,2^{\lambda_1-\mu_1} \\ 2^{\lambda_2-t}\,3^{t} \end{array}.$$

If $\mu_3 - t < e$ then since $e \mid (\lambda_2 - \mu_3 + 1)$ we have that $e \mid \binom{\lambda_2 - t}{\mu_3 - t}$ and so $\hat{\Theta}\psi_{2,t} = 0$. If $\mu_3 - t \geq e$ then $\mu_1 + \mu_2 - t > \lambda_1$ and $\hat{\Theta}\psi_{2,t} = 0$. Now fix t with $0 \leq t < \mu_2$. Then

$$\hat{\Theta}\psi_{1,t} = \binom{\lambda_1 - t}{\mu_2 - t} \begin{array}{l} 1^{\mu_1+\mu_2-t}\,2^{t-\lambda_2+\mu_3} \\ 2^{\lambda_2-\mu_3}\,3^{\mu_3} \end{array}$$

from the proof of Proposition 3.3.6. If $\mu_2 - t < e$ then since $e \mid (\lambda_1 - \mu_2 + 1)$ we have that $e \mid \begin{pmatrix} \lambda_1 - t \\ \mu_2 - t \end{pmatrix}$ and so $\hat{\Theta}\psi_{1,t} - 0$. If $\mu_2 - t \geq e$ then $\mu_1 + \mu_2 - t > \lambda_1$ and so $\hat{\Theta}\psi_{1,t} = 0$. □

Proposition 3.3.14. *Suppose that $e \leq \mu_3 < 2e$ and $l = 0$ and $k > s$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3, \mu_2)$.*

Proof. Assume that λ, μ satisfy the conditions of Proposition 3.3.14. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\begin{aligned}
\hat{\Theta} &= \sum_{i=0}^{\mu_3} (-1)^i \frac{i!(\mu_2 - i)!}{(\mu_2 - e)!} \begin{matrix} 1^{\mu_1}\, 2^{\lambda_1 - \mu_1 + i}\, 3^{\mu_3 - i} \\ 2^{\lambda_2 - i}\, 3^i \end{matrix} \\
&= \sum_{i=0}^{\mu_3} \alpha_i \begin{matrix} 1^{\mu_1}\, 2^{\lambda_1 - \mu_1 + i}\, 3^{\mu_3 - i} \\ 2^{\lambda_2 - i}\, 3^i \end{matrix}
\end{aligned}$$

where

$$\alpha_i = (-1)^i \frac{i!\,(\mu_2 - i)!}{(\mu_2 - e)!} = \begin{cases} (-1)^i\, i!\,(\mu_2 - e + 1)(\mu_2 - e + 2)\dots(\mu_2 - i) & i < e \\ (-1)^i\, i! & i = e \\ (-1)^i\, i!\,(\mu_2 - i + 1)^{-1}(\mu_2 - i + 2)^{-1}\dots(\mu_2 - e)^{-1} & i > e \end{cases}.$$

Note that this is well defined. If $i > e$ then $e \nmid \mu_2 - a$ for $e \leq a < i \leq \mu_3 < 2e$. Note also that $\hat{\Theta} \neq 0$; in particular if $i = e - 1$ then $\alpha_i = (-1)^{e-1}(e-1)!(\mu_2 - e + 1)$. However for $i \geq e$ we have that $e \mid i!$, hence $\alpha_i = 0$.

The proof that $\hat{\Theta}\psi_{2,t} = 0$ for $0 \leq t < \mu_3$ is identical to the proof of the same statement in Proposition 3.3.7. Now fix t with $0 \leq t < \mu_2$. Then

$$\begin{aligned}
\hat{\Theta}\psi_{1,t} &= \sum_{i=0}^{\mu_3} \sum_{\substack{r = max\{0, \\ \mu_2 - t - i\}}}^{\substack{min\{\mu_2 - t, \\ \mu_2 - i\}}} \alpha_i \begin{pmatrix} \mu_1 + \mu_2 - t - r \\ \mu_1 \end{pmatrix} \begin{matrix} 1^{\mu_1 + \mu_2 - t - r}\, 2^{t - \mu_2 + r + i}\, 3^{\mu_3 - i} \\ 1^r\, 2^{\lambda_2 - r - i}\, 3^i \end{matrix} \\
&\equiv \sum_{i=0}^{e-1} \sum_{\substack{r = max\{0, \\ \mu_2 - t - i\}}}^{\substack{min\{\mu_2 - t, \\ \mu_2 - i\}}} \alpha_i \begin{pmatrix} \mu_1 + \mu_2 - t - r \\ \mu_1 \end{pmatrix} \begin{matrix} 1^{\mu_1 + \mu_2 - t - r}\, 2^{t - \mu_2 + r + i}\, 3^{\mu_3 - i} \\ 1^r\, 2^{\lambda_2 - r - i}\, 3^i \end{matrix}
\end{aligned}$$

since $\alpha_i \equiv 0$ for $i \geq e$.

However, $\mu_1 \equiv -2$ mod e and $\mu_2 - t - e < \mu_2 - t - i \leq r \leq \mu_2 - t$. Hence by Lemma 3.2.21

$$\begin{pmatrix} \mu_1 + \mu_2 - t - r \\ \mu_1 \end{pmatrix} \equiv \begin{cases} 1 & r = \mu_2 - t \\ -1 & r = \mu_2 - t - 1 \\ 0 & \text{otherwise} \end{cases}.$$

The remainder of the proof is then identical to the proof of Proposition 3.3.7.

□

Remarks 3.3.15. In Remarks 3.3.11, we considered, for $\mu_3 < e$, the cases in Lemma 2.3.13 which we could not reprove. We now look at the same results for $e \leq \mu_3 < 2e-1$.

- If $\mu_3 = e$ and $k > l > 0$ then S^μ has a composition factor $D^{(\mu_1+\mu_3,\mu_2)}$. For example, if $e = 3$ then $S^{(5,4,3)}$ has a composition factor $D^{(8,4)}$, but there is no non–zero map from $S^{(8,4)}$ into $S^{(5,4,3)}$.

- If $\mu_3 = e$ and $k > l > 0$ and $\mu_1 - \mu_2 < e-1$ then S^μ has a composition factor $D^{(\mu_1-l+k,\mu_2+\mu_3+l-k)}$. For example, if $e = 3$ then $S^{(5,4,3)}$ has a composition factor $D^{(6,6)}$, but there is no non–zero map from $S^{(6,6)}$ into $S^{(5,4,3)}$.

- If $\mu_3 = e$ and $k > l > 0$ and $\mu_2 > 2e-1$ then S^μ has a composition factor $D^{(\mu_1-l+k+e,\mu_2+\mu_3+l-k-e)}$. For example, if $e = 3$ then $S^{(6,6,3)}$ has a composition factor $D^{(8,7)}$, but there is no non–zero map from $S^{(8,7)}$ into $S^{(6,6,3)}$.

- If $\mu_3 = e$ and $k = 0$ and $l > 0$ then S^μ has a composition factor $D^{(\mu_1+\mu_3-l,\mu_2+l)}$. For example, if $e = 3$ then $S^{(6,5,3)}$ has a composition factor $D^{(7,7)}$, but there is no non–zero map from $S^{(7,7)}$ into $S^{(6,5,3)}$.

- If $\mu_3 \geq e$ and $k = 0$ and $l > s$ then S^μ has a composition factor $D^{(\mu_1+\mu_3-l,\mu_2+l)}$. For example, if $e = 3$ then $S^{(6,5,4)}$ has a composition factor $D^{(8,7)}$, but there is no non–zero map from $S^{(8,7)}$ into $S^{(6,5,4)}$.

Remark 3.3.16. So far, we have only looked at Lemma 2.3.13 rather than Theorem 2.3.12. However, it is known (Theorem 2.3.7) that if $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\lambda_1,\lambda_2)}$ then $S^{(\mu_1+a,\mu_2+a,\mu_3+a)}$ has a composition factor $D^{(\lambda_1+a,\lambda_2+a,a)}$ for any positive integer a. In fact, we will later show

Theorem 3.3.17. *Suppose there exists*

$$0 \neq \hat{\Theta} : S^{(\lambda_1,\lambda_2)} \to M^{(\mu_1,\mu_2,\mu_3)}$$

such that $\operatorname{Im}\theta \subseteq S^{(\mu_1,\mu_2,\mu_3)}$. *Then there exists*

$$0 \neq \hat{\Theta}' : S^{(\lambda_1+a,\lambda_2+a,a)} \to M^{(\mu_1+a,\mu_2+a,\mu_3+a)}$$

such that $\operatorname{Im}\hat{\Theta}' \subseteq S^{(\mu_1+a,\mu_2+a,\mu_3+a)}$.

We will prove Theorem 3.3.17 in greater generality in Chapter 5.

3.4 A look at the case $e = 2$

Let $e = 2$. In Lemma 2.3.15, we discovered some of the composition factors of the Specht module $S(\mu_1, \mu_2, \mu_3)$ where $\mu_3 \leq 2$. However, the composition factors of these modules where $\mu_3 \leq 1$ were explicitly determined by James [13]. Rather than go over old ground, we will take a brief look at the case that $\mu_3 = 2$, and consider the adjustment matrices introduced in Section 3.1. The notation will be as usual, with $\mu = (\mu_1, \mu_2, \mu_3)$ assumed to be a partition of n and $\lambda = (\lambda_1, \lambda_2)$ a 2-regular partition of n.

Proposition 3.4.1. *Suppose that* $\mu_3 = 2$ *and* $\mu_1, \mu_2 \equiv 0 \mod 2$*. Then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* D^λ *where* $\lambda = (\mu_1 + 2, \mu_2)$*.*

Proof. This follows from Lemma 2.3.15; however we present an alternative proof. Assume μ, λ satisfy the conditions of Proposition 3.4.1. Define $\hat{\Theta} : S^\lambda \to M^\mu$ by

$$\hat{\Theta} = \begin{array}{l} 1^{\mu_1}\,2\,3 \\ 2^{\mu_2-1}\,3 \end{array}.$$

Then we have

$$\begin{aligned}
\hat{\Theta}\psi_{2,0} &= 2\ \begin{array}{l} 1^{\mu_1}\,2^2 \\ 2^{\mu_2} \end{array} \\
&= 0 \\
\hat{\Theta}\psi_{2,1} &= 2\ \begin{array}{l} 1^{\mu_1}\,2^2 \\ 2^{\mu_2-1}\,3 \end{array} + \mu_2\ \begin{array}{l} 1^{\mu_1}\,2\,3 \\ 2^{\mu_2} \end{array} \\
&= 0. \\
\hat{\Theta}\psi_{1,\mu_2-1} &= (\mu_2+1)\ \begin{array}{l} 1^{\mu_1+1}\,3 \\ 2^{\mu_2-1}\,3 \end{array} + \begin{array}{l} 1^{\mu_1}\,2\,3 \\ 1\,2^{\mu_2-2}\,3 \end{array} \\
&= \begin{array}{l} 1^{\mu_1+1}\,3 \\ 2^{\mu_2-1}\,3 \end{array} + \begin{array}{l} 1^{\mu_1+1}\,3 \\ 2^{\mu_2-1}\,3 \end{array} \\
&= 0 \\
\hat{\Theta}\psi_{1,\mu_2-2} &= (\mu_2+1)\ \begin{array}{l} 1^{\mu_1+1}\,3 \\ 1\,2^{\mu_2-2}\,3 \end{array} + \begin{array}{l} 1^{\mu_1}\,2\,3 \\ 1^2\,2^{\mu_2-3}\,3 \end{array} \\
&= 0 + 0
\end{aligned}$$

and for $0 \leq t < \mu_2 - 2$, Lemma 3.2.19 gives that $\hat{\Theta}\psi_{d,t} = 0$, so that $\operatorname{Im}\hat{\Theta} \subseteq S^{\mu}$. $\square$

Proposition 3.4.2. *Suppose that* $\mu_3 = 2$ *and* $\mu_1 \equiv 1 \mod 2$ *and that* $\mu_2 \equiv 3 \mod 4$. *Then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* D^{λ} *where* $\lambda = (\mu_1 + 1, \mu_2 + 1)$.

Proof. This follows from Lemma 2.3.15; however we present an alternative proof. Assume μ, λ satisfy the conditions of Proposition 3.4.1. Define $\hat{\Theta} : S^{(\mu_1+1,\mu_2)} \to M^{\mu}$ by

$$\hat{\Theta} = \begin{array}{l} 1^{\mu_1}\,3 \\ 2^{\mu_2}\,3 \end{array} + \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2-1}\,3^2 \end{array}.$$

Then we have

$$\begin{aligned}
\hat{\Theta}\psi_{2,0} &= (\mu_2 - 1) \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2+1} \end{array} + \binom{\mu_2+1}{2} \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2+1} \end{array} \\
&= 0 \text{ since } 4 \mid \mu_2 + 1. \\
\hat{\Theta}\psi_{2,1} &= \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2}\,3 \end{array} + (\mu_2 + 1) \begin{array}{l} 1^{\mu_1}\,3 \\ 2^{\mu_2+1} \end{array} + \mu_2 \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2}\,3 \end{array} \\
&= 0. \\
\hat{\Theta}\psi_{1,\mu_2-1} &= \begin{array}{l} 1^{\mu_1}\,3 \\ 1\,2^{\mu_2-1}\,3 \end{array} + (\mu_2 + 1) \begin{array}{l} 1^{\mu_1+1} \\ 2^{\mu_2-1}\,3^2 \end{array} + \mu_2 \begin{array}{l} 1^{\mu_1}\,2 \\ 12^{\mu_2-2}\,3^2 \end{array} \\
&= 0
\end{aligned}$$

and for $0 \leq t < \mu_2 - 2$, Lemma 3.2.19 gives that $\hat{\Theta}\psi_{d,t} = 0$, so that $\operatorname{Im}\hat{\Theta} \subseteq S^{\mu}$. $\square$

Proposition 3.4.3. *Suppose that* $\mu_3 = 2$ *and* $\mu_1 \equiv 1 \mod 2$ *and that* $\mu_2 \equiv 1 \mod 4$. *Then* $S^{(\mu_1,\mu_2,\mu_3)}$ *has a composition factor* D^{λ} *where* $\lambda = (\mu_1+1, \mu_2+1)$, *but there is no non-zero map* $\hat{\Theta} : S^{\lambda} \to M^{\mu}$ *such that* $\operatorname{Im}\hat{\Theta} \subseteq S^{\mu}$.

Proof. The first part of the statement follows from 2.3.15. Suppose that λ, μ satisfy the conditions of Proposition 3.4.3 and that there exists $\hat{\Theta} : S^{\lambda} \to M^{\mu}$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^{\mu}$. Then

$$\hat{\Theta} = a_1 \begin{array}{l} 1^{\mu_1}\,3 \\ 2^{\mu_2}\,3 \end{array} + a_2 \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2-1}\,3^2 \end{array}$$

for some coefficients a_1, a_2. Letting $\equiv$ denote equivalence $\mod 2$, we have

$$\begin{aligned}
\hat{\Theta}\psi_{2,0} &= a_1(\mu_2 - 1) \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2+1} \end{array} + a_2 \binom{\mu_2+1}{2} \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2+1} \end{array} \\
&\equiv a_2 \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2+1} \end{array}
\end{aligned}$$

since $4 \nmid (\mu_2 + 1)$. Hence $a_2 = 0$. Then

$$\hat{\Theta}\psi_{2,1} = a_1 \begin{array}{l} 1^{\mu_1}\,2 \\ 2^{\mu_2}\,3 \end{array} + a_1(\mu_2 + 1) \begin{array}{l} 1^{\mu_1}\,3 \\ 2^{\mu_2+1} \end{array} .$$

So a_1 and hence $\hat{\Theta}$ are zero. $\square$

Now suppose that $2 \mid n$. Then the part of the decomposition matrix of $\mathcal{H}_{\mathbb{C},-1}(\mathfrak{S}_n)$ indexed by 1 and 2 part partitions is given by the following $(\frac{n}{2}+1) \times \frac{n}{2}$ matrix, which we will call a Type A matrix.

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ & 1 & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 1 & 1 \end{pmatrix}$$

The corresponding part of the decomposition matrix of $F\mathfrak{S}_n$ is given by an $(\frac{n}{2}+1) \times \frac{n}{2}$ matrix which James [13] describes a Type II matrix (without the extra 1):

$$\begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & 1 & & 1 & 1 & \\ & & & 1 & 1 & 1 \\ & & & 1 & & 1 \end{pmatrix}$$

Thus the top part of the adjustment matrix is, in James's notation, an $\frac{n}{2} \times \frac{n}{2}$ Type I matrix:

$$\begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & 1 & & 1 & & \\ & & & & 1 & \\ & & & 1 & & 1 \end{pmatrix}$$

(Similarly, the adjustment matrix when n is odd is a $\frac{n+1}{2} \times \frac{n+1}{2}$ Type I matrix.) We now consider the part of the $(\frac{n}{2} - 1) \times \frac{n}{2}$ matrix whose rows are indexed by partitions $(n-a-2, a, 2)$ and whose columns are indexed by partitions $(n-b, b)$. Using Lemma

2.3.15, it can be seen that it is of the form

$$\begin{pmatrix} 0 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ddots & \\ & & & & & & & \ddots & \end{pmatrix};$$

that is it can be formed from a Type A matrix by removing the top 3 rows, and replacing the entries in columns $2, 4, 6, \ldots$ with 0. Now, if we multiply this by part of our adjustment matrix, we produce an $(\frac{n}{2} - 1) \times \frac{n}{2}$ matrix

$$\begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & 0 & 0 & & & \\ & 0 & 0 & 1 & & \\ & 0 & 0 & 0 & 0 & \\ & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \ddots & & & & & \\ & 0 & & & & \\ & 0 & 1 & & & \\ & 0 & 0 & 0 & & \\ & 0 & 1 & 0 & 1 & \\ & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 \end{pmatrix};$$

which may be considered as being formed from a Type I matrix (without the extra 1) by removing the top 3 rows and replacing the enties in columns $2, 4, 6, \ldots$ with 0.

This produces the following theorem.

Theorem 3.4.4. *Consider $F\mathfrak{S}_n$-modules. Suppose $2 \mid b$ and that $n-2b+1$ contains $a-b+1$ to base 2. Then $S^{(n-a-1,a+1,2)}$ contains a composition factor $D^{(n-b,b)}$.*

Proof. Recall [17] that $S^{(n-k,k)}$ contains a composition factor $D^{(n-j,j)}$ (with multiplicity 1) if and only if $n-2j+1$ contains $k-j$ to base 2, and notice from the above remarks that if $2 \mid b$ and $S^{(n-a-1,a+1)}$ contains a composition factor $D^{(n-b,b)}$ then $S^{(n-a-1,a+1,2)}$ contains a composition factor $D^{(n-b,b)}$. □

4 The general linear groups

4.1 Decomposition matrices

Let V be an n dimensional vector space over the field $\mathbb{F}_q$ with basis $\{e_1, e_2, \ldots, e_n\}$. For subspaces A and B of V, we will use the notation $A > B$ to mean B is a subspace of A, possibly equal to A. We associate the general linear group $\mathrm{GL}_n(q)$ with $\mathrm{GL}(V)$. Suppose K is a field of characteristic p such that $p \nmid q$. Define e to be the least positive integer such that

$$(1 + q + q^2 + \ldots + q^{e-1})1_K = 0$$

with $e = \infty$ if no such integer exists; henceforth we will assume that e is finite. We wish to consider modules of the algebra $K\mathrm{GL}_n(q)$; the main reference will be [18].

Definition 4.1.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_r) \models n$. A λ-flag

$$X = (X_0 > X_1 > \ldots > X_r)$$

is a set of subspaces of V such that $\dim(X_{i-1}/X_i) = \lambda_i$ for $1 \leq i \leq r$. Note that this forces $X_0 = V$ and $X_r = 0$. The λ-flags are acted upon in a natural way by $\mathrm{GL}_n(q)$. We define the $K\mathrm{GL}_n(q)$-module M^λ to be the module whose basis elements, as a vector space, are given by the set of λ-flags, together with the natural action of $\mathrm{GL}_n(q)$. Define a bilinear form $\langle\ ,\ \rangle$ on M^λ by setting

$$\langle X, Y\rangle = \begin{cases} 1 & X = Y \\ 0 & \text{otherwise} \end{cases}$$

and extending bilinearly.

Definition 4.1.2. Suppose $\lambda \vdash n$. Let d be a positive integer, and choose t such that $0 \leq t < \lambda_{d+1}$. Define $\nu = (\nu_1, \nu_2, \ldots) \models n$ by

$$\nu_i = \begin{cases} \lambda_i + \lambda_{i+1} - t & i = d \\ t & i = d+1 \\ \lambda_i & \text{otherwise} \end{cases}$$

and define a homomorphism

$$\psi_{d,t} : M^\lambda \to M^\nu$$

as follows. Let $X = (X_0 > X_1 > \ldots > X_r)$. Then

$$X\psi_{d,t} = \sum \big\{ W = (W_0 > W_1 > \ldots > W_r) \mid W \text{ is a } \nu\text{-flag such that} \\ W_i = X_i \text{ for } i \neq d \text{ and } X_d > W_d \big\}.$$

Now define the Specht module $S^\lambda \subseteq M^\lambda$ by

$$S^\lambda = \bigcap_{d \geq 1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker(\psi_{d,t})$$

and set

$$D^\lambda = \frac{S^\lambda}{S^\lambda \cap S^{\lambda^\perp}}.$$

Then S^λ is cyclic as a $\mathrm{GL}_n(q)$-module, and a generator of S^λ can always be described, but for most partitions λ, no explicit basis for S^λ as a vector space is known.

The following results then hold.

1. Suppose $K = \mathbb{C}$. Then $\{S^\lambda \mid \lambda \vdash n\}$ form the complete set of non-isomorphic irreducible unipotent $K\mathrm{GL}_n(q)$-modules.

2. When K is an arbitrary field, $D^\lambda \neq 0$ for all $\lambda \vdash n$ and $\{D^\lambda \mid \lambda \vdash n\}$ form the complete set of non-isomorphic irreducible unipotent $K\mathrm{GL}_n(q)$-modules.

As in Chapter 2, we consider the decomposition matrix of $K\mathrm{GL}_n(q)$. For $\lambda, \mu \vdash n$ set $d_{\mu\lambda} = [S^\mu : D^\lambda]$ to be the composition multiplicity of D^λ in S^μ. The matrix $D = (d_{\mu\lambda})$ where $\lambda, \mu \vdash n$ is then the unipotent part of the decomposition matrix of $K\mathrm{GL}_n(q)$. These further facts then hold.

1. $d_{\lambda\lambda} = 1$ for all $\lambda \vdash n$.

2. $d_{\mu\lambda} = 0$ unless $\lambda \trianglerighteq \mu$.

Hence by arranging the rows and columns of D in a suitable order, we get that D is a square lower unitriangular matrix. We now reconsider the algebra $\mathcal{H}_0 =$

$\mathcal{H}_{\mathbb{C},\omega}(\mathfrak{S}_n)$ where ω is a primitive e^{th} root of unity in $\mathbb{C}$. The decomposition matrix of $\mathcal{H}_0$ is actually a submatrix of the decomposition matrix, D_0 say, of the q-Schur algebra $\mathcal{S}_{\mathbb{C},q}$, which has the same structure as our matrix D. Furthermore it can be shown that

$$D = D_0 A$$

where A is a suitably sized lower unitriangular matrix whose entries are non-negative integers [26]. (In fact it has been conjectured [19] that if $ep > n$ then A is the identity matrix.) This gives the following corollary.

Corollary 4.1.3. *Suppose the $\mathcal{H}_0$-module S^μ has a composition factor D^λ. Then the $K\,GL_n(q)$-module S^μ has a composition factor D^λ.*

Many other the results from the work on $\mathfrak{S}_n$ also hold for $\mathrm{GL}_n(q)$. The following are of particular use.

- Submodule Theorem [18, Theorem 11.2]: Suppose $M \subseteq M^\lambda$ is a $K\mathrm{GL}_n(q)$-module. Then $M \supseteq S^\lambda$ or $M \subseteq S^{\lambda^\perp}$.
- Quotient Theorem [18]: $M^\lambda/S^{\lambda^\perp} \cong (S^\lambda)^*$ where, for a module M, M^* represents the dual of M.
- M^μ has a composition factor D^λ only if $\lambda \trianglerighteq \mu$ and if $\lambda \ntrianglerighteq \mu$ then $\mathrm{Hom}(S^\lambda, M^\mu) = 0$ [18].

Thus there exists the following analogue of Corollary 3.2.4; the proof is identical to the proof of Corollary 3.2.4.

Corollary 4.1.4. *Let $\lambda, \mu \vdash n$. Then*

1. *Suppose we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\ker\Theta \not\supseteq S^\lambda$ and $\mathrm{Im}\,\Theta \subseteq S^\mu$. Then every composition factor of S^λ is a composition factor of S^μ.*

2. *Suppose λ is e-regular and we can find a map $0 \neq \Theta : S^\lambda \to M^\mu$ such that $\mathrm{Im}\,\Theta \subseteq S^\mu$. Then D^λ is a composition factor of S^μ.*

Note that the maps that we consider in this chapter will always be $K\mathrm{GL}_n(q)$-homomorphisms. However, there is no known analogue of the semistandard homomorphism theorem (Theorem 3.2.9), although it is conjectured that one exists. Now suppose that $\Theta : M^\lambda \to M^\mu$. Note that $\mathrm{Im}\,\Theta \subseteq S^\mu$ if and only if $\Theta\psi_{d,t} = 0$ for all $d \geq 1$ and $0 \leq t < \mu_{d+1}$. The map $\Theta\psi_{d,t}$ is expressed in a manner similar to that of Lemma 3.2.13, using the counting argument described in Lemma 4.1.7.

We end this section with an analogue of binomial coefficients: Gaussian coefficients.

Definition 4.1.5. If $m \geq 1$, define

$$[m] = [m]_q = 1 + q + \ldots + q^{m-1}$$

and

$$[m]! = [m][m-1]\ldots[1]$$

where $[0] = [0]! = 1$. Now for $n \geq m \geq 0$, define

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}.$$

It can be shown that these are polynomials in q.

Lemma 4.1.6. *[18, Theorem 19.5] Suppose $1 \leq b < e$ and $e \mid A - b + 1$. Then*

$$p \mid \begin{bmatrix} A \\ b \end{bmatrix}, \begin{bmatrix} A-1 \\ b-1 \end{bmatrix}, \ldots, \begin{bmatrix} A-b+1 \\ 1 \end{bmatrix}.$$

Lemma 4.1.7. *[18, Theorem 3.1] Let V_1, V_2 be subspaces of V with $\dim V_1 = a$, $\dim V_2 = b$ and $V_1 \cap V_2 = 0$. Then the number of m-dimensional subspaces Y of V such that $Y \cap V_1 = 0$ and $Y > V_2$ is given by*

$$q^{a(m-b)} \begin{bmatrix} n-a-b \\ m-b \end{bmatrix}.$$

An easy corollary of Lemma 4.1.7 is the following lemma.

Lemma 4.1.8. *Let V_1 be a subspace of V with $\dim V_1 = a$. Then the number of m dimensional subspaces of V_1 is given by*

$$\begin{bmatrix} a \\ m \end{bmatrix}.$$

Lemma 4.1.9. *Let X, W be subspaces of V with $\dim X = a, \dim W = b$ and $\dim(X \cap W) = r$. Then the number of m-dimensional subspaces Y of V such that $Y > W$ and $\dim(Y \cap X) = k$ is given by*

$$q^{(a-k)(m-k-b+r)} \begin{bmatrix} a-r \\ k-r \end{bmatrix} \begin{bmatrix} n-a-b+r \\ m-k-b+r \end{bmatrix}.$$

Proof. Assume we have the conditions of Lemma 4.1.9. The number of k-dimensional subspaces, Z say, satisfying $X > Z > (X \cap W)$ is

$$\begin{bmatrix} a-r \\ k-r \end{bmatrix}.$$

Now fix such a Z. We wish to know the number of m-dimensional subspaces, Y say, of V such that $(X \cap Y) = Z$ and $Y > W$ where $\dim(Z+W) = k+b-r$.

Without loss of generality, assume that

$$\begin{aligned} W &= \langle e_1, e_2, \ldots, e_b \rangle \\ X &= \langle e_1, e_2, \ldots, e_r, e_{b+1}, e_{b+2}, \ldots, e_{b+a-r} \rangle \\ Z &= \langle e_1, e_2, \ldots, e_r e_{b+1}, e_{b+2}, \ldots, e_{b+k-r} \rangle. \end{aligned}$$

Note that $Y > Z$. The number of choices of Y is therefore the number of ways of choosing $m-b-k+r$ independent vectors from $\langle e_{b+k-r+1}, e_{b+k-r+2}, \ldots, e_n \rangle$ such that the intersection with $\langle e_{b+k-r+1}, e_{b+k-r+2}, \ldots, e_{b+a-r} \rangle$ is 0, that is

$$q^{(a-k)(m-b-k+r)} \begin{bmatrix} n-a-b+r \\ m-k-b+r \end{bmatrix}$$

by Lemma 4.1.7.

Thus the total number of subspaces Y of V satisfying the conditions of Lemma 4.1.9 is given by

$$q^{(a-k)(m-k-b+r)} \begin{bmatrix} a-r \\ k-r \end{bmatrix} \begin{bmatrix} n-a-b+r \\ m-k-b+r \end{bmatrix}.$$

□

Lemma 4.1.10. *[18, Theorem 3.4]*

$$\begin{aligned} \begin{bmatrix} n \\ m \end{bmatrix} &= \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m \end{bmatrix} \\ &= \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \end{aligned}$$

Lemma 4.1.11. *[18, Theorem 3.5]*

$$(1+x)(1+xq)\ldots(1+xq^{n-1}) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} x^j.$$

Hence putting $x = -1$,

$$\sum_{j=0}^{n} (-1)^j q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix} = 0$$

for $n \geq 1$.

Lemma 4.1.12. *Suppose $A \geq 0$. Then*

$$\sum_{a=0}^{A} q^{a(F+a)} \begin{bmatrix} A \\ a \end{bmatrix} \begin{bmatrix} E \\ F+a \end{bmatrix} = \begin{bmatrix} A+E \\ A+F \end{bmatrix}.$$

Proof. The proof is by induction on A, the case $A = 0$ being trivial. Assume that Lemma 4.1.12 is true for A. Then

$$\begin{aligned}
&\sum_{a=0}^{A+1} q^{a(F+a)} \begin{bmatrix} A+1 \\ a \end{bmatrix} \begin{bmatrix} E \\ F+a \end{bmatrix} \\
&\qquad = \sum_{a=0}^{A+1} q^{a(F+a)} \left(\begin{bmatrix} A \\ a-1 \end{bmatrix} + q^a \begin{bmatrix} A \\ a \end{bmatrix} \right) \begin{bmatrix} E \\ F+a \end{bmatrix} \\
&\qquad = \sum_{a=0}^{A+1} q^{a(F+a)} \begin{bmatrix} A \\ a-1 \end{bmatrix} \begin{bmatrix} E \\ F+a \end{bmatrix} + \sum_{a=0}^{A+1} q^a q^{a(F+a)} \begin{bmatrix} A \\ a \end{bmatrix} \begin{bmatrix} E \\ F+a \end{bmatrix} \\
&\qquad = \sum_{a'=0}^{A} q^{(a'+1)(F+a'+1)} \begin{bmatrix} A \\ a' \end{bmatrix} \begin{bmatrix} E \\ F+a'+1 \end{bmatrix} + \sum_{a=0}^{A} q^{a(F+a+1)} \begin{bmatrix} A \\ a \end{bmatrix} \begin{bmatrix} E \\ F+a \end{bmatrix} \\
&\qquad = \sum_{a=0}^{A} q^{a(F+a+1)} \begin{bmatrix} A \\ a \end{bmatrix} \left(q^{F+a+1} \begin{bmatrix} E \\ F+a+1 \end{bmatrix} + \begin{bmatrix} E \\ F+a \end{bmatrix} \right) \\
&\qquad = \sum_{a=0}^{A} q^{a(F+a+1)} \begin{bmatrix} A \\ a \end{bmatrix} \begin{bmatrix} E+1 \\ F+a+1 \end{bmatrix} \\
&\qquad = \begin{bmatrix} A+E+1 \\ A+F+1 \end{bmatrix}
\end{aligned}$$

by the induction hypothesis. □

4.2 Two part partitions

Suppose $\lambda = (n-b, b) \vdash n$. Then M^λ is the vector space whose basis elements are the b-dimensional subspaces of V. We will adopt the notation of James [18] to describe a such a subspace.

Definition 4.2.1. Write
$$\begin{bmatrix} a_1^1 & a_2^1 & a_3^1 & \dots & a_n^1 \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \vdots & & & & \vdots \\ a_1^b & a_2^b & a_3^b & \dots & a_n^b \end{bmatrix}$$
to denote the subspace with basis vectors $\{f_1, f_2, \dots, f_b\}$ where $f_j = \sum_{i=1}^{n} a_i^j e_i$.

Proposition 4.2.2. *[18, Examples 11.17] The Specht module $S^{(n-b,b)}$ is generated by the element*

$$G = \sum_{\bullet \in \{0,1\}} (-1)^{n(\bullet)} \sum_{* \in \mathbb{F}_q} \begin{bmatrix} \bullet & 1 & & & & & & & & & \\ * & & \bullet & 1 & & & & & & & \\ * & & * & & \bullet & 1 & & & & & \\ \vdots & & & & \vdots & & & & & & \\ * & & * & & * & & & & \bullet & 1 & \end{bmatrix}$$

where any blank spaces are assumed to contain 0, and $n(\bullet)$ is the number of $\bullet$ taking the value 1. The sum is taken over each individual entry of $*$ and $\bullet$, so that the generator contains $2^b q^{b(b-1)/2}$ entries.

Example 4.2.3. Let $b = 2$ and $q = 3$. Then the element

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$
$$- \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

is a generator of $S^{(n-b,b)}$.

Definition 4.2.4. Let $\mu = (n-a, a) \vdash n$ where $a > b$ and let $0 \leq j \leq b$. Suppose X is a b-dimensional subspace of V. We define the maps $\Theta_j, \Phi_j : M^\lambda \to M^\mu$ by

$$X\Theta_j = \sum \{Y \mid \dim Y = a \text{ and } \dim(Y \cap X) = j\}$$
$$X\Phi_j = \sum \{Y \mid \dim Y = a \text{ and } \dim(Y \cap X) \geq j\}.$$

Let $\hat{\Theta}_j$ and $\hat{\Phi}_j$ denote the restrictions to S^λ.

It is easy to see that $\{\Theta_j \mid 0 \leq j \leq b\}$, and therefore $\{\Phi_j \mid 0 \leq j \leq b\}$, form bases of $\mathrm{Hom}(M^\lambda, M^\mu)$.

Theorem 4.2.5. *Assume the conditions of Definition 4.2.4 and let* $0 \le j \le b$. *Then*

$$\hat{\Phi}_j = (-1)^{b-j} q^{(b-j)(b-j+1)/2} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} \hat{\Phi}_b.$$

Proof. A more detailed proof is given in [28]. Assume the conditions of Definition 4.2.5 and let $0 \le j < b$. Suppose $\sum_X a_X X \in S^{(n-b,b)}$. Then

$$\left(\sum_X a_X X\right) \hat{\Phi}_j = \sum_X \sum_Y \{a_X Y \mid \dim Y = b \text{ and } \dim(X \cap Y) \ge j\}.$$

Given an a-dimensional subspace W, the coefficient of W in this sum is given by

$$\begin{aligned}
&\sum_X \{a_X \mid \dim(X \cap W) \ge j\} \\
&\quad = \sum_{\dim W_j = j} \sum_X \{a_X \mid (X \cap W) > W_j\} \\
&\qquad - \sum_{i=j+1}^{b} \left(\begin{bmatrix} i \\ j \end{bmatrix} - 1\right) \sum \{a_X \mid \dim(X \cap W) = i\} \\
&\quad = - \sum_{i=j+1}^{b} \left(\begin{bmatrix} i \\ j \end{bmatrix} - 1\right) \Big(\sum \{a_X \mid \dim(X \cap W) \ge i\} \\
&\qquad - \sum \{a_X \mid \dim(X \cap W) \ge i+1\}\Big)
\end{aligned}$$

since $\sum_X a_X X \in S^{(n-b,b)}$

$$\begin{aligned}
&\quad = - \sum_{i=j+1}^{b} \left(\begin{bmatrix} i \\ j \end{bmatrix} - \begin{bmatrix} i-1 \\ j \end{bmatrix}\right) \sum \{a_X \mid \dim(X \cap W) \ge i\} \\
&\quad = - \sum_{i=j+1}^{b} \left(q^{i-j} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix}\right) \sum \{a_X \mid \dim(X \cap W) \ge i\}.
\end{aligned}$$

Hence

$$\hat{\Phi}_j = - \sum_{i=j+1}^{b} q^{i-j} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \hat{\Phi}_i.$$

Note that $\hat{\Phi}_0 = 0$ as expected.

We now prove Theorem 4.2.5 by reverse induction; the case $l = b$ being trivially true. Suppose that Theorem 4.2.5 holds for $0 < l+1 \le j \le b$. Since it holds for

$l = 0$, we assume that $l > 0$.

$$\begin{aligned}\hat{\Phi}_l &= -\sum_{j=l+1}^{b} q^{j-l} \begin{bmatrix} j-1 \\ l-1 \end{bmatrix} \hat{\Phi}_j \\ &= -\sum_{j=l+1}^{b} q^{j-l} \begin{bmatrix} j-1 \\ l-1 \end{bmatrix} (-1)^{b-j} q^{(b-j)(b-j+1)/2} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} \hat{\Phi}_b\end{aligned}$$

by the induction hypothesis

$$\begin{aligned}&= -\sum_{j=l}^{b} q^{j-l} \begin{bmatrix} j-1 \\ l-1 \end{bmatrix} (-1)^{b-j} q^{(b-j)(b-j+1)/2} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} \hat{\Phi}_b \\ &\quad + (-1)^{b-l} q^{(b-l)(b-l+1)/2} \begin{bmatrix} b-1 \\ l-1 \end{bmatrix} \Phi_b \\ &= (-1)^{b-l} q^{(b-l)(b-l+1)/2} \begin{bmatrix} b-1 \\ l-1 \end{bmatrix} \Phi_b\end{aligned}$$

as required, since we have that

$$\begin{aligned}-\sum_{j=l}^{b} q^{j-l} &\begin{bmatrix} j-1 \\ l-1 \end{bmatrix} (-1)^{b-j} q^{(b-j)(b-j+1)/2} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} \\ &= q^{b-l} \sum_{j'=0}^{b-l} (-1)^{j'} q^{j'(j'-1)/2} \begin{bmatrix} b-j'-1 \\ b-l-j' \end{bmatrix} \begin{bmatrix} b-1 \\ j' \end{bmatrix}\end{aligned}$$

by the change of limits $j = b - j'$. We expand this expression, noting that since $l \geq 1$ all terms are non-negative, and working over $\mathbb{Q}$.

$$\begin{aligned}\sum_{j=0}^{b-l} (-1)^j q^{j(j-1)/2} &\begin{bmatrix} b-j-1 \\ b-l-j \end{bmatrix} \begin{bmatrix} b-1 \\ j \end{bmatrix} \\ &= \sum_{j=0}^{b-l} (-1)^j q^{j(j-1)/2} \frac{[b-j-1][b-1]}{[b-l-j][l-1][j][b-j-1]} \\ &= \frac{[b-1]}{[l-1][b-l]} \sum_{j=0}^{b-l} (-1)^j q^{j(j-1)/2} \begin{bmatrix} b-l \\ j \end{bmatrix} \\ &= 0 \text{ by Lemma 4.1.11, since } b-l > 0.\end{aligned}$$

□

Corollary 4.2.6. *Note that $\hat{\Phi}_b = \hat{\Theta}_b$ and $\hat{\Theta}_j = \hat{\Phi}_j - \hat{\Phi}_{j+1}$ for $0 \leq j < b$. Let*

$0 \le j < b$. *Then*

$$\begin{aligned}\hat{\Theta}_j &= \hat{\Phi}_j - \hat{\Phi}_{j+1}\\ &= (-1)^{b-j} q^{(b-j)(b-j+1)/2} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} \hat{\Phi}_b - (-1)^{b-j-1} q^{(b-j-1)(b-j)/2} \begin{bmatrix} b-1 \\ j \end{bmatrix} \hat{\Phi}_b \\ &= (-1)^{b-j} q^{(b-j-1)(b-j)/2} \left(q^{b-j} \begin{bmatrix} b-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} b-1 \\ j \end{bmatrix} \right) \hat{\Theta}_b \\ &= (-1)^{b-j} q^{(b-j-1)(b-j)/2} \begin{bmatrix} b \\ j \end{bmatrix} \hat{\Theta}_b.\end{aligned}$$

Proposition 4.2.7. *Assume the conditions of Definition 4.2.4. Then* $\hat{\Theta}_b \neq 0$.

Proof. Consider the action of $\hat{\Theta}_b$ on the generator G of S^λ described in Proposition 4.2.2. The subspace $W = \langle e_2, e_4, \ldots, e_{2b}, e_{2b+2}, \ldots, e_{2a} \rangle$ occurs in $G\hat{\Theta}_b$ with multiplicity 1. □

This suggests that there may be an analogue of the semistandard homomorphism theorem (Theorem 3.2.9). However, it may be that there are maps $\hat{\Theta} : S^\lambda \to M^\mu$ which cannot be extended to M^λ. As in Chapter 3, we shall therefore use tableaux to denote homomorphisms. Using the notation of Definition 4.2.4, if T_j is the tableau

$$T_j = \begin{array}{l} 1^{n-b-a+j}\, 2^{a-j} \\ 1^{b-j}\, 2^{j} \end{array}$$

then $\Theta_{T_j} = \Theta_j$.

4.3 Accounting for composition factors

As in Chapter 3, we will now reprove some of the results determined by the LLT algorithm; specifically those of Lemma 2.3.13 and Corollary 4.1.3. Throughout this section, we fix $e \ge 3$.

We first look at two part partitions. As before, if $\mu = (\mu_1, \mu_2)$ then in fact the composition factors of S^μ are known [18]. However, this will serve as an introduction to the more complicated combinatorics involved with 3 part partitions.

Proposition 4.3.1. *Suppose* $0 < a - b < e$ *and* $e \mid (n - a - b + 1)$. *Then* $S^{(n-a,a)}$ *has a composition factor* $D^{(n-b,b)}$.

Proof. Suppose the conditions above hold. Let $\Theta = \Theta_0 : M^{(n-b,b)} \to M^{(n-a,a)}$ be as defined in Definition 4.2.4, and let $\hat{\Theta}_0$ denote the restriction to $S^{(n-b,b)}$. By Corollary 4.2.6

$$\hat{\Theta}_0 = (-1)^b q^{b(b-1)/2} \hat{\Theta}_b \neq 0.$$

Now let $0 \le t < a$. Then

$$\Theta\psi_{1,t} = q^{b(a-t)} \begin{bmatrix} n-b-t \\ a-t \end{bmatrix} \begin{array}{l} 1^{n-b-t}2^t \\ 1^b \end{array} .$$

Now by Lemma 4.1.6 we know that if $a-t<e$ then $p \mid \begin{bmatrix} n-b-t \\ a-t \end{bmatrix}$; that is $\Theta\psi_{1,t} = 0$. Hence if $a < e$ then there exists a map $\Theta : M^{(n-b,b)} \longrightarrow M^{(n-a,a)}$ such that $\operatorname{Im}\Theta \subseteq S^{(n-a,a)}$ and $S^{(n-b,b)} \not\subseteq \ker\Theta$. By Remarks 4.1.4, every composition factor of $S^{(n-b,b)}$ is a composition factor of $S^{(n-a,a)}$; in particular $D^{(n-b,b)}$ is a composition factor of $S^{(n-a,a)}$.

Now suppose $a \ge e$. By the above remarks, we have that $\hat{\Theta}\psi_{1,t} = 0$ if $a-t<e$. But if $a-t \ge e$ then $(n-b,b) \ntrianglerighteq (n-t,t)$ so that $\operatorname{Hom}(S^{(n-b,b)}, M^{(n-t,t)}) = 0$ and again $\hat{\Theta}\psi_{1,t} = 0$.

Hence $\operatorname{Im}\hat{\Theta} \subseteq S^{(n-a,a)}$ and by Remarks 4.1.4 we have that $S^{(n-a,a)}$ has a composition factor $D^{(n-b,b)}$ as required. □

We now look at three part partitions. Assume that $\mu_3 > 0$.

Proposition 4.3.2. *(See Proposition 3.3.3.) Suppose $\mu_3 < e$ and $k=0$. Then*

- *If $l > s > 0$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1+\mu_3-l+e, \mu_2+l-e)$.*

- *If $s > l > 0$ and $\mu_2 = e-1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1+\mu_3-l+e, \mu_2+l-e)$.*

- *If $s > l > 0$ and $\mu_1-\mu_2 > e-1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1+\mu_3-l, \mu_2+l)$.*

- *If $l = s > 0$ and $\mu_2 = e-1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1+e, \mu_2+\mu_3-e)$.*

- *If $l = s > 0$ and $\mu_1-\mu_2 > e-1$ then S^μ has a composition factor D^λ where $\lambda = (\mu_1, \mu_2+\mu_3)$.*

Proof. Assume that λ, μ satisfy one of the conditions of Proposition 3.3.3. Let $\dim X = \lambda_2$ and define $\Theta : M^\lambda \to M^\mu$ by

$$X \to \sum \{Y = (Y_0 > Y_1 > Y_2 > Y_3) \mid Y \text{ is a } \mu\text{-flag such that } Y_1 \cap X = 0\}.$$

Let $\hat{\Theta}$ denote the restriction of Θ to S^λ. Let G be the generator of the Specht module as defined in Proposition 4.2.2. We consider the μ-flag

$$W = (V > W_1 > W_2 > 0)$$

where

$$\begin{aligned} W_2 &= \langle e_2, e_4, \ldots, e_{2\mu_3} \rangle \\ W_1 &= \langle e_2, e_4, \ldots, e_{2\lambda_2}, e_{2\lambda_2+1}, e_{2\lambda_2+2}, \ldots, e_{\mu_2+\mu_3+\lambda_2} \rangle. \end{aligned}$$

(Note that $\mu_2 + \mu_3 + \lambda_2 \leq n$.) Then the only λ_2-dimensional subspaces that occur in G and which intersect with W_1 in 0 are of the form

$$\begin{bmatrix} 1 & 1 & & & & & & & & & \\ * & & 1 & 1 & & & & & & & \\ * & & * & & 1 & 1 & & & & & \\ \vdots & & \vdots & & \vdots & & & & & & \\ * & & * & & * & & & & 1 & 1 & \end{bmatrix}$$

where $* \in \mathbb{F}_q$. Hence the coefficient of W in $G\hat{\Theta}$ is $(-1)^{\lambda_2} q^{\lambda_2(\lambda_2-1)/2}$ and $\hat{\Theta} \neq 0$.

Fix t with $0 \leq t < \mu_3$ and let $\nu = (\mu_1, \mu_2 + \mu_3 - t, t)$. Then

$$X(\Theta\psi_{2,t}) = \begin{bmatrix} \mu_2 + \mu_3 - t \\ \mu_3 - t \end{bmatrix} \sum \{ Z = (Z_0 > Z_1 > Z_2 > Z_3) \mid Z \text{ is a } \nu\text{-flag} \\ \text{such that } Z_1 \cap X = 0 \}.$$

Now $\mu_3 - t < e$ and $e \mid \mu_2 + 1$. By Lemma 4.1.6, we have that $p \mid \begin{bmatrix} \mu_2 + \mu_3 - t \\ \mu_3 - t \end{bmatrix}$, and so $\Theta\psi_{2,t} = 0$.

Now fix t with $0 \leq t < \mu_2$ and let $\nu = (\mu_1 + \mu_2 - t, t, \mu_3)$. Then

$$X(\Theta\psi_{1,t}) = \begin{bmatrix} \mu_1 + \mu_2 - t \\ \mu_2 - t \end{bmatrix} \sum \{ Z = (Z_0 > Z_1 > Z_2 > Z_3) \mid Z \text{ is a } \nu\text{-flag} \\ \text{such that } Z_1 \cap X = 0 \}.$$

If $\mu_2 < e$ then $e \mid \mu_1 - \lambda_2 + 1$ and so $p \mid \begin{bmatrix} \mu_1 + \mu_2 - \lambda_2 - t \\ \mu_2 - t \end{bmatrix}$ for $0 \leq t < \mu_2$. Thus $\Theta\psi_{1,t} = 0$.

If $\mu_2 \geq e$ then $p \mid \begin{bmatrix} \mu_1 + \mu_2 - \lambda_2 - t \\ \mu_2 - t \end{bmatrix}$ for $\mu_2 - e < t < \mu_2$. But if $0 \leq t \leq \mu_2 - e$

then, assuming the conditions of Proposition 4.3.2, $\mu_1 + \mu_2 - t > \lambda_1$ and so we have that $\hat{\Theta}\psi_{1,t} = 0$.

Thus if $\mu_2 < e$ we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\operatorname{Im}\Theta \subseteq S^\mu$ and $\ker\Theta \not\supseteq S^\lambda$, and otherwise there exists $0 \neq \hat{\Theta} : S^\lambda \to M^\mu$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$. □

Proposition 4.3.3. *(See Proposition 3.3.4.) Suppose that $\mu_3 < e$ and and $k = 0$ and that either*

- *$l = 0$ or*
- *$s > l > 0$ and $\mu_1 - \mu_2 > e - 1$ or*
- *$l > s$.*

Then S^μ has a composition factor D^λ where $\lambda = (\mu_1, \mu_2 + \mu_3)$.

Proof. Assume that λ, μ satisfy one of the conditions of Proposition 3.3.4. Let $\dim X = \lambda_2$ and define $\Theta : M^\lambda \to M^\mu$ by

$$X \to \sum\{Y = (Y_0 > Y_1 > Y_2 > Y_3) \mid Y \text{ is a } \mu\text{-flag such that } Y_1 = X\}.$$

Let $\hat{\Theta}$ denote the restriction of Θ to S^λ. Let G be the generator of the Specht module as defined in Proposition 4.2.2 and consider the coefficient of the μ-flag

$$W = (V > \langle e_2, e_4, \ldots, e_{2(\mu_3+\mu_2)}\rangle > \langle e_2, e_4, \ldots, e_{2\mu_3}\rangle > 0)$$

in $G\hat{\Theta}$. Then there is exactly one λ_2-dimensional subspace that occurs in G and is equal to $\langle e_2, e_4, \ldots, e_{2(\mu_3+\mu_2)}\rangle$ so that the coefficient of W in $G\hat{\Theta}$ is 1 and $\hat{\Theta} \neq 0$.

Fix t with $0 \leq t < \mu_3$ and let $\nu = (\mu_1, \mu_2 + \mu_3 - t, t)$. Then

$$X(\Theta\psi_{2,t}) = \begin{bmatrix} \mu_2 + \mu_3 - t \\ \mu_3 - t \end{bmatrix} \sum\{Z = (Z_0 > Z_1 > Z_2 > Z_3) \mid Z \text{ is a } \nu\text{-flag}$$
$$\text{such that } Z_1 = X\}.$$

Now $\mu_3 - t < e$ and $e \mid \mu_2 + 1$. By Lemma 4.1.6, we have that $p \mid \begin{bmatrix} \mu_2 + \mu_3 - t \\ \mu_3 - t \end{bmatrix}$, and so $\Theta\psi_{2,t} = 0$. Also for $0 \leq t < \mu_2$ it is true that $\mu_1 + \mu_2 - t > \lambda_1$ so that $\hat{\Theta}\psi_{1,t} = 0$.

Thus $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$ as required. □

Proposition 4.3.4. *(See Proposition 3.3.5.) Suppose that $\mu_3 < e$ and $l = 0$ and $s \geq k$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + \mu_3, \mu_2)$.*

Proof. Assume that λ, μ satisfy one of the conditions of Proposition 3.3.5. Let $\dim X = \lambda_2$ and define $\Theta : M^\lambda \to M^\mu$ by

$$X \to \sum\{Y = (Y_0 > Y_1 > Y_2 > Y_3) \mid Y \text{ is a } \mu\text{-flag}$$
$$\text{such that } Y_2 \cap X = 0 \text{ and } \dim(Y_1 \cap X) = k\}.$$

Let $\hat{\Theta}$ denote the restriction of Θ to S^λ. Let G be the generator of the Specht module as defined in Proposition 4.2.2. We consider the μ-flag

$$W = (V > W_1 > W_2 > 0)$$

where

$$W_2 = \langle e_2, e_4, \ldots, e_{2k}, e_{2\mu_2+1}, e_{2\mu_2+2}, \ldots, e_{2\mu_2+\mu_3-k}\rangle$$
$$W_1 = \langle e_1, e_2, \ldots, e_{2k}, e_{2(k+1)}, e_{2(k+2)}, \ldots, e_{2\mu_2}, e_{2\mu_2+1}, e_{2\mu_2+2}, \ldots, e_{2\mu_2+\mu_3-k}\rangle$$

Then the only λ_2-dimensional subspaces that occur in G and which intersect with W_2 in 0 and W_1 in a k-dimensional space are of the form

$$\begin{bmatrix} 1 & 1 & & & & & & & & \\ * & & 1 & 1 & & & & & & \\ * & & * & & 1 & 1 & & & & \\ \vdots & & \vdots & & \vdots & & & & & \\ * & & * & & * & & & 1 & 1 & \end{bmatrix}$$

where $* \in \mathbb{F}_q$. Hence the coefficient of W in $G\hat{\Theta}$ is $(-1)^{\lambda_2} q^{\lambda_2(\lambda_2-1)/2}$ and $\hat{\Theta} \neq 0$.

Fix t with $0 \leq t < \mu_3$ and let $\nu = (\mu_1, \mu_2 + \mu_3 - t, t)$. Then

$$X(\Theta\psi_{2,t}) = q^{k(\mu_3-t)} \begin{bmatrix} \mu_2+\mu_3-k-t \\ \mu_3-t \end{bmatrix} \sum\{Z = (Z_0 > Z_1 > Z_2 > Z_3) \mid Z \text{ is a}$$
$$\nu\text{-flag such that } Z_2 \cap X = 0 \text{ and } \dim(Z_1 \cap X) = k\}.$$

Now $\mu_3 - t < e$ and $e \mid \mu_2 - k + 1$. By Lemma 4.1.6, we have that $p \mid \begin{bmatrix} \mu_2+\mu_3-k-t \\ \mu_3-t \end{bmatrix}$, and so $\Theta\psi_{2,t} = 0$.

Now fix t with $0 \leq t < \mu_2$ and let $\nu = (\mu_1 + \mu_2 - t, t, \mu_3)$. Then

$$X(\Theta\psi_{2,t}) = \sum_{r=0}^{k} q^{(\lambda_2-k)(\mu_2-k-t+r)} \begin{bmatrix} \lambda_2 - r \\ k - r \end{bmatrix} \begin{bmatrix} \mu_1 + \mu_2 - t - \lambda_2 + r \\ \mu_2 - k + r - t \end{bmatrix}$$
$$\sum \{Z = (Z_0 > Z_1 > Z_2 > Z_3) \mid Z \text{ is a } \nu\text{-flag}$$
$$\text{such that } Z_2 \cap X = 0 \text{ and } \dim(W_1 \cap X) = r\}$$

by Lemma 4.1.9.

Now if $r \neq k$ then $p \mid \begin{bmatrix} \lambda_2 - r \\ k - r \end{bmatrix}$; and if $r = k$ and $\mu_2 - t < e$ then $p \mid \begin{bmatrix} \mu_1 - t + r \\ \mu_2 - t - k + r \end{bmatrix}$. If $\mu_2 - t \geq e$ then $\mu_1 + \mu_2 - t > \lambda_1$. Hence in all cases $\hat{\Theta}\psi_{1,t} = 0$.

Then if $\mu_2 < e$ we can find a map $\Theta : M^\lambda \to M^\mu$ such that $\operatorname{Im}\Theta \subseteq S^\mu$ and $\ker\theta \not\supseteq S^\lambda$, and otherwise there exists $0 \neq \hat{\Theta} : S^\lambda \to M^\mu$ such that $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$. $\square$

Proposition 4.3.5. *Suppose $\mu_3 < e$ and $l = k = 0$ and $\mu_2 = e - 1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$*

Proof. Suppose $\mu_3 < e$ and $l = k = 0$ and $\mu_2 = e - 1$. Then by the proof of Proposition 4.3.4, there exists a map

$$\Theta : M^{(\mu_1+\mu_3,\mu_2)} \to M^\mu$$

such that

$$\ker\Theta \not\supseteq S^{(\mu_1+\mu_3,\mu_2)} \text{ and } \operatorname{Im}\Theta \subseteq S^\mu.$$

So by Corollary 4.1.4, every composition factor of $S^{(\mu_1+\mu_3,\mu_2)}$ is a composition factor of S^μ. In particular, by Proposition 4.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3,\mu_2)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^μ. $\square$

Proposition 4.3.6. *Suppose $\mu_3 < e$ and $k = 0$ and $s > l > 0$ and $\mu_2 = e - 1$ and $\mu_1 - \mu_2 > e - 1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$*

Proof. Suppose $\mu_3 < e$ and $k = 0$ and $s > l > 0$ and $\mu_2 = e - 1$ and $\mu_1 - \mu_2 > e - 1$. Then by the proof of Proposition 4.3.2, there exists a map

$$\Theta : M^{(\mu_1+\mu_3-l,\mu_2+l)} \to M^\mu$$

such that

$$\ker\Theta \not\supseteq S^{(\mu_1+\mu_3-l,\mu_2+l)} \text{ and } \operatorname{Im}\Theta \subseteq S^\mu.$$

So by Corollary 4.1.4, every composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$ is a composition factor of S^μ. In particular, by Proposition 4.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^μ. □

Proposition 4.3.7. *Suppose $\mu_3 < e$ and $k = 0$ and $l > s$ and $\mu_2 = e - 1$. Then S^μ has a composition factor D^λ where $\lambda = (\mu_1 + e, \mu_2 + \mu_3 - e)$.*

Proof. Suppose $\mu_3 < e$ and $k = 0$ and $l > s$ and $\mu_2 = e - 1$. Then by the proof of Proposition 4.3.2, there exists a map

$$\Theta : M^{(\mu_1+\mu_3-l,\mu_2+l)} \to M^\mu$$

such that

$$\ker\Theta \not\supseteq S^{(\mu_1+\mu_3-l,\mu_2+l)} \text{ and } \operatorname{Im}\Theta \subseteq S^\mu.$$

So by Corollary 4.1.4, every composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$ is a composition factor of S^μ. In particular, by Proposition 4.3.1, $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of $S^{(\mu_1+\mu_3-l,\mu_2+l)}$. Hence $D^{(\mu_1+e,\mu_2+\mu_3-e)}$ is a composition factor of S^μ. □

Note that without a semistandard homomorphism theorem, it is hard to produce an analogue of Corollaries 3.2.15 and 3.2.17 and so more complicated maps do not carry over in such a convenient way.

Remark 4.3.8. An obvious analogue to Theorem 2.3.7 (the column addition theorem) holds for the decomposition matrices of the general linear groups. Thus if $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\lambda_1,\lambda_2)}$ then $S^{(\mu_1+a,\mu_2+a,\mu_3+a)}$ has a composition factor $D^{(\lambda_1+a,\lambda_2+a,a)}$ for any positive integer a. This enables us to determine more composition factors.

4.4 A look at the case $e = 2$

Suppose $e = 2$. We now consider some of the results obtained by Lemma 2.3.15 and Corollary 4.1.3. Suppose that $\mu_3 = 1$. Then the following propositions hold.

Proposition 4.4.1. *If $\mu_1 \equiv 1$ and $\mu_2 = 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+2,\mu_2-1)}$.*

Proof. This follows from the proof of Proposition 4.3.2. □

Proposition 4.4.2. *If $\mu_1 \equiv \mu_2 \equiv 1$ and $\mu_1 - \mu_2 > 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1,\mu_2+1)}$.*

Proof. This follows from the proof of Proposition 4.3.2. □

Proposition 4.4.3. *If $\mu_1 \equiv 0$ and $\mu_2 \equiv 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1,\mu_2+1)}$. This result holds even if $(\mu_1, \mu_2 + 1)$ is not 2-regular.*

Proof. This follows from the proof of Proposition 4.3.3. $\square$

Proposition 4.4.4. *If $\mu_1 \equiv 0$ and $\mu_2 \equiv 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+1,\mu_2)}$.*

Proof. This follows from the proof of Proposition 4.3.4. $\square$

Proposition 4.4.5. *If $\mu_1 \equiv \mu_2 \equiv 0$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+1,\mu_2)}$.*

Proof. This follows from the proof of Proposition 4.3.4. $\square$

Proposition 4.4.6. *If $\mu_1 \equiv 0$ and $\mu_2 = 1$ then $S^{(\mu_1,\mu_2,\mu_3)}$ has a composition factor $D^{(\mu_1+2,\mu_2-1)}$.*

Proof. This follows from the proof of Proposition 4.3.5. $\square$

5 Some results concerning the Specht modules of the symmetric groups

In this chapter, we consider some interesting results about the Specht modules of the symmetric groups. The work contained has been submitted for publication.

5.1 The reducible Specht modules

We wish to consider which ordinary irreducible representations of the symmetric group $\mathfrak{S}_n$ remain irreducible modulo a prime p; this is the same as asking which partitions $\lambda \vdash n$ have the property that the corresponding Specht module S^λ is reducible over a field of characteristic p. If λ or the conjugate partition λ' is p-regular then the answer is known [15, 22]; it is also known in the case $p = 2$ [21]. (Note that S^λ is reducible if and only if $S^{\lambda'}$ is reducible.) For $p \geq 3$, James and Mathas have presented the following conjecture [26].

The Specht module S^λ is reducible if and only if $[\lambda]$ contains a node (a,b) such that $\nu_p(h(a,b)) > 0$ and nodes (x,b) and (a,y) such that $\nu_p(h(x,b)) \neq \nu_p(h(a,b))$ and $\nu_p(h(a,y)) \neq \nu_p(h(a,b))$

where $\nu_p(k)$ is the largest integer such that $p^{\nu_p(k)} \mid k$. This section discusses some of the reducible Specht modules. Throughout, p will be an odd prime and λ a partition of some integer n. We will identify λ with its Young diagram $[\lambda]$. If λ has a removable node of residue i, write λ^i for the partition obtained by removing all removable nodes of residue i from λ. Let $[h_\lambda(a,b)]$ denote the hook indexed by the node (a,b) in λ and $h_\lambda(a,b)$ the length of this hook. All other notation is as described in [2, 14].

Condition A:

Say that λ satisfies Condition A if λ contains a node (a,b) such that $p \mid h_\lambda(a,b)$ and nodes (x,b) and (a,y) such that $p \nmid h_\lambda(x,b)$ and $p \nmid h_\lambda(a,y)$.

Condition B:

Say that λ satisfies Condition B if for some i, λ has a removable node of residue i and λ^i satisfies Condition A.

We will show that if λ satisfies Condition A then S^λ is reducible.

Lemma 5.1.1. *Suppose λ satisfies Condition A with notation as above, and that $[h_\lambda(a,b)]$ has hand of residue r and foot of residue $r+1$. If λ contains a removable node of residue k where $k \neq r, r+1$ then λ^k satisfies Condition A, and hence λ satisfies Condition B.*

Proof. Suppose λ is of the form described in the hypothesis of Lemma 5.1.1 and λ contains a removable node of residue k where $k \neq r, r+1$. Consider λ^k. Since we have removed neither the foot nor the hand from $[h_\lambda(a,b)]$, it is true that $p \mid h_{\lambda^k}(a,b)$.

Now consider the node (x,b). By definition, $p \nmid h_\lambda(x,b)$ and we have not removed the foot of $[h_\lambda(x,b)]$. So if $p \mid h_{\lambda^k}(x,b)$ then we must have removed the hand of $[h_\lambda(x,b)]$, and furthermore, since $[h_{\lambda^k}(x,b)]$ has foot of residue $r+1$, this removed hand must have been of residue $r+1$. But this contadicts our assumption that $k \neq r+1$. Hence $p \nmid h_{\lambda^k}(x,b)$.

A similar argument works to show that $p \nmid h_{\lambda^k}(a,y)$, and so λ^k satisfies Condition A. ☐

Lemma 5.1.2. *Suppose that all removable nodes of λ are of residue i and λ satisfies Condition A. Then λ^i satisfies Condition A.*

Proof. Suppose that λ satisfies Condition A and that all removable nodes are of residue i, but that λ^i does not satisfy Condition A. Then λ contains a node (a,b) such that $p \mid h_\lambda(a,b)$ and nodes (x,b) and (a,y) such that $p \nmid h_\lambda(x,b)$ and $p \nmid h_\lambda(a,y)$. If neither the hand or the foot of $[h_\lambda(a,b)]$ is of residue i then λ^i satisfies Condition A by Lemma 5.1.1, so assume this is not the case.

Suppose the foot of $[h_\lambda(a,b)]$ is of residue i. Then the hand is of residue $i-1$ and is therefore not removable. However there must be a removable node in the same column as this hand, and this must be of residue i. Suppose that this is in position (c, λ_c). Then by construction, $\lambda_c = \lambda_{c-1} = \ldots = \lambda_{c-p+1} = \lambda_a$.

Claim that all of the columns $1, 2, \ldots, \lambda_c$ have lowest node of residue i. We have shown this is true for column λ_c so assume that it is true for column $s+1$. Suppose that $\lambda'_s = \lambda'_{s+1}$. The node (λ'_s, s) then has residue $i-1$. Now consider the hook $[h_\lambda(c-p+2, s)]$. It has hand $(c-p+2, \lambda_c)$ of residue $i-2$ and foot (λ'_s, s) of residue $i-1$. So $p \mid h_\lambda(c-p+2, s)$ but $p \nmid h_\lambda(c-p+2, s+1)$ and $p \nmid h_\lambda(\lambda'_s, s)$. Since $p \geq 3$, it follows from Lemma 5.1.1 that λ^i satisfies Condition A, contradicting our initial assumption. So all of the columns $1, 2, \dots, \lambda_c$ do indeed contain a removable node of residue i.

But if all of the columns $1, 2, \dots, \lambda_c$ have lowest node of residue i then $p \mid h_\lambda(a,1), h_\lambda(a,2), \dots, h_\lambda(a,\lambda_c)$, contradicting our choice of (a,b).

Hence if the foot of $[h_\lambda(a,b)]$ is of residue i, we can find a contradiction to our initial assumption that λ^i does not satisfy Condition A. If the hand of $[h_\lambda(a,b)]$ is of residue i, we can either use a similar argument, or consider the conjugate partition λ'. Therefore if λ satisfies Condition A, λ^i must also satisfy Condition A.

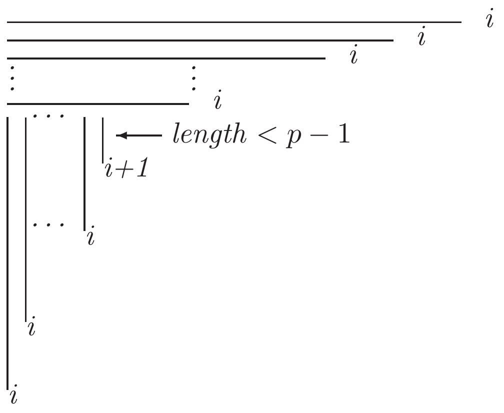
□

Corollary 5.1.3. *Suppose λ satisfies Condition A, but does not satisfy Condition B. Then λ has one of the following forms.*

- *λ has top and bottom removable nodes both of residue i. All removable nodes are of residue i or $i+1$ and λ contains at least one removable node of residue $i+1$.*
- *λ has top and bottom removable nodes both of residue i. All removable nodes are of residue i or $i-1$ and λ contains at least one removable node of residue $i-1$.*
- *λ has top removable node of residue i and bottom removable node of residue $i+1$. All removable nodes are of residue i or $i+1$.*
- *λ has top removable node of residue i and bottom removable node of residue $i-1$. All removable nodes are of residue i or $i-1$.*

Lemma 5.1.4. *Suppose λ satisfies Condition A but not Condition B and λ has top and bottom removable nodes of residue i where all removable nodes are of residue i or $i+1$. Then λ is of the following form.*

- *For some $r > 0$, the nodes $(1,\lambda_1), (2,\lambda_2), \dots, (r,\lambda_r)$ are removable of residue i.*
- *For some $t > 0$, the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \dots, (\lambda'_t, t)$ are removable of residue i.*
- *The node $(\lambda'_{t+1}, t+1)$ is removable of residue $i+1$ and $\lambda'_{t+1} - \lambda'_{t+2} < p-1$.*
- *λ has no removable nodes other than those listed above.*



Say that λ is of Type I.

Proof. Suppose λ is of the form described in the hypothesis of Lemma 5.1.4. Since λ has at least one removable node of residue $i+1$ and the bottom removable node is of residue i, let $(\lambda'_{t+1}, t+1)$ be the highest removable node of residue $i+1$. Suppose that $\lambda'_{t+1} = \lambda'_t$ and consider λ^i. Now λ has a top removable node of residue i in position (c, λ_c) say. So $p \mid h_{\lambda^i}(c, t)$ but $p \nmid h_{\lambda^i}(c, t+1)$ and $p \nmid h_{\lambda^i}(\lambda'_t, t)$ so that λ^i satisfies Condition A.

So $\lambda'_{t+1} \neq \lambda'_t$ and hence column t contains a removable node. By assumption, this is of residue i or $i+1$. Suppose it is of residue $i+1$. Now λ has bottom removable node of residue i, in position (d, λ_d) say. Consider $[h_\lambda(\lambda'_{t+1}+1, \lambda_d)]$. It has foot (d, λ_d) which has residue i and hook length 1, and it has hand $(\lambda'_{t+1}+1, t)$ of residue $i-1$ and hook length congruent to $-1 \mod p$. Therefore by Lemma 5.1.1, λ^{i+1} satisfies Condition A. Column t must therefore contain a removable node of residue i.

Now claim that each of the columns $1, 2, \dots t$ has a removable node of residue i. We have shown that this is true for t. Suppose it holds for column $s+1$. If $\lambda'_{s+1} = \lambda'_s$ then $[h_\lambda(\lambda'_t - p + 2, s)]$ has foot (λ'_s, s) of residue $i-1$ and hook length 2 and has hand $(\lambda'_t - p + 2, t)$ of residue $i-2$ and hook length $p-1$. So by Lemma 5.1.1, λ^i satisfies Condition A.

Column s therefore has a removable node. Suppose it is of residue $i+1$. Then $p \mid h_{\lambda^{i+1}}(\lambda'_t - 1, s)$ but $p \nmid h_{\lambda^{i+1}}(c, s)$ and $p \nmid h_{\lambda^{i+1}}(\lambda'_t - 1, t)$ so that λ^{i+1} satisfies Condition A. Therefore column s must have a removable node of residue i and hence by induction, columns $1, 2, \dots, t$ all have removable nodes of residue i.

Claim that $\lambda'_{t+1} - \lambda'_{t+2} < p-1$. If $\lambda'_{t+1} - \lambda'_{t+2} \geq p-1$ then $[h_\lambda(\lambda'_{t+1} - p + 2, \lambda_d)]$ has hand $(\lambda'_{t+1} - p + 2, t+1)$ of residue $i-1$ and hook length $p-1$ and has foot (d, λ_d) of residue i and hook length 1. So by Lemma 5.1.1 we have that λ^{i+1} satisfies Condition A.

All the removable nodes above $(\lambda'_{t+1}, t+1)$ were assumed to be of residue i and there is at least one of them. Let the lowest removable node above $(\lambda'_{t+1}, t+1)$ lie in position (r, λ_r). Claim that each of the rows $1, 2, \ldots, r$ has a removable node of residue i. This is true for r, so suppose true for $s+1$. If $\lambda_s = \lambda_{s+1}$ then $p \mid h_{\lambda^{i+1}}(s, t+1)$ but $p \nmid h_{\lambda^{i+1}}(s, \lambda_s)$ and $p \nmid h_{\lambda^{i+1}}(s+1, t+1)$ and so λ^{i+1} satisfies Condition A. So row s must have a removable node, which must be of residue i. $\square$

Lemma 5.1.5. *Suppose λ satisfies Condition A but not Condition B and λ has top and bottom removable nodes of residue i where all removable nodes are of residue i or $i-1$. Then λ is of the following form.*

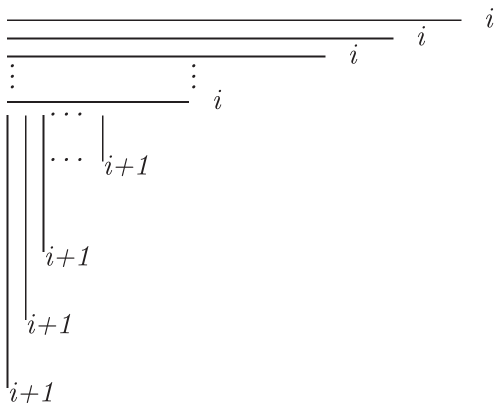
- *For some $r > 0$, the nodes $(1, \lambda_1), (2, \lambda_2), \ldots, (r, \lambda_r)$ are removable of residue i.*
- *The node $(r+1, \lambda_{r+1})$ is removable of residue $i-1$ and $\lambda_{r+1} - \lambda_{r+2} < p-1$.*
- *For some $t > 0$, the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \ldots, (\lambda'_t, t)$ are removable of residue i.*
- *λ has no removable nodes other than those listed above.*

Say that λ is of Type II.

Proof. If λ satisfies the conditions in Lemma 5.1.5 then λ' satisfies the conditions in Lemma 5.1.4. Hence λ' is of Type I and so λ is of Type II. $\square$

Lemma 5.1.6. *Suppose λ satisfies Condition A but not Condition B and λ has top removable node of residue i and bottom removable node of residue $i+1$ where all removable nodes are of residue i or $i+1$. Then λ is of the following form.*

- *For some $r > 0$, the nodes $(1, \lambda_1), (2, \lambda_2), \ldots, (r, \lambda_r)$ are removable of residue i.*
- *For some $t > 0$, the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \ldots, (\lambda'_t, t)$ are removable of residue $i+1$.*
- *λ has no removable nodes other than those listed above.*



Say that λ is of Type III.

Proof. Suppose λ is of the form described the hypothesis of Lemma 5.1.6. Claim that if (s, λ_s) is a removable node of residue i where $s \neq 1$ then $(s-1, \lambda_{s-1})$ is a removable node of residue i.

Suppose the bottom removable node, of residue $i+1$, is in position (d, λ_d). If $\lambda_s = \lambda_{s-1}$ then $p \mid h_{\lambda^{i+1}}(s-1, \lambda_d)$ but $p \nmid h_{\lambda^{i+1}}(s, \lambda_d)$ and $p \nmid h_{\lambda^{i+1}}(s-1, \lambda_{s-1})$ so that λ^{i+1} satisfies Condition A.

So row $s-1$ has a removable node. Suppose it is of residue $i+1$ so that $\lambda_{s-1} \geq \lambda_s + p$ and that the highest removable node of λ, of residue i, is in position (c, λ_c). Then $p \mid h_{\lambda^i}(c, \lambda_{s-1} - 1)$ but $p \nmid h_{\lambda^i}(c, \lambda_{s-1})$ and $p \nmid h_{\lambda^i}(s-1, \lambda_{s-1} - 1)$ so that λ^i satisfies Condition A.

Therefore row $s-1$ has a removable node of residue i, and so for some $r > 0$, the nodes $(1, \lambda_1), (2, \lambda_2), \ldots, (r, \lambda_r)$ are removable of residue i, and there are no other removable nodes of residue i. By considering the conjugate partition λ', we also get that for some $t > 0$, the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \ldots, (\lambda'_t, t)$ are removable of residue $i+1$ and there are no other removable nodes of residue $i+1$. □

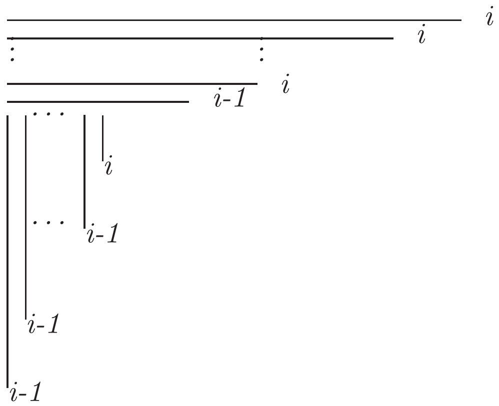
Lemma 5.1.7. *Suppose λ satisfies Condition A but not Condition B and λ has top removable node of residue i and bottom removable node of residue $i-1$ where all removable nodes are of residue i or $i-1$. Then λ is of the following form.*

- *For some $r > 0$, the nodes $(1, \lambda_1), (2, \lambda_2), \ldots, (r, \lambda_r)$ are removable of residue i.*

- *For some $t > 0$, the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \ldots, (\lambda'_t, t)$ are removable of residue $i-1$.*

- *The node $(r+1, \lambda_{r+1})$ is removable of residue $i-1$ and $\lambda_{r+1} - \lambda_{r+2} < p-1$.*

- *The node $(\lambda'_{t+1}, t+1)$ is removable of residue i and $\lambda'_{t+1} - \lambda'_{t+2} < p-1$.*
- *λ has no removable nodes other than those listed above.*



Say that λ is of Type IV.

Proof. Suppose λ is of the form described in the hypothesis of Lemma 5.1.7. Since λ satisfies Condition A, there is a node (a, b) with the usual properties. Suppose it has hand of residue j and foot of residue $j+1$. Then unless $j = i-1$, either λ^i or λ^{i-1} satisfies Condition A by Lemma 5.1.1. Suppose that $j = i-1$ and λ has bottom removable node of residue $i-1$ in position (d, λ_d). Suppose $\lambda_{a+1} = \lambda_a$. Then $p \mid h_\lambda(a+1, \lambda_d)$ where this hook has foot of residue $i-1$ and hand of residue $i-2$, and $p \nmid h_\lambda(d, \lambda_d)$ and $p \nmid h_\lambda(a+1, b)$. So by Lemma 5.1.1, λ^i satisfies Condition A. Hence $\lambda_{a+1} \neq \lambda_a$ and the node (a, λ_a) is removable. A similar argument shows that the node (λ'_b, b) is removable.

So λ has at least one hook with hand of residue i and foot of residue $i-1$ where both hand and foot are removable. Since the node indexing this hook does not lie in the first row or the first column, let $(\lambda'_{t+1}, t+1)$ be a removable i-node such that the lowest removable node above it, $(r+1, \lambda_{r+1})$ say, is of residue $i-1$. Claim that the nodes $(\lambda'_1, 1), (\lambda'_2, 2), \ldots, (\lambda'_t, t)$ are all removable of residue $i-1$ and that $\lambda'_{t+1} - \lambda'_{t+2} < p-1$; this proof follows exactly along the lines of the proof of Lemma 5.1.4. A similar proof, or consideration of the conjugate partition λ', which also has properties as in the hypothesis of Lemma 5.1.7, gives that the nodes $(1, \lambda_1), (2, \lambda_2), \ldots, (r, \lambda_r)$ are all removable of residue i and that $\lambda_{r+1} - \lambda_{r+2} < p-1$. □

Corollary 5.1.8. *If λ satisfies Condition A but does not satisfy Condition B then λ is of Type I, II, III or IV as defined above.*

Lemma 5.1.9. *The Specht module S^λ has a composition factor $D^{\lambda_{reg}}$ where λ_{reg} is formed from λ by moving all nodes as high as possible on their p-ladders. Moreover,*

λ_{reg} is such that if D^ν is a composition factor of S^λ then $\nu \trianglerighteq \lambda_{reg}$. The proof is given in [12].

Lemma 5.1.10. *If λ is of Type I then S^λ is reducible.*

Proof. Consider the partition μ formed by moving the node $(\lambda'_{t+1}, t+1)$ to the position $(r, \lambda_r + 1)$. By a theorem of Carter and Payne [4, 5], there exists $0 \neq \hat{\Theta} : S^\mu \to S^\lambda$. Hence S^μ and S^λ have a common composition factor, D^ν say, such that $\nu \trianglerighteq \mu_{reg}$.

S^λ also has a composition factor $D^{\lambda_{reg}}$. Consider the p-ladder containing the node $(\lambda'_{t+1}, t+1)$. In λ this ladder is full since $\lambda'_{t+1} - \lambda'_{t+2} < p-1$ but since we then remove a node, the ladder is not full in μ.

This means that λ_{reg} has strictly more parts than μ_{reg}; in particular $\lambda_{reg} \ntrianglerighteq \mu_{reg}$ and hence $\lambda_{reg} \neq \nu$. Therefore S^λ has distinct composition factors $D^{\lambda_{reg}}$ and D^ν and hence is reducible. □

Lemma 5.1.11. *If λ is of Type II then S^λ is reducible.*

Proof. If λ is of Type II then λ' is of Type I and $S^{\lambda'}$ is reducible. Hence S^λ is reducible. □

Lemma 5.1.12. *If λ is of Type III then S^λ is reducible.*

Proof. Assume that $\lambda_1 \geq \lambda'_1$ since, if not, λ' is a partition of Type III where the first row is at least as long as the first column. Then $\lambda_1 = (\lambda_{reg})_1$. Form the partition μ by moving the node $(\lambda'_1, 1)$ to the position $(1, \lambda_1 + 1)$ and repeat the proof of Lemma 5.1.10. □

Lemma 5.1.13. *If λ is of Type IV then S^λ is reducible.*

Proof. Consider the partition μ formed by moving the node $(\lambda'_{t+1}, t+1)$ to the position $(r+1, \lambda_{r+1} + 1)$ and then use the same proof as in Lemma 5.1.10. □

Corollary 5.1.14. *If λ satisfies Condition A but does not satisfy Condition B then S^λ is reducible.*

Lemma 5.1.15. *Suppose λ has a removable node of residue i such that S^{λ^i} is reducible. Then S^λ is reducible.*

Proof. This follows from a result by Brundan and Kleshchev [2, Lemma 2.13]. □

Theorem 5.1.16. *If λ satisfies Condition A then S^λ is reducible.*

Proof. The proof is by induction on n, where the case $n = 0$ is trivially true. Assume that Theorem 5.1.16 holds for $m < n$ and let $\lambda \vdash n$ satisfy Condition A. If λ satisfies Condition B then for some i, we know that λ has a removable node of residue i and λ^i satisfies Condition A. By induction hypothesis, S^{λ^i} is reducible. Therefore S^λ is reducible by Lemma 5.1.15. If λ does not satisfy Condition B then, by Corollary 5.1.14, S^λ is reducible.

Hence Theorem 5.1.16 holds for all n.

□

We end this section by mentioning a conjecture that arose while considering reducible Specht modules; it provided somewhat weaker results than those of Theorem 5.1.16, and has not been proved. However, if true, it may have some interesting corollaries.

Definition 5.1.17. Suppose $\lambda, \nu \vdash n$ such that ν is p-regular. Let ν^M denote the image of ν under the Mullineux map [27] and let λ_{reg} denote the regularisation of λ as defined in Lemma 5.1.9. Then $S^{\lambda'}$ has composition factors $D^{(\lambda')_{reg}}$ and $D^{(\lambda_{reg})^M}$ [10].

Then if $(\lambda')_{reg} \neq (\lambda_{reg})^M$, we have that $S^{\lambda'}$, and hence S^λ, is reducible.

Conjecture 5.1.18. *Let $\lambda \vdash n$. Then $(\lambda')_{reg} \neq (\lambda_{reg})^M$ if and only if $[\lambda]$ contains a node (i, j) such that $h(i, j) = ap$ for some integer a where*

$$a \leq \min \begin{cases} \textit{arm length } [h(i,j)] \\ \textit{leg length } [h(i,j)] \end{cases} .$$

5.2 Homomorphisms between Specht modules I

Section 5.2 consists of joint work with Matthew Fayers [9].

In [16], James proved two theorems regarding the decomposition numbers for the symmetric groups; these concern the removal of a row or a column from a partition diagram. These results were generalised by Donkin in [8]. We are interested in the space of $F\mathfrak{S}_n$-homomorphisms between the Specht modules S^λ and S^μ, for λ and μ partitions of n and shall prove the following two theorems which are direct analogues of Donkin's results.

Throughout this section, we assume that F is a field of characteristic not two.

Theorem 5.2.1. *Let λ and μ be partitions of n, and suppose that for some s we have*

$$\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s = n - m.$$

Define

$$\begin{aligned}
\lambda^{\mathrm{T}} &= (\lambda_1, \dots, \lambda_s),\\
\lambda^{\mathrm{B}} &= (\lambda_{s+1}, \lambda_{s+2}, \dots),\\
\mu^{\mathrm{T}} &= (\mu_1, \dots, \mu_s),\\
\mu^{\mathrm{B}} &= (\mu_{s+1}, \mu_{s+2}, \dots).
\end{aligned}$$

Then we have

$$\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$$

as F-vector spaces.

Theorem 5.2.2. *Let λ and μ be partitions of n, and suppose that for some r we have*

$$\lambda'_1 + \cdots + \lambda'_r = \mu'_1 + \cdots + \mu'_r = n - l.$$

Define

$$\begin{aligned}
\lambda^{\mathrm{L}} &= \big(\min(\lambda_1, r), \min(\lambda_2, r), \dots\big),\\
\lambda^{\mathrm{R}} &= \big(\max(\lambda_1 - r, 0), \max(\lambda_2 - r, 0), \dots\big),\\
\mu^{\mathrm{L}} &= \big(\min(\mu_1, r), \min(\mu_2, r), \dots\big),\\
\mu^{\mathrm{R}} &= \big(\max(\mu_1 - r, 0), \max(\mu_2 - r, 0), \dots\big).
\end{aligned}$$

Then we have

$$\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{F\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{L}}}, S^{\mu^{\mathrm{L}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}})$$

as F-vector spaces.

Proposition 5.2.3. *Theorems 5.2.1 and 5.2.2 are equivalent.*

Proof. Let sgn denote the one-dimensional sign representation, and $(S^\lambda)^*$ the dual of S^λ. Recall [17, Theorem 8.15]:

$$S^{\lambda'} \cong (S^\lambda)^* \otimes \mathrm{sgn}\,.$$

Note also that, since sgn is one-dimensional, we have

$$\mathrm{Hom}_{F\mathfrak{S}_n}(M \otimes \mathrm{sgn}, N \otimes \mathrm{sgn}) \cong \mathrm{Hom}_{F\mathfrak{S}_n}(M, N)$$

as vector spaces, for any $F\mathfrak{S}_n$-modules M and N.

Now if λ and μ satisfy the conditions of Theorem 5.2.1, then λ' and μ' satisfy the conditions of Theorem 5.2.2, with $r = s$, $l = m$ and with

$$\begin{aligned}(\lambda')^{\mathrm{L}} &= (\lambda^{\mathrm{T}})',\\ (\lambda')^{\mathrm{R}} &= (\lambda^{\mathrm{B}})',\\ (\mu')^{\mathrm{L}} &= (\mu^{\mathrm{T}})',\\ (\mu')^{\mathrm{R}} &= (\mu^{\mathrm{B}})'.\end{aligned}$$

So we have

$$\begin{aligned}\mathrm{Hom}_{F\mathfrak{S}_n}(S^{\mu'}, S^{\lambda'}) &\cong \mathrm{Hom}_{F\mathfrak{S}_n}\big((S^{\mu})^* \otimes \mathrm{sgn}, (S^{\lambda})^* \otimes \mathrm{sgn}\big)\\ &\cong \mathrm{Hom}_{F\mathfrak{S}_n}\big((S^{\mu})^*, (S^{\lambda})^*\big)\\ &\cong \mathrm{Hom}_{F\mathfrak{S}_n}(S^{\lambda}, S^{\mu})\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\mu'^{\mathrm{L}}}, S^{\lambda'^{\mathrm{L}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\mu'^{\mathrm{R}}}, S^{\lambda'^{\mathrm{R}}})\\ &\qquad \cong \mathrm{Hom}_{F\mathfrak{S}_{n-m}}\big((S^{\mu^{\mathrm{T}}})^* \otimes \mathrm{sgn}, (S^{\lambda^{\mathrm{T}}})^* \otimes \mathrm{sgn}\big)\\ &\qquad\quad \otimes \mathrm{Hom}_{F\mathfrak{S}_m}\big((S^{\mu^{\mathrm{B}}})^* \otimes \mathrm{sgn}, (S^{\lambda^{\mathrm{B}}})^* \otimes \mathrm{sgn}\big)\\ &\qquad \cong \mathrm{Hom}_{F\mathfrak{S}_{n-m}}\big((S^{\mu^{\mathrm{T}}})^*, (S^{\lambda^{\mathrm{T}}})^*\big)\\ &\qquad\quad \otimes \mathrm{Hom}_{F\mathfrak{S}_m}\big((S^{\mu^{\mathrm{B}}})^*, (S^{\lambda^{\mathrm{B}}})^*\big)\\ &\qquad \cong \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}}).\end{aligned}$$

□

We shall prove Theorem 5.2.1 in the following manner: given homomorphisms $V : S^{\lambda^{\mathrm{T}}} \to S^{\mu^{\mathrm{T}}}$ and $W : S^{\lambda^{\mathrm{B}}} \to S^{\mu^{\mathrm{B}}}$, we shall define a homomorphism $(V, W) : S^{\lambda} \to M^{\mu}$ and show that $(V, W)\psi_{d,t} = 0$ for all d and t, so that the image of (V, W) lies in S^{μ}; we proceed similarly in the other direction.

Let λ be a partition of n and ν a composition of n with $\lambda_1 + \cdots + \lambda_s = \nu_1 + \cdots + \nu_s = n - m$. Define λ^{T} and λ^{B} as in Theorem 5.2.1, and ν^{T} and ν^{B} analogously.

Definition 5.2.4. Recall that $\mathcal{T}(\lambda, \nu)$ form the set of λ-tableaux of type ν and that $\mathcal{T}_0(\lambda, \nu)$ form the set of semistandard λ-tableaux of type ν. Suppose that $R \in \mathcal{T}(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ has entry x^i_j in position i of row j and $S \in \mathcal{T}(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$ has entry y^i_j in position i of row j. Define (R, S) to be the λ-tableau of type ν with entry z^i_j in position i of row j, where

$$z_j^i = \begin{cases} x_j^i & j \leq s \\ y_j^i + s & j > s \end{cases}.$$

Lemma 5.2.5. *If R and S are semistandard then so is (R, S), and $\{(R, S) \mid R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}}), S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \mu^{\mathrm{B}})\}$ is precisely the set of semistandard λ-tableaux of type ν.*

Proof. The first statement is clear. Suppose T is a semistandard λ-tableau of type ν with entry z_j^i in position i of row j. Then the entries of T strictly increase down the columns, so that for $j > s$ we have that $z_j^i > s$. Since the number, $n - m$, of entries less than or equal to s equals the number of positions in rows $1, \ldots, s$, it also is true that $z_j^i \leq s$ for $j \leq s$. So the top s rows of T constitute a λ^{T}-tableau of type ν^{T} (which is clearly semistandard), and a similar statement holds for the lower rows. □

Definition 5.2.6. Suppose

$$V = \sum_{R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})} a_R \hat{\Theta}_R \in \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\nu^{\mathrm{T}}})$$

and

$$W = \sum_{S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})} b_S \hat{\Theta}_S \in \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, M^{\nu^{\mathrm{B}}}).$$

Define $(V, W) \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\nu)$ by

$$(V, W) = \sum_R \sum_S a_R b_S \hat{\Theta}_{(R,S)}.$$

Lemma 5.2.7. *Suppose $U \in \mathcal{T}(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ and $S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$. Then*

$$\hat{\Theta}_{(U,S)} = (\hat{\Theta}_U, \hat{\Theta}_S).$$

Proof. By Theorem 3.2.9, we may write

$$\hat{\Theta}_U = \sum_{i=1}^{a} c_i \hat{\Theta}_{R_i}$$

with $R_1, \ldots, R_a$ semistandard. We then need to prove that

$$\hat{\Theta}_{(U,S)} = \sum_{i=1}^{a} c_i \hat{\Theta}_{(R_i,S)}.$$

Apply both homomorphisms to an arbitrary λ-polytabloid e_t, and compare coefficients of an arbitrary ν-tabloid ρ. Note that if π is in the column stabiliser of t then

$$e_{t\pi}\hat{\Theta}_{(U,S)} = (\epsilon(\pi))e_t\hat{\Theta}_{(U,S)} = (e_t\hat{\Theta}_{(U,S)})\pi$$

and

$$e_{t\pi}(\hat{\Theta}_U, \hat{\Theta}_S) = (\epsilon(\pi))e_t(\hat{\Theta}_U, \hat{\Theta}_S) = (e_t(\hat{\Theta}_U, \hat{\Theta}_S))\pi.$$

Suppose ρ is a ν-tabloid which contains distinct entries, a and b, from the same column of t in one row. Then the coefficient of ρ in the image of e_t under both maps is equal to its coefficient in the image of $e_{t(a\,b)}$, and hence must be zero. We may therefore assume that the entries of any column of t occur in distinct rows of ρ. Now suppose column k of t contains the numbers $t_{k,1}, t_{k,2}, \dots$ where $t_{k,1} < t_{k,2} < \dots$. We can choose π in the column stabiliser of t such that, in $\rho\pi$, the entry $t_{k,i}$ occurs in a lower row than $t_{k,j}$ for any $j < i$ and any k. Then, in $\rho\pi$, the entries $t_{k,s+1}, t_{k,s+2}, \dots$ cannot occur in rows $1, \dots, s$. They must lie in rows $s+1, s+2, \dots$, and therefore they fill rows $s+1, s+2, \dots$. Then the top s entries of each column of t occur in the top s rows of $\rho\pi$. By replacing ρ with $\rho\pi$, we may simply assume that ρ has this property.

Let t^{T} be the λ^{T}-tableau consisting of the first s rows of t, and t^{B} the λ^{B}-tableau consisting of the remaining rows, and define ρ^{T} and ρ^{B} similarly. Then ρ^{T} is a tabloid with entries in the same set as t^{T}, and ρ^{B} is a tabloid with entries in the same set as t^{B}. So the coefficient of ρ in $e_t\hat{\Theta}_{(U,S)}$ equals

$$\begin{aligned}
&(\text{coefficient of } \rho^{\mathrm{T}} \text{ in } e_{t^{\mathrm{T}}}\hat{\Theta}_U) \times (\text{coefficient of } \rho^{\mathrm{B}} \text{ in } e_{t^{\mathrm{B}}}\hat{\Theta}_S)\\
&= \sum_{i=1}^{a} c_i(\text{coefficient of } \rho^{\mathrm{T}} \text{ in } e_{t^{\mathrm{T}}}\hat{\Theta}_{R_i}) \times (\text{coefficient of } \rho^{\mathrm{B}} \text{ in } e_{t^{\mathrm{B}}}\hat{\Theta}_S)\\
&= \text{coefficient of } \rho \text{ in } e_t\left(\sum_{i=1}^{a} c_i\hat{\Theta}_{(R_i,S)}\right).
\end{aligned}$$

□

The following result is then easy to see.

Lemma 5.2.8. *Suppose $U \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \nu^{\mathrm{T}})$ and $S \in \mathcal{T}(\lambda^{\mathrm{B}}, \nu^{\mathrm{B}})$. Then*

$$\hat{\Theta}_{(U,S)} = (\hat{\Theta}_U, \hat{\Theta}_S).$$

Now we suppose that λ and μ are as in Theorem 5.2.1.

Lemma 5.2.9. *If* $V \in \operatorname{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\mu^{\mathrm{T}}})$ *and* $W \in \operatorname{Hom}_{F\mathfrak{S}_n}(S^{\lambda^{\mathrm{B}}}, M^{\mu^{\mathrm{B}}})$*, then*

$$(V,W)\psi_{d,t} = \begin{cases} (V\psi_{d,t}, W) & d < s \\ (V, W\psi_{d-s,t}) & d > s \end{cases}.$$

Proof. We deal with the case where $d < s$; the other case follows similarly. By linearity, we need only show that

$$(\hat{\Theta}_R, \hat{\Theta}_S)\psi_{d,t} = (\hat{\Theta}_R\psi_{d,t}, \hat{\Theta}_S)$$

for $R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})$, $S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \mu^{\mathrm{B}})$. Let $U_{d,t}$ be the set of λ^{T}-tableaux which may be obtained from R by replacing $\lambda_{d+1} - t$ entries equal to $d+1$ with ds. Then

$$(\hat{\Theta}_R\psi_{d,t}, \hat{\Theta}_S) = \left(\sum_{U \in U_{d,t}} \hat{\Theta}_U, \hat{\Theta}_S \right) = \sum_{U \in U_{d,t}} \hat{\Theta}_{(U,S)}$$

by Lemma 5.2.7 and

$$(\hat{\Theta}_R, \hat{\Theta}_S)\psi_{d,t} = \hat{\Theta}_{(R,S)}\psi_{d,t} = \sum_{U \in U_{d,t}} \hat{\Theta}_{(U,S)}$$

as required. $\square$

Lemma 5.2.10. *Suppose* $V \in \operatorname{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}})$ *and* $W \in \operatorname{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$*. Then* $(V,W) \in \operatorname{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu)$*.*

Proof. We have that $(V,W) \in \operatorname{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu)$ if $(V,W)\psi_{d,t} = 0$ for all d and t. Let ν be the composition such that $\psi_{d,t} : M^\mu \to M^\nu$.

If $d = s$ then (since $\lambda_1 + \cdots + \lambda_s = \mu_1 + \cdots + \mu_s$) we have $\lambda \not\trianglerighteq \nu$ and so $\operatorname{Hom}(S^\lambda, M^\nu) = 0$. Hence $(V,W)\psi_{d,t} = 0$.

If $d < s$ then by Lemma 5.2.9 we have

$$(V,W)\psi_{d,t} = (V\psi_{d,t}, W),$$

which is zero, since the image of V lies in $S^{\mu^{\mathrm{T}}}$. Similarly for $d > s$. $\square$

Definition 5.2.11. Let

$$\phi : \operatorname{Hom}_{\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, M^{\mu^{\mathrm{T}}}) \otimes \operatorname{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, M^{\mu^{\mathrm{B}}}) \longrightarrow \operatorname{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu)$$

be given by

$$V \otimes W \longrightarrow (V,W).$$

Note that ϕ is a linear bijection.

Theorem 5.2.12. *Let $\hat{\phi}$ denote the restriction of ϕ to* $\mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$*. Then $\hat{\phi}$ is a bijective mapping between* $\mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$ *and* $\mathrm{Hom}_{F\mathfrak{S}_n}(S^{\lambda}, S^{\mu})$*.*

Proof. Suppose that $V \otimes W \in \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$. Then by Lemma 5.2.10, $(V \otimes W)\hat{\phi} \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^{\lambda}, S^{\mu})$. Also since ϕ is injective, we have that $\hat{\phi}$ is injective. It remains only to show that $\hat{\phi}$ is surjective.

Suppose that $\{A_i \mid i \in I\}$ is a basis of $\mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}})$ and that $\{A_j \mid j \in J\}$ is a basis of $\mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$. Let

$$Z = \sum_{R \in \mathcal{T}_0(\lambda^{\mathrm{T}}, \mu^{\mathrm{T}})} \sum_{S \in \mathcal{T}_0(\lambda^{\mathrm{B}}, \mu^{\mathrm{B}})} c_{R,S}(\hat{\Theta}_R, \hat{\Theta}_S) \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^{\lambda}, S^{\mu}).$$

Then $\left(\sum\limits_R \sum\limits_S c_{R,S}(\hat{\Theta}_R, \hat{\Theta}_S)\right) \psi_{d,t} = 0$ for all d. When $d > s$, this gives

$$\sum_R \sum_S (\hat{\Theta}_R, c_{R,S}\hat{\Theta}_S \psi_{d-s,t}) = 0$$

for $d > s$. Since ϕ is injective, we get

$$\sum_S c_{R,S}\hat{\Theta}_S \psi_{d-s,t} = 0$$

for every R, so that $\sum\limits_S c_{R,S}\hat{\Theta}_S \in \mathrm{Hom}_{F\mathfrak{S}_m}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$ for every R. Writing

$$\sum_S c_{R,S}\hat{\Theta}_S = \sum_{j \in J} \alpha_{R,j} A_j$$

for $\alpha_{R,j} \in F$, we have

$$\left(\sum_{j \in J} \sum_R \alpha_{R,j}(\hat{\Theta}_R, A_j)\right) \psi_{d,t} = 0$$

for all $d < s$. The injectivity of ϕ gives

$$\sum_R \alpha_{R,j} \hat{\Theta}_R \psi_{d,t} = 0$$

for every j when $d < s$, so that

$$\sum_R \alpha_{R,j} \hat{\Theta}_R \in \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}).$$

Writing $\sum_R \alpha_{R,j}\hat{\Theta}_R = \sum_{i\in I} \beta_{i,j} A_i$ for each j, we have

$$Z = \sum_{i\in I}\sum_{j\in J} \beta_{i,j}(A_i, A_j) = Z'\hat{\phi},$$

where

$$Z' = \sum_{i,j} \beta_{i,j}(A_i \otimes A_j).$$

This completes the proof. □

Remark 5.2.13. It is not clear whether Theorem 5.2.1 is true if F has characteristic 2. In characteristic 2, Theorem 3.2.9 does not generally hold if λ is 2-singular, and it is not certain whether Lemmas 5.2.7 and 5.2.8 hold.

We now consider Theorem 5.2.2. By Proposition 5.2.3, we have

$$\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \mathrm{Hom}_{F\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{L}}}, S^{\mu^{\mathrm{L}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}})$$

as F-vector spaces. We wish to exibit such a linear bijection. Assume now that λ and μ satisfy the conditions of Theorem 5.2.2 and that $\lambda^{\mathrm{L}} \trianglerighteq \mu^{\mathrm{L}}$ and $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$; there is no loss in doing this, since otherwise all the homomorphism spaces involved are zero.

Definition 5.2.14. Let λ^{L}, λ^{R}, μ^{L} and μ^{R} be as in Theorem 5.2.2. Given a λ^{L} tableau T^{L} of type μ^{L} and a λ^{R}-tableau T^{R} of type μ^{R}, define the λ-tableau $T^{\mathrm{L}}|T^{\mathrm{R}}$ (of type μ) simply by juxtaposing T^{L} and T^{R}.

Lemma 5.2.15. *Suppose T^{L} and T^{R} are semistandard tableaux as described in Definition 5.2.14. Then $T^{\mathrm{L}}|T^{\mathrm{R}}$ is semistandard, and $\{T^{\mathrm{L}}|T^{\mathrm{R}} \mid T^{\mathrm{L}} \in \mathcal{T}_0(\lambda^{\mathrm{L}}, \mu^{\mathrm{L}}), T^{\mathrm{R}} \in \mathcal{T}_0(\lambda^{\mathrm{R}}, \mu^{\mathrm{R}})\}$ is precisely the set of semistandard λ-tableaux of type μ.*

Proof. Suppose T^{L} is a semistandard λ^{L} tableau of type μ^{L} and T^{R} is a semistandard λ^{R} tableau of type μ^{R}. Then the entries of $T^{\mathrm{L}}|T^{\mathrm{R}}$ strictly increase down the columns and increase across the rows from column 1 to r and from column $r+1$ to λ_1. Now suppose that μ^{R} has exactly s non-zero parts. Then $\mu^{\mathrm{L}} = (r^s, \mu_{s+1}, \mu_{s+2}\ldots)$ and since $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$ we have that $\lambda^{\mathrm{L}} = (r^s, \lambda_{s+1}, \lambda_{s+2}\ldots)$. Consider T^{L}. For $1 \le j \le s$, row j consists of entries of j (since T^{L} is semistandard), and if x lies in row j of T^{R} then $x \ge j$ from the fact that T^{R} is semistandard. Furthermore, $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$ and so λ^{R} has at most s non-zero parts. Thus $T^{\mathrm{L}}|T^{\mathrm{R}}$ is semistandard as required.

Now suppose that T is a semistandard λ tableau of type μ. Define T^{L} and T^{R} in the obvious manner. Now $(\lambda^{\mathrm{L}})_1 + (\lambda^{\mathrm{L}})_2 + \ldots + (\lambda^{\mathrm{L}})_s = (\mu^{\mathrm{L}})_1 + (\mu^{\mathrm{L}})_2 + \ldots + (\mu^{\mathrm{L}})_s$ and so $(\lambda^{\mathrm{L}})_{s+1} + (\lambda^{\mathrm{L}})_{s+2} + \ldots = (\mu^{\mathrm{L}})_{s+1} + (\mu^{\mathrm{L}})_{s+2} + \ldots = \lambda_{s+1} + \lambda_{s+2} + \ldots =$

$\lambda_{s+1} + \lambda_{s+2} + \dots$. Then for $j > s$, the entries in row j must be greater than s. But the number of entries greater than s is equal to the number of positions in rows $s+1, s+2, \dots$. Hence the numbers $s+1, s+2, \dots$ are all contained in the rows $s+1, s+2, \dots$, and so for $1 \leq j \leq s$, the entries in row j of T^{L} must all be equal to j. Hence T^{L} is a λ^{L} tableau of type μ^{L} and T^{R} is a λ^{R} tableau of type μ^{R} as required. □

Now for λ^{L}- and λ^{R}-tabloids u^{L} and u^{R}, define $u^{\mathrm{L}}|u^{\mathrm{R}}$ similarly, and extend this to the whole of M^λ, so that if $x^{\mathrm{L}} = \sum_{u^{\mathrm{L}}} c_{u^{\mathrm{L}}} u^{\mathrm{L}}$ is a linear combination of λ^{L}-tabloids on $\{1, \dots, n-l\}$, and $x^{\mathrm{R}} = \sum_{u^{\mathrm{R}}} c_{u^{\mathrm{R}}} u^{\mathrm{R}}$ is a linear combination of λ^{R}-tabloids on $\{n-1+1, \dots, n\}$, then we define

$$x^{\mathrm{L}}|x^{\mathrm{R}} = \sum_{u^{\mathrm{L}}, u^{\mathrm{R}}} c_{u^{\mathrm{L}}} c_{u^{\mathrm{R}}} \left(u^{\mathrm{L}}|u^{\mathrm{R}}\right);$$

we make similar definitions for μ^{L}- and μ^{R}-tableaux and tabloids. It is then clear that for a μ^{L}-tableau t^{L} on $\{1, \dots, n-l\}$, and a μ^{R}-tableau t^{R} on $\{n-l+1, \dots, n\}$, we have

$$e_{t^{\mathrm{L}}|t^{\mathrm{R}}} = e_{t^{\mathrm{L}}}|e_{t^{\mathrm{R}}}.$$

Lemma 5.2.16. *Suppose T^{L} is a semistandard λ^{L}-tableau of type μ^{L} and that T^{R} is a semistandard λ^{R}-tableau of type μ^{R}. Suppose also that $S \sim_{row} T^{\mathrm{L}}|T^{\mathrm{R}}$ is such that each column has distinct entries. Then $S = U^{\mathrm{L}}|U^{\mathrm{R}}$ for some U^{L} and U^{R} such that $U^{\mathrm{L}} \sim_{row} T^{\mathrm{L}}$ and $U^{\mathrm{R}} \sim_{row} T^{\mathrm{R}}$.*

Proof. Let s be the number of non-zero parts of μ^{R}. Then, since we are assuming $\lambda^{\mathrm{L}} \trianglerighteq \mu^{\mathrm{L}}$ and $\lambda^{\mathrm{R}} \trianglerighteq \mu^{\mathrm{R}}$, we have

$$\lambda_1^{\mathrm{L}} = \dots = \lambda_s^{\mathrm{L}} = \mu_1^{\mathrm{L}} = \dots = \mu_s^{\mathrm{L}} = r,$$

and we know that the number of non-zero parts of λ^{R} is at most s. So, for $1 \leq i \leq s$, the ith row of T^{L} consists entirely of is. We must therefore show that the ith row of S begins with r is, for all such i. Suppose we have shown this for $i = s, s-1, \dots, j+1$, and consider row j. The entries of T^{R} are all at most s, while the entries in row j of S are all at least j (since T^{L} and T^{R} are semistandard). So the first r entries of row j of S all lie between j and s. But none of them can be greater than j, since then S would have equal entries in some column. The result follows. □

Now suppose T^{L} is a semistandard λ^{L}-tableau of type μ^{L}, and T^{R} a semistandard λ^{R}-tableau of type μ^{R}. In order to calculate the image of a λ-polytabloid under $\hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}}$, we need only consider row permutations of $T^{\mathrm{L}}|T^{\mathrm{R}}$ in which each column has distinct entries. By Lemma 5.2.16, any such row permutation is of the form

$U^{\mathrm{L}}|U^{\mathrm{R}}$, where U^{L} and U^{R} are row permutations of T^{L} and T^{R} respectively. For a λ^{L}-tableau t^{L} on $\{1,\ldots,n-l\}$ and a λ^{R}-tableau t^{R} on $\{n-l+1,\ldots,n\}$, this gives us

$$(e_{t^{\mathrm{L}}|t^{\mathrm{R}}})\hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}} = \big((e_{t^{\mathrm{L}}})\hat{\Theta}_{T^{\mathrm{L}}}\big)\big|\big((e_{t^{\mathrm{R}}})\hat{\Theta}_{T^{\mathrm{R}}}\big).$$

Hence we have the following.

Theorem 5.2.17. *Let t^{L} be a λ^{L}-tableau on $\{1,\ldots,n-l\}$, and t^{R} a λ^{R}-tableau on $\{n-l+1,\ldots,n\}$. Suppose that*

$$\hat{\Theta}^{\mathrm{L}} = \sum_{T^{\mathrm{L}}\in\mathcal{T}_0(\lambda^{\mathrm{L}},\mu^{\mathrm{L}})} c_{T^{\mathrm{L}}}\hat{\Theta}_{T^{\mathrm{L}}}$$

is a homomorphism from $S^{\lambda^{\mathrm{L}}}$ to $M^{\mu^{\mathrm{L}}}$ with

$$(e_{t^{\mathrm{L}}})\hat{\Theta}^{\mathrm{L}} = \sum_{s^{\mathrm{L}}} d_{s^{\mathrm{L}}}e_{s^{\mathrm{L}}},$$

for some μ^{L}-tableaux s^{L} on $\{1,\ldots,n-l\}$, and some $c_{T^{\mathrm{L}}}, d_{s^{\mathrm{L}}}\in F$, and similarly suppose that

$$\hat{\Theta}^{\mathrm{R}} = \sum_{T^{\mathrm{R}}\in\mathcal{T}_0(\lambda^{\mathrm{R}},\mu^{\mathrm{R}})} c_{T^{\mathrm{R}}}\hat{\Theta}_{T^{\mathrm{R}}}$$

is a homomorphism from $S^{\lambda^{\mathrm{R}}}$ to $M^{\mu^{\mathrm{R}}}$ with

$$(e_{t^{\mathrm{R}}})\hat{\Theta}^{\mathrm{R}} = \sum_{s^{\mathrm{R}}} d_{s^{\mathrm{R}}}e_{s^{\mathrm{R}}},$$

for μ^{R}-tableaux s^{R} on $\{n-l+1,\ldots,n\}$.

Then the homomorphism

$$\hat{\Theta} = \sum_{T^{\mathrm{L}}\in\mathcal{T}_0(\lambda^{\mathrm{L}},\mu^{\mathrm{L}})}\ \sum_{T^{\mathrm{R}}\in\mathcal{T}_0(\lambda^{\mathrm{R}},\mu^{\mathrm{R}})} c_{T^{\mathrm{L}}}c_{T^{\mathrm{R}}}\hat{\Theta}_{T^{\mathrm{L}}|T^{\mathrm{R}}}$$

satisfies

$$(e_{t^{\mathrm{L}}|t^{\mathrm{R}}})\hat{\Theta} = \sum_{s^{\mathrm{L}}}\sum_{s^{\mathrm{R}}} d_{s^{\mathrm{L}}}d_{s^{\mathrm{R}}}e_{s^{\mathrm{L}}|s^{\mathrm{R}}};$$

in particular, the image of $\hat{\Theta}$ lies in the Specht module S^{μ}.

Definition 5.2.18. Suppose

$$V = \sum_{T^{\mathrm{L}}\in\mathcal{T}_0(\lambda^{\mathrm{L}},\mu^{\mathrm{L}})} a_{T^{\mathrm{L}}}\hat{\Theta}_{\mathrm{L}} \in \mathrm{Hom}_{F\mathfrak{S}_{n-m}}(S^{\lambda^{\mathrm{L}}}, M^{\mu^{\mathrm{L}}})$$

and

$$W = \sum_{T^{\mathrm{R}} \in \mathcal{T}_0(\lambda^{\mathrm{R}}, \mu^{\mathrm{R}})} b_{T^{\mathrm{R}}} \hat{\Theta}_{\mathrm{R}} \in \mathrm{Hom}_{F\mathfrak{S}_m}(\lambda^{\mathrm{R}}, M^{\mu^{\mathrm{R}}}).$$

Define $V|W \in \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu)$ by

$$V|W = \sum_R \sum_S a_R b_S \hat{\Theta}_{R|S}.$$

Definition 5.2.19. Let

$$\phi : \mathrm{Hom}_{\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{T}}}, M^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_l}(S^{\lambda^{\mathrm{B}}}, M^{\mu^{\mathrm{B}}}) \longrightarrow \mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, M^\mu)$$

be given by

$$V \otimes W \longrightarrow V|W.$$

Note that ϕ is a linear bijection.

Proposition 5.2.20. *Now let $\hat{\phi}$ denote the restriction of ϕ to $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu)$. Then $\hat{\phi}$ is a linear bijection between $\mathrm{Hom}_{F\mathfrak{S}_n}(S^\lambda, S^\mu)$ and $\mathrm{Hom}_{F\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{L}}}, S^{\mu^{\mathrm{L}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}})$.*

Proof. The image of $\hat{\phi}$ lies in $\mathrm{Hom}_{F\mathfrak{S}_{n-l}}(S^{\lambda^{\mathrm{L}}}, S^{\mu^{\mathrm{L}}}) \otimes \mathrm{Hom}_{F\mathfrak{S}_l}(S^{\lambda^{\mathrm{R}}}, S^{\mu^{\mathrm{R}}})$ by Theorem 5.2.17, and ϕ is an injective linear mapping. By Proposition 5.2.3, $\hat{\phi}$ is a linear bijection. $\square$

6 Some results concerning the Specht modules of the general linear groups

In this chapter, we consider some interesting results about the Specht modules of the general linear groups. Fix K to be a field of characteristic p where $p \nmid q$.

6.1 Homomorphisms between Specht modules II

We would like to prove an analogue of Theorem 5.2.1 for $K\mathrm{GL}_n(q)$-modules. However, this situation is more complicated than that of $\mathfrak{S}_n$-modules, since for the general linear groups, a basis of $\mathrm{Hom}_{K\mathrm{GL}_n(q)}$ is not generally known. Nevertheless, we can go some way to proving such an analogue; in particular, given the notation of Theorem 5.2.1 we will define an injective linear mapping

$$\begin{aligned}\mathrm{Hom}_{K\mathrm{GL}_{n-m}(q)}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{K\mathrm{GL}_m(q)}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}}) \\ \longrightarrow \mathrm{Hom}_{K\mathrm{GL}_n(q)}(S^{\lambda}, S^{\mu}).\end{aligned}$$

We begin by stating some notation.

Definition 6.1.1. Suppose $\nu = (\nu_1, \dots, \nu_s, \nu_{s+1}, \dots, \nu_{s+r})$ is a partition of n such that $(\nu_{s+1}, \dots, \nu_{s+r})$ is a partition of m. We define the ν-tableau t^ν to be the tableau with entries $\nu'_1 + \nu'_2 + \dots \nu'_{i-1} + 1, \nu'_1 + \nu'_2 + \dots \nu'_{i-1} + 2, \dots, \nu'_1 + \nu'_2 + \dots \nu'_i$ in increasing order down column i. Now let $\mathcal{A}^{\mathrm{T}}$ denote the set which contains the integers from rows $1, 2, \dots, s$ of t^ν and $\mathcal{A}^{\mathrm{B}}$ the set containing the integers from rows $s+1, s+2, \dots, s+r$ of t^ν. Set

$$\begin{aligned}\nu^{\mathrm{T}} &= (\nu_1, \dots, \nu_s) \\ \nu^{\mathrm{B}} &= (\nu_{s+1}, \dots, \nu_{s+r})\end{aligned}$$

and define

$$\begin{aligned}V &= \langle e_1, e_2, \dots, e_n \rangle \\ V^{\mathrm{B}} &= \langle \{e_i \mid i \in \mathcal{A}^{\mathrm{B}}\} \rangle\end{aligned}$$

$$V^{\mathrm{T}} = \langle \{e_i \mid i \in \mathcal{A}^{\mathrm{T}}\}\rangle$$

to be vector spaces of dimension n, m and $n - m$ respectively over the field $\mathbb{F}_q$. We will associate $\mathrm{GL}_n(q)$ with $\mathrm{GL}(V)$, $\mathrm{GL}_m(q)$ with $\mathrm{GL}(V^{\mathrm{B}})$ and $\mathrm{GL}_{n-m}(q)$ with $\mathrm{GL}(V^{\mathrm{T}})$. Let M^ν be the vector space whose basis elements are ν-flags on V; and let $M^{\nu^{\mathrm{T}}}$ and $M^{\nu^{\mathrm{B}}}$ be the vector spaces whose basis elements are respectively the ν^{T}-flags on V^{T} and the ν^{B}-flags on V^{B}. Write

$$S^\nu = \bigcap_{d=1}^{s+r-1} \bigcap_{t=0}^{\nu_{d+1}-1} \ker \psi^\nu_{d,t}$$

and write $S^{\nu^{\mathrm{T}}}$ and $S^{\nu^{\mathrm{B}}}$ similarly.

Definition 6.1.2. A ν^{T}_+-flag

$$X = (X_0 > X_1 > \ldots > X_s)$$

is a set of subspaces of V such that $\dim(X_{i-1}/X_i) = \nu_i$ for $1 \le i \le s$ where $X_0 = V$ and $\dim X_s = m$.

Now suppose that $X = (X_0 > X_1 > \ldots > X_{s+r})$ is a ν-flag. Set

$$\begin{aligned} X^{\mathrm{T}} &= (X_0 > X_1 > \ldots > X_s) \\ X^{\mathrm{B}} &= (X_s > X_{s+1} > \ldots > X_{s+r}). \end{aligned}$$

Note that X^{B} is a ν^{B}-flag and that X^{T} is a ν^{T}_+-flag. Write $X = (X^{\mathrm{T}} > X^{\mathrm{B}})$. Similarly, if

$$\begin{aligned} X^{\mathrm{T}} &= (X^{\mathrm{T}}_0 > \ldots > X^{\mathrm{T}}_s) \\ X^{\mathrm{B}} &= (X^{\mathrm{B}}_0 > \ldots > X^{\mathrm{B}}_r) \end{aligned}$$

are respectively ν^{T}_+- and ν^{B}-flags such that $X^{\mathrm{T}}_s = X^{\mathrm{B}}_0$, we define the ν-flag

$$X = (X^{\mathrm{T}} > X^{\mathrm{B}})$$

in the obvious manner. Now suppose that

$$\begin{aligned} Y^{\mathrm{T}} &= \sum \left\{ A_{X^{\mathrm{T}}} X^{\mathrm{T}} \mid X^{\mathrm{T}} \text{ is a } \nu^{\mathrm{T}}_+\text{-flag} \right\} \\ Y^{\mathrm{B}} &= \sum \left\{ A_{X^{\mathrm{B}}} X^{\mathrm{B}} \mid X^{\mathrm{B}} \text{ is a } \nu^{\mathrm{B}}\text{-flag} \right\} \end{aligned}$$

are such that $X_s^{\mathrm{T}} = X_0^{\mathrm{B}}$ for all X^{T} and X^{B}. Define

$$(Y^{\mathrm{T}} > Y^{\mathrm{B}}) = \sum_{X^{\mathrm{T}}} \sum_{X^{\mathrm{B}}} A_{X^{\mathrm{T}}} A_{X^{\mathrm{B}}} (X^{\mathrm{T}} > X^{\mathrm{B}}).$$

Definition 6.1.3. We now look at a particular type of flag. Suppose

$$X^{\mathrm{T}} = (X_0^{\mathrm{T}} > X_1^{\mathrm{T}} > \ldots > X_s^{\mathrm{T}})$$

is a ν_+^{T}-flag such that $X_s^{\mathrm{T}} = V^{\mathrm{B}}$. Define

$$\widehat{X^{\mathrm{T}}} = (\widehat{X_0^{\mathrm{T}}} > \widehat{X_1^{\mathrm{T}}} \ldots > \widehat{X_s^{\mathrm{T}}})$$

where each $\widehat{X_i^{\mathrm{T}}}$ is such that

$$V^{\mathrm{T}} > \widehat{X_i^{\mathrm{T}}} \text{ and } \widehat{X_i^{\mathrm{T}}} \oplus V^{\mathrm{B}} = X_i^{\mathrm{T}}.$$

Note that $\widehat{X^{\mathrm{T}}}$ is a ν^{T}-flag.

Similarly, if $X^{\mathrm{T}} = (X_0 > \ldots > X_s)$ is a ν^{T}-flag such that $X_0 = V^{\mathrm{T}}$, we define the ν_+^{T}-flag

$$\overline{X^{\mathrm{T}}} = (\overline{X_0^{\mathrm{T}}} > \overline{X_1^{\mathrm{T}}} > \ldots > \overline{X_s^{\mathrm{T}}})$$

where

$$\overline{X_i^{\mathrm{T}}} = X_i^{\mathrm{T}} \oplus V^{\mathrm{B}}.$$

Definition 6.1.4. We may use the notation of Definition 4.2.1 to describe a ν-flag rather than a subspace. (See [18, Examples 11.17].) Suppose $\nu = (\nu_1, \nu_2, \ldots, \nu_{s+r}) \vdash n$. Write this as a stack of vectors, with the lowest ν_{s+r} rows representing the subspace corresponding to ν_{s+r}, the lowest $\nu_{s+r} + \nu_{s+r-1}$ rows representing the subspace corresponding to ν_{s+r-1}, and so on. See Example 6.1.6 for an example of this notation.

Proposition 6.1.5. *We may now describe a generator, G, of the Specht module S^ν; an example follows. The notation is similar to that of Proposition 4.2.2. We will describe an $n \times n$ matrix M whose entries consist of $0, 1, *$ and $\bullet$. This matrix is divided into blocks $M_1, M_2, \ldots, M_{s+r}$ where each block M_i consists of rows $\nu_{i-1}-1$ to ν_i of M, so that by substituting values of $* \in \mathbb{F}_q$ and $\bullet \in \{0,1\}$ into M, we obtain a ν-flag. To find the generator, we will take the sum of all such possible flags, together* *with a sign depending on the number of $\bullet$ that take the value 1. So we could write*

$$G = \sum_{\bullet \in \{0,1\}} \sum_{\ast \in \mathbb{F}_q} (-1)^{n(\bullet)} M.$$

We will now recursively describe the rows of M. Consider row k of block M_i. Assume that M_j has been defined for $j > i$ and suppose that the tableau t^ν contains the numbers $t^i_1, t^i_2, \ldots, t^i_{\nu_i}$ in row i. Now assume that row l of M_i has been defined for $l < k$. Write row k of M_i in the following way.

- *Put '1' in position t^i_k of row k.*
- *If $t^i_k \neq 1$, and no '1' has previously been written in column $t^i_k - 1$ (in any row), then put '$\bullet$' in position $t^i_k - 1$ of row k.*
- *For $1 \leq a < t^i_k - 1$, if no '1' has previously been written in column a (in any row), then put '$\ast$' in position a of row k.*
- *Now put 0 in any empty spaces.*

Example 6.1.6. Let $\nu = (3, 3, 2, 2)$. Then

$$t^\nu = \begin{array}{lll} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & \\ 4 & 8 & \end{array}$$

and S^ν is generated by

$$G = \sum_{\bullet \in \{0,1\}} \sum_{\ast \in \mathbb{F}_q} (-1)^{n(\bullet)} \left(\begin{array}{cccccccccc} 1 &&&&&&&&& \\ 0 & 0 & 0 & 0 & 1 &&&&& \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ \hline \bullet & 1 &&&&&&&& \\ \ast & 0 & 0 & 0 & \bullet & 1 &&&& \\ \ast & 0 & 0 & 0 & \ast & 0 & 0 & 0 & \bullet & 1 \\ \hline \ast & \bullet & 1 &&&&&&& \\ \ast & \ast & 0 & 0 & \ast & \bullet & 1 &&& \\ \hline \ast & \ast & \bullet & 1 &&&&&& \\ \ast & \ast & \ast & 0 & \ast & \ast & \bullet & 1 && \end{array} \right)$$

where any empty space is assumed to contain 0.

We now return to the notation of Theorem 5.2.1. Suppose that

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{s+r})$$
$$\mu = (\mu_1, \mu_2, \ldots, \mu_{s+r'})$$

are partitions of n such that λ^{B} and μ^{B} are partitions of m. Suppose there exist homomorphisms

$$\hat{\Theta}^{\mathrm{T}} : S^{\lambda^{\mathrm{T}}} \to M^{\mu^{\mathrm{T}}}$$
$$\hat{\Theta}^{\mathrm{B}} : S^{\lambda^{\mathrm{B}}} \to M^{\mu^{\mathrm{B}}}$$

such that $\operatorname{Im} \hat{\Theta}^{\mathrm{T}} \subseteq S^{\mu^{\mathrm{T}}}$ and $\operatorname{Im} \hat{\Theta}^{\mathrm{B}} \subseteq S^{\mu^{\mathrm{B}}}$. We will define a map $\hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}})$ where $\hat{\Theta} : S^{\lambda} \to M^{\mu}$ is such that $\operatorname{Im} \hat{\Theta} \subseteq S^{\mu}$.

Lemma 6.1.7. *Suppose that $Z \in S^{\lambda}$. So*

$$Z = \sum_{\dim W = m} \sum_{X_W^{\mathrm{T}}} \sum_{X_W^{\mathrm{B}}} \Big\{ A^W_{X^{\mathrm{T}}X^{\mathrm{B}}} (X_W^{\mathrm{T}} > X_W^{\mathrm{B}}) \mid X_W^{\mathrm{T}} \text{ is a } \lambda_+^{\mathrm{T}}\text{-flag and } X_W^{\mathrm{B}} \text{ is a } \lambda^{\mathrm{B}}\text{-flag such that } (X_W^{\mathrm{T}})_s = (X_W^{\mathrm{B}})_0 = W \Big\}.$$

It is then possible to write

$$Z = \sum_{\dim W = m} \sum_i \sum_j C^W_{ij} (R_i^W > S_j^W)$$

(where each S_j^W is a sum of λ^{B}-flags and each R_j^W is a sum of λ_+^{T}-flags) such that, when we fix W and omit the corresponding superscript, we have that for all i and j,

$$(R_i > S_j) \in \bigcap_{d=1}^{s-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^{\lambda}_{d,t} \quad \bigcap \quad \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^{\lambda}_{d,t}.$$

Proof. Consider

$$Z = \sum_{\dim W = m} \sum_{X_W^{\mathrm{T}}} \sum_{X_W^{\mathrm{B}}} A^W_{X^{\mathrm{T}}X^{\mathrm{B}}} (X_W^{\mathrm{T}} > X_W^{\mathrm{B}})$$

as in Lemma 6.1.7. Fix W and consider

$$Z' = \sum_{X^{\mathrm{T}}} \sum_{X^{\mathrm{B}}} A_{X^{\mathrm{T}}X^{\mathrm{B}}} (X^{\mathrm{T}} > X^{\mathrm{B}}).$$

Now fix X^{T} so that for $s+1 \leq d < s+r$ and $0 \leq t < \lambda_{d+1}$

$$\left(\sum_{X^{\mathrm{B}}} A_{X^{\mathrm{T}}X^{\mathrm{B}}}(X^{\mathrm{T}} > X^{\mathrm{B}})\right) \psi^{\lambda}_{d,t} = 0.$$

Let $S^{\lambda^{\mathrm{B}}}_W \cong S^{\lambda^{\mathrm{B}}}$ be the vector space whose elements are sums of λ-flags on W lying inside

$$\bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t}$$

and suppose $\{S_j \mid j \in J\}$ form a basis of $S^{\lambda^{\mathrm{B}}}_W$. Then

$$\sum_{X^{\mathrm{B}}} A_{X^{\mathrm{T}}X^{\mathrm{B}}} X^{\mathrm{B}} = \sum_j B_{X^{\mathrm{T}}j} S_j$$

for some coefficients $B_{X^{\mathrm{T}}j} \in K$. Hence

$$Z' = \sum_{Y^{\mathrm{T}}} \sum_j B_{X^{\mathrm{T}}j}(X^{\mathrm{T}} > S_j).$$

Now fix S_j. For $1 \leq d < s$ and $0 \leq t < \lambda_{d+1}$

$$\left(\sum_{X^{\mathrm{T}}} B_{X^{\mathrm{T}}j}(X^{\mathrm{T}} > S_j)\right) \psi^{\lambda}_{d,t} = 0.$$

Hence

$$Z' = \sum_i \sum_j C_{ij}(R_i > S_j)$$

where for fixed S_j, each $(R_i > S_j)$ is such that

$$(R_i > S_j) \in \bigcap_{d=1}^{s-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^{\lambda}_{d,t}$$

and for fixed R_i, each $(R_i > S_j)$ is such that

$$(R_i > S_j) \in \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^{\lambda}_{d,t}.$$

□

Definition 6.1.8. We define a subspace H^{λ} of M^{λ} in the following manner. Let

$$Y = \sum_X A_X(X_0 > X_1 > \ldots > X_{s+r}) \in M^{\lambda}.$$

Then $Y \in H^\lambda$ if and only if

$$X_s = V^{\mathrm{B}} \text{ for all } X$$

and

$$Y \in \bigcap_{d=1}^{s-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t}^{\lambda} \quad \bigcap \quad \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t}^{\lambda}.$$

Note that H^λ is a $\mathrm{GL}_{n-m}(q)\times\mathrm{GL}_m(q)$-module.

Definition 6.1.9. We use the homomorphisms $\hat{\Theta}^{\mathrm{T}}$ and $\hat{\Theta}^{\mathrm{B}}$ as previously defined to define a map from H^λ to M^μ. Note that by the proof of Lemma 6.1.7, if $Y \in H^\lambda$ then

$$Y = \sum_i \sum_j B_{ij}(R_i > S_j)$$

where for fixed i and j,

$$(R_i > S_j) \in \bigcap_{d=1}^{s-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t}^{\lambda} \quad \bigcap \quad \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi_{d,t}^{\lambda}.$$

Now fix i and j, and suppose that

$$(R_i > S_j) = \sum_{X^{\mathrm{T}}} \sum_{X^{\mathrm{B}}} C_{X^{\mathrm{T}}} C_{X^{\mathrm{B}}} (X^{\mathrm{T}} > X^{\mathrm{B}})$$

and note that

$$\sum_{X^{\mathrm{B}}} C_{X^{\mathrm{B}}} X^{\mathrm{B}} \in S^{\lambda^{\mathrm{B}}}$$

$$\sum_{X^{\mathrm{T}}} C_{X^{\mathrm{T}}} \widehat{X^{\mathrm{T}}} \in S^{\lambda^{\mathrm{T}}}.$$

Suppose that

$$\left(\sum_{X^{\mathrm{B}}} C_{X^{\mathrm{B}}} X^{\mathrm{B}}\right) \hat{\Theta}^{\mathrm{B}} = \sum_{G^{\mathrm{B}}} D_{G^{\mathrm{B}}} G^{\mathrm{B}}$$

$$\left(\sum_{X^{\mathrm{T}}} C_{X^{\mathrm{T}}} \widehat{X^{\mathrm{T}}}\right) \hat{\Theta}^{\mathrm{T}} = \sum_{G^{\mathrm{T}}} D_{G^{\mathrm{T}}} G^{\mathrm{T}}.$$

Then define the map $\phi : H^\lambda \to M^\mu$ by

$$(R_i > S_j)\phi = \sum_{G^{\mathrm{T}}} \sum_{G^{\mathrm{B}}} D_{G^{\mathrm{T}}} D_{G^{\mathrm{B}}} (\overline{G^{\mathrm{T}}} > G^{\mathrm{B}})$$

and extending linearly. Note that ϕ is a $\mathrm{GL}_{n-m}(q)\times\mathrm{GL}_m(q)$-homomorphism.

We now wish to extend the map ϕ.

Definition 6.1.10. Let $Z \in S^\lambda$, and express Z in the form of Lemma 6.1.7. Fix i, j and W and consider the corresponding set of flags $(R_i > S_j)$. Suppose $W = \langle f_1, f_2, \ldots, f_m\rangle$ and that $V = \langle f_1, \ldots, f_m, f_{m+1} \ldots, f_n\rangle$. Let $\pi \in \mathrm{GL}_n(q)$ be defined by

$$\pi : f_i \to e_i.$$

Now define a map $\overline{\Theta}$ by specifying that

$$(R_i > S_j)\overline{\Theta} = (R_i > S_j)\pi\phi\pi^{-1}$$

and extending linearly.

Let $\hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}})$ be the restriction of $\overline{\Theta}$ to S^λ.

Theorem 6.1.11. *The map $\overline{\Theta}$, and hence the map $\hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}})$, is a well defined $K\,GL_n(q)$-homomorphism.*

Proof. Fix $(R_i > S_j)$ as in Definition 6.1.10. Suppose

$$W = \langle f_1, f_2, \ldots, f_m\rangle = \langle g_1, g_2, \ldots, g_m\rangle$$
$$V = \langle f_1, \ldots, f_m, f_{m+1} \ldots, f_n\rangle = \langle g_1, \ldots, g_m, g_{m+1} \ldots, g_n\rangle$$

Define the maps

$$\pi_1 : f_i \to e_i$$
$$\pi_2 : g_i \to e_i.$$

Then $\pi_1 = \pi_2\tau$ for some $\tau \in \mathrm{GL}_{n-m}(q)\times\mathrm{GL}_m(q)$. Thus

$$\begin{aligned}(R_i > S_j)\pi_1\phi\pi_1^{-1} &= (R_i > S_j)(\pi_2\tau)\phi(\pi_2\tau)^{-1}\\ &= (R_i > S_j)\pi_2(\tau\phi\tau^{-1})\pi_2^{-1}\\ &= (R_i > S_j)\pi_2\phi\pi_2^{-1}\end{aligned}$$

since ϕ is a $\mathrm{GL}_{n-m}(q)\times\mathrm{GL}_m(q)$-homomorphism.

This proof also demonstrates that $\hat{\Theta}$ is a $K\mathrm{GL}_n(q)$ homomorphism. ∎

Theorem 6.1.12. *Suppose $\hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}})$. Then $\operatorname{Im}\hat{\Theta} \subseteq S^\mu$.*

Proof. To show that

$$\operatorname{Im}\hat{\Theta} \subseteq \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker\psi^\mu_{d,t}$$

it is sufficient to show that for any $(R_i > S_j) \in H^\lambda$, we have that

$$(R_i > S_j)\phi \in \bigcap_{d=s+1}^{s+r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^\mu_{d,t}.$$

But

$$(R_i > S_j)\phi = \sum_{W^{\mathrm{T}}} \sum_{W^{\mathrm{B}}} D_{W^{\mathrm{T}}} D_{W^{\mathrm{B}}} (\overline{W^{\mathrm{T}}} > W^{\mathrm{B}})$$

where

$$\sum_{W^{\mathrm{B}}} D_{W^{\mathrm{B}}} W^{\mathrm{B}} \in \bigcap_{d=1}^{r-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^{\mu^{\mathrm{B}}}_{d,t}$$

and so this is clearly true.

Similarly, we can show that

$$\operatorname{Im} \hat{\Theta} \subseteq \bigcap_{d=1}^{s-1} \bigcap_{t=0}^{\lambda_{d+1}-1} \ker \psi^\mu_{d,t}.$$

Finally, if $d = s$ and $\operatorname{Im} \psi^\mu_{d,t} \subseteq M^\nu$ then $\lambda \not\trianglerighteq \nu$ and so $\operatorname{Hom}(S^\lambda, M^\nu) = 0$ and $\operatorname{Im} \hat{\Theta} \subseteq \ker \psi_{d,t}$. □

Proposition 6.1.13. *Suppose that $\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}} \neq 0$. Then $\hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}}) \neq 0$.*

Proof. Suppose $\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}} \neq 0$ and consider the action of $\hat{\Theta}$ on the generator G of S^λ described in Proposition 6.1.5. By Lemma 6.1.7, we may write

$$G = \sum_W \sum_i \sum_j C^W_{ij} (R^W_i > S^W_j).$$

Note that for fixed W, the image of $(R_i > S_j)$ under $\overline{\Theta}$ consists of a sums of flags where each flag, A say, satisfies $A_s = W$. Now fix $W = V^{\mathrm{B}}$. Then by studying G, it can be seen that

$$\sum_i \sum_j C_{ij}(R_i > S_j) = \sum_{X^{\mathrm{T}}} \sum_{X^{\mathrm{B}}} \{A_{X^{\mathrm{T}}} A_{X^{\mathrm{B}}} (X^{\mathrm{T}} > X^{\mathrm{B}}) \mid X^{\mathrm{T}} \text{ is a } \lambda^{\mathrm{T}}_+\text{-flag and } X^{\mathrm{B}} \text{ is a } \lambda^{\mathrm{B}}\text{-flag}\}$$

where

$$Y^{\mathrm{T}} = \sum_{X^{\mathrm{T}}} A_{X^{\mathrm{T}}} \widehat{X^{\mathrm{T}}}$$

generates $S^{\lambda^{\mathrm{T}}}$ and

$$Y^{\mathrm{B}} = \sum_{X^{\mathrm{B}}} A_{X^{\mathrm{B}}} X^{\mathrm{B}}$$

generates $S^{\lambda^{\mathrm{B}}}$. From the definition,

$$(Y^{\mathrm{T}} > Y^{\mathrm{B}})\overline{\Theta} = (Y^{\mathrm{T}} > Y^{\mathrm{B}})\phi = (\widehat{Y^{\mathrm{T}}\Theta^{\mathrm{T}}} > Y^{\mathrm{B}}\Theta^{\mathrm{B}}) \neq 0.$$

□

Proposition 6.1.14. *The mapping*

$$\hat{\Theta}^{\mathrm{T}} \otimes \hat{\Theta}^{\mathrm{B}} \to \hat{\Theta} = (\hat{\Theta}^{\mathrm{T}}, \hat{\Theta}^{\mathrm{B}})$$

is a linear bijection between

$$\mathrm{Hom}_{K\,GL_{n-m}(q)}(S^{\lambda^{\mathrm{T}}}, S^{\mu^{\mathrm{T}}}) \otimes \mathrm{Hom}_{K\,GL_{m}(q)}(S^{\lambda^{\mathrm{B}}}, S^{\mu^{\mathrm{B}}})$$

and a subspace N where

$$\mathrm{Hom}_{K\,GL_{n}(q)}(S^{\lambda}, S^{\mu}) > N.$$

Proof. The mapping is clearly linear; it is injective by Proposition 6.1.13 and its image lies within $\mathrm{Hom}_{K\mathrm{GL}_n(q)}(S^{\lambda}, S^{\mu})$ by Theorem 6.1.12. However it is not clear whether in fact we have $N = \mathrm{Hom}_{K\mathrm{GL}_n(q)}(S^{\lambda}, S^{\mu})$. □

6.2 An example of a homomorphism between Specht modules

We would also like to produce an analogue of Theorem 5.2.2 for the general linear groups. However, Proposition 5.2.3 does not transfer. Furthermore, the combinatorics involved are more complicated than in the analogue of Theorem 5.2.1. We consider one very specific example.

Let $\lambda = (\lambda_1)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be partitions of n and set

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) = (\lambda_1 + r, r, \ldots, r)$$
$$\beta = (\beta_1, \beta_2, \ldots, \beta_k) = (\mu_1 + r, \ldots, \mu_s + r, r, \ldots, r)$$

so that α, β are partitions of $n + kr$. We shall prove the following theorem.

Theorem 6.2.1. *Suppose there exists*

$$0 \neq \hat{\Theta}_0 : S^\lambda \to M^\mu$$

such that $\operatorname{Im} \hat{\Theta}_0 \subseteq S^\mu$. *Then there exists*

$$0 \neq \hat{\Theta} : S^\alpha \to M^\beta$$

such that $\operatorname{Im} \hat{\Theta} \subseteq S^\beta$.

Lemma 6.2.2. *Suppose such a map* $\hat{\Theta}_0$ *exists. Then*

$$\begin{bmatrix} \mu_d + \mu_{d+1} - t \\ \mu_{d+1} - t \end{bmatrix} = 0$$

for all $1 \leq d < s$ *and* $0 \leq t < \mu_{d+1}$.

Proof. Let X be a λ-flag and note that X generates S^λ. If a map $\hat{\Theta}_0$ as described above exists, then it is a linear multiple of the map

$$X \to \sum \{Y \mid Y \text{ is a } \mu\text{-flag}\}.$$

Fix d and t such that $1 \leq d < s$ and $0 \leq t < \mu_{d+1}$ and suppose $\operatorname{Im} \psi_{d,t} \subseteq M^\nu$. Then $\hat{\Theta}_0 \psi_{d,t} = 0$ since $\operatorname{Im} \hat{\Theta}_0 \subseteq S^\mu$. But

$$X(\hat{\Theta}_0 \psi_{d,t}) = \begin{bmatrix} \mu_d + \mu_{d+1} - t \\ \mu_{d+1} - t \end{bmatrix} \sum \{W \mid W \text{ is a } \nu\text{-flag } \}.$$

□

Definition 6.2.3. Define

$$\Theta : M^\alpha \to M^\beta$$

by

$$(X_0 > X_1 > \ldots > X_k) \to \sum \{(Y_0 > Y_1 > \ldots > Y_k) \mid Y_i > X_i \text{ for all } i\}$$

and let $\hat{\Theta}$ denote the restriction to S^α.

Lemma 6.2.4. *It can be shown that* $\hat{\Theta} \neq 0$.

Proof. Consider the generator G of S^α as described in Proposition 6.1.5 and let $R = (R_0 > R_1 > \ldots > R_k)$ be the β-flag given by

$R_k = 0$

$$
\begin{aligned}
R_{k-1} &= R_k \oplus \langle e_k, e_{2k}, \ldots, e_{rk}, e_{rk+1}, e_{rk+2}, \ldots, e_{rk+\mu_k} \rangle \\
R_{k-2} &= R_{k-1} \oplus \langle e_{k-1}, e_{2k-1}, \ldots, e_{rk-1}, e_{rk+\mu_k+1}, e_{rk+\mu_k+2}, \ldots, e_{rk+\mu_k+\mu_{k-1}} \rangle \\
R_{k-3} &= R_{k-2} \oplus \langle e_{k-2}, e_{2k-2}, \ldots, e_{rk-2}, e_{rk+\mu_k+\mu_{k-1}+1}, \ldots, e_{rk+\mu_r+\mu_{r-1}+\mu_{r-2}} \rangle \\
&\vdots \quad \vdots \\
R_1 &= R_2 \oplus e_2, e_{k+2}, \ldots, e_{(r-1)k+2}, e_{rk+\mu_k+\mu_{k-1}+\ldots+\mu_3+1}, \ldots, e_{rk+\mu_k+\ldots\mu_2} \rangle \\
R_0 &= R_1 \oplus e_1, e_{k+1}, \ldots, e_{(r-1)k+1}, e_{rk+\mu_k+\ldots\mu_2+1} \cdots e_{rk+\mu_k+\ldots\mu_1} \rangle.
\end{aligned}
$$

Then the coefficient of R in $G\hat{\Theta}$ is 1. □

We now wish to show that $\operatorname{Im}\hat{\Theta} \subseteq S^\beta$, that is that $\hat{\Theta}\psi_{d,t} = 0$ for $1 \le d < k$ and $0 \le t < \beta_{d+1}$. Fix such a d and t and suppose $\operatorname{Im}\psi_{d,t} \subseteq M^\nu$.

If $d \ge s$ or $t < r$ then $\lambda \not\trianglerighteq \nu$ and so $\hat{\Theta}\psi_{d,t} = 0$ as required. We now suppose that $1 \le d < s$ and $t = t' + r$ where $0 \le t' < \mu_{d+1}$.

Lemma 6.2.5. *Let $Z \in S^\alpha$ be given by*

$$Z = \sum_{X_1}\sum_{X_2}\cdots\sum_{X_k} A_{X_1X_2\ldots X_k}(X_0 > X_1 > \ldots > X_k)$$

and let $T = (T_0 > T_1 > \ldots > T_k) \in M^\nu$. Set $N_j = \alpha_j + \alpha_{j+1} + \ldots + \alpha_k$. The coefficient of T in $Z\hat{\Theta}\psi_{d,t}$ is given by

$$
\begin{aligned}
\sum_{X_1}\sum_{X_2}\cdots\sum_{X_k} \Bigg(& \begin{bmatrix} \beta_d + \beta_{d+1} - t - \alpha_{d+1} \\ \beta_{d+1} - t - \alpha_{d+1} \end{bmatrix} \{A_{X_1X_2\ldots X_k} \mid T_i > X_i \ \ for\ i \neq d\} \\
& + q^{\beta_{d+1}-t-\alpha_{d+1}} \sum_{s=1}^{\alpha_{d+1}} q^s \begin{bmatrix} \beta_d + \beta_{d+1} - t - \alpha_{d+1} + s - 1 \\ \beta_{d+1} - t - \alpha_{d+1} + s \end{bmatrix} \\
& \{A_{X_1X_2\ldots X_k} \mid T_i > X_i \ for\ i \neq d\ and\ \dim(X_d \cap T_d) \ge N_{d+2} + s\} \Bigg).
\end{aligned}
$$

Proof. Recall that

$$(X_0 > X_1 > \ldots > X_k)\Theta \to \sum\{(Y_0 > Y_1 > \ldots > Y_k) \mid Y_i > X_i \text{ for all } i\}$$

and

$$(Y_0 > Y_1 > \ldots > Y_k)\psi_{d,t} \to \sum\{(Y_0 > \ldots > Y_{d-1} > W_d > Y_{d+1} > \ldots > Y_k \mid Y_d > W_d)\}.$$

Hence

$$
\begin{aligned}
X(\Theta\psi_{d,t}) &= \sum_{s=0}^{\alpha_{d+1}} \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \\
&\qquad \text{for all } i \neq d \text{ and } \dim(W_d \cap X_d) = N_{d+2}+s\} \\
&= \sum_{s=0}^{\alpha_{d+1}} \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \\
&\qquad \text{for all } i \neq d \text{ and } \dim(W_d \cap X_d) \geq N_{d+2}+s\} \\
&\quad - \sum_{s=0}^{\alpha_{d+1}} \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \\
&\qquad \text{for all } i \neq d \text{ and } \dim(W_d \cap X_d) \geq N_{d+2}+s+1\} \\
&= \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1} \\ \beta_{d+1}-t-\alpha_{d+1} \end{bmatrix} \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \text{ for } i \neq d\} \\
&\quad \sum_{s=1}^{\alpha_{d+1}} \left(\begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} - \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s-1 \\ \beta_{d+1}-t-\alpha_{d+1}+s-1 \end{bmatrix} \right) \\
&\quad \sum\{(W_0 > \ldots > W_k \mid W_i > X_i \text{ for all } i \neq d \\
&\qquad \text{and } \dim(W_d \cap X_d) \geq N_{d+2}+s\} \\
&= \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1} \\ \beta_{d+1}-t-\alpha_{d+1} \end{bmatrix} \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \text{ for all } i \neq d\} \\
&\quad + q^{\beta_{d+1}-t-\alpha_{d+1}} \sum_{s=1}^{\alpha_{d+1}} q^s \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s-1 \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} \\
&\quad \sum\{(W_0 > \ldots > W_k) \mid W_i > X_i \text{ for all } i \neq d \\
&\qquad \text{and } \dim(W_d \cap X_d) \geq N_{d+2}+s\}.
\end{aligned}
$$

The identity follows. ☐

Remarks 6.2.6. We now fix $X_1, \ldots, X_{d-1}, X_{d+1}, \ldots, X_k$ and consider

$$
\begin{aligned}
(\dagger) = &\begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1} \\ \beta_{d+1}-t-\alpha_{d+1} \end{bmatrix} \sum_{X_d}\{A_{X_1\ldots X_k} \mid T_i > X_i \text{ for all } i \neq d\} \\
&+ q^{\beta_{d+1}-t-\alpha_{d+1}} \sum_{s=1}^{\alpha_{d+1}} q^s \begin{bmatrix} \beta_d+\beta_{d+1}-t-\alpha_{d+1}+s-1 \\ \beta_{d+1}-t-\alpha_{d+1}+s \end{bmatrix} \\
&\sum_{X_d}\{A_{X_1X_2\ldots X_k} \mid T_i > X_i \text{ for } i \neq d \text{ and } \dim(X_d \cap T_d) \geq N_{d+2}+s\}.
\end{aligned}
$$

If $T_i \not> X_i$ for some $i \neq d$ then the identity $(\dagger)$ is clearly zero. Therefore fix

$X_1, \ldots, X_k$ such that $T_i > X_i$ for all $i \neq d$ and write $A_{X_1 X_2 \ldots X_k} = A_{X_d}$. Now set

$$C = \beta_d + \beta_{d+1} - t - \alpha_{d+1} - 1$$
$$D = \beta_{d+1} - t - \alpha_{d+1}$$

so that (†) is equal to

$$\begin{bmatrix} C+1 \\ D+1 \end{bmatrix} \sum_{X_d} A_{X_d} + q^D \sum_{s=1}^{\alpha_{d+1}} q^s \begin{bmatrix} C+s \\ D+s \end{bmatrix} \sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}.$$

Then the first term in the expression of above is 0 because $Z \in S^\alpha$. Hence

$$(\dagger) = q^D \sum_{s=1}^{\alpha_{d+1}} q^s \begin{bmatrix} C+s \\ D+s \end{bmatrix} \sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}.$$

Now suppose $1 \leq r \leq \alpha_d$. Then

$$\begin{aligned} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + r\} &= \sum_{T_d > T^r > X_{d+1}} \sum_{X_d} \{A_{X_d} \mid (X_d \cap T_d) = T^r\} \\ - \sum_{s=r+1}^{\alpha_{d+1}} \left(\begin{bmatrix} s \\ s-r \end{bmatrix} - \begin{bmatrix} s-1 \\ s-r-1 \end{bmatrix} \right) &\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\} \end{aligned}$$

where $\{T^r\}$ are the set of all r-dimensional subspaces of T_d. Since $Z \in S^\alpha$, the first term in the equation above is zero, and hence

$$\begin{aligned} &\{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + r\} \\ &\qquad = - \sum_{s=r+1}^{\alpha_{d+1}} q^{s-r} \begin{bmatrix} s-1 \\ s-r \end{bmatrix} \sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}. \end{aligned}$$

Lemma 6.2.7. *Suppose $1 \leq r < \alpha_{d+1}$. Then*

$$\begin{aligned} (\ddagger) := &\left(\begin{bmatrix} C+r \\ D+r \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} r-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) \\ &\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + r\} \\ = - &\left(\sum_{s=r+1}^{\alpha_{d+1}} q^{s-r} q^{(r-1)(D+r)} \begin{bmatrix} s-1 \\ s-r \end{bmatrix} \begin{bmatrix} C+1 \\ D+r \end{bmatrix} \right) \\ &\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}. \end{aligned}$$

Proof.

$$(\ddagger) = -\sum_{s=r+1}^{\alpha_{d+1}} q^{s-r} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} \left(\begin{bmatrix} C+r \\ D+r \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} r-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right)$$
$$\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}$$

by Remarks 6.2.6

$$= -\sum_{s=r+1}^{\alpha_{d+1}} q^{s-r} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} \left(\sum_{a=0}^{r-1} q^{a(D+a+1)} \begin{bmatrix} r-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right.$$
$$\left. - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} r-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) \sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}$$

by Lemma 4.1.12

$$= -\sum_{s=r+1}^{\alpha_{d+1}} q^{s-r} q^{(r-1)(D+r)} \begin{bmatrix} s-1 \\ r-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+r \end{bmatrix} \sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}.$$

□

Lemma 6.2.8. *For $1 \leq r \leq \alpha_{d+1}$ we have that $q^{-D}(\dagger)$ is equal to*

$$\sum_{s=r}^{\alpha_{d+1}} q^s \left(\begin{bmatrix} C+s \\ D+s \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} s-1 \\ s-a-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right)$$
$$\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq s\}.$$

Proof. The proof is by induction on r, the case $r = 1$ being trivially true. Assume that Lemma 6.2.8 holds for r. Then

$$q^{-D}(\dagger) = \sum_{s=r}^{\alpha_{d+1}} q^s \left(\begin{bmatrix} C+s \\ D+s \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} s-1 \\ s-a-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right)$$
$$\sum_{X_d} \{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2} + s\}$$

by the induction hypothesis

$$= \sum_{s=r+1}^{\alpha_{d+1}} q^s \left(\begin{bmatrix} C+s \\ D+s \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} s-1 \\ s-a-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right)$$

$$
\begin{aligned}
&\sum_{X_d}\{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2}+s\} \\
&+ q^r \left(\begin{bmatrix} C+r \\ D+r \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} r-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) \\
&\sum_{X_d}\{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2}+r\} \\
= &\sum_{s=r+1}^{\alpha_{d+1}} q^s \left(\begin{bmatrix} C+s \\ D+s \end{bmatrix} - \sum_{a=0}^{r-2} q^{a(D+a+1)} \begin{bmatrix} s-1 \\ s-a-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) \\
&\sum_{X_d}\{A_{X_d} \mid \dim(X_a \cap T_d) \geq N_{d+2}+s\} \\
&- q^r \left(\sum_{s=r+1}^{\alpha_{d+1}-1} q^{s-r} q^{(r-1)(D+r)} \begin{bmatrix} s-1 \\ s-r \end{bmatrix} \begin{bmatrix} C+1 \\ D+r \end{bmatrix} \right) \\
&\sum_{X_d}\{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2}+s\}
\end{aligned}
$$

by Lemma 6.2.7

$$
\begin{aligned}
= &\sum_{s=r+1}^{\alpha_{d+1}} q^s \left(\begin{bmatrix} C+s \\ D+s \end{bmatrix} - \sum_{a=0}^{r-1} q^{a(D+a+1)} \begin{bmatrix} s-1 \\ s-a-1 \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) \\
&\sum_{X_d}\{A_{X_d} \mid \dim(X_d \cap T_d) \geq N_{d+2}+s\}
\end{aligned}
$$

as required. □

Lemma 6.2.9. *The sum* (†) *is equal to 0.*

Proof. Setting $r = \alpha_{d+1}$ in Lemma 6.2.8, we get that

$$
\begin{aligned}
q^{-D}(\dagger) &= q^{\alpha_{d+1}} \left(\begin{bmatrix} C+\alpha_{d+1} \\ D+\alpha_{d+1} \end{bmatrix} - \sum_{a=0}^{\alpha_{d+1}-2} q^{a(D+a+1)} \begin{bmatrix} \alpha_{d+1}-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) A_{X_d} \\
&= q^{\alpha_{d+1}} \left(\sum_{a=0}^{\alpha_{d+1}-1} q^{a(D+a+1)} \begin{bmatrix} \alpha_{d+1} \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right. \\
&\qquad \left. - \sum_{a=0}^{\alpha_{d+1}-2} q^{a(D+a+1)} \begin{bmatrix} \alpha_{d+1}-1 \\ a \end{bmatrix} \begin{bmatrix} C+1 \\ D+a+1 \end{bmatrix} \right) A_{X_d}
\end{aligned}
$$

by Lemma 4.1.12. Let $E = D + \alpha_{d+1} + (\alpha_{d+1} - 1)(D + \alpha + 1)$. Then

$$
(\dagger) = q^E \begin{bmatrix} C+1 \\ D+\alpha_{d+1} \end{bmatrix} A_{X_d}
$$

$$
\begin{aligned}
&= q^E \begin{bmatrix} \beta_d + \beta_{d+1} - t - \alpha_{d+1} \\ \beta_{d+1} - t \end{bmatrix} A_{X_d} \\
&= q^E \begin{bmatrix} \mu_d + r + \mu_{d+1} + r - t - r \\ \mu_{d+1} + r - t \end{bmatrix} A_{X_d} \\
&= q^E \begin{bmatrix} \mu_d + \mu_{d+1} - t' \\ \mu_{d+1} - t' \end{bmatrix} A_{X_d}.
\end{aligned}
$$

□

Thus by Lemma 6.2.2 and previous comments, we have that $\hat{\Theta}\psi_{d,t} = 0$ for all $1 \leq d < k$ and all $0 \leq t < \beta_{d+1}$, and so $\operatorname{Im}\hat{\Theta} \subseteq S^\beta$ as required.

6.3 A useful property of the Specht modules

We consider $GL_n(q)$ acting on a vector space $V = \langle e_1, e_2, \ldots, e_n \rangle$ over the field $\mathbb{F}_q$, as described in Chapter 4. The results that we present in this section may have applications in determining a basis for the Specht modules; they have been submitted for publication. Recall that

$$S^\lambda = \bigcap_{i \geq 2} \bigcap_{t=0}^{\lambda_i - 1} \ker \psi_{i-1,t}.$$

An analogous definition holds when we consider the Specht modules of the symmetric group [17]. Then each element of the Specht module is the sum of tabloids, and it can be shown that if we arrange them according to a particular partial order then the last tabloid is standard. We will show that, in a certain sense, the last flag in an element of an $K\mathrm{GL}_n(q)$-module S^λ also corresponds to a standard tabloid.

We begin by looking at the case where λ is a 2 part partition, and then generalise for arbitrary partitions.

6.3.1 Two part partitions

Definition 6.3.1. Let $\lambda = (n-m, m) \vdash n$. Suppose that Y is an m-dimensional subspace of V. We may assume that $Y = \langle f_1, f_2, \ldots, f_m \rangle$ where

$$f_d = \sum_{j=1}^{s_d} a_{d,j} e_j$$

where $a_{d,s_d} \neq 0$ for all d and $s_1 < s_2 < \ldots < s_m$.

Say that the λ-tabloid corresponding to Y, denoted $T(Y)$, is given by

$$\begin{array}{c}\hline s_1 s_2 \dots s_m\end{array} = \begin{array}{c}\hline r_1 r_2 \dots r_{n-m} \\ \hline s_1 s_2 \dots s_m \\ \hline\end{array}.$$

We say that a tabloid $\{t\}$ is standard if we can find a tableau t' such that $\{t'\} = \{t\}$ and t' is standard. Define a partial order $\trianglelefteq$ on the set of $(n-m, m)$-tabloids by

$$\begin{array}{c}\hline s_1 s_2 \dots s_m\end{array} \trianglelefteq \begin{array}{c}\hline s'_1 s'_2 \dots s'_m\end{array}$$

if and only if $s_d \leq s'_d$ for all d, where we assume that $s_1 < s_2 < \ldots < s_m$ and $s'_1 < s'_2 < \ldots < s'_m$.

Theorem 6.3.2. *Let*

$$R = \sum_{\dim Y = m} W_Y Y \in S^{(n-m,m)}$$

where $W_Y \in K$ for all Y. Say that a flag Y occurs in R if $W_Y \neq 0$.

Suppose that the flag Y' occurs in R and is such that for every flag Y with $Y \neq Y'$ and $T(Y') \trianglelefteq T(Y)$ we have that Y does not occur in R. Then the tabloid $T(Y')$ is standard.

Proof. Suppose there exists a flag Y' of the form described in Theorem 6.3.2 where $T(Y')$ is not standard. Suppose $Y' = \langle f_1, f_2, \ldots, f_m \rangle$ is such that

$$f_d = \sum_{j=1}^{s_d} a_{d,j} e_j$$

where $a_{d,s_d} \neq 0$ for all d and $s_1 < s_2 < \ldots < s_m$. Then

$$T(Y') = \begin{array}{c}\hline r_1 r_2 \dots r_{n-m} \\ \hline s_1 s_2 \dots s_m \\ \hline\end{array}$$

where we assume $r_1 < r_2 < \ldots < r_{n-m}$. Hence for some i with $1 \leq i \leq m$ we have that $s_i < r_i$. Now set

$$Z := \langle e_{r_1}, e_{r_2}, \ldots e_{r_{i-1}} \rangle$$

and consider the sum

$$\begin{aligned}(\ddagger) &= \sum_{k=0}^{i-1} (-1)^k q^{\frac{1}{2}k(k-1)} \sum_{\substack{Z > A \\ \dim A = k}} \sum_{\dim X = m} \Big\{ W_X \big| X > \big(A \oplus \langle f_{i+1}, f_{i+2} \ldots f_m \rangle \big) \Big\} \\ &= 0 \text{ since } R \in S^{(n-m,m)}.\end{aligned}$$

Then we claim that the following two statements hold.

Claim 1:

$$(\ddagger) = \sum_{\dim X = m} \Big\{ W_X \big| X \cap Z = 0 \text{ and } X > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle \Big\}.$$

Claim 2: If X satisfies $\dim X = m$ and $X \cap Z = 0$ and $X > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$ then $T(Y') \trianglelefteq T(X)$.

Note that Y' satisfies $Y' \cap Z = 0$ and $Y' > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$. Furthermore, if $T(Y') \trianglelefteq T(X)$ and $X \neq Y'$ then $W_X = 0$ by the nature of Y'. Hence $(\ddagger) = W_{Y'} = 0$, contradicting our choice of Y' as an occurring flag. Thus the proof of Theorem 6.3.2 will be complete once we have proved these claims.

We now prove the claims above.

Claim 1: Consider X such that $\dim X = m$ and $X > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$ and $X \cap Z = 0$. If $k = 0$ and $\dim A = k$ then X satisfies $X > A \oplus \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$ and so W_X is counted once. There does not exist A such that $\dim A > 0$, where $Z > A$ and $X > A$. Hence W_X is counted exactly once.

Now suppose $\dim X = m$ and $X > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$ and $\dim(X \cap Z) = b > 0$. For fixed k, the number of times that W_X is counted is equal to

$$\begin{aligned} &(-1)^k q^{\frac{1}{2}k(k-1)} (\#A \text{ such that } \dim A = k \text{ where } Z > A \text{ and } X > A) \\ &\quad = (-1)^k q^{\frac{1}{2}k(k-1)} (\#A \text{ such that } \dim A = k \text{ and } Z \cap X > A) \\ &\quad = (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} b \\ k \end{bmatrix} \end{aligned}$$

Hence the total number of times that W_X is counted is equal to

$$\sum_{k=0}^{i-1} (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} b \\ k \end{bmatrix} = 0$$

by Lemma 4.1.11, since $0 < b \leq \dim Z = i - 1$.

Claim 2: Suppose that $\dim X = m$ and $X > \langle f_{i+1}, f_{i+2}, \ldots, f_m \rangle$ and $\dim(X \cap Z) = 0$. Write

$$X = \langle g_1, g_2, \ldots, g_i, f_{i+1}, \ldots, f_m \rangle$$

where for $1 \leq d \leq i$ we have that

$$g_d = \sum_{j=1}^{u_d} b_{d,j} e_j$$

where $b_{d,u_d} \neq 0$ and $u_1 < u_2 < \ldots < u_i$ and $u_d \neq s_a$ for all a with $i < a \leq m$ and all d with $1 \leq d \leq i$.

Claim that $u_d \geq s_d$ for all $d \leq i$. Suppose for a contradiction that $u_{d^*} < s_{d^*}$ for some $d^* \leq i$. For $1 \leq d \leq i$, write

$$g_d = \sum_{l=1}^{m} \alpha_{d,l} e_{s_l} + \sum_{l=1}^{n-m} \beta_{d,l} e_{r_l}$$

for coefficients $\alpha_{d,l}, \beta_{d,l} \in K$ and let

$$h_d = \sum_{l=1}^{m} \alpha_{d,l} e_{s_l}$$

noting that for $d \leq d^*$ we have that

$$\begin{aligned} h_d &= \sum_{l=1}^{d^*-1} \alpha_{d,l} e_{s_l} \\ g_d &= \sum_{l=1}^{d^*-1} \alpha_{d,l} e_{s_l} + \sum_{l=1}^{i-1} \beta_{d,l} e_{r_l} \end{aligned}$$

since $u_d \leq u_{d^*} < s_{d^*} \leq s_i < r_i$. Then there are coefficients x_d, not all zero, such that

$$\sum_{d=1}^{d^*} x_d h_d = 0$$

and so

$$\sum_{d=1}^{d^*} x_d g_d \in Z$$

and is non–zero since $\{g_d\}$ are linearly independent. Hence $X \cap Z \neq 0$, giving the required contradiction. Thus

$$\overline{s_1 s_2 \ldots s_m} \trianglelefteq \overline{u_1 u_2 \ldots u_i s_{i+1} \ldots s_m} ,$$

completing the proof. □

We now look at the case where λ has more than 2 parts.

6.3.2 Arbitrary partitions

Definition 6.3.3. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c) \vdash n$ and

$$Y = (Y_0 > Y_1 > \ldots > Y_{c-1} > Y_c)$$

is a λ-flag. We will define a corresponding λ-tabloid $T(Y)$.

Recursively, for $c \geq k \geq 1$, starting with $k = c$, we obtain numbers $s_1^k, s_2^k, \ldots, s_{\lambda_k}^k$ in the following way. Suppose that

$$Y_{k-1} = Y_k \oplus \langle f_1^k, f_2^k, \ldots f_{\lambda_k}^k \rangle$$

where each

$$f_d^k = \sum_{j=1}^{s_d^k} a_{d,j}^k e_j$$

is such that $a^k_{d,s^k_d} \neq 0$ for all d, and $s_d^k \neq s_a^j$ for $j > k$ and $1 \leq a \leq \lambda_j$, and $s_1^k < s_2^k \ldots < s_{\lambda_k}^k$. We define the tabloid $T(Y)$ to be

$$T(Y) = \begin{array}{c} \hline s_1^1 s_2^1 \ldots s_{\lambda_1}^1 \\ \hline s_1^2 s_2^2 \ldots s_{\lambda_2}^2 \\ \hline \vdots \qquad\quad \vdots \\ \hline s_1^c s_2^c \ldots s_{\lambda_c}^c \\ \hline \end{array}.$$

We now extend the partial order $\trianglelefteq$ to the set of λ-tabloids as follows. For a tabloid $\{t\}$ and positive integers a and b define

$$m_{a,b}\{t\} := \#\text{ entries } \leq a \text{ in the first } b \text{ rows of } \{t\}.$$

Say that

$$\{t_1\} \trianglelefteq \{t_2\}$$

if and only if

$$m_{a,b}\{t_1\} \leq m_{a,b}\{t_2\}$$

for all a, b. (Note that if λ has exactly two parts, this is the same partial order as previously defined.)

Now suppose that t_1 and t_2 are tableaux whose entries increase along the rows such that all entries of t_1 and t_2 are equal except in possibly rows b and $b+1$ for some integer b, and that for $1 \leq i \leq \lambda_{b+1}$, the ith entry of row $b+1$ in t_2 is at least as large as the ith entry of row $b+1$ in t_1. Then note that $\{t_1\} \trianglelefteq \{t_2\}$.

Theorem 6.3.4. *Let*
$$R = \sum_Y W_Y Y \in S^\lambda$$
where for all Y, we have that Y is a λ-flag and $W_Y \in K$.

Suppose that the flag Y' occurs in R and is such that for every flag Y with $Y \neq Y'$ and $T(Y') \trianglelefteq T(Y)$ we have that Y does not occur in R. Then the tabloid $T(Y')$ is standard.

Proof. Note that the result is trivially true if λ has only one part. Hence assume $c \geq 2$. Suppose there exists a flag Y' of the form described in Theorem 6.3.4 where $T(Y')$ is not standard. Let $T(Y')$ be given by

$$T(Y') = \begin{array}{l} \hline s_1^1 s_2^1 \ldots s_{\lambda_1}^1 \\ \hline s_1^2 s_2^2 \ldots s_{\lambda_2}^2 \\ \hline \vdots \qquad\quad \vdots \\ \hline s_1^c s_2^c \ldots s_{\lambda_c}^c \\ \hline \end{array}$$

where s_d^k is determined by the vector f_d^k with notation exactly as in Definition 6.3.3. Then for some i, m with $2 \leq m \leq c$ and $1 \leq i \leq \lambda_m$ we have that $s_i^{m-1} > s_i^m$. Set

$$Z := \langle f_1^{m-1}, f_2^{m-1}, \ldots, f_{i-1}^{m-1} \rangle$$

and consider the sum

$$\begin{aligned} (\ddagger) = \sum_{k=0}^{i-1} (-1)^k q^{\frac{1}{2}k(k-1)} \sum_{\substack{Z>A \\ \dim A = k}} \sum_X \Big\{ W_X \Big| X = (X_0 > X_1 > \ldots > X_c) \text{ is a} \\ \lambda\text{-flag such that } X_j = Y'_j \text{ for } j \neq m-1 \text{ and } X_{m-1} > \big(A \oplus \langle f_{i+1}^m, \ldots, f_{\lambda_m}^m \rangle\big) \Big\} \\ = 0 \text{ since } R \in S^\lambda. \end{aligned}$$

Note that the subspaces

$$\begin{aligned} Z =& \langle f_1^{m-1}, f_2^{m-1}, \ldots, f_{i-1}^{m-1} \rangle, \\ C =& \langle f_{i+1}^m, \ldots, f_{\lambda_m}^m \rangle \end{aligned}$$

and

$$Y'_m = \langle f_1^c, \ldots, f_{\lambda_c}^c, f_1^{c-1}, \ldots, f_{\lambda_{c-1}}^{c-1}, \ldots, f_1^{m+1}, \ldots, f_{\lambda_{m+1}}^{m+1} \rangle$$

all pairwise intersect in 0. We claim that the following two statements hold.

Claim 1:

$$(\ddagger) = \sum_X \Big\{ W_X \big| X = (X_0 > X_1 \ldots > X_c) \text{ where } X \text{ is a } \lambda\text{-flag such that } X_j = Y'_j \text{ for } j \neq m-1 \text{ and } X_{m-1} > \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle \text{ and } X_{m-1} \cap Z = 0 \Big\}.$$

Claim 2: If $X = (X_0 > X_1 \ldots > X_c)$ is a λ-flag such that $X_j = Y'_j$ for $j \neq m-1$ and $X_{m-1} > \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ and $X_{m-1} \cap Z = 0$ then we have that $T(Y') \trianglelefteq T(X)$.

Note that $Y' = (Y'_0 > Y'_1 \ldots > Y'_c)$ satisfies $Y'_m > \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ and $Y'_{m-1} \cap Z = 0$. Furthermore, if $T(Y') \trianglelefteq T(X)$ and $X \neq Y'$ then $W_X = 0$ by the nature of Y'. Hence $(\ddagger) = W_{Y'} = 0$ contradicting our choice of Y' as an occurring flag. Thus proving these claims will complete the proof of Theorem 6.3.4.

We now prove the claims above.

Claim 1: Consider a λ-flag $X = (X_0 > X_1 \ldots > X_c)$ such that $X_j = Y'_j$ for $j \neq m-1$ and $X_{m-1} > \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ and $X_{m-1} \cap Z = 0$. If $k = 0$ and $\dim A = k$ then X_{m-1} satisfies $X_{m-1} > A \oplus \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ and so W_X is counted once. There does not exist A such that $\dim A > 0$ where $Z > A$ and $X_{m-1} > A$. Hence W_X is counted exactly once.

Now suppose X as above except that $\dim(X_{m-1} \cap Z) = b > 0$. For fixed k, the number of times W_X is counted is equal to

$$\begin{aligned} &(-1)^k q^{\frac{1}{2}k(k-1)} (\#A \text{ such that } \dim A = k \text{ where } Z > A \text{ and } X_{m-1} > A) \\ &\quad = (-1)^k q^{\frac{1}{2}k(k-1)} (\#A \text{ such that } \dim A = k, \text{ and } Z \cap X_{m-1} > A) \\ &\quad = (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} b \\ k \end{bmatrix} \end{aligned}$$

Hence the total number of times that $W(X)$ is counted is equal to

$$\sum_{k=0}^{i-1} (-1)^k q^{\frac{1}{2}k(k-1)} \begin{bmatrix} b \\ k \end{bmatrix} = 0$$

since $0 < b \leq \dim Z = i - 1$.

Claim 2: Suppose that $X = (X_0 > X_1 > \ldots X_c)$ is a λ-flag such that $X_j = Y'_j$ for $j \neq m-1$ and $X_{m-1} > \langle f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ and $X_{m-1} \cap Z = 0$. All rows of $T(X)$ are equal to the rows of $T(Y')$ except for possibly rows $m-1$ and m.

Suppose that $X_{m-1} = X_m \oplus \langle g^m_1, g^m_2, \ldots, g^m_i, f^m_{i+1}, \ldots, f^m_{\lambda_m} \rangle$ where for $1 \leq d \leq i$

we have that

$$g_d^m = \sum_{j=1}^{u_d^m} b_{d,j}^m e_j$$

where

$b_{d,u_d^m}^m \neq 0$

$u_1^m < u_2^m < \ldots < u_i^m$

$u_d^m \neq s_a^m$ for all a with $i < a \leq \lambda_j$, and all d with $1 \leq d \leq i$

$u_d^m \neq s_a^j$ for all $j > m$ and all a with $1 < a \leq \lambda_j$, and all d with $1 \leq d \leq i$.

Then row m of $T(X)$ contains the entries

$$u_1^m, u_2^m, \ldots, u_i^m, s_{i+1}^m, \ldots, s_{\lambda_m}^m.$$

Note that

$$X_{m-1} < X_{m-2} = \langle f_1^c, \ldots, f_{\lambda_c}^c, f_1^{c-1}, \ldots, f_{\lambda_{c-1}}^{c-1}, \ldots, f_1^{m-1}, \ldots, f_{\lambda_{m-1}}^{m-1} \rangle$$

so that

$$g_d^m = \sum_{j=m+1}^{c} \sum_{l=1}^{\lambda_j} \beta_{d,l}^{j,m} f_l^j + \sum_{l=1}^{\lambda_m} \beta_{d,l}^m f_l^m + \sum_{l=1}^{\lambda_{m-1}} \beta_{d,l}^{m-1} f_l^{m-1}.$$

Now suppose we replace each g_d^m with the vector

$$\tilde{g}_d^m = g_d^m - \sum_{j=m+2}^{c} \sum_{l=1}^{\lambda_j} \beta_{d,l}^{j,m} f_l^j.$$

Since

$$\sum_{j=m+2}^{c} \sum_{l=1}^{\lambda_j} \beta_{d,l}^{j} f_l^j \in X_m,$$

we still have that

$$X_{m-1} = X_m \oplus \langle \tilde{g}_1^m, \ldots, \tilde{g}_i^m, f_{i+1}^m, \ldots, f_{\lambda_m}^m \rangle$$

and

$$\tilde{g}_d^m = \sum_{j=1}^{u_d^m} c_{d,j}^m e_j$$

where $c^m_{d,u^m_d} \neq 0$. Hence we can assume that g^m_d is of the form

$$g^m_d = \sum_{l=1}^{\lambda_m} \beta^m_{d,l} f^m_l + \sum_{l=1}^{\lambda_{m-1}} \beta^{m-1}_{d,l} f^{m-1}_l.$$

Claim that $u^m_d \geq s^m_d$ for all $d \leq i$. Suppose for a contradiction that $u^m_{d^*} < s^m_{d^*}$ for some $d^* \leq i$. Then for $d \leq d^*$ we have that

$$g^m_d = \sum_{l=1}^{d^*-1} \beta^m_{d,l} f^m_l + \sum_{l=1}^{i-1} \beta^{m-1}_{d,l} f^{m-1}_l$$

since $u^m_d \leq u^m_{d^*} < s^m_{d^*} \leq s^m_i < s^{m-1}_i$. Let

$$h^m_d = \sum_{l=1}^{d^*-1} \beta^m_{d,l} f^m_l.$$

Then there are coefficients x_d, not all zero, such that

$$\sum_{d=1}^{d^*} x_d h^m_d = 0$$

and so

$$\sum_{d=1}^{d^*} x_d g^m_d \in Z$$

and is non–zero since $\{g^m_d\}$ are linearly independent. Hence $X_{m-1} \cap Z \neq 0$, giving the required contradiction.

So $T(Y') \trianglelefteq T(X)$, completing the proof. $\square$

A Some results relating to partitions with at most 4 parts

A.1 Notation

Suppose λ is a partition of n with at most 3 parts. Then $B_4'(\lambda)$, as defined in Definition 2.3.1, is given by the tables overleaf; the notation is that of Section 2.4. To obtain the entries of the decomposition matrix, we set $v = 1$. Note that the partitions in each table are arranged according to the length–lexicographic order, except in a few cases which are discussed below.

The results are, purely for convenience, arranged in the following order.

1. The centre: $a, b, c \geq 1$
2. Wall $a : a = 0$ and $b, c > 0$
3. Wall $b : b = 0$ and $a, c > 0$
4. Wall $c : c = 0$ and $a, b > 0$
5. Spoke $a : b = c = 0$ and $a > 0$
6. Spoke $b : a = c = 0$ and $b > 0$
7. Spoke $c : a = b = 0$

We then consider each of the regions A–Z as in Definition 2.4.1. Most of the tables are straightforward to read.

Example A.1.1. Consider case B, that is $j = k = 0$ and $i > 0$, when $a, b, c \geq 1$. The following table occurs in Section A.2.

		$\mathbf{a, b, c \geq 1}$
B	$(\mathbf{m_1 + i, m_2, m_3})$	1
	$(\mathbf{m_1, m_2 + i, m_3})$	v
	$(\mathbf{m_1, m_2, m_3 + i})$	v^2
	$(\mathbf{m_1, m_2, m_3, i})$	v^3

This should be taken to mean that for the given case

$$\begin{aligned}B_4'(m_1+i,m_2,m_3) =&(m_1+i,m_2,m_3)+v(m_1,m_2+i,m_3)+v^2(m_1,m_2,m_3+i)\\&+v^3(m_1,m_2,m_3,i).\end{aligned}$$

In cases Y and Z of the centre, a row may look similar to the following line.

$$(\mu_1,\mu_2,\mu_3,\mu_4) \qquad\qquad v^2 \iff a>2.$$

This simply means that the partition $(\mu_1,\mu_2,\mu_3,\mu_4)$ occurs with coefficient v^2 if and only if $a>2$, with no further conditions on b and c. More complicated notation sometimes occurs in the spokes.

Example A.1.2. Consider the case I, that is $j=0$ and $i+k=e$, when $a=c=0$. The following table occurs in Section A.2.

I				
$\mathbf{b=2x+y+2}$ where $\mathbf{0\le y<2}$	$\mathbf{b=1}$	$\mathbf{y=0}$	$\mathbf{y=1}$	
$(\mathbf{m_1+e,m_2+k,m_3+k})$	1	1	1	
$(\mathbf{m_1+k,m_2,m_3+e+k})$		v	v	
$(\mathbf{m_1+e,m_2-e+k,m_3+e,k})$		v	v	
$(\mathbf{m_1+k,m_2,m_3+e,k})$		v^2	v^2	
$(\mathbf{m_1-ze+k,m_2-(z+1)e,m_3+(z+1)e+k,(z+1)e})$		v	v	$0\le z<x$
$(\mathbf{m_1-ze,m_2-(z+1)e+k,m_3+(z+2)e,ze+k})$		v	v	$0\le z<x$
$(\mathbf{m_1-ze,m_2-(z+1)e+k,m_3+(z+1)e+k,(z+1)e})$		$2v^2$	$2v^2$	$0\le z<x$
$(\mathbf{m_1-ze,m_2-(z+2)e+k,m_3+(z+2)e,(z+1)e+k})$		v	v	$0\le z<x$
$(\mathbf{m_1-(z+1)e+k,m_2-(z+1)e,m_3+(z+2)e+k,(z+1)e})$		v	v	$0\le z<x$
$(\mathbf{m_1-(z+1)e+k,m_2-(z+1)e,m_3+(z+2)e,(z+1)e+k})$		$2v^2$	$2v^2$	$0\le z<x$
$(\mathbf{m_1-xe+k,m_2-(x+1)e,m_3+(x+1)e+k,(x+1)e})$			v	
$(\mathbf{m_1-xe,m_2-(x+1)e+k,m_3+(x+2)e,xe+k})$			v	
$(\mathbf{m_1-xe,m_2-(x+1)e+k,m_3+(x+1)e+k,(x+1)e})$			$2v^2$	

This should be read in the following way. If $b=1$ then

$$B_4'(m_1+e,m_2+k,m_3+k)=(m_1+e,m_2+k,m_3+k).$$

Otherwise write

$$b=2x+y+2$$

where $0\le y<2$. If $b>1$ and $y=0$ then

$$\begin{aligned}B_4'(m_1+e,m_2+k,m_3+k)&=(m_1+e,m_2+k,m_3+k)\\&+v(m_1+k,m_2,m_3+e+k)+v(m_1+e,m_2+k-e,m_3+e,k)\\&+v^2(m_1+k,m_2,m_3+e,k)\end{aligned}$$

$$
\begin{aligned}
&+ v(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e) \\
&+ v(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + k) \\
&+ 2v^2(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e) \\
&+ v(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + k) \\
&+ v(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e) \\
&+ 2v^2(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + k)
\end{aligned}
$$

for all values of z where $0 \le z < x$. If $b > 1$ and $y = 1$ then

$$
\begin{aligned}
B_4'(m_1 + e, m_2 + k, m_3 + k) = {}& (m_1 + e, m_2 + k, m_3 + k) \\
&+ v(m_1 + k, m_2, m_3 + e + k) + v(m_1 + e, m_2 + k - e, m_3 + e, k) \\
&+ v^2(m_1 + k, m_2, m_3 + e, k) \\
&+ v(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e) \\
&+ v(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + k) \\
&+ 2v^2(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e) \\
&+ v(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + k) \\
&+ v(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e) \\
&+ 2v^2(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + k) \\
&+ (m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e) \\
&+ (m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + k) \\
&+ (m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + k, (x+1)e)
\end{aligned}
$$

for all values of z where $0 \le z < x$.

When we use this notation, we do not arrange the partitions according to the length–lexicographic order.

A.2 Tables

The Centre: $a, b, c > 0$

A	
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3})$	1

B	
$(\mathbf{m_1}+\mathbf{i}, \mathbf{m_2}, \mathbf{m_3})$	1
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{i}, \mathbf{m_3})$	v
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{i})$	v^2
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{i})$	v^3

C	
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3})$	1
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j})$	v
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{j})$	v^2
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{j})$	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}, \mathbf{j})$	v^3
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j}, \mathbf{j})$	v^4

D	
$(\mathbf{m_1}+\mathbf{i}+\mathbf{j}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3})$	1
$(\mathbf{m_1}+\mathbf{i}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j})$	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{i}+\mathbf{j}, \mathbf{m_3})$	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{i}+\mathbf{j})$	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{i}+\mathbf{j}, \mathbf{m_3}+\mathbf{j})$	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{i}+\mathbf{j})$	v^3
$(\mathbf{m_1}+\mathbf{i}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{j})$	v^2
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{i}+\mathbf{j})$	v^3
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{i}+\mathbf{j}, \mathbf{m_3}, \mathbf{j})$	v^3
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}, \mathbf{i}+\mathbf{j})$	v^4
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{i}+\mathbf{j}, \mathbf{j})$	v^4
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j}, \mathbf{i}+\mathbf{j})$	v^5

E								
a	**1**	**1**	**1**	**1**	$>$ **1**	$>$ **1**	$>$ **1**	$>$ **1**
b	**1**	**1**	$>$ **1**	$>$ **1**	**1**	**1**	$>$ **1**	$>$ **1**
c	**1**	$>$ **1**	**1**	$>$ **1**	**1**	$>$ **1**	**1**	$>$ **1**
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{e})$			v	v			v	v
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3}+\mathbf{j})$					v	v	v	v
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{j})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{e})$		v		v		v		v
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3}, \mathbf{j})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}, \mathbf{e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{e}, \mathbf{j})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j}, \mathbf{e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3

F								
a	**1**	**1**	**1**	**1**	$>$ **1**	$>$ **1**	$>$ **1**	$>$ **1**
b	**1**	**1**	$>$ **1**	$>$ **1**	**1**	**1**	$>$ **1**	$>$ **1**
c	**1**	$>$ **1**	**1**	$>$ **1**	**1**	$>$ **1**	**1**	$>$ **1**
$(\mathbf{m_1}+\mathbf{e}+\mathbf{I}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1}+\mathbf{e}+\mathbf{I}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}+\mathbf{I}, \mathbf{m_3}+\mathbf{j})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{e}+\mathbf{I}, \mathbf{m_3})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3}+\mathbf{I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{I}, \mathbf{m_3}+\mathbf{e})$			v	v			v	v
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{e}+\mathbf{I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1}+\mathbf{I}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3}+\mathbf{j})$					v	v	v	v
$(\mathbf{m_1}+\mathbf{I}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{e}+\mathbf{I}, \mathbf{m_3}+\mathbf{j})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}+\mathbf{e}+\mathbf{I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1}+\mathbf{e}+\mathbf{I}, \mathbf{m_2}, \mathbf{m_3}, \mathbf{j})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}+\mathbf{j}, \mathbf{m_3}, \mathbf{I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}+\mathbf{I}, \mathbf{m_3}, \mathbf{j})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{j}, \mathbf{I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1}+\mathbf{e}, \mathbf{m_2}, \mathbf{m_3}+\mathbf{I}, \mathbf{j})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1}+\mathbf{j}, \mathbf{m_2}+\mathbf{e}, \mathbf{m_3}, \mathbf{I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3

$(\mathbf{m_1+j, m_2+I, m_3, e})$		v		v		v		v
$(\mathbf{m_1+j, m_2, m_3+e, I})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+j, m_2, m_3+I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j, m_2, m_3, e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+I, m_2+e, m_3, j})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+j, m_3, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+I, m_2, m_3+e, j})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2, m_3+j, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3, j})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+j, I})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+I, j})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j, m_3+e, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j, m_3+I, e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1, m_2+j, m_3, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1, m_2+I, m_3+e, j})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1, m_2+I, m_3+j, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2, m_3+e+I, j})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2, m_3+j, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

G

$(\mathbf{m_1+k, m_2+k, m_3+k})$	1
$(\mathbf{m_1+k, m_2+k, m_3, k})$	v
$(\mathbf{m_1+k, m_2, m_3+k, k})$	v^2
$(\mathbf{m_1, m_2+k, m_3+k, k})$	v^3

H

$(\mathbf{m_1+i+k, m_2+k, m_3+k})$	1
$(\mathbf{m_1+k, m_2+i+k, m_3+k})$	v
$(\mathbf{m_1+k, m_2+k, m_3+i+k})$	v^2
$(\mathbf{m_1+i+k, m_2+k, m_3, k})$	v
$(\mathbf{m_1+i+k, m_2, m_3+k, k})$	v^2
$(\mathbf{m_1+k, m_2+i+k, m_3, k})$	v^2
$(\mathbf{m_1+k, m_2+k, m_3, i+k})$	v^3
$(\mathbf{m_1+k, m_2, m_3+i+k, k})$	v^3
$(\mathbf{m_1+k, m_2, m_3+k, i+k})$	v^4
$(\mathbf{m_1, m_2+i+k, m_3+k, k})$	v^3
$(\mathbf{m_1, m_2+k, m_3+i+k, k})$	v^4
$(\mathbf{m_1, m_2+k, m_3+k, i+k})$	v^5

I

a	**1**	**1**	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
b	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$
c	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$
$(\mathbf{m_1+e, m_2+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2+e, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+k, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2, m_3+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+k, m_3, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+k, m_2, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k, m_2, m_3+k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+k, m_3+e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+k, m_3+k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4

J

a	**1**	**1**	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
b	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$
c	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$
$(\mathbf{m_1+e+I, m_2+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+k, m_3+e+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2, m_3+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+k, I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3+I, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+k, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+k, m_3+e, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+k, m_3+I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+k, m_2+k, m_3, e+I})$		v^3		v^3		v^3		v^3

$(\mathbf{m_1 + k, m_2 + I, m_3 + e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1 + k, m_2 + I, m_3 + k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + k, m_2, m_3 + e + I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1 + k, m_2, m_3 + k, e + I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1 + I, m_2 + e, m_3 + k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1 + I, m_2 + k, m_3 + e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + I, m_2 + k, m_3 + k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2 + e + I, m_3 + k, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2 + k, m_3 + e + I, k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2 + k, m_3 + k, e + I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

K

$(\mathbf{m_1 + j + k, m_2 + j + k, m_3 + k})$	1
$(\mathbf{m_1 + j + k, m_2 + k, m_3 + j + k})$	v
$(\mathbf{m_1 + k, m_2 + j + k, m_3 + j + k})$	v^2
$(\mathbf{m_1 + j + k, m_2 + j + k, m_3, k})$	v
$(\mathbf{m_1 + j + k, m_2 + k, m_3, j + k})$	v^2
$(\mathbf{m_1 + j + k, m_2, m_3 + j + k, k})$	v^2
$(\mathbf{m_1 + j + k, m_2, m_3 + k, j + k})$	v^3
$(\mathbf{m_1 + k, m_2 + j + k, m_3, j + k})$	v^3
$(\mathbf{m_1 + k, m_2, m_3 + j + k, j + k})$	v^4
$(\mathbf{m_1, m_2 + j + k, m_3 + j + k, k})$	v^3
$(\mathbf{m_1, m_2 + j + k, m_3 + k, j + k})$	v^4
$(\mathbf{m_1, m_2 + k, m_3 + j + k, j + k})$	v^5

L

a	**1**	**1**	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$
b	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
c	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
$(\mathbf{m_1 + e, m_2 + e, m_3 + k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1 + e, m_2 + k, m_3 + e})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + k, m_2 + e, m_3 + e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + e, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e, m_2 + k, m_3, e})$		v		v		v		v
$(\mathbf{m_1 + e, m_2, m_3 + e, k})$			v	v			v	v
$(\mathbf{m_1 + e, m_2, m_3 + k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + k, m_2 + e, m_3, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1 + k, m_2, m_3 + e, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1, m_2 + e, m_3 + e, k})$					v	v	v	v
$(\mathbf{m_1, m_2 + e, m_3 + k, e})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2 + k, m_3 + e, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3

M

a	**1**	**1**	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$
b	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
c	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
$(\mathbf{m_1 + e + I, m_2 + e + I, m_3 + k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + e + I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + k, m_2 + e + I, m_3 + e + I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e + I, m_2 + e + I, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + k, I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + I, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + e, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + I, e})$		v		v		v		v
$(\mathbf{m_1 + e + I, m_2 + k, m_3, e + I})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1 + e + I, m_2 + I, m_3 + e, k})$			v	v			v	v
$(\mathbf{m_1 + e + I, m_2 + I, m_3 + k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e + I, m_2, m_3 + e + I, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1 + e + I, m_2, m_3 + k, e + I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + e, m_2 + e + I, m_3 + k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + I, m_3 + I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + e, m_2 + k, m_3 + e + I, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + e, m_2 + k, m_3 + I, e + I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1 + e, m_2 + I, m_3 + e + I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1 + e, m_2 + I, m_3 + k, e + I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1 + k, m_2 + e + I, m_3 + e, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + k, m_2 + e + I, m_3 + I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1 + k, m_2 + e + I, m_3, e + I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1 + k, m_2 + e, m_3 + e + I, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1 + k, m_2 + e, m_3 + I, e + I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1 + k, m_2 + I, m_3 + e + I, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1 + k, m_2 + I, m_3 + e, e + I})$			v^3	v^3			v^3	v^3

$(\mathbf{m_1+k, m_2, m_3+e+I, e+I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+I, m_2+e+I, m_3+e, k})$					v	v	v	v
$(\mathbf{m_1+I, m_2+e+I, m_3+k, e})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+e+I, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, e+I})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+k, m_3+e+I, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+k, m_3+e, e+I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+I, m_3+e+I, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, e+I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e+I, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

N

$(\mathbf{m_1+i+j+k, m_2+j+k, m_3+k})$	1
$(\mathbf{m_1+i+j+k, m_2+k, m_3+j+k})$	v
$(\mathbf{m_1+j+k, m_2+i+j+k, m_3+k})$	v
$(\mathbf{m_1+j+k, m_2+k, m_3+i+j+k})$	v^2
$(\mathbf{m_1+k, m_2+i+j+k, m_3+j+k})$	v^2
$(\mathbf{m_1+k, m_2+j+k, m_3+i+j+k})$	v^3
$(\mathbf{m_1+i+j+k, m_2+j+k, m_3, k})$	v
$(\mathbf{m_1+i+j+k, m_2+k, m_3, j+k})$	v^2
$(\mathbf{m_1+i+j+k, m_2, m_3+j+k, k})$	v^2
$(\mathbf{m_1+i+j+k, m_2, m_3+k, j+k})$	v^3
$(\mathbf{m_1+j+k, m_2+i+j+k, m_3, k})$	v^2
$(\mathbf{m_1+j+k, m_2+k, m_3, i+j+k})$	v^3
$(\mathbf{m_1+j+k, m_2, m_3+i+j+k, k})$	v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, i+j+k})$	v^4
$(\mathbf{m_1+k, m_2+i+j+k, m_3, j+k})$	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3, i+j+k})$	v^4
$(\mathbf{m_1+k, m_2, m_3+i+j+k, j+k})$	v^4
$(\mathbf{m_1+k, m_2, m_3+j+k, i+j+k})$	v^5
$(\mathbf{m_1, m_2+i+j+k, m_3+j+k, k})$	v^3
$(\mathbf{m_1, m_2+i+j+k, m_3+k, j+k})$	v^4
$(\mathbf{m_1, m_2+j+k, m_3+i+j+k, k})$	v^4
$(\mathbf{m_1, m_2+j+k, m_3+k, i+j+k})$	v^5
$(\mathbf{m_1, m_2+k, m_3+i+j+k, j+k})$	v^5
$(\mathbf{m_1, m_2+k, m_3+j+k, i+j+k})$	v^6

O

a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
c	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e, m_2+j+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e, m_2+k, m_3+j+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+j+k, m_3+e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+j+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+k, m_3, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+j+k, m_2, m_3+k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3, e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2, m_3+e, j+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+j+k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e, m_3+j+k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+e, m_3+k, j+k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+j+k, m_3+e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+j+k, m_3+k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+j+k, e})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

$\mathbf{P}$								
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+j+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3+j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+j+k, m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+j+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+I, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+j+k, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+k, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+j+k, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+k, m_3+e, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+k, m_3+I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2+k, m_3, e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+j+k, m_2+I, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+j+k, m_2+I, m_3+k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+e+I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, e+I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e+I, m_3, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+j+k, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+I, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+j+k, m_3+e, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+j+k, m_3+I, e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+j+k, m_3, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2+I, m_3+e, j+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2+I, m_3+j+k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+e+I, j+k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+j+k, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+I, m_2+e, m_3+j+k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, j+k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+j+k, m_3+e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+j+k, m_3+k, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2+k, m_3+e, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2+k, m_3+j+k, e})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+e+I, m_3+j+k, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, j+k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+e+I, k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+k, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+e+I, j+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+j+k, e+I})$	v^6	v^6	v^6	v^6	v^6	v^6	v^6	v^6

$\mathbf{Q}$								
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+k, m_2+j+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+k, m_2+k, m_3+j+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e+k, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+j+k})$					v	v	v	v
$(\mathbf{m_1+k, m_2+j+k, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+j+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+j+k, m_3+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+j+k, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+k, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+j+k, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+k, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3

$(\mathbf{m_1+j+k, m_2+k, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+j+k, m_2+k, m_3+k, e})$		v		v		v		v
$(\mathbf{m_1+j+k, m_2+k, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2, m_3+e+k, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+e+k, m_3, j+k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+j+k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+k, j+k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3+e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3+k, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+k, m_2+j+k, m_3, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+k, m_3+e, j+k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k, m_2+k, m_3+j+k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+k, j+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+j+k, e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+k, m_3+j+k, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+k, j+k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+e+k, k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+k, e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1, m_2+k, m_3+e+k, j+k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+j+k, e+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

$\mathbf{R}$								
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$
$\mathbf{c}$	$\mathbf{1}$	$>\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$	$\mathbf{1}$	$>\mathbf{1}$
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+j+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+j+k, m_3+I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+I, m_3+j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e+k, m_3+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+I, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e+I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+j+k})$					v	v	v	v
$(\mathbf{m_1+I, m_2+j+k, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3+j+k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+j+k, m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+j+k, m_3, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, j+k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+j+k, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+I, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+j+k, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+k, j+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+j+k, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+I, j+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+j+k, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e, m_3+k, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+j+k, m_2+I, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+j+k, m_2+I, m_3+k, e})$		v		v		v		v
$(\mathbf{m_1+j+k, m_2+I, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2+k, m_3+e, I})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+j+k, m_2+k, m_3+I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2+k, m_3, e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+j+k, m_2, m_3+e+I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+e+k, I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+j+k, m_2, m_3+I, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+I, m_2+e+k, m_3, j+k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+j+k, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, j+k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+j+k, m_3+e, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+j+k, m_3+k, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+I, m_2+j+k, m_3, e+k})$		v^3		v^3		v^3		v^3

$(\mathbf{m_1+I,m_2+k,m_3+e,j+k})$			v	v			v	v
$(\mathbf{m_1+I,m_2+k,m_3+j+k,e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I,m_2,m_3+e+k,j+k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I,m_2,m_3+j+k,e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2+e+I,m_3,j+k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+k,m_2+e,m_3+j+k,I})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+k,m_2+e,m_3+I,j+k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2+j+k,m_3+e,I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2+j+k,m_3+I,e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k,m_2+j+k,m_3,e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k,m_2+I,m_3+e,j+k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k,m_2+I,m_3+j+k,e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2,m_3+e+I,j+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k,m_2,m_3+j+k,e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1,m_2+e+I,m_3+j+k,k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1,m_2+e+I,m_3+k,j+k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1,m_2+e+k,m_3+j+k,I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1,m_2+e+k,m_3+I,j+k})$					v^5	v^5	v^5	v^5
$(\mathbf{m_1,m_2+j+k,m_3+e+I,k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1,m_2+j+k,m_3+e+k,I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1,m_2+j+k,m_3+I,e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1,m_2+j+k,m_3+k,e+I})$		v^5		v^5		v^5		v^5
$(\mathbf{m_1,m_2+I,m_3+e+k,j+k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1,m_2+I,m_3+j+k,e+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1,m_2+k,m_3+e+I,j+k})$			v^5	v^5			v^5	v^5
$(\mathbf{m_1,m_2+k,m_3+j+k,e+I})$	v^6	v^6	v^6	v^6	v^6	v^6	v^6	v^6

S								
a	**1**	**1**	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
b	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$
c	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$
$(\mathbf{m_1+e+I,m_2+e,m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I,m_2+k,m_3+e})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e,m_2+e+I,m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e,m_2+k,m_3+e+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k,m_2+e+I,m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k,m_2+e,m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I,m_2+e,m_3,k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I,m_2+k,m_3,e})$		v		v		v		v
$(\mathbf{m_1+e+I,m_2,m_3+e,k})$			v	v			v	v
$(\mathbf{m_1+e+I,m_2,m_3+k,e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e,m_2+e+I,m_3,k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e,m_2+e,m_3+k,I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e,m_2+e,m_3+I,k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e,m_2+k,m_3+e,I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e,m_2+k,m_3+I,e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e,m_2+k,m_3,e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e,m_2+I,m_3+e,k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e,m_2+I,m_3+k,e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e,m_2,m_3+e+I,k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e,m_2,m_3+k,e+I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2+e+I,m_3,e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+k,m_2+e,m_3+e,I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k,m_2+e,m_3+I,e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k,m_2+e,m_3,e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k,m_2+I,m_3+e,e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k,m_2,m_3+e+I,e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k,m_2,m_3+e,e+I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+I,m_2+e,m_3+e,k})$					v	v	v	v
$(\mathbf{m_1+I,m_2+e,m_3+k,e})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I,m_2+k,m_3+e,e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1,m_2+e+I,m_3+e,k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1,m_2+e+I,m_3+k,e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1,m_2+e,m_3+e+I,k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1,m_2+e,m_3+k,e+I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1,m_2+k,m_3+e+I,e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1,m_2+k,m_3+e,e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5

T								
a	**1**	**1**	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$
b	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
c	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
$(\mathbf{m_1+e+k, m_2+e, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+k, m_2+k, m_3+e})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+e+k, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+e})$					v	v	v	v
$(\mathbf{m_1+k, m_2+e, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, e})$		v		v		v		v
$(\mathbf{m_1+e+k, m_2, m_3+e, k})$			v	v			v	v
$(\mathbf{m_1+e+k, m_2, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e, m_3+k, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+k, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e, m_2+k, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e, m_2+k, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e, m_2, m_3+e+k, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+k, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+e+k, m_3, e})$						v		v
$(\mathbf{m_1+k, m_2+e, m_3+e, k})$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+k, m_2+e, m_3+k, e})$		v^2		v^2	v^2	$2v^2$	v^2	$2v^2$
$(\mathbf{m_1+k, m_2+e, m_3, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+k, m_3+e, e})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+k, m_2, m_3+e+k, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k, m_2, m_3+e, e+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+e, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+e+k, m_3+k, e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+e+k, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+k, e+k})$						v^3		v^3
$(\mathbf{m_1, m_2+k, m_3+e+k, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1, m_2+k, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4

U								
a	**1**	**1**	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$	$> \mathbf{1}$
b	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
c	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
$(\mathbf{m_1+e+I, m_2+e, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+e})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+I, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+e+k, m_3+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+e, m_2+k, m_3+e+I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+e})$					v	v	v	v
$(\mathbf{m_1+I, m_2+e, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3+e})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I, m_2+e, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e})$		v		v		v		v
$(\mathbf{m_1+e+I, m_2, m_3+e, k})$			v	v			v	v
$(\mathbf{m_1+e+I, m_2, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+k, m_2, m_3+e, I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+I, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e, m_3+I, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+e, m_3+k, I})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+I, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e, m_2+I, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e, m_2+I, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e, m_2+k, m_3+e, I})$	v^3	v^3	$2v^3$	$2v^3$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+I, e})$	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3, e+I})$				v^3		v^3		v^3
$(\mathbf{m_1+e, m_2, m_3+e+I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+e+k, I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+e, m_2, m_3+I, e+k})$		v^3		v^3		v^3		v^3

$(\mathbf{m_1+e, m_2, m_3+k, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+I, m_2+e+k, m_3, e})$						v		v
$(\mathbf{m_1+I, m_2+e, m_3+e, k})$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2+e, m_3+k, e})$		v^2		v^2	v^2	$2v^2$	v^2	$2v^2$
$(\mathbf{m_1+I, m_2+e, m_3, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+I, m_2+k, m_3+e, e})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2, m_3+e+k, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2, m_3+e, e+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3, e})$						v^2		v^2
$(\mathbf{m_1+k, m_2+e, m_3+e, I})$	v^4	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+e, m_3+I, e})$		v^3		v^3	v^3	$2v^3$	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2+I, m_3+e, e})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2, m_3+e+I, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e, e+I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2+e+I, m_3+e, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+e+I, m_3+k, e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+e, I})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+I, e})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e, m_3+e+I, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e, m_3+e+k, I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e, m_3+I, e+k})$						v^3		v^3
$(\mathbf{m_1, m_2+e, m_3+k, e+I})$						v^4		v^4
$(\mathbf{m_1, m_2+I, m_3+e+k, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1, m_2+I, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e+I, e})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
V								
a	**1**	**1**	**1**	**1**	**> 1**	**> 1**	**> 1**	**> 1**
b	**1**	**1**	**> 1**	**> 1**	**1**	**1**	**> 1**	**> 1**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
$(\mathbf{m_1+e+I, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+e+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+J, e})$		v		v		v		v
$(\mathbf{m_1+e+I, m_2+k, m_3, e+J})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+I, m_2+J, m_3+e, k})$			v	v			v	v
$(\mathbf{m_1+e+I, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+e+J, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+e, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+I, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+J, m_2+k, m_3, e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e+J, m_2+I, m_3+e, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+J, m_2+I, m_3+k, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, e+I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+e+I, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3+J, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+k, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+I, k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+e+I, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+I, e+J})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e, m_2+k, m_3+J, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+e, m_2+I, m_3+e+J, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+k, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+I, k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+k, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+k, m_2+e+I, m_3+e, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3+J, e})$		v^2		v^2		v^2		v^2

$(\mathbf{m_1+k, m_2+e+I, m_3, e+J})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+e+J, m_3+e, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e+J, m_3+I, e})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+e+J, m_2, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_3+e, m_2+e+I, J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_3+e, m_2+e+J, I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+k, m_3+e, m_2+I, e+J})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_3+e, m_2+J, e+I})$		v^5		v^5		v^5		v^5
$(\mathbf{m_1+k, m_3+I, m_2+e+J, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k, m_3+I, m_2+e, e+J})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_3+J, m_2+e+I, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_3+J, m_2+e, e+I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_3, m_2+e+I, e+J})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_3, m_2+e+J, e+I})$			v^5	v^5			v^5	v^5
$(\mathbf{m_1+I, m_3+e+J, m_2+e, k})$					v	v	v	v
$(\mathbf{m_1+I, m_3+e+J, m_2+k, e})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_3+e, m_2+e+J, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_3+e, m_2+k, e+J})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_3+k, m_2+e+J, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+k, m_3+e, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+e+I, m_3+e, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+J, m_2+e+I, m_3+k, e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+I, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+k, m_3+e+I, e})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+k, m_3+e, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+e+I, m_3+e+J, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, e+J})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+I, k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+I})$					v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+e+I, e+J})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+e+J, e+I})$	v^6	v^6	v^6	v^6	v^6	v^6	v^6	v^6
W								
a	**1**	**1**	**1**	**1**	$>\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$	$>\mathbf{1}$
b	**1**	**1**	$>\mathbf{1}$	$>\mathbf{1}$	**1**	**1**	$>\mathbf{1}$	$>\mathbf{1}$
c	**1**	$>\mathbf{1}$	**1**	$>\mathbf{1}$	**1**	$>\mathbf{1}$	**1**	$>\mathbf{1}$
$(\mathbf{m_1+e+k, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+k, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+k, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+e+J})$					v	v	v	v
$(\mathbf{m_1+k, m_2+e+J, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+J, e})$		v		v		v		v
$(\mathbf{m_1+e+k, m_2+k, m_3, e+J})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+k, m_2+J, m_3+e, k})$			v	v			v	v
$(\mathbf{m_1+e+k, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+e+J, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+k, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+k, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+k, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+J, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+k, k})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3+e+k, J})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+e+J, k})$	v^3	v^3	$2v^3$	$2v^3$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+k, e+J})$	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3+J, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e, m_2+J, m_3+e+k, k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+k, e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2+e+k, m_3+e, J})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+k, m_3+J, e})$						v		v
$(\mathbf{m_1+k, m_2+e+k, m_3, e+J})$						v^2		v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+e, k})$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$

$(\mathbf{m_1+k, m_2+e+J, m_3+k, e})$		v^2		v^2	v^2	$2v^2$	v^2	$2v^2$
$(\mathbf{m_1+k, m_2+e+J, m_3, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2+e, m_3+e+k, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+e+J, k})$	v^4	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+e, m_3+k, e+J})$		v^3		v^3	v^3	$2v^3$	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3+J, e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2+k, m_3+e+J, e})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+k, m_2+k, m_3+e, e+J})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+J, m_3+e+k, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+k, m_2+J, m_3+e, e+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+k, e+J})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+J, e+k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+J, m_2+e+k, m_3+e, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+J, m_2+e+k, m_3+k, e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+k, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+k})$						v^3		v^3
$(\mathbf{m_1+J, m_2+k, m_3+e+k, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+J, m_2+k, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+k, m_3+e+J, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+k, e+J})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+k, k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+k})$						v^4		v^4
$(\mathbf{m_1, m_2+k, m_3+e+k, e+J})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e+J, e+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
X								
a	**1**	**1**	**1**	**1**	**> 1**	**> 1**	**> 1**	**> 1**
b	**1**	**1**	**> 1**	**> 1**	**1**	**1**	**> 1**	**> 1**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
$(\mathbf{m_1+e+I, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3+I})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+I, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+k, m_3+I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+I, m_3+e+k})$			v	v			v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+e+I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+e+J})$					v	v	v	v
$(\mathbf{m_1+I, m_2+e+J, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3+e+J})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+e+I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+I, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+J, e})$		v		v		v		v
$(\mathbf{m_1+e+I, m_2+k, m_3, e+J})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+I, m_2+J, m_3+e, k})$			v	v			v	v
$(\mathbf{m_1+e+I, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+e+J, k})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e+J, m_3, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3+I, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3+J, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+I, m_3+e, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+I, m_3+J, e})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e+J})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e+k, m_2+J, m_3+e, I})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e+k, m_2+J, m_3+I, e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+e+J, I})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+I, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+J, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e, m_3+I, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+e, m_3+k, I})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+J, m_2+I, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+I, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e+J, m_2+I, m_3, e+k})$		v^2		v^2		v^2		v^2
$(\mathbf{m_1+e+J, m_2+k, m_3+e, I})$	v^3	v^3	$2v^3$	$2v^3$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3+I, e})$	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e+J, m_2+k, m_3, e+I})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+I, k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+k, I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+e+J, m_2, m_3+I, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, e+I})$		v^4		v^4		v^4		v^4

$(\mathbf{m_1+e, m_2+e+I, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3+J, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+I, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+J, I})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+e+J, m_3+I, k})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+J, m_3+k, I})$	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+e, m_2+I, m_3+e+k, J})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+e+J, k})$	v^3	v^3	$2v^3$	$2v^3$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+I, m_3+k, e+J})$	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+I, m_3+J, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+I, J})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, I})$	v^4	v^4	$2v^3$	$2v^3$	v^4	v^4	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+I, e+J})$	v^5	v^3+v^5	v^5	v^3+v^5	v^5	v^3+v^5	v^5	v^3+v^5
$(\mathbf{m_1+e, m_2+k, m_3+J, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+I, k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+k, I})$			v^5	v^5			v^5	v^5
$(\mathbf{m_1+e, m_2+J, m_3+I, e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+e, m_2+J, m_3+k, e+I})$		v^5		v^5		v^5		v^5
$(\mathbf{m_1+I, m_2+e+k, m_3+e, J})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+J, e})$						v		v
$(\mathbf{m_1+I, m_2+e+k, m_3, e+J})$						v^2		v^2
$(\mathbf{m_1+I, m_2+e+J, m_3+e, k})$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2+e+J, m_3+k, e})$		v^2		v^2	v^2	$2v^2$	v^2	$2v^2$
$(\mathbf{m_1+I, m_2+e+J, m_3, e+k})$		v^3		v^3		v^3		v^3
$(\mathbf{m_1+I, m_2+e, m_3+e+k, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+e, m_3+e+J, k})$	v^4	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+I, m_2+e, m_3+k, e+J})$		v^3		v^3	v^3	$2v^3$	v^3	$2v^3$
$(\mathbf{m_1+I, m_2+e, m_3+J, e+k})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+I, m_2+k, m_3+e+J, e})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2+k, m_3+e, e+J})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+I, m_2+J, m_3+e+k, e})$			v^2	v^2			v^2	v^2
$(\mathbf{m_1+I, m_2+J, m_3+e, e+k})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+I, m_2, m_3+e+k, e+J})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+I, m_2, m_3+e+J, e+k})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_2+e+I, m_3+e, J})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3+J, e})$						v^2		v^2
$(\mathbf{m_1+k, m_2+e+I, m_3, e+J})$						v^3		v^3
$(\mathbf{m_1+k, m_2+e+J, m_3+e, I})$	v^4	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+e+J, m_3+I, e})$		v^3		v^3	v^3	$2v^3$	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e+J, m_3, e+I})$		v^4		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2+e, m_3+e+I, J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+e+J, I})$	v^5	v^5	v^5	v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+k, m_2+e, m_3+I, e+J})$		v^4		v^4	v^4	$2v^3$	v^4	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3+J, e+I})$		v^5		v^5		v^5		v^5
$(\mathbf{m_1+k, m_2+I, m_3+e+J, e})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+I, m_3+e, e+J})$	v^5	v^5	v^3+v^5	v^3+v^5	v^5	v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+k, m_2+J, m_3+e+I, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+k, m_2+J, m_3+e, e+I})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+e+I, e+J})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+e+J, e+I})$			v^5	v^5			v^5	v^5
$(\mathbf{m_1+J, m_2+e+I, m_3+e, k})$					v^2	v^2	v^2	v^2
$(\mathbf{m_1+J, m_2+e+I, m_3+k, e})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e+k, m_3+e, I})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e+k, m_3+I, e})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+e, m_3+e+I, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+k, I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+e, m_3+I, e+k})$						v^3		v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+I})$						v^4		v^4
$(\mathbf{m_1+J, m_2+I, m_3+e+k, e})$			v^3	v^3			v^3	v^3
$(\mathbf{m_1+J, m_2+I, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+k, m_3+e+I, e})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1+J, m_2+k, m_3+e, e+I})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+e+I, m_3+e+J, k})$					v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, e+J})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+k, m_3+e+J, I})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+k, m_3+I, e+J})$					v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+e+J, m_3+e+I, k})$					v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+k, I})$					v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+e+J, m_3+I, e+k})$						v^4		v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+I})$						v^5		v^5
$(\mathbf{m_1, m_2+I, m_3+e+k, e+J})$			v^4	v^4			v^4	v^4
$(\mathbf{m_1, m_2+I, m_3+e+J, e+k})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+e+I, e+J})$			v^5	v^5			v^5	v^5
$(\mathbf{m_1, m_2+k, m_3+e+J, e+I})$	v^6	v^6	v^6	v^6	v^6	v^6	v^6	v^6

$\mathbf{Y}$								
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{(m_1+2e, m_2+e+J, m_3+k)}$	1	1	1	1	1	1	1	1
$\mathbf{(m_1+2e, m_2+k, m_3+e+J)}$	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e+k, m_2+e+J, m_3+e)}$	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e+k, m_2+e, m_3+e+J)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+2e, m_3+k)}$	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e+J, m_2+e+k, m_3+e)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+e, m_3+e+k)}$			v	v			v	v
$\mathbf{(m_1+e+J, m_2+k, m_3+2e)}$			v^2	v^2			v^2	v^2
$\mathbf{(m_1+e, m_2+e+k, m_3+e+J)}$					v	v	v	v
$\mathbf{(m_1+e, m_2+e+J, m_3+e+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+2e, m_3+e+J)}$					v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+J, m_3+2e)}$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+2e, m_2+e+J, m_3, k)}$	v	v	v	v	v	v	v	v
$\mathbf{(m_1+2e, m_2+e, m_3+k, J)}$	v	v	v	v	v	v	v	v
$\mathbf{(m_1+2e, m_2+e, m_3+J, k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2+k, m_3+e, J)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2+k, m_3+J, e)}$		v		v		v		v
$\mathbf{(m_1+2e, m_2+k, m_3, e+J)}$		v^2		v^2		v^2		v^2
$\mathbf{(m_1+2e, m_2+J, m_3+e, k)}$			v	v			v	v
$\mathbf{(m_1+2e, m_2+J, m_3+k, e)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2, m_3+e+J, k)}$			v^2	v^2			v^2	v^2
$\mathbf{(m_1+2e, m_2, m_3+k, e+J)}$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+k, m_2+e+J, m_3, e)}$		v		v		v		v
$\mathbf{(m_1+e+k, m_2+e, m_3+e, J)}$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$\mathbf{(m_1+e+k, m_2+e, m_3+J, e)}$	v^2	$2v^2$	v^2	$2v^2$	v^2	$2v^2$	v^2	$2v^2$
$\mathbf{(m_1+e+k, m_2+e, m_3, e+J)}$		v^3		v^3		v^3		v^3
$\mathbf{(m_1+e+k, m_2+J, m_3+e, e)}$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$\mathbf{(m_1+e+k, m_2, m_3+e+J, e)}$			v^2	v^2			v^2	v^2
$\mathbf{(m_1+e+k, m_2, m_3+e, e+J)}$			v^3	v^3			v^3	v^3
$\mathbf{(m_1+e+J, m_2+2e, m_3, k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+e+k, m_3, e)}$		v^2		v^2		v^2		v^2
$\mathbf{(m_1+e+J, m_2+e, m_3+e, k)}$	$2v^2$	$2v^2$	$3v^2$	$3v^2$	$2v^2$	$2v^2$	$3v^2$	$3v^2$
$\mathbf{(m_1+e+J, m_2+e, m_3+k, e)}$	$2v^3$	$v+2v^3$	$2v^3$	$v+2v^3$	$2v^3$	$v+2v^3$	$2v^3$	$v+2v^3$
$\mathbf{(m_1+e+J, m_2+e, m_3, e+k)}$		v^2		v^2		v^2		v^2
$\mathbf{(m_1+e+J, m_2+k, m_3+e, e)}$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$
$\mathbf{(m_1+e+J, m_2+k, m_3, 2e)}$								$v^2 \iff c>2$
$\mathbf{(m_1+e+J, m_2, m_3+2e, k)}$								$v^2 \iff b>2$
$\mathbf{(m_1+e+J, m_2, m_3+e+k, e)}$			v^3	v^3			v^3	v^3
$\mathbf{(m_1+e+J, m_2, m_3+e, e+k)}$				v^2				v^2
$\mathbf{(m_1+e+J, m_2, m_3+k, 2e)}$		v^3		v^3		v^3		v^3
$\mathbf{(m_1+e, m_2+2e, m_3+k, J)}$					v	v	v	v
$\mathbf{(m_1+e, m_2+2e, m_3+J, k)}$					v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+e+k, m_3+e, J)}$	v^2	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$\mathbf{(m_1+e, m_2+e+k, m_3+J, e)}$	v^3	v^3	v^3	v^3	v^3	$v+v^3$	v^3	$v+v^3$
$\mathbf{(m_1+e, m_2+e+k, m_3, e+J)}$						v^2		v^2
$\mathbf{(m_1+e, m_2+e+J, m_3+e, k)}$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$v+2v^3$	$v+2v^3$	$v+2v^3$	$v+2v^3$
$\mathbf{(m_1+e, m_2+e+J, m_3+k, e)}$	v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^2+v^4	$2v^2+v^4$
$\mathbf{(m_1+e, m_2+e+J, m_3, e+k)}$		v^3		v^3		v^3		v^3
$\mathbf{(m_1+e, m_2+e, m_3+e+k, J)}$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$
$\mathbf{(m_1+e, m_2+e, m_3+e+J, k)}$	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$
$\mathbf{(m_1+e, m_2+e, m_3+k, e+J)}$		$v+v^3$		$v+v^3$	v^3	$v+2v^3$	v^3	$v+2v^3$
$\mathbf{(m_1+e, m_2+e, m_3+J, e+k)}$		v^2+v^4		v^2+v^4		v^2+v^4		v^2+v^4
$\mathbf{(m_1+e, m_2+k, m_2+2e, J)}$			v^2	v^2			v^2	v^2
$\mathbf{(m_1+e, m_3+k, m_2+e+J, e)}$	v^3	v^3	$v+2v^3$	$v+2v^3$	v^3	v^3	$v+2v^3$	$v+2v^3$
$\mathbf{(m_1+e, m_3+k, m_2+e, e+J)}$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$
$\mathbf{(m_1+e, m_2+k, m_3+J, 2e)}$		v^3		v^3		v^3		v^3
$\mathbf{(m_1+e, m_2+J, m_3+2e, k)}$			v^3	v^3			v^3	v^3
$\mathbf{(m_1+e, m_2+J, m_3+e+k, e)}$			v^2+v^4	v^2+v^4			v^2+v^4	v^2+v^4
$\mathbf{(m_1+e, m_2+J, m_3+e, e+k)}$		v^3	v^3	$2v^3$		v^3	v^3	$2v^3$
$\mathbf{(m_1+e, m_2+J, m_3+k, 2e)}$		v^4		v^4		v^4		v^4
$\mathbf{(m_1+e, m_2, m_3+e+k, e+J)}$			v^3	v^3			v^3	v^3
$\mathbf{(m_1+e, m_2, m_3+e+J, e+k)}$			v^4	v^4			v^4	v^4
$\mathbf{(m_1+k, m_2+2e, m_3+e, J)}$					v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+2e, m_3+J, e)}$						v^2		v^2
$\mathbf{(m_1+k, m_2+2e, m_3, e+J)}$						v^3		v^3
$\mathbf{(m_1+k, m_2+e+J, m_3+e, e)}$		v^3		v^3	$v+v^3$	$v+2v^3$	$v+v^3$	$v+2v^3$
$\mathbf{(m_1+k, m_2+e+J, m_3, 2e)}$								$v^3 \iff c>2$
$\mathbf{(m_1+k, m_2+e, m_3+2e, J)}$			v^3	v^3			v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+e+J, e)}$	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$
$\mathbf{(m_1+k, m_2+e, m_3+e, e+J)}$		v^3	v^3	$2v^3$	v^3	$2v^3$	$2v^3$	$3v^3$

$(\mathbf{m_1+k, m_2+e, m_3+J, 2e})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+k, m_2+J, m_3+2e, e})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+k, m_2+J, m_3+e, 2e})$				v^3				v^3	
$(\mathbf{m_1+k, m_2, m_3+2e, e+J})$								v^3	$\iff b>2$
$(\mathbf{m_1+k, m_2, m_3+e+J, 2e})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+J, m_2+2e, m_3+e, k})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+J, m_2+2e, m_3+k, e})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+J, m_2+e+k, m_3+e, e})$					v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	
$(\mathbf{m_1+J, m_2+e, m_3+2e, k})$							v^2	v^2	
$(\mathbf{m_1+J, m_2+e, m_3+e+k, e})$			v^3	v^3	v^3	v^3	$2v^3$	$2v^3$	
$(\mathbf{m_1+J, m_2+e, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^2+v^4	v^4	v^2+v^4	
$(\mathbf{m_1+J, m_2+e, m_3+k, 2e})$						v^3		v^3	
$(\mathbf{m_1+J, m_2+k, m_3+2e, e})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+J, m_2+k, m_3+e, 2e})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1, m_2+2e, m_3+e+J, k})$								v^2	$\iff a>2$
$(\mathbf{m_1, m_2+2e, m_3+k, e+J})$								v^3	$\iff a>2$
$(\mathbf{m_1, m_2+e+k, m_3+e+J, e})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1, m_2+e+k, m_3+e, e+J})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1, m_2+e+J, m_3+2e, k})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1, m_2+e+J, m_3+e+k, e})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1, m_2+e+J, m_3+e, e+k})$						v^3		v^3	
$(\mathbf{m_1, m_2+e+J, m_3+k, 2e})$						v^4		v^4	
$(\mathbf{m_1, m_2+e, m_3+e+k, e+J})$							v^3	v^3	
$(\mathbf{m_1, m_2+e, m_3+e+J, e+k})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1, m_2+k, m_3+2e, e+J})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1, m_2+k, m_3+e+J, 2e})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5	
Z									
a	**1**	**1**	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	$\mathbf{>1}$	
b	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$	
c	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$	
$(\mathbf{m_1+2e+H, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	
$(\mathbf{m_1+2e+H, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e+H})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+J, m_2+2e+H, m_3+k})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e+H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e+k})$			v	v			v	v	
$(\mathbf{m_1+e+J, m_2+k, m_3+2e+H})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+e+H, m_2+e+k, m_3+e+J})$					v	v	v	v	
$(\mathbf{m_1+e+H, m_2+e+J, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+k, m_2+2e+H, m_3+e+J})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+k, m_2+e+J, m_3+2e+H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+2e+H, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+2e+H, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+2e+H, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e+H, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e+H, m_2+k, m_3+J, e})$		v		v		v		v	
$(\mathbf{m_1+2e+H, m_2+k, m_3, e+J})$		v^2		v^2		v^2		v^2	
$(\mathbf{m_1+2e+H, m_2+J, m_3+e, k})$			v	v			v	v	
$(\mathbf{m_1+2e+H, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e+H, m_2, m_3+e+J, k})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+2e+H, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+2e, m_2+e+J, m_3+k, H})$	v	v	v	v	v	v	v	v	
$(\mathbf{m_1+2e, m_2+e+J, m_3+H, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e, m_2+e+H, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e, m_2+e+H, m_3+J, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+2e, m_2+k, m_3+e+J, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+2e, m_2+k, m_3+e+H, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+2e, m_2+k, m_3+J, e+H})$		v^2		v^2		v^2		v^2	
$(\mathbf{m_1+2e, m_2+k, m_3+H, e+J})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+2e, m_2+J, m_3+e+H, k})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+2e, m_2+J, m_3+k, e+H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+2e, m_2+H, m_3+e+J, k})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+2e, m_2+H, m_3+k, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4	
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+k, m_2+e+J, m_3+H, e})$		v		v		v		v	
$(\mathbf{m_1+e+k, m_2+e+J, m_3, e+H})$		v^2		v^2		v^2		v^2	
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e, J})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	
$(\mathbf{m_1+e+k, m_2+e+H, m_3+J, e})$	v^2	$2v^2$	v^2	$2v^2$	v^2	$2v^2$	v^2	$2v^2$	
$(\mathbf{m_1+e+k, m_2+e+H, m_3, e+J})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+k, m_2+e, m_3+e+J, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+e+k, m_2+e, m_3+e+H, J})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	
$(\mathbf{m_1+e+k, m_2+e, m_3+J, e+H})$	v^3	$2v^3$	v^3	$2v^3$	v^3	$2v^3$	v^3	$2v^3$	

$(\mathbf{m_1+e+k, m_2+e, m_3+H, e+J})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+e+k, m_2+J, m_3+e+H, e})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$	
$(\mathbf{m_1+e+k, m_2+J, m_3+e, e+H})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4	
$(\mathbf{m_1+e+k, m_2+H, m_3+e+J, e})$			v	v			v	v	
$(\mathbf{m_1+e+k, m_2+H, m_3+e, e+J})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+e+k, m_2, m_3+e+J, e+H})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+e+k, m_2, m_3+e+H, e+J})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+e+J, m_2+2e+H, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+J, m_2+2e, m_3+k, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+J, m_2+2e, m_3+H, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+e+J, m_2+e+k, m_3+H, e})$		v^2		v^2		v^2		v^2	
$(\mathbf{m_1+e+J, m_2+e+k, m_3, e+H})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$	v^2	v^2	$2v^2$	$2v^2$	
$(\mathbf{m_1+e+J, m_2+e+H, m_3+k, e})$	$v+v^3$	$v+2v^3$	$v+v^3$	$v+2v^3$	$v+v^3$	$v+2v^3$	$v+v^3$	$v+2v^3$	
$(\mathbf{m_1+e+J, m_2+e+H, m_3, e+k})$		v^2		v^2		v^2		v^2	
$(\mathbf{m_1+e+J, m_2+e, m_3+e+k, H})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+e+J, m_2+e, m_3+e+H, k})$	v^3	v^3	$2v^3$	$2v^3$	v^3	v^3	$2v^3$	$2v^3$	
$(\mathbf{m_1+e+J, m_2+e, m_3+k, e+H})$	v^2+v^4	v^2+2v^4	v^2+v^4	v^2+2v^4	v^2+v^4	v^2+2v^4	v^2+v^4	v^2+2v^4	
$(\mathbf{m_1+e+J, m_2+e, m_3+H, e+k})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+J, m_2+k, m_3+2e, H})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+e+J, m_2+k, m_3+e+H, e})$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	
$(\mathbf{m_1+e+J, m_2+k, m_3+e, e+H})$	v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	
$(\mathbf{m_1+e+J, m_2+k, m_3+H, 2e})$								v^2	$\iff c>2$
$(\mathbf{m_1+e+J, m_2+k, m_3, 2e+H})$								v^3	$\iff c>2$
$(\mathbf{m_1+e+J, m_2+H, m_3+2e, k})$								v^2	$\iff b>2$
$(\mathbf{m_1+e+J, m_2+H, m_3+e+k, e})$			v^3	v^3			v^3		
$(\mathbf{m_1+e+J, m_2+H, m_3+e, e+k})$				v^2				v^2	
$(\mathbf{m_1+e+J, m_2+H, m_3+k, 2e})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+J, m_2, m_3+2e+H, k})$								v^3	$\iff b>2$
$(\mathbf{m_1+e+J, m_2, m_3+e+k, e+H})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+e+J, m_2, m_3+e+H, e+k})$				v^3				v^3	
$(\mathbf{m_1+e+J, m_2, m_3+k, 2+H})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+e+H, m_2+2e, m_3+k, J})$					v	v	v	v	
$(\mathbf{m_1+e+H, m_2+2e, m_3+J, k})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e+H, m_2+e+k, m_3+e, J})$	v^2	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	
$(\mathbf{m_1+e+H, m_2+e+k, m_3+J, e})$	v^3	v^3	v^3	v^3	v^3	$v+v^3$	v^3	$v+v^3$	
$(\mathbf{m_1+e+H, m_2+e+k, m_3, e+J})$						v^2		v^2	
$(\mathbf{m_1+e+H, m_2+e+J, m_3+e, k})$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$v+2v^3$	$v+2v^3$	$v+2v^3$	$v+2v^3$	
$(\mathbf{m_1+e+H, m_2+e+J, m_3+k, e})$	v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^2+v^4	$2v^2+v^4$	
$(\mathbf{m_1+e+H, m_2+e+J, m_3, e+k})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+H, m_2+e, m_3+e+k, J})$	v^3	v^3	$v+v^3$	$v+v^3$	v^3	v^3	$v+v^3$	$v+v^3$	
$(\mathbf{m_1+e+H, m_2+e, m_3+e+J, k})$	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	
$(\mathbf{m_1+e+H, m_2+e, m_3+k, e+J})$		$v+v^3$		$v+v^3$	v^3	$v+2v^3$	v^3	$v+2v^3$	
$(\mathbf{m_1+e+H, m_2+e, m_3+J, e+k})$		v^2+v^4		v^2+v^4		v^2+v^4		v^2+v^4	
$(\mathbf{m_1+e+H, m_2+k, m_3+2e, J})$			v^2	v^2			v^2	v^2	
$(\mathbf{m_1+e+H, m_2+k, m_3+e+J, e})$	v^3	v^3	$v+2v^3$	$v+2v^3$	v^3	v^3	$v+2v^3$	$v+2v^3$	
$(\mathbf{m_1+e+H, m_2+k, m_3+e, e+J})$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	
$(\mathbf{m_1+e+H, m_2+k, m_3+J, 2e})$		v^3		v^3		v^3		v^3	
$(\mathbf{m_1+e+H, m_2+J, m_3+2e, k})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+e+H, m_2+J, m_3+e+k, e})$			v^2+v^4	v^2+v^4			v^2+v^4	v^2+v^4	
$(\mathbf{m_1+e+H, m_2+J, m_3+e, e+k})$		v^3	v^3	$2v^3$		v^3	v^3	$2v^3$	
$(\mathbf{m_1+e+H, m_2+J, m_3+k, 2e})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+e+H, m_2, m_3+e+k, e+J})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+e+H, m_2, m_3+e+J, e+k})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+e, m_2+2e+H, m_3+k, J})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e, m_2+2e+H, m_3+J, k})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+e, m_2+e+k, m_3+e+J, H})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+e, m_2+e+k, m_3+e+H, J})$	v^3	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	
$(\mathbf{m_1+e, m_2+e+k, m_3+J, e+H})$	v^4	v^4	v^4	v^4	v^4	v^2+v^4	v^4	v^2+v^4	
$(\mathbf{m_1+e, m_2+e+k, m_3+H, e+J})$						v^3		v^3	
$(\mathbf{m_1+e, m_2+e+J, m_3+e+k, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	
$(\mathbf{m_1+e, m_2+e+J, m_3+e+H, k})$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	v^2+2v^4	v^2+2v^4	v^2+2v^4	v^2+2v^4	
$(\mathbf{m_1+e, m_2+e+J, m_3+k, e+H})$	v^5	v^3+v^5	v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	v^3+v^5	$2v^3+v^5$	
$(\mathbf{m_1+e, m_2+e+J, m_3+H, e+k})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+e, m_2+e+H, m_3+e+k, J})$	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^4	v^2+v^4	v^2+v^4	
$(\mathbf{m_1+e, m_2+e+H, m_3+e+J, k})$	v^5	v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	$2v^3+v^5$	
$(\mathbf{m_1+e, m_2+e+H, m_3+k, e+J})$		v^2+v^4		v^2+v^4		v^2+v^4		v^2+v^4	
$(\mathbf{m_1+e, m_2+e+H, m_3+J, e+k})$		v^3+v^5		v^3+v^5		v^3+v^5		v^3+v^5	
$(\mathbf{m_1+e, m_2+k, m_3+2e+H, J})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+e, m_2+k, m_3+e+J, e+H})$	v^4	v^4	v^2+2v^4	v^2+2v^4	v^4	v^4	v^2+2v^4	v^2+2v^4	
$(\mathbf{m_1+e, m_2+k, m_3+e+H, e+J})$	v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	
$(\mathbf{m_1+e, m_2+k, m_3+J, 2e+H})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+e, m_2+J, m_3+2e+H, k})$			v^4	v^4			v^4	v^4	

$(\mathbf{m_1+e, m_2+J, m_3+e+k, e+H})$			v^3+v^5	v^3+v^5			v^3+v^5	v^3+v^5	
$(\mathbf{m_1+e, m_2+J, m_3+e+H, e+k})$		v^4	v^4	$2v^3$		v^4	v^4	$2v^3$	
$(\mathbf{m_1+e, m_2+J, m_3+k, 2e+H})$		v^5		v^5		v^5		v^5	
$(\mathbf{m_1+e, m_2+H, m_3+e+k, e+J})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+e, m_2+H, m_3+e+J, e+k})$			v^5	v^5			v^5	v^5	
$(\mathbf{m_1+k, m_2+2e+H, m_3+e, J})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+k, m_2+2e+H, m_3+J, e})$						v^2		v^2	
$(\mathbf{m_1+k, m_2+2e+H, m_3, e+J})$						v^3		v^3	
$(\mathbf{m_1+k, m_2+2e, m_3+e+J, H})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+k, m_2+2e, m_3+e+H, J})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1+k, m_2+2e, m_3+J, e+H})$						v^3		v^3	
$(\mathbf{m_1+k, m_2+2e, m_3+H, e+J})$						v^4		v^4	
$(\mathbf{m_1+k, m_2+e+J, m_3+2e, H})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4	
$(\mathbf{m_1+k, m_2+e+J, m_3+e+H, e})$		v^3		v^3	$v+v^3$	$v+2v^3$	$v+v^3$	$v+2v^3$	
$(\mathbf{m_1+k, m_2+e+J, m_3+e, e+H})$		v^4		v^4	v^2+v^4	v^2+2v^4	v^2+v^4	v^2+2v^4	
$(\mathbf{m_1+k, m_2+e+J, m_3+H, 2e})$								v^3	$\iff c>2$
$(\mathbf{m_1+k, m_2+e+J, m_3, 2e+H})$								v^4	$\iff c>2$
$(\mathbf{m_1+k, m_2+e+H, m_3+2e, J})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+k, m_2+e+H, m_3+e+J, e})$	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	
$(\mathbf{m_1+k, m_2+e+H, m_3+e, e+J})$		v^3	v^3	$2v^3$	v^3	$2v^3$	v^3	$3v^3$	
$(\mathbf{m_1+k, m_2+e+H, m_3+J, 2e})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+k, m_2+e, m_3+2e+H, J})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+k, m_2+e, m_3+e+J, e+H})$	v^5	v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	$2v^3+v^5$	$2v^3+v^5$	
$(\mathbf{m_1+k, m_2+e, m_3+e+H, e+J})$		v^4	v^4	$2v^3$	v^4	$2v^3$	v^4	$3v^4$	
$(\mathbf{m_1+k, m_2+e, m_3+J, 2e+H})$		v^5		v^5		v^5		v^5	
$(\mathbf{m_1+k, m_2+J, m_3+2e+H, e})$			v^3	v^3			v^3	v^3	
$(\mathbf{m_1+k, m_2+J, m_3+2e, e+H})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+k, m_2+J, m_3+e+H, 2e})$				v^3				v^3	
$(\mathbf{m_1+k, m_2+J, m_3+e, 2e+H})$				v^4				v^4	
$(\mathbf{m_1+k, m_2+H, m_3+2e, e+J})$								v^3	$\iff b>2$
$(\mathbf{m_1+k, m_2+H, m_3+e+J, 2e})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+k, m_2, m_3+2e+H, e+J})$								v^4	$\iff b>2$
$(\mathbf{m_1+k, m_2, m_3+e+J, 2e+H})$			v^5	v^5			v^5	v^5	
$(\mathbf{m_1+J, m_2+2e+H, m_3+e, k})$					v^2	v^2	v^2	v^2	
$(\mathbf{m_1+J, m_2+2e+H, m_3+k, e})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+J, m_2+2e, m_3+e+H, k})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+J, m_2+2e, m_3+k, e+H})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1+J, m_2+e+k, m_3+e+H, e})$					v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	
$(\mathbf{m_1+J, m_2+e+k, m_3+e, e+H})$					v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	
$(\mathbf{m_1+J, m_2+e+H, m_3+2e, k})$							v^2	v^2	
$(\mathbf{m_1+J, m_2+e+H, m_3+e+k, e})$			v^3	v^3	v^3	v^3	$2v^3$	$2v^3$	
$(\mathbf{m_1+J, m_2+e+H, m_3+e, e+k})$	v^4	v^4	v^4	v^4	v^4	v^2+v^4	v^4	v^2+v^4	
$(\mathbf{m_1+J, m_2+e+H, m_3+k, 2e})$						v^3		v^3	
$(\mathbf{m_1+J, m_2+e, m_3+2e+H, k})$							v^3	v^3	
$(\mathbf{m_1+J, m_2+e, m_3+e+k, e+H})$			v^4	v^4	v^4	v^4	$2v^3$	$2v^4$	
$(\mathbf{m_1+J, m_2+e, m_3+e+H, e+k})$	v^5	v^5	v^5	v^5	v^5	v^3+v^5	v^5	v^3+v^5	
$(\mathbf{m_1+J, m_2+e, m_3+k, 2e+H})$						v^4		v^4	
$(\mathbf{m_1+J, m_2+k, m_3+2e+H, e})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+J, m_2+k, m_3+2e, e+H})$			v^5	v^5			v^5	v^5	
$(\mathbf{m_1+J, m_2+k, m_3+e+H, 2e})$		v^4		v^4		v^4		v^4	
$(\mathbf{m_1+J, m_2+k, m_3+e, 2e+H})$		v^5		v^5		v^5		v^5	
$(\mathbf{m_1+H, m_2+2e, m_3+e+J, k})$								v^2	$\iff a>2$
$(\mathbf{m_1+H, m_2+2e, m_3+k, e+J})$								v^3	$\iff a>2$
$(\mathbf{m_1+H, m_2+e+k, m_3+e+J, e})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+H, m_2+e+k, m_3+e, e+J})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1+H, m_2+e+J, m_3+2e, k})$					v^3	v^3	v^3	v^3	
$(\mathbf{m_1+H, m_2+e+J, m_3+e+k, e})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1+H, m_2+e+J, m_3+e, e+k})$						v^3		v^3	
$(\mathbf{m_1+H, m_2+e+J, m_3+k, 2e})$						v^4		v^4	
$(\mathbf{m_1+H, m_2+e, m_3+e+k, e+J})$							v^3	v^3	
$(\mathbf{m_1+H, m_2+e, m_3+e+J, e+k})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1+H, m_2+k, m_3+2e, e+J})$			v^4	v^4			v^4	v^4	
$(\mathbf{m_1+H, m_2+k, m_3+e+J, 2e})$	v^5	v^5	v^5	v^5	v^5	v^5	v^5	v^5	
$(\mathbf{m_1, m_2+2e+H, m_3+e+J, k})$								v^3	$\iff a>2$
$(\mathbf{m_1, m_2+2e+H, m_3+k, e+J})$								v^4	$\iff a>2$
$(\mathbf{m_1, m_2+e+k, m_3+e+J, e+H})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1, m_2+e+k, m_3+e+H, e+J})$					v^5	v^5	v^5	v^5	
$(\mathbf{m_1, m_2+e+J, m_3+2e+H, k})$					v^4	v^4	v^4	v^4	
$(\mathbf{m_1, m_2+e+J, m_3+e+k, e+H})$					v^5	v^5	v^5	v^5	
$(\mathbf{m_1, m_2+e+J, m_3+e+H, e+k})$						v^4		v^4	
$(\mathbf{m_1, m_2+e+J, m_3+k, 2e+H})$						v^5		v^5	
$(\mathbf{m_1, m_2+e+H, m_3+e+k, e+J})$							v^4	v^4	
$(\mathbf{m_1, m_2+e+H, m_3+e+J, e+k})$					v^5	v^5	v^5	v^5	
$(\mathbf{m_1, m_2+k, m_3+2e+H, e+J})$			v^5	v^5			v^5	v^5	
$(\mathbf{m_1, m_2+k, m_3+e+J, 2e+H})$	v^6	v^6	v^6	v^6	v^6	v^6	v^6	v^6	

Wall $a : b = c = 0$

B

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+i, m_2, m_3})$	1	1	1	1	1	1
$(\mathbf{m_1, m_2-e+i, m_3+e})$			v	v	v	v
$(\mathbf{m_1, m_2-e+i, m_3, e})$		v		v		v
$(\mathbf{m_1, m_2-e, m_3+e, i})$					v	v
$(\mathbf{m_1, m_2-e, m_3+i, e})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1-e+i, m_2-e, m_3+e, e})$					v^2	v^2

D

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+i+j, m_2+j, m_3})$	1	1	1	1	1	1
$(\mathbf{m_1+i+j, m_2, m_3+j})$	v	v	v	v	v	v
$(\mathbf{m_1+j, m_2-e+i+j, m_3+e})$			v	v	v	v
$(\mathbf{m_1, m_2-e+i+j, m_3+e+j})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+i+j, m_2, m_3, j})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j, m_2-e+i+j, m_3, e})$		v		v		v
$(\mathbf{m_1+j, m_2-e, m_3+e, i+j})$					v	v
$(\mathbf{m_1+j, m_2-e, m_3+i+j, e})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2-e+i+j, m_3+e, j})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2-e+i+j, m_3+j, e})$		v^2		v^2		v^2
$(\mathbf{m_1, m_2-e+i+j, m_3, e+j})$		v^3		v^3		v^3
$(\mathbf{m_1, m_2-e+j, m_3+e, i+j})$					v^2	v^2
$(\mathbf{m_1, m_2-e+j, m_3+i+j, e})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2-e, m_3+e+j, i+j})$					v^3	v^3
$(\mathbf{m_1, m_2-e, m_3+i+j, e+j})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1-e+i+j, m_2-e+j, m_3+e, e})$					v^2	v^2
$(\mathbf{m_1-e+i+j, m_2-e, m_3+e+j, e})$					v^3	v^3
$(\mathbf{m_1-e+i+j, m_2-e, m_3+e, e+j})$					v^4	v^4

E

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e, m_2+j, m_3})$	1	1	1	1
$(\mathbf{m_1+e, m_2, m_3+j})$	v	v	v	v
$(\mathbf{m_1+j, m_2, m_3+e})$			v	v
$(\mathbf{m_1+e, m_2, m_3, j})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j, m_2, m_3, e})$		v		v
$(\mathbf{m_1, m_2-e+j, m_3+e, e})$			v^2	v^2

F

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+I, m_2+j, m_3})$	1	1	1	1
$(\mathbf{m_1+e+I, m_2, m_3+j})$	v	v	v	v
$(\mathbf{m_1+e, m_2+j, m_3+I})$	v	v	v	v
$(\mathbf{m_1+e, m_2+I, m_3+j})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j, m_2+I, m_3+e})$			v	v
$(\mathbf{m_1+j, m_2, m_3+e+I})$			v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3, j})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+j, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3, j})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+j, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+I, j})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+j, m_2+I, m_3, e})$		v		v
$(\mathbf{m_1+j, m_2, m_3+e, I})$			v^3	v^3
$(\mathbf{m_1+j, m_2, m_3+I, e})$		v^2		v^2
$(\mathbf{m_1+j, m_2, m_3, e+I})$		v^3		v^3
$(\mathbf{m_1+I, m_2-e+j, m_3+e, e})$			v^2	v^2
$(\mathbf{m_1, m_2-e+j, m_3+e+I, e})$			v^3	v^3
$(\mathbf{m_1, m_2-e+j, m_3+e, e+I})$			v^4	v^4

H

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
(m_1+i+k, m_2+k, m_3+k)	1	1	1	1	1	1
$(m_1+k, m_2-e+i+k, m_3+e+k)$			v	v	v	v
(m_1+i+k, m_2+k, m_3, k)	v	v	v	v	v	v
(m_1+i+k, m_2, m_3+k, k)	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+k, m_2-e+i+k, m_3+e, k)$			v^2	v^2	v^2	v^2
$(m_1+k, m_2-e+i+k, m_3+k, e)$		v		v		v
$(m_1+k, m_2-e+i+k, m_3, e+k)$		v^2		v^2		v^2
$(m_1+k, m_2-e+k, m_3+e, i+k)$					v	v
$(m_1+k, m_2-e+k, m_3+i+k, e)$			v^2	v^2	v^2	v^2
$(m_1+k, m_2-e, m_3+e+k, i+k)$					v^2	v^2
$(m_1+k, m_2-e, m_3+i+k, e+k)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+i+k, m_3+e+k, k)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+i+k, m_3+k, e+k)$		v^3		v^3		v^3
$(m_1, m_2-e+k, m_3+e+k, i+k)$					v^3	v^3
$(m_1, m_2-e+k, m_3+i+k, e+k)$			v^4	v^4	v^4	v^4
$(m_1-e+i+k, m_2-e+k, m_3+e+k, e)$					v^2	v^2
$(m_1-e+i+k, m_2-e+k, m_3+e, e+k)$					v^3	v^3
$(m_1-e+i+k, m_2-e, m_3+e+k, e+k)$					v^4	v^4

I

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
(m_1+e, m_2+k, m_3+k)	1	1	1	1	1	1
(m_1+k, m_2, m_3+e+k)			v	v	v	v
(m_1+e, m_2+k, m_3, k)	v	v	v	v	v	v
(m_1+e, m_2, m_3+k, k)	v^2	v^2	v^2	v^2	v^2	v^2
(m_1+k, m_2, m_3+e, k)			v^2	v^2	v^2	v^2
(m_1+k, m_2, m_3+k, e)		v		v		v
$(m_1+k, m_2, m_3, e+k)$		v^2		v^2		v^2
$(m_1+k, m_2-e+k, m_3+e, e)$			v^2	v^2	v^2	v^2
$(m_1, m_2-e+k, m_3+e+k, e)$					v^2	v^2
$(m_1, m_2-e+k, m_3+e, e+k)$			v^3	v^3	v^3	v^3

J

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
(m_1+e+I, m_2+k, m_3+k)	1	1	1	1	1	1
(m_1+k, m_2+I, m_3+e+k)			v	v	v	v
(m_1+e+I, m_2+k, m_3, k)	v	v	v	v	v	v
(m_1+e+I, m_2, m_3+k, k)	v^2	v^2	v^2	v^2	v^2	v^2
(m_1+e, m_2+k, m_3+k, I)	v	v	v	v	v	v
(m_1+e, m_2+k, m_3+I, k)	v^2	v^2	v^2	v^2	v^2	v^2
(m_1+e, m_2+I, m_3+k, k)	v^3	v^3	v^3	v^3	v^3	v^3
(m_1+k, m_2+I, m_3+e, k)			v^2	v^2	v^2	v^2
(m_1+k, m_2+I, m_3+k, e)		v		v		v
$(m_1+k, m_2+I, m_3, e+k)$		v^2		v^2		v^2
(m_1+k, m_2, m_3+e+k, I)			v^2	v^2	v^2	v^2
(m_1+k, m_2, m_3+e+I, k)			v^3	v^3	v^3	v^3
$(m_1+k, m_2, m_3+k, e+I)$		v^2		v^2		v^2
$(m_1+k, m_2, m_3+I, e+k)$		v^3		v^3		v^3
$(m_1+k, m_2-e+k, m_3+e+I, e)$			v^2	v^2	v^2	v^2
$(m_1+k, m_2-e+k, m_3+e, e+I)$			v^3	v^3	v^3	v^3
$(m_1+I, m_2-e+k, m_3+e+k, e)$					v^2	v^2
$(m_1+I, m_2-e+k, m_3+e, e+k)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+k, m_3+e+k, e+I)$					v^3	v^3
$(m_1, m_2-e+k, m_3+e+I, e+k)$			v^4	v^4	v^4	v^4

N

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
$(m_1+i+j+k, m_2+j+k, m_3+k)$	1	1	1	1	1	1
$(m_1+i+j+k, m_2+k, m_3+j+k)$	v	v	v	v	v	v
$(m_1+j+k, m_2-e+i+j+k, m_3+e+k)$			v	v	v	v
$(m_1+k, m_2-e+i+j+k, m_3+e+j+k)$			v^2	v^2	v^2	v^2
$(m_1+i+j+k, m_2+j+k, m_3, k)$	v	v	v	v	v	v
$(m_1+i+j+k, m_2+k, m_3, j+k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+i+j+k, m_2, m_3+j+k, k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+i+j+k, m_2, m_3+k, j+k)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+j+k, m_2-e+i+j+k, m_3+e, k)$			v^2	v^2	v^2	v^2

$(m_1+j+k, m_2-e+i+j+k, m_3+k, e)$		v		v		v
$(m_1+j+k, m_2-e+i+j+k, m_3, e+k)$		v^2		v^2		v^2
$(m_1+j+k, m_2-e+k, m_3+e, i+j+k)$					v	v
$(m_1+j+k, m_2-e+k, m_3+i+j+k, e)$			v^2	v^2	v^2	v^2
$(m_1+j+k, m_2-e, m_3+e+k, i+j+k)$					v^2	v^2
$(m_1+j+k, m_2-e, m_3+i+j+k, e+k)$			v^3	v^3	v^3	v^3
$(m_1+k, m_2-e+i+j+k, m_3+e, j+k)$			v^3	v^3	v^3	v^3
$(m_1+k, m_2-e+i+j+k, m_3+j+k, e)$		v^2		v^2		v^2
$(m_1+k, m_2-e+i+j+k, m_3, e+j+k)$		v^3		v^3		v^3
$(m_1+k, m_2-e+j+k, m_3+e, i+j+k)$					v^2	v^2
$(m_1+k, m_2-e+j+k, m_3+i+j+k, e)$			v^3	v^3	v^3	v^3
$(m_1+k, m_2-e, m_3+e+j+k, i+j+k)$					v^3	v^3
$(m_1+k, m_2-e, m_3+i+j+k, e+j+k)$			v^4	v^4	v^4	v^4
$(m_1, m_2-e+i+j+k, m_3+e+j+k, k)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+i+j+k, m_3+j+k, e+k)$		v^3		v^3		v^3
$(m_1, m_2-e+i+j+k, m_3+e+k, j+k)$			v^4	v^4	v^4	v^4
$(m_1, m_2-e+i+j+k, m_3+k, e+j+k)$		v^4		v^4		v^4
$(m_1, m_2-e+j+k, m_3+e+k, i+j+k)$					v^3	v^3
$(m_1, m_2-e+j+k, m_3+i+j+k, e+k)$			v^4	v^4	v^4	v^4
$(m_1, m_2-e+k, m_3+e+j+k, i+j+k)$					v^4	v^4
$(m_1, m_2-e+k, m_3+i+j+k, e+j+k)$			v^5	v^5	v^5	v^5
$(m_1-e+i+j+k, m_2-e+j+k, m_3+e+k, e)$					v^2	v^2
$(m_1-e+i+j+k, m_2-e+j+k, m_3+e, e+k)$					v^3	v^3
$(m_1-e+i+j+k, m_2-e+k, m_3+e+j+k, e)$					v^3	v^3
$(m_1-e+i+j+k, m_2-e+k, m_3+e, e+j+k)$					v^4	v^4
$(m_1-e+i+j+k, m_2-e, m_3+e+j+k, e+k)$					v^4	v^4
$(m_1-e+i+j+k, m_2-e, m_3+e+k, e+j+k)$					v^5	v^5

O

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
(m_1+e, m_2+j+k, m_3+k)	1	1	1	1	1	1
(m_1+e, m_2+k, m_3+j+k)	v	v	v	v	v	v
(m_1+j+k, m_2, m_3+e+k)			v	v	v	v
$(m_1+k, m_2, m_3+e+j+k)$			v^2	v^2	v^2	v^2
(m_1+e, m_2+j+k, m_3, k)	v	v	v	v	v	v
$(m_1+e, m_2+k, m_3, j+k)$	v^2	v^2	v^2	v^2	v^2	v^2
(m_1+e, m_2, m_3+j+k, k)	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e, m_2, m_3+k, j+k)$	v^3	v^3	v^3	v^3	v^3	v^3
(m_1+j+k, m_2, m_3+e, k)			v^2	v^2	v^2	v^2
(m_1+j+k, m_2, m_3+k, e)		v		v		v
$(m_1+j+k, m_2, m_3, e+k)$		v^2		v^2		v^2
$(m_1+j+k, m_2-e+k, m_3+e, e)$			v^2	v^2	v^2	v^2
$(m_1+k, m_2, m_3+e, j+k)$			v^3	v^3	v^3	v^3
(m_1+k, m_2, m_3+j+k, e)		v^2		v^2		v^2
$(m_1+k, m_2, m_3, e+j+k)$		v^3		v^3		v^3
$(m_1+k, m_2-e+j+k, m_3+e, e)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+j+k, m_3+e+k, e)$					v^2	v^2
$(m_1, m_2-e+j+k, m_3+e, e+k)$			v^3	v^3	v^3	v^3
$(m_1, m_2-e+k, m_3+e+j+k, e)$					v^3	v^3
$(m_1, m_2-e+k, m_3+e, e+j+k)$			v^4	v^4	v^4	v^4

P

b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
c	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
$(m_1+e+I, m_2+j+k, m_3+k)$	1	1	1	1	1	1
$(m_1+e+I, m_2+k, m_3+j+k)$	v	v	v	v	v	v
$(m_1+j+k, m_2+I, m_3+e+k)$			v	v	v	v
$(m_1+k, m_2+I, m_3+e+j+k)$			v^2	v^2	v^2	v^2
$(m_1+e+I, m_2+j+k, m_3, k)$	v	v	v	v	v	v
$(m_1+e+I, m_2+k, m_3, j+k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+I, m_2, m_3+j+k, k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+I, m_2, m_3+k, j+k)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e, m_2+j+k, m_3+k, I)$	v	v	v	v	v	v
$(m_1+e, m_2+j+k, m_3+I, k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e, m_2+k, m_3+j+k, I)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e, m_2+k, m_3+I, j+k)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e, m_2+I, m_3+j+k, k)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e, m_2+I, m_3+k, j+k)$	v^4	v^4	v^4	v^4	v^4	v^4
$(m_1+j+k, m_2+I, m_3+e, k)$			v^2	v^2	v^2	v^2
$(m_1+j+k, m_2+I, m_3+k, e)$		v		v		v

$(\mathbf{m_1+j+k, m_2+I, m_3, e+k})$		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2, m_3+e+k, I})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2, m_3+e+I, k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, e+I})$		v^2		v^2		v^2
$(\mathbf{m_1+j+k, m_2, m_3+I, e+k})$		v^3		v^3		v^3
$(\mathbf{m_1+j+k, m_2-e+k, m_3+e+I, e})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2-e+k, m_3+e, e+I})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+I, m_3+e, j+k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+I, m_3+j+k, e})$		v^2		v^2		v^2
$(\mathbf{m_1+k, m_2+I, m_3, e+j+k})$		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2, m_3+e+j+k, I})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+I, j+k})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+j+k, e+I})$		v^3		v^3		v^3
$(\mathbf{m_1+k, m_2, m_3+I, e+j+k})$		v^4		v^4		v^4
$(\mathbf{m_1+k, m_2-e+j+k, m_3+e+I, e})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2-e+j+k, m_3+e, e+I})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2-e+j+k, m_3+e+k, e})$					v^2	v^2
$(\mathbf{m_1+I, m_2-e+j+k, m_3+e, e+k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2-e+k, m_3+e+j+k, e})$					v^3	v^3
$(\mathbf{m_1+I, m_2-e+k, m_3+e, e+j+k})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2-e+j+k, m_3+e+k, e+I})$					v^3	v^3
$(\mathbf{m_1, m_2-e+j+k, m_3+e+I, e+k})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2-e+k, m_3+e+j+k, e+I})$					v^4	v^4
$(\mathbf{m_1, m_2-e+k, m_3+e+I, e+j+k})$			v^5	v^5	v^5	v^5

Q

b	**1**	**1**	$>\mathbf{1}$	$>\mathbf{1}$
c	**1**	$>\mathbf{1}$	**1**	$>\mathbf{1}$
$(\mathbf{m_1+e+k, m_2+j+k, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+k, m_2+k, m_3+j+k})$	v	v	v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e+k})$			v	v
$(\mathbf{m_1+e+k, m_2+j+k, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+j+k, m_3+k, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+j+k, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+k, j+k})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+j+k, m_2+k, m_3+e, k})$			v^2	v^2
$(\mathbf{m_1+j+k, m_2+k, m_3+k, e})$		v		v
$(\mathbf{m_1+j+k, m_2+k, m_3, e+k})$		v^2		v^2
$(\mathbf{m_1+j+k, m_2, m_3+e+k, k})$			v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+k, e+k})$		v^3		v^3
$(\mathbf{m_1+k, m_2-e+j+k, m_3+e+k, e})$			v^2	v^2
$(\mathbf{m_1+k, m_2-e+j+k, m_3+e, e+k})$			v^3	v^3
$(\mathbf{m_1, m_2-e+j+k, m_3+e+k, e+k})$			v^4	v^4

R

b	**1**	**1**	$>\mathbf{1}$	$>\mathbf{1}$
c	**1**	$>\mathbf{1}$	**1**	$>\mathbf{1}$
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+j+k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+j+k, m_3+I})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+I, m_3+j+k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+I, m_3+e+k})$			v	v
$(\mathbf{m_1+j+k, m_2+k, m_3+e+I})$			v^2	v^2
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, j+k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+j+k, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, j+k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+j+k, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, j+k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+j+k, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+I, j+k})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+j+k, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+k, j+k})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+j+k, I})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+I, j+k})$	v^5	v^5	v^5	v^5
$(\mathbf{m_1+j+k, m_2+I, m_3+e, k})$			v^2	v^2
$(\mathbf{m_1+j+k, m_2+I, m_3+k, e})$		v		v
$(\mathbf{m_1+j+k, m_2+I, m_3, e+k})$		v^2		v^2
$(\mathbf{m_1+j+k, m_2+k, m_3+e, I})$			v^3	v^3
$(\mathbf{m_1+j+k, m_2+k, m_3+I, e})$		v^2		v^2

$(\mathbf{m_1 + j + k, m_2 + k, m_3, e + I})$		v^3		v^3
$(\mathbf{m_1 + j + k, m_2, m_3 + e + I, k})$			v^3	v^3
$(\mathbf{m_1 + j + k, m_2, m_3 + e + k, I})$			v^4	v^4
$(\mathbf{m_1 + j + k, m_2, m_3 + I, e + k})$		v^3		v^3
$(\mathbf{m_1 + j + k, m_2, m_3 + k, e + I})$		v^4		v^4
$(\mathbf{m_1 + I, m_2 - e + j + k, m_3 + e + k, e})$			v^2	v^2
$(\mathbf{m_1 + I, m_2 - e + j + k, m_3 + e, e + k})$			v^3	v^3
$(\mathbf{m_1 + k, m_2 - e + j + k, m_3 + e + I, e})$			v^3	v^3
$(\mathbf{m_1 + k, m_2 - e + j + k, m_3 + e, e + I})$			v^4	v^4
$(\mathbf{m_1, m_2 - e + j + k, m_3 + e + I, e + k})$			v^4	v^4
$(\mathbf{m_1, m_2 - e + j + k, m_3 + e + k, e + I})$			v^5	v^5

S

b	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{> 2}$	$\mathbf{> 2}$	$\mathbf{> 2}$
c	**1**	**2**	$\mathbf{> 2}$	**1**	**2**	$\mathbf{> 2}$	**1**	**2**	$\mathbf{> 2}$
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + e})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e, m_2 + I, m_3 + e + k})$				v	v	v	v	v	v
$(\mathbf{m_1 + k, m_2 + I, m_3 + 2e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e + I, m_2 + e, m_3, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e + I, m_2 + k, m_3, e})$		v	v		v	v		v	v
$(\mathbf{m_1 + e + I, m_2, m_3 + e, k})$				v	v	v	v	v	v
$(\mathbf{m_1 + e + I, m_2, m_3 + k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + k, m_3 + e, I})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1 + e, m_2 + k, m_3 + I, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + I, m_3 + e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1 + e, m_2 + I, m_3 + k, e})$	v^3	$v + v^3$	$v + v^3$	v^3	$v + v^3$	$v + v^3$	v^3	$v + v^3$	$v + v^3$
$(\mathbf{m_1 + e, m_2 + I, m_3, e + k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1 + e, m_2, m_3 + e + k, I})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2, m_3 + e + I, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + e, m_2, m_3 + k, e + I})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1 + e, m_2, m_3 + I, e + k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1 + e, m_2 - e + k, m_3 + e + I, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 - e + k, m_3 + e, e + I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + k, m_2 + I, m_3 + e, e})$		v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1 + k, m_2 + I, m_3, 2e})$			v^2			v^2			v^2
$(\mathbf{m_1 + k, m_2, m_3 + 2e, I})$							v^2	v^2	v^2
$(\mathbf{m_1 + k, m_2, m_3 + e + I, e})$				$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1 + k, m_2, m_3 + e, e + I})$				v^4	$v^2 + v^4$	$v^2 + v^4$	v^4	$v^2 + v^4$	$v^2 + v^4$
$(\mathbf{m_1 + k, m_2, m_3 + I, 2e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1 + I, m_2, m_3 + e + k, e})$							v^2	v^2	v^2
$(\mathbf{m_1 + I, m_2, m_3 + e, e + k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + I, m_2 - e + k, m_3 + 2e, e})$							v^3	v^3	v^3
$(\mathbf{m_1 + I, m_2 - e + k, m_3 + e, 2e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2 - e + k, m_3 + 2e, e + I})$							v^3	v^3	v^3
$(\mathbf{m_1, m_2 - e + k, m_3 + e + I, 2e})$				v^4	v^4	v^4	v^4	v^4	v^4

T

b	**1**	**1**	$\mathbf{> 1}$	$\mathbf{> 1}$
c	**1**	$\mathbf{> 1}$	**1**	$\mathbf{> 1}$
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + k})$	1	1	1	1
$(\mathbf{m_1 + e + k, m_2 + k, m_3 + e})$	v	v	v	v
$(\mathbf{m_1 + e, m_2 + k, m_3 + e + k})$			v	v
$(\mathbf{m_1 + e + k, m_2 + e, m_3, k})$	v	v	v	v
$(\mathbf{m_1 + e + k, m_2 + k, m_3, e})$		v		v
$(\mathbf{m_1 + e + k, m_2, m_3 + e, k})$			v	v
$(\mathbf{m_1 + e + k, m_2, m_3 + k, e})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + k, m_3 + e, k})$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1 + e, m_2 + k, m_3 + k, e})$	v^3	$v + v^3$	v^3	$v + v^3$
$(\mathbf{m_1 + e, m_2 + k, m_3, e + k})$		v^2		v^2
$(\mathbf{m_1 + e, m_2, m_3 + e + k, k})$			v^3	v^3
$(\mathbf{m_1 + e, m_2, m_3 + k, e + k})$		v^3		v^3
$(\mathbf{m_1 + k, m_2, m_3 + e + k, e})$			v^2	v^2
$(\mathbf{m_1 + k, m_2, m_3 + e, e + k})$			v^3	v^3

U

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+I, m_2+e, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+e})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+I})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+I, m_3+e})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+e+k})$			v	v
$(\mathbf{m_1+e, m_2+k, m_3+e+I})$			v^2	v^2
$(\mathbf{m_1+e+I, m_2+e, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e})$		v		v
$(\mathbf{m_1+e+I, m_2, m_3+e, k})$			v	v
$(\mathbf{m_1+e+I, m_2, m_3+k, e})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e})$		v^2		v^2
$(\mathbf{m_1+e+k, m_2, m_3+e, I})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+I, e})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e, m_2+I, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e, m_2+I, m_3, e+k})$		v^2		v^2
$(\mathbf{m_1+e, m_2+k, m_3+e, I})$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+I, e})$	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3, e+I})$		v^3		v^3
$(\mathbf{m_1+e, m_2, m_3+e+I, k})$			v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+e+k, I})$			v^4	v^4
$(\mathbf{m_1+e, m_2, m_3+I, e+k})$		v^3		v^3
$(\mathbf{m_1+e, m_2, m_3+k, e+I})$		v^4		v^4
$(\mathbf{m_1+I, m_2, m_3+e+k, e})$			v^2	v^2
$(\mathbf{m_1+I, m_2, m_3+e, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+I, e})$			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e, e+I})$			v^4	v^4

V

$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{>2}$
$(\mathbf{m_1+e+I, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+e+I, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+I, m_3+e+k})$				v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+I, m_3+2e+J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+J, e})$		v	v		v	v		v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e+J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+I, m_2+J, m_3+e, k})$				v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+e+J, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+e, I})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+I, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+I, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+I, m_3+k, e})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+I, m_3, e+k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+k, I})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+I, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, e+I})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+I, e+k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+J, m_2-e+k, m_3+e+I, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2-e+k, m_3+e, e+I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+e+k, J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+e+J, k})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+I, m_3+k, e+J})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+I, m_3+J, e+k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+I, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+J, m_3+e+I, k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+k, I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+J, m_3+k, e+I})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e, m_2+J, m_3+I, e+k})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e, m_2-e+k, m_3+e+I, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3

$(\mathbf{m_1+e, m_2-e+k, m_3+e+J, e+I})$			v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+I, m_3+2e, J})$			v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+I, m_3+e+J, e})$	v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+k, m_2+I, m_3+e, e+J})$	v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+k, m_2+I, m_3+J, 2e})$		v^2			v^2			v^2
$(\mathbf{m_1+k, m_2+I, m_3, 2e+J})$		v^3			v^3			v^3
$(\mathbf{m_1+k, m_2+J, m_3+2e, I})$						v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+J, m_3+e+I, e})$			$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+k, m_2+J, m_3+e, e+I})$			v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+J, m_3+I, 2e})$	v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+2e+J, I})$						v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+I, e+J})$			$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$
$(\mathbf{m_1+k, m_2, m_3+e+J, e+I})$			v^5	v^3+v^5	v^3+v^5	v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+k, m_2, m_3+I, 2e+J})$	v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+I, m_2+J, m_3+e+k, e})$						v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+J, m_3+e, e+k})$			v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2, m_3+e+k, e+J})$						v^3	v^3	v^3
$(\mathbf{m_1+I, m_2, m_3+e+J, e+k})$			v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2-e+k, m_3+2e+J, e})$						v^3	v^3	v^3
$(\mathbf{m_1+I, m_2-e+k, m_3+2e, e+J})$						v^4	v^4	v^4
$(\mathbf{m_1+I, m_2-e+k, m_3+e+J, 2e})$				v^3	v^3		v^3	v^3
$(\mathbf{m_1+I, m_2-e+k, m_3+e, 2e+J})$				v^4	v^4		v^4	v^4
$(\mathbf{m_1+J, m_2-e+k, m_3+2e, e+I})$						v^3	v^3	v^3
$(\mathbf{m_1+J, m_2-e+k, m_3+e+I, 2e})$			v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2-e+k, m_3+2e+J, e+I})$						v^4	v^4	v^4
$(\mathbf{m_1, m_2-e+k, m_3+e+I, 2e+J})$			v^5	v^5	v^5	v^5	v^5	v^5

$\mathbf{W}$				
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+k, m_2+e+J, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+k, m_2+k, m_3+e+J})$	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+e+k})$			v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+k, J})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+J, e})$		v		v
$(\mathbf{m_1+e+k, m_2+k, m_3, e+J})$		v^2		v^2
$(\mathbf{m_1+e+k, m_2+J, m_3+e, k})$			v	v
$(\mathbf{m_1+e+k, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+e+J, k})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+k, m_3+k, e})$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3, e+k})$		v^2		v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+k, k})$			v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, e+k})$		v^3		v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+k, J})$			v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+e+J, k})$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+k, e+J})$	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3+J, e+k})$		v^3		v^3
$(\mathbf{m_1+e, m_2+J, m_3+e+k, k})$			v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+k, e+k})$		v^4		v^4
$(\mathbf{m_1+k, m_2+J, m_3+e+k, e})$			v^2	v^2
$(\mathbf{m_1+k, m_2+J, m_3+e, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+k, e+J})$			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+J, e+k})$			v^4	v^4

$\mathbf{X}$				
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+I, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+e+J)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+e+J, m_3+I)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+I, m_3+e+J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+I, m_3+e+k)}$			v	v
$\mathbf{(m_1+e+J, m_2+k, m_3+e+I)}$			v^2	v^2
$\mathbf{(m_1+e+I, m_2+e+J, m_3, k)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+J, k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+k, m_3+e, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+k, m_3+J, e)}$		v		v
$\mathbf{(m_1+e+I, m_2+k, m_3, e+J)}$		v^2		v^2
$\mathbf{(m_1+e+I, m_2+J, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+I, m_2+J, m_3+k, e)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2, m_3+e+J, k)}$			v^2	v^2
$\mathbf{(m_1+e+I, m_2, m_3+k, e+J)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+k, m_2+e+J, m_3, I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+e, m_3+I, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+e, m_3+J, I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+k, m_2+I, m_3+e, J)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+k, m_2+I, m_3+J, e)}$		v^2		v^2
$\mathbf{(m_1+e+k, m_2+I, m_3, e+J)}$		v^3		v^3
$\mathbf{(m_1+e+k, m_2+J, m_3+e, I)}$			v^2	v^2
$\mathbf{(m_1+e+k, m_2+J, m_3+I, e)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+k, m_2, m_3+e+J, I)}$			v^3	v^3
$\mathbf{(m_1+e+k, m_2, m_3+I, e+J)}$	v^4	v^4	v^4	v^4
$\mathbf{(m_1+e+J, m_2+I, m_3+e, k)}$	v^2	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e+J, m_2+I, m_3+k, e)}$	v^3	$v+v^3$	v^3	$v+v^3$
$\mathbf{(m_1+e+J, m_2+I, m_3, e+k)}$		v^2		v^2
$\mathbf{(m_1+e+J, m_2+k, m_3+e, I)}$	v^3	v^3	$2v^3$	$2v^3$
$\mathbf{(m_1+e+J, m_2+k, m_3+I, e)}$	v^4	v^2+v^4	v^3	v^2+v^4
$\mathbf{(m_1+e+J, m_2+k, m_3, e+I)}$		v^3		v^3
$\mathbf{(m_1+e+J, m_2, m_3+e+I, k)}$			v^3	v^3
$\mathbf{(m_1+e+J, m_2, m_3+e+k, I)}$			v^4	v^4
$\mathbf{(m_1+e+J, m_2, m_3+I, e+k)}$		v^3		v^3
$\mathbf{(m_1+e+J, m_2, m_3+k, e+I)}$		v^4		v^4
$\mathbf{(m_1+e, m_2+I, m_3+e+k, J)}$			v^2	v^2
$\mathbf{(m_1+e, m_2+I, m_3+e+J, k)}$	v^3	v^3	$2v^3$	$2v^3$
$\mathbf{(m_1+e, m_2+I, m_3+k, e+J)}$	v^4	v^2+v^4	v^4	v^2+v^4
$\mathbf{(m_1+e, m_2+I, m_3+J, e+k)}$		v^3		v^3
$\mathbf{(m_1+e, m_2+k, m_3+e+I, J)}$			v^3	v^3
$\mathbf{(m_1+e, m_2+k, m_3+e+J, I)}$	v^4	v^4	$2v^4$	$2v^4$
$\mathbf{(m_1+e, m_2+k, m_3+I, e+J)}$	v^5	v^3+v^5	v^5	v^3+v^5
$\mathbf{(m_1+e, m_2+k, m_3+J, e+I)}$		v^4		v^4
$\mathbf{(m_1+e, m_2+J, m_3+e+I, k)}$			v^4	v^4
$\mathbf{(m_1+e, m_2+J, m_3+e+k, I)}$			v^5	v^5
$\mathbf{(m_1+e, m_2+J, m_3+I, e+k)}$		v^4		v^4
$\mathbf{(m_1+e, m_2+J, m_3+k, e+I)}$		v^5		v^5
$\mathbf{(m_1+I, m_2+J, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1+I, m_2+J, m_3+e, e+k)}$			v^3	v^3
$\mathbf{(m_1+I, m_2, m_3+e+k, e+J)}$			v^3	v^3
$\mathbf{(m_1+I, m_2, m_3+e+J, e+k)}$			v^4	v^4
$\mathbf{(m_1+k, m_2+J, m_3+e+I, e)}$			v^3	v^3
$\mathbf{(m_1+k, m_2+J, m_3+e, e+I)}$			v^4	v^4
$\mathbf{(m_1+k, m_2, m_3+e+I, e+J)}$			v^4	v^4
$\mathbf{(m_1+k, m_2, m_3+e+J, e+I)}$			v^5	v^5

Y

b	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
c	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
$(\mathbf{m_1+2e, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+2e, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e+k})$				v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+2e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+J, e})$		v	v		v	v		v	v
$(\mathbf{m_1+2e, m_2+k, m_3, e+J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+2e, m_2+J, m_3+e, k})$				v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2, m_3+e+J, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e+J, m_3, e})$		v	v		v	v		v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+e, J})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2+e, m_3+J, e})$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+k, m_2+e, m_3, e+J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+k, m_2+J, m_3+e, e})$	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2, m_3+e+J, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+e, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+e, m_3+k, e})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+e, m_3, e+k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2+k, m_3+e, e})$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(\mathbf{m_1+e+J, m_2+k, m_3, 2e})$			v^2			v^2			v^2
$(\mathbf{m_1+e+J, m_2, m_3+2e, k})$							v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+k, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e, e+k})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+k, 2e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+2e, J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+e+J, e})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+e, e+J})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3+J, 2e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e, m_2+J, m_3+2e, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+J, m_3+e+k, e})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+J, m_3+e, e+k})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+J, m_3+k, 2e})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e, m_2, m_3+e+k, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2, m_3+e+J, e+k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+J, m_3+2e, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+J, m_3+e, 2e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+2e, e+J})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+J, 2e})$				v^4	v^4	v^4	v^4	v^4	v^4

Z

b	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
c	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
$(\mathbf{m_1+2e+H, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+2e+H, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e+H})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e+k})$				v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+2e+H})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2+e+J, m_3, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e+H, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e+H, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2+k, m_3+J, e})$		v	v		v	v		v	v
$(\mathbf{m_1+2e+H, m_2+k, m_3, e+J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+2e+H, m_2+J, m_3+e, k})$				v	v	v	v	v	v
$(\mathbf{m_1+2e+H, m_2+J, m_3+k, e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2, m_3+e+J, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2, m_3+k, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+e+J, m_3+k, H})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+e+J, m_3+H, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+e+H, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2

$(\mathbf{m_1+2e, m_2+e+H, m_3+J, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+k, m_3+e+J, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+e+H, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+k, m_3+J, e+H})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+H, e+J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+2e, m_2+J, m_3+e+H, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+J, m_3+k, e+H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+H, m_3+e+J, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+H, m_3+k, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e+J, m_3+H, e})$		v	v		v	v		v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3, e+H})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e, J})$	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2+e+H, m_3+J, e})$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+k, m_2+e+H, m_3, e+J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+k, m_2+e, m_3+e+J, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e, m_3+e+H, J})$	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+k, m_2+e, m_3+J, e+H})$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e+k, m_2+e, m_3+H, e+J})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e+k, m_2+J, m_3+e+H, e})$	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2+J, m_3+e, e+H})$	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+k, m_2+H, m_3+e+J, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+H, m_3+e, e+J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+e+J, e+H})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2, m_3+e+H, e+J})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+e+H, m_3+k, e})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+e+H, m_3, e+k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e+k, H})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e+H, k})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e+J, m_2+e, m_3+k, e+H})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+J, m_2+e, m_3+H, e+k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+2e, H})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+e+H, e})$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(\mathbf{m_1+e+J, m_2+k, m_3+e, e+H})$	v^3+v^5	$2v^3+v^5$	$2v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$
$(\mathbf{m_1+e+J, m_2+k, m_3+H, 2e})$			v^2			v^2			v^2
$(\mathbf{m_1+e+J, m_2+k, m_3, 2e+H})$			v^3			v^3			v^3
$(\mathbf{m_1+e+J, m_2+H, m_3+2e, k})$							v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+H, m_3+e+k, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+H, m_3+e, e+k})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+J, m_2+H, m_3+k, 2e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+2e+H, k})$							v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+k, e+H})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+J, m_2, m_3+e+H, e+k})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+k, 2e+H})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e+H, m_2+k, m_3+2e, J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+H, m_2+k, m_3+e+J, e})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e+H, m_2+k, m_3+e, e+J})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+H, m_2+k, m_3+J, 2e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+H, m_2+J, m_3+2e, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+H, m_2+J, m_3+e+k, e})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+H, m_2+J, m_3+e, e+k})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e+H, m_2+J, m_3+k, 2e})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e+H, m_2, m_3+e+k, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+H, m_2, m_3+e+J, e+k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+2e+H, J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, e+H})$	v^4	v^4	v^4	$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$
$(\mathbf{m_1+e, m_2+k, m_3+e+H, e+J})$	v^5	v^3+v^5	v^3+v^5	v^5	v^3+v^5	v^3+v^5	v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+e, m_2+k, m_3+J, 2e+H})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+2e+H, k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+k, e+H})$				v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+e, m_2+J, m_3+e+H, e+k})$		v^4	v^4	v^4	$2v^4$	$2v^4$	v^4	$2v^4$	$2v^4$
$(\mathbf{m_1+e, m_2+J, m_3+k, 2e+H})$		v^5	v^5		v^5	v^5		v^5	v^5
$(\mathbf{m_1+e, m_2+H, m_3+e+k, e+J})$					v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+H, m_3+e+J, e+k})$				v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+k, m_2+J, m_3+2e+H, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+J, m_3+2e, e+H})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+J, m_3+e+H, 2e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+J, m_3+e, 2e+H})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1+k, m_2+H, m_3+2e, e+J})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+H, m_3+e+J, 2e})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+2e+H, e+J})$							v^4	v^4	v^4
$(\mathbf{m_1+k, m_2, m_3+e+J, 2e+H})$				v^5	v^5	v^5	v^5	v^5	v^5

Wall $b : a = c = 0$

C

a	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
c	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
$(\mathbf{m_1 + j, m_2 + j, m_3})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1 - e + j, m_2 + e, m_3 + j})$		v	v		v	v		v	v
$(\mathbf{m_1 + j, m_2, m_3 - e + j, e})$				v	v	v	v	v	v
$(\mathbf{m_1, m_2 + j, m_3, j})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1, m_2 + j, m_3 - e + j, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2, m_3 - e + j, e + j})$							v^2	v^2	v^2
$(\mathbf{m_1 - e + j, m_2 + e, m_3, j})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1 - e + j, m_2 + j, m_3, e})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1 - e, m_2 + e, m_3 + j, j})$			v^2			v^2			v^2
$(\mathbf{m_1 - e, m_2 + j, m_3, e + j})$					v^3	v^3		v^3	v^3

D

a	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
c	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
$(\mathbf{m_1 + i + j, m_2 + j, m_3})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1 + j, m_2 + i + j, m_3})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1 - e + i + j, m_2 + e, m_3 + j})$		v	v		v	v		v	v
$(\mathbf{m_1 - e + j, m_2 + e, m_3 + i + j})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1 + i + j, m_2, m_3 - e + j, e})$				v	v	v	v	v	v
$(\mathbf{m_1 + j, m_2, m_3 - e + i + j, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2 + i + j, m_3, j})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1, m_2 + i + j, m_3 - e + j, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2 + j, m_3, i + j})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2 + j, m_3 - e + i + j, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2, m_3 - e + i + j, e + j})$							v^2	v^2	v^2
$(\mathbf{m_1, m_2, m_3 - e + j, e + i + j})$							v^3	v^3	v^3
$(\mathbf{m_1 - e + i + j, m_2 + e, m_3, j})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1 - e + i + j, m_2 + j, m_3, e})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1 - e + j, m_2 + e, m_3, i + j})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1 - e + j, m_2 + i + j, m_3, e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1 - e, m_2 + e, m_3 + i + j, j})$			v^2			v^2			v^2
$(\mathbf{m_1 - e, m_2 + e, m_3 + j, i + j})$			v^3			v^3			v^3
$(\mathbf{m_1 - e, m_2 + i + j, m_3, e + j})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1 - e, m_2 + j, m_3, e + i + j})$					v^4	v^4		v^4	v^4

E

a	**1**	**> 1**	**1**	**> 1**
c	**1**	**1**	**> 1**	**> 1**
$(\mathbf{m_1 + e, m_2 + j, m_3})$	1	1	1	1
$(\mathbf{m_1 + j, m_2 + e, m_3})$	v	v	v	v
$(\mathbf{m_1, m_2 + e, m_3 + j})$		v		v
$(\mathbf{m_1 + e, m_2, m_3 - e + j, e})$			v	v
$(\mathbf{m_1, m_2 + e, m_3, j})$		v^2		v^2
$(\mathbf{m_1, m_2 + j, m_3, e})$			v^2	v^2

F

a	**1**	**> 1**	**1**	**> 1**
c	**1**	**1**	**> 1**	**> 1**
$(\mathbf{m_1 + e + I, m_2 + j, m_3})$	1	1	1	1
$(\mathbf{m_1 + e, m_2 + j, m_3 + I})$	v	v	v	v
$(\mathbf{m_1 + j, m_2 + e + I, m_3})$	v	v	v	v
$(\mathbf{m_1 + j, m_2 + e, m_3 + I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + I, m_2 + e, m_3 + j})$		v		v
$(\mathbf{m_1, m_2 + e + I, m_3 + j})$		v^2		v^2
$(\mathbf{m_1 + e + I, m_2, m_3 - e + j, e})$			v	v
$(\mathbf{m_1 + e, m_2 + j, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + I, m_3 - e + j, e})$			v^2	v^2
$(\mathbf{m_1 + e, m_2, m_3 - e + j, e + I})$			v^3	v^3
$(\mathbf{m_1 + j, m_2 + e, m_3, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1 + I, m_2 + e, m_3, j})$		v^2		v^2
$(\mathbf{m_1 + I, m_2 + j, m_3, e})$			v^2	v^2
$(\mathbf{m_1, m_2 + e + I, m_3, j})$		v^3		v^3
$(\mathbf{m_1, m_2 + e, m_3 + j, I})$		v^3		v^3
$(\mathbf{m_1, m_2 + e, m_3 + I, j})$		v^4		v^4
$(\mathbf{m_1, m_2 + j, m_3 + I, e})$			v^3	v^3
$(\mathbf{m_1, m_2 + j, m_3, e + I})$			v^4	v^4

K									
a	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$
c	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+j+k, m_2+j+k, m_3+k)}$	1	1	1	1	1	1	1	1	1
$\mathbf{(m_1+j+k, m_2+j+k, m_3, k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3+j+k)}$		v	v		v	v		v	v
$\mathbf{(m_1+j+k, m_2+k, m_3-e+j+k, e)}$				v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2, m_3-e+j+k, e+k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3, j+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+k, m_2+j+k, m_3-e+j+k, e)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2, m_3-e+j+k, e+j+k)}$							v^2	v^2	v^2
$\mathbf{(m_1, m_2+j+k, m_3+k, j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1, m_2+j+k, m_3-e+j+k, e+k)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1, m_2+k, m_3-e+j+k, e+j+k)}$							v^3	v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+j+k, m_3+k, e)}$					v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+j+k, m_2+j+k, m_3, e+k)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3, j+k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+j+k, m_2+e, m_3+j+k, k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+j+k, m_2+e, m_3+k, j+k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+k, m_2+e, m_3+j+k, j+k)}$			v^2			v^2			v^2
$\mathbf{(m_1-e+k, m_2+j+k, m_3, e+j+k)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e, m_2+e+k, m_3+j+k, j+k)}$			v^3			v^3			v^3
$\mathbf{(m_1-e, m_2+j+k, m_3+k, e+j+k)}$					v^4	v^4		v^4	v^4

L				
a	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$
c	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e, m_2+e, m_3+k)}$	1	1	1	1
$\mathbf{(m_1, m_2+e+k, m_3+e)}$		v		v
$\mathbf{(m_1+e, m_2+e, m_3, k)}$	v	v	v	v
$\mathbf{(m_1+e, m_2+k, m_3, e)}$			v	v
$\mathbf{(m_1+k, m_2+e, m_3, e)}$			v^2	v^2
$\mathbf{(m_1, m_2+e, m_3+k, e)}$		v^2		v^2

M				
a	**1**	$\mathbf{>1}$	**1**	$\mathbf{>1}$
c	**1**	**1**	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+I, m_2+e+I, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+I, m_2+e+k, m_3+e+I)}$		v		v
$\mathbf{(m_1+e+I, m_2+e+I, m_3, k)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+k, I)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+I, k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+k, m_3+I, e)}$			v	v
$\mathbf{(m_1+e+I, m_2+k, m_3, e+I)}$			v^2	v^2
$\mathbf{(m_1+e, m_2+e+I, m_3+k, I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+e+I, m_3+I, k)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e, m_2+k, m_3+I, e+I)}$			v^3	v^3
$\mathbf{(m_1+I, m_2+e+I, m_3+k, e)}$		v^2		v^2
$\mathbf{(m_1+I, m_2+e+k, m_3+e, I)}$		v^2		v^2
$\mathbf{(m_1+I, m_2+e, m_3+k, e+I)}$		v^3		v^3
$\mathbf{(m_1+k, m_2+e+I, m_3+I, e)}$			v^2	v^2
$\mathbf{(m_1+k, m_2+e+I, m_3, e+I)}$			v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+I, e+I)}$			v^4	v^4
$\mathbf{(m_1, m_2+e+I, m_3+k, e+I)}$		v^4		v^4
$\mathbf{(m_1, m_2+e+k, m_3+e+I, I)}$		v^3		v^3

N									
a	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$
c	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+i+j+k, m_2+j+k, m_3+k)}$	1	1	1	1	1	1	1	1	1
$\mathbf{(m_1+j+k, m_2+i+j+k, m_3+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1-e+i+j+k, m_2+e+k, m_3+j+k)}$		v	v		v	v		v	v
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3+i+j+k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1+i+j+k, m_2+j+k, m_3, k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+i+j+k, m_2+k, m_3-e+j+k, e)}$				v	v	v	v	v	v
$\mathbf{(m_1+i+j+k, m_2, m_3-e+j+k, e+k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+i+j+k, m_3, k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+k, m_3-e+i+j+k, e)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2, m_3-e+i+j+k, e+k)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+i+j+k, m_3, j+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+k, m_2+i+j+k, m_3-e+j+k, e)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3, i+j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2

$\mathbf{(m_1+k, m_2+j+k, m_3-e+i+j+k, e)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2, m_3-e+i+j+k, e+j+k)}$							v^2	v^2	v^2
$\mathbf{(m_1+k, m_2, m_3-e+j+k, e+i+j+k)}$							v^3	v^3	v^3
$\mathbf{(m_1, m_2+i+j+k, m_3+k, j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1, m_2+i+j+k, m_3-e+j+k, e+k)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1, m_2+j+k, m_3+k, i+j+k)}$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1, m_2+j+k, m_3-e+i+j+k, e+k)}$				v^4	v^4	v^4	v^4	v^4	v^4
$\mathbf{(m_1, m_2+k, m_3-e+i+j+k, e+j+k)}$							v^3	v^3	v^3
$\mathbf{(m_1, m_2+k, m_3-e+j+k, e+i+j+k)}$							v^4	v^4	v^4
$\mathbf{(m_1-e+i+j+k, m_2+e+k, m_3, j+k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+i+j+k, m_2+e, m_3+j+k, k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+i+j+k, m_2+e, m_3+k, j+k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+i+j+k, m_2+j+k, m_3+k, e)}$					v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+i+j+k, m_2+j+k, m_3, e+k)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3, i+j+k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e, m_3+i+j+k, k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e, m_3+k, i+j+k)}$		v^4	v^4		v^4	v^4		v^4	v^4
$\mathbf{(m_1-e+j+k, m_2+i+j+k, m_3+k, e)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+i+j+k, m_3, e+k)}$					v^4	v^4		v^4	v^4
$\mathbf{(m_1-e+k, m_2+e, m_3+i+j+k, j+k)}$			v^2			v^2			v^2
$\mathbf{(m_1-e+k, m_2+e, m_3+j+k, i+j+k)}$			v^3			v^3			v^3
$\mathbf{(m_1-e+k, m_2+i+j+k, m_3, e+j+k)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+k, m_2+j+k, m_3, e+i+j+k)}$					v^4	v^4		v^4	v^4
$\mathbf{(m_1-e, m_2+e+k, m_3+i+j+k, j+k)}$			v^3			v^3			v^3
$\mathbf{(m_1-e, m_2+e+k, m_3+j+k, i+j+k)}$			v^4			v^4			v^4
$\mathbf{(m_1-e, m_2+i+j+k, m_3+k, e+j+k)}$					v^4	v^4		v^4	v^4
$\mathbf{(m_1-e, m_2+j+k, m_3+k, e+i+j+k)}$					v^5	v^5		v^5	v^5

O									
a	**1**	**2**	$>$ **2**	**1**	**2**	$>$ **2**	**1**	**2**	$>$ **2**
c	**1**	**1**	**1**	**2**	**2**	**2**	$>$ **2**	$>$ **2**	$>$ **2**
$\mathbf{(m_1+e, m_2+j+k, m_3+k)}$	1	1	1	1	1	1	1	1	1
$\mathbf{(m_1+j+k, m_2+e, m_3+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1, m_2+e+k, m_3+j+k)}$		v	v		v	v		v	v
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3+e)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1+e, m_2+j+k, m_3, k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e, m_2+k, m_3-e+j+k, e)}$				v	v	v	v	v	v
$\mathbf{(m_1+e, m_2, m_3-e+j+k, e+k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+e, m_3, k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+k, m_3, e)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e, m_3, j+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+k, m_2+e, m_3-e+j+k, e)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3, e)}$				$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$\mathbf{(m_1+k, m_2, m_3-e+j+k, 2e)}$							v^2	v^2	v^2
$\mathbf{(m_1, m_2+e+k, m_3, j+k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1, m_2+e, m_3+j+k, k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1, m_2+e, m_3+k, j+k)}$		$v+v^3$	$v+v^3$		$v+v^3$	$v+v^3$		$v+v^3$	$v+v^3$
$\mathbf{(m_1, m_2+e, m_3-e+j+k, e+k)}$					v^2	v^2		v^2	v^2
$\mathbf{(m_1, m_2+j+k, m_3+k, e)}$	v^2	v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1, m_2+j+k, m_3, e+k)}$				v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$\mathbf{(m_1, m_2+k, m_3, e+j+k)}$							v^2	v^2	v^2
$\mathbf{(m_1, m_2+k, m_3-e+j+k, 2e)}$							v^3	v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3, e)}$					v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+j+k, m_2+e, m_3+k, e)}$		v^3	v^3		$2v^3$	$2v^3$		$2v^3$	$2v^3$
$\mathbf{(m_1-e+j+k, m_2+e, m_3, e+k)}$					v^4	v^4		v^4	v^4
$\mathbf{(m_1-e+k, m_2+e, m_3+j+k, e)}$			v^2			v^2			v^2
$\mathbf{(m_1-e+k, m_2+e, m_3, e+j+k)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+k, m_2+j+k, m_3, 2e)}$								v^3	v^3
$\mathbf{(m_1-e, m_2+e+k, m_3+e, j+k)}$			v^2			v^2			v^2
$\mathbf{(m_1-e, m_2+e+k, m_3+j+k, e)}$			v^3			v^3			v^3
$\mathbf{(m_1-e, m_2+e, m_3+k, e+j+k)}$						v^3			v^3
$\mathbf{(m_1-e, m_2+j+k, m_3+k, 2e)}$					v^4	v^4		v^4	v^4

P									
a	**1**	**2**	$>$ **2**	**1**	**2**	$>$ **2**	**1**	**2**	$>$ **2**
c	**1**	**1**	**1**	**2**	**2**	**2**	$>$ **2**	$>$ **2**	$>$ **2**
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+j+k, m_2+e+I, m_3+k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+I, m_2+e+k, m_3+j+k})$		v	v		v	v		v	v
$(\mathbf{m_1-e+j+k, m_2+e+k, m_3+e+I})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3-e+j+k, e})$				v	v	v	v	v	v
$(\mathbf{m_1+e+I, m_2, m_3-e+j+k, e+k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3-e+j+k, e+I})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3-e+j+k, e+k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+k, m_3+I, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+k, m_3, e+I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3, j+k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+e+I, m_3-e+j+k, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+I, j+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3-e+j+k, e+I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3+I, e})$				$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+k, m_2+j+k, m_3, e+I})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+I, m_3-e+j+k, 2e})$							v^2	v^2	v^2
$(\mathbf{m_1+k, m_2, m_3-e+j+k, 2e+I})$							v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+e+k, m_3, j+k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+j+k, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, j+k})$		$v+v^3$	$v+v^3$		$v+v^3$	$v+v^3$		$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2+e, m_3-e+j+k, e+k})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+I, m_2+j+k, m_3+k, e})$	v^2	v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+I, m_2+j+k, m_3, e+k})$				v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+I, m_2+k, m_3, e+j+k})$							v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+k, m_3-e+j+k, 2e})$							v^3	v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+j+k, I})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1, m_2+e+k, m_3+I, j+k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+j+k, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, j+k})$		v^2+v^4	v^2+v^4		v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1, m_2+e+I, m_3-e+j+k, e+k})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+j+k, m_3+k, e+I})$	v^3	v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1, m_2+j+k, m_3+I, e+k})$				v^4	$2v^3$	$2v^3$	v^4	$2v^3$	$2v^3$
$(\mathbf{m_1, m_2+k, m_3+I, e+j+k})$							v^3	v^3	v^3
$(\mathbf{m_1, m_2+k, m_3-e+j+k, 2e+I})$							v^4	v^4	v^4
$(\mathbf{m_1-e+j+k, m_2+e+k, m_3+e, I})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+j+k, m_2+e+k, m_3+I, e})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1-e+j+k, m_2+e+k, m_3, e+I})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+j+k, m_2+e+I, m_3+k, e})$		v^3	v^3		$2v^3$	$2v^3$		$2v^3$	$2v^3$
$(\mathbf{m_1-e+j+k, m_2+e+I, m_3, e+k})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1-e+j+k, m_2+e, m_3+k, e+I})$		v^4	v^4		$2v^3$	$2v^3$		$2v^3$	$2v^3$
$(\mathbf{m_1-e+j+k, m_2+e, m_3+I, e+k})$					v^5	v^5		v^5	v^5
$(\mathbf{m_1-e+k, m_2+e+I, m_3+j+k, e})$			v^2			v^2			v^2
$(\mathbf{m_1-e+k, m_2+e+I, m_3, e+j+k})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+k, m_2+e, m_3+j+k, e+I})$			v^3			v^3			v^3
$(\mathbf{m_1-e+k, m_2+e, m_3+I, e+j+k})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1-e+k, m_2+j+k, m_3+I, 2e})$								v^3	v^3
$(\mathbf{m_1-e+k, m_2+j+k, m_3, 2e+I})$								v^4	v^4
$(\mathbf{m_1-e+I, m_2+e+k, m_3+e, j+k})$			v^2			v^2			v^2
$(\mathbf{m_1-e+I, m_2+e+k, m_3+j+k, e})$			v^3			v^3			v^3
$(\mathbf{m_1-e+I, m_2+e, m_3+k, e+j+k})$						v^3			v^3
$(\mathbf{m_1-e+I, m_2+j+k, m_3+k, 2e})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1-e, m_2+e+k, m_3+j+k, e+I})$			v^4			v^4			v^4
$(\mathbf{m_1-e, m_2+e+k, m_3+e+I, j+k})$			v^3			v^3			v^3
$(\mathbf{m_1-e, m_2+e+I, m_3+k, e+j+k})$						v^4			v^4
$(\mathbf{m_1-e, m_2+j+k, m_3+k, 2e+I})$					v^5	v^5		v^5	v^5

Q				
a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+k, m_2+j+k, m_3+k})$	1	1	1	1
$(\mathbf{m_1+j+k, m_2+e+k, m_3+k})$	v	v	v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+j+k})$		v		v
$(\mathbf{m_1+e+k, m_2+j+k, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+k, m_3-e+j+k, e})$			v	v
$(\mathbf{m_1+e+k, m_2, m_3-e+j+k, e+k})$			v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3-e+j+k, e+k})$			v^3	v^3
$(\mathbf{m_1+j+k, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e, m_3+k, j})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+k, m_3, j+k})$		v^2		v^2
$(\mathbf{m_1+k, m_2+e, m_3+j+k, k})$		v^2		v^2
$(\mathbf{m_1+k, m_2+e, m_3+J, j+k})$		v^3		v^3
$(\mathbf{m_1+k, m_2+j+k, m_3+k, e})$			v^2	v^2
$(\mathbf{m_1+k, m_2+j+k, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+j+k, k})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+k, j+k})$		v^4		v^4
$(\mathbf{m_1, m_2+j+k, m_3+k, e+k})$			v^4	v^4

R				
a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+k, m_2+j+k, m_3+I})$	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e+I, m_3+k})$	v	v	v	v
$(\mathbf{m_1+j+k, m_2+e+k, m_3+I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+j+k})$		v		v
$(\mathbf{m_1+k, m_2+e+I, m_3+j+k})$		v^2		v^2
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3-e+j+k, e})$			v	v
$(\mathbf{m_1+e+I, m_2, m_3-e+j+k, e+k})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2+j+k, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3-e+j+k, e})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3-e+j+k, e+I})$			v^3	v^3
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3-e+j+k, e+k})$			v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3-e+j+k, e+I})$			v^4	v^4
$(\mathbf{m_1+j+k, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+j+k, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e, m_3+I, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+e, m_3+k, I})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2+e+k, m_3, j+k})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e, m_3+j+k, k})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, j+k})$		v^3		v^3
$(\mathbf{m_1+I, m_2+j+k, m_3+k, e})$			v^2	v^2
$(\mathbf{m_1+I, m_2+j+k, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3, j+k})$		v^3		v^3
$(\mathbf{m_1+k, m_2+e, m_3+j+k, I})$		v^3		v^3
$(\mathbf{m_1+k, m_2+e, m_3+I, j+k})$		v^4		v^4
$(\mathbf{m_1+k, m_2+j+k, m_3+I, e})$			v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3, e+I})$			v^4	v^4
$(\mathbf{m_1, m_2+e+I, m_3+j+k, k})$		v^3		v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, j+k})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+j+k, I})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+I, j+k})$		v^5		v^5
$(\mathbf{m_1, m_2+j+k, m_3+I, e+k})$			v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+k, e+I})$			v^5	v^5

S				
a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+i, m_2+e, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e, m_2+e+i, m_3+k})$	v	v	v	v
$(\mathbf{m_1+i, m_2+e+k, m_3+e})$		v		v
$(\mathbf{m_1, m_2+e+k, m_3+e+i})$		v^2		v^2
$(\mathbf{m_1+e+i, m_2+e, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+i, m_2+k, m_3, e})$			v	v
$(\mathbf{m_1+e, m_2+e+i, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e, m_3+k, i})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e, m_3+i, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+i, e})$			v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3, e+i})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+i, m_3, e})$			v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+i, e})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3, e+i})$			v^4	v^4
$(\mathbf{m_1+i, m_2+e, m_3+k, e})$		v^2		v^2
$(\mathbf{m_1, m_2+e+i, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1, m_2+e, m_3+k, e+i})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+e, i})$		v^3		v^3

T				
a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+k, m_2+e, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e, m_2+e+k, m_3+k})$	v	v	v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+e})$		v		v
$(\mathbf{m_1+e+k, m_2+e, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, e})$			v	v
$(\mathbf{m_1+e, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e, m_3+k, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+k, m_3, e+k})$			v^2	v^2
$(\mathbf{m_1+k, m_2+e+k, m_3, e})$				v
$(\mathbf{m_1+k, m_2+e, m_3+k, e})$		v^2	v^2	$2v^2$
$(\mathbf{m_1+k, m_2+e, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1, m_2+e+k, m_3+e, k})$		v^2		v^2
$(\mathbf{m_1, m_2+e+k, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1, m_2+e, m_3+k, e+k})$				v^3

U				
a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I, m_2+e, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+k, m_2+e, m_3+I})$	v	v	v	v
$(\mathbf{m_1+e, m_2+e+I, m_3+k})$	v	v	v	v
$(\mathbf{m_1+e, m_2+e+k, m_3+I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+e})$		v		v
$(\mathbf{m_1+k, m_2+e+I, m_3+e})$		v^2		v^2
$(\mathbf{m_1+e+I, m_2+e, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e})$			v	v
$(\mathbf{m_1+e+k, m_2+e, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e})$			v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e, m_3+I, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+e, m_3+k, I})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+I, m_3, e+k})$			v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3, e+I})$				$2v^3$
$(\mathbf{m_1+I, m_2+e+k, m_3, e})$				v
$(\mathbf{m_1+I, m_2+e, m_3+k, e})$		v^2	v^2	$2v^2$
$(\mathbf{m_1+I, m_2+e, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3, e})$				v^2
$(\mathbf{m_1+k, m_2+e, m_3+I, e})$		v^3	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3, e+I})$			v^4	v^4
$(\mathbf{m_1, m_2+e+I, m_3+e, k})$		v^2		v^2
$(\mathbf{m_1, m_2+e+I, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+e, I})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+I, e})$		v^4		v^4
$(\mathbf{m_1, m_2+e, m_3+I, e+k})$				v^3
$(\mathbf{m_1, m_2+e, m_3+k, e+I})$				v^4

V

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I, m_2+e+J, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+J, m_2+e+I, m_3+k})$	v	v	v	v
$(\mathbf{m_1+I, m_2+e+k, m_3+e+J})$		v		v
$(\mathbf{m_1+J, m_2+e+k, m_3+e+I})$		v^2		v^2
$(\mathbf{m_1+e+I, m_2+e+J, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+k, J})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+J, e})$			v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e+J})$			v^2	v^2
$(\mathbf{m_1+e+J, m_3+e+I, m_2, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_3+e, m_2+I, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_3+k, m_2+I, e})$			v^2	v^2
$(\mathbf{m_1+e+J, m_3+e, m_2+k, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_3+k, m_2, e+I})$			v^3	v^3
$(\mathbf{m_1+e, m_3+e+I, m_3+k, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3+J, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+k, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+I, k})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+k, m_3+I, e+J})$			v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+J, e+I})$			v^4	v^4
$(\mathbf{m_1+k, m_2+e+I, m_3+J, e})$			v^2	v^2
$(\mathbf{m_1+k, m_2+e+I, m_3, e+J})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+J, m_3+I, e})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+J, m_3, e+I})$			v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+I, e+J})$			v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+J, e+I})$			v^5	v^5
$(\mathbf{m_1+I, m_2+e+k, m_3+e, J})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e+J, m_3+k, e})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e, m_3+k, e+J})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e+k, m_3+e, I})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e+I, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+I})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+e+I, J})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+e+J, I})$		v^4		v^4
$(\mathbf{m_1, m_2+e+I, m_3+k, e+J})$		v^4		v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+I})$		v^5		v^5

W

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
c	**1**	**1**	$> \mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+k, m_2+e+J, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+J, m_2+e+k, m_3+k})$	v	v	v	v
$(\mathbf{m_1+k, m_2+e+k, m_3+e+J})$		v		v
$(\mathbf{m_1+e+k, m_2+e+J, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+k, J})$	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+J, e})$			v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, e+J})$			v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+k, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+k, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3, e+k})$			v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+k, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+J, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+k, k})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+k, m_3+J, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+k, m_3+e, J})$		v^2		v^2
$(\mathbf{m_1+k, m_2+e+k, m_3+J, e})$				v
$(\mathbf{m_1+k, m_2+e+k, m_3, e+J})$				v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+k, e})$		v^2	v^2	$2v^2$
$(\mathbf{m_1+k, m_2+e+J, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+k, e+J})$		v^3	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3+J, e+k})$			v^4	v^4
$(\mathbf{m_1+J, m_2+e+k, m_3+e, k})$		v^2		v^2
$(\mathbf{m_1+J, m_2+e+k, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+k})$				v^3
$(\mathbf{m_1, m_2+e+k, m_3+e+J, k})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+k, e+J})$		v^4		v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+k})$				v^4

$\mathbf{X}$				
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{c}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$(\mathbf{m_1+e+I, m_2+e+J, m_3+k})$	1	1	1	1
$(\mathbf{m_1+e+k, m_2+e+J, m_3+I})$	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+I, m_3+k})$	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+k, m_3+I})$		v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+e+J})$		v		v
$(\mathbf{m_1+k, m_2+e+I, m_3+e+J})$		v^2		v^2
$(\mathbf{m_1+e+I, m_2+e+J, m_3, k})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+k, J})$	v	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3+J, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2+k, m_3+J, e})$			v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e+J})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2+e+J, m_3, I})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3+I, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3+J, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+I, m_3+J, e})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e+J})$			v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+I, m_3, k})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+k, m_3, I})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e, m_3+I, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+e, m_3+k, I})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+J, m_2+I, m_3, e+k})$			v^2	v^2
$(\mathbf{m_1+e+J, m_2+k, m_3, e+I})$			v^3	v^3
$(\mathbf{m_1+e, m_2+e+I, m_3+k, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+I, m_3+J, k})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+I, J})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+J, I})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+e+J, m_3+I, k})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+J, m_3+k, I})$	v^3+v^5	v^3+v^5	v^3+v^5	v^3+v^5
$(\mathbf{m_1+e, m_2+I, m_3+J, e+k})$			v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+J, e+I})$			v^4	v^4
$(\mathbf{m_1+I, m_2+e+k, m_3+e, J})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e+k, m_3+J, e})$				v
$(\mathbf{m_1+I, m_2+e+k, m_3, e+J})$				v^2
$(\mathbf{m_1+I, m_2+e+J, m_3+k, e})$		v^2	v^2	$2v^2$
$(\mathbf{m_1+I, m_2+e+J, m_3, e+k})$			v^3	v^3
$(\mathbf{m_1+I, m_2+e, m_3+k, e+J})$		v^3	v^3	$2v^3$
$(\mathbf{m_1+I, m_2+e, m_3+J, e+k})$			v^4	v^4
$(\mathbf{m_1+k, m_2+e+I, m_3+e, J})$		v^3		v^3
$(\mathbf{m_1+k, m_2+e+I, m_3+J, e})$				v^2
$(\mathbf{m_1+k, m_2+e+I, m_3, e+J})$				v^3
$(\mathbf{m_1+k, m_2+e+J, m_3+I, e})$		v^3	v^3	$2v^3$
$(\mathbf{m_1+k, m_2+e+J, m_3, e+I})$			v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+I, e+J})$		v^4	v^4	$2v^3$
$(\mathbf{m_1+k, m_2+e, m_3+J, e+I})$			v^5	v^5
$(\mathbf{m_1+J, m_2+e+I, m_3+e, k})$		v^2		v^2
$(\mathbf{m_1+J, m_2+e+I, m_3+k, e})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e+k, m_3+e, I})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e+k, m_3+I, e})$		v^4		v^4
$(\mathbf{m_1+J, m_2+e, m_3+I, e+k})$				v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, e+I})$				v^4
$(\mathbf{m_1, m_2+e+I, m_3+e+J, k})$		v^3		v^3
$(\mathbf{m_1, m_2+e+I, m_3+k, e+J})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+e+J, I})$		v^4		v^4
$(\mathbf{m_1, m_2+e+k, m_3+I, e+J})$		v^5		v^5
$(\mathbf{m_1, m_2+e+J, m_3+I, e+k})$				v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, e+I})$				v^5

Y

a	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
c	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
(m_1+2e, m_2+e+J, m_3+k)	1	1	1	1	1	1	1	1	1
$(m_1+e+k, m_2+e+J, m_3+e)$	v	v	v	v	v	v	v	v	v
(m_1+e+J, m_2+2e, m_3+k)	v	v	v	v	v	v	v	v	v
$(m_1+e+J, m_2+e+k, m_3+e)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e, m_2+e+k, m_3+e+J)$		v	v		v	v		v	v
(m_1+k, m_2+2e, m_3+e+J)		v^2	v^2		v^2	v^2		v^2	v^2
$(m_1+2e, m_2+e+J, m_3, k)$	v	v	v	v	v	v	v	v	v
$(m_1+2e, m_2+e, m_3+k, J)$	v	v	v	v	v	v	v	v	v
$(m_1+2e, m_2+e, m_3+J, k)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e, m_2+k, m_3+J, e)$				v	v	v	v	v	v
$(m_1+2e, m_2+k, m_3, e+J)$				v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+k, m_2+e+J, m_3, e)$				v	v	v	v	v	v
$(m_1+e+k, m_2+e, m_3+J, e)$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(m_1+e+k, m_2+e, m_3, e+J)$				v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e+J, m_2+2e, m_3, k)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+J, m_2+e+k, m_3, e)$				v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+J, m_2+e, m_3+k, e)$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(m_1+e+J, m_2+e, m_3, e+k)$				v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+J, m_2+k, m_3, 2e)$							v^2	v^2	v^2
$(m_1+e, m_2+2e, m_3+k, J)$		v	v		v	v		v	v
$(m_1+e, m_2+2e, m_3+J, k)$		v^2	v^2		v^2	v^2		v^2	v^2
$(m_1+e, m_2+e+k, m_3+e, J)$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(m_1+e, m_2+e+k, m_3+J, e)$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(m_1+e, m_2+e+k, m_3, e+J)$					v^2	v^2		v^2	v^2
$(m_1+e, m_2+e+J, m_3+e, k)$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(m_1+e, m_2+e+J, m_3+k, e)$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(m_1+e, m_2+e+J, m_3, e+k)$				v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e, m_2+e, m_3+k, e+J)$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(m_1+e, m_2+e, m_3+J, e+k)$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(m_1+e, m_2+k, m_3+J, 2e)$				v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+k, m_2+2e, m_3+e, J)$		v^3	v^3		v^3	v^3		v^3	v^3
$(m_1+k, m_2+2e, m_3+J, e)$					v^2	v^2		v^2	v^2
$(m_1+k, m_2+2e, m_3, e+J)$					v^3	v^3		v^3	v^3
$(m_1+k, m_2+e+J, m_3+e, e)$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(m_1+k, m_2+e+J, m_3, 2e)$							v^3	v^3	v^3
$(m_1+k, m_2+e, m_3+J, 2e)$				v^4	v^4	v^4	v^4	v^4	v^4
$(m_1+J, m_2+2e, m_3+e, k)$		v^2	v^2		v^2	v^2		v^2	v^2
$(m_1+J, m_2+2e, m_3+k, e)$		v^3	v^3		v^3	v^3		v^3	v^3
$(m_1+J, m_2+e+k, m_3+e, e)$		v^2+v^4	v^2+v^4		v^2+v^4	v^2+v^4		v^2+v^4	v^2+v^4
$(m_1+J, m_2+e, m_3+k, 2e)$					v^3	v^3		v^3	v^3
$(m_1, m_2+2e, m_3+e+J, k)$			v^2			v^2			v^2
$(m_1, m_2+2e, m_3+k, e+J)$			v^3			v^3			v^3
$(m_1, m_2+e+k, m_3+e+J, e)$		v^3	v^3		v^3	v^3		v^3	v^3
$(m_1, m_2+e+k, m_3+e, e+J)$		v^4	v^4		v^4	v^4		v^4	v^4
$(m_1, m_2+e+J, m_3+e, e+k)$					v^3	v^3		v^3	v^3
$(m_1, m_2+e+J, m_3+k, 2e)$					v^4	v^4		v^4	v^4

Z

a	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
c	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
$(m_1+2e+H, m_2+e+J, m_3+k)$	1	1	1	1	1	1	1	1	1
$(m_1+e+k, m_2+e+J, m_3+e+H)$	v	v	v	v	v	v	v	v	v
$(m_1+e+J, m_2+2e+H, m_3+k)$	v	v	v	v	v	v	v	v	v
$(m_1+e+J, m_2+e+k, m_3+e+H)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+H, m_2+e+k, m_3+e+J)$		v	v		v	v		v	v
$(m_1+k, m_2+2e+H, m_3+e+J)$		v^2	v^2		v^2	v^2		v^2	v^2
$(m_1+2e+H, m_2+e+J, m_3, k)$	v	v	v	v	v	v	v	v	v
$(m_1+2e+H, m_2+e, m_3+k, J)$	v	v	v	v	v	v	v	v	v
$(m_1+2e+H, m_2+e, m_3+J, k)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e+H, m_2+k, m_3+J, e)$				v	v	v	v	v	v
$(m_1+2e+H, m_2+k, m_3, e+J)$				v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e, m_2+e+J, m_3+k, H)$	v	v	v	v	v	v	v	v	v
$(m_1+2e, m_2+e+J, m_3+H, k)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e, m_2+e+H, m_3+k, J)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e, m_2+e+H, m_3+J, k)$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+2e, m_2+k, m_3+J, e+H)$				v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+2e, m_2+k, m_3+H, e+J)$				v^3	v^3	v^3	v^3	v^3	v^3
$(m_1+e+k, m_2+e+J, m_3+H, e)$				v	v	v	v	v	v
$(m_1+e+k, m_2+e+J, m_3+e, H)$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1+e+k, m_2+e+J, m_3, e+H)$					v^2	v^2		v^2	v^2

$(\mathbf{m_1+e+k, m_2+e+H, m_3+J, e})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+k, m_2+e+H, m_3, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e, m_3+J, e+H})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e+k, m_2+e, m_3+H, e+J})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+J, m_2+2e+H, m_3, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+2e, m_3+k, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+2e, m_3+H, k})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+k, m_3+H, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+k, m_3, e+H})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+H, m_3+k, e})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+J, m_2+e+H, m_3, e+k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+k, e+H})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+J, m_2+e, m_3+H, e+k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+k, m_3+H, 2e})$							v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+k, m_3, 2e+H})$							v^3	v^3	
$(\mathbf{m_1+e+H, m_2+2e, m_3+k, J})$		v	v		v	v		v	v
$(\mathbf{m_1+e+H, m_2+2e, m_3+J, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+H, m_2+e+k, m_3+e, J})$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+H, m_2+e+k, m_3+J, e})$	v^3	v^3	v^3	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+H, m_2+e+k, m_3, e+J})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+e+H, m_2+e+J, m_3+e, k})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+H, m_2+e+J, m_3+k, e})$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(\mathbf{m_1+e+H, m_2+e+J, m_3, e+k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+H, m_2+e, m_3+k, e+J})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e+H, m_2+e, m_3+J, e+k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+2e+H, m_3+k, J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e, m_2+2e+H, m_3+J, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+e, m_2+e+k, m_3+e+J, H})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+e+H, J})$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+e+k, m_3+J, e+H})$	v^4	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+k, m_3+H, e+J})$					v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+e+H, k})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+J, m_3+k, e+H})$	v^3+v^5	$2v^3+v^5$	$2v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$
$(\mathbf{m_1+e, m_2+e+J, m_3+H, e+k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+e+H, m_3+k, e+J})$		v^4	v^4	v^4	$2v^3$	$2v^3$	v^4	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+e+H, m_3+J, e+k})$				v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+k, m_2+2e+H, m_3+e, J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e+H, m_3+J, e})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+k, m_2+2e+H, m_3, e+J})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+e+J, H})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+e+H, J})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+k, m_2+2e, m_3+J, e+H})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+H, e+J})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1+k, m_2+e+J, m_3+e+H, e})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+k, m_2+e+J, m_3+e, e+H})$		v^4	v^4	v^4	$2v^3$	$2v^3$	v^4	$2v^3$	$2v^4$
$(\mathbf{m_1+k, m_2+e+J, m_3+H, 2e})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+J, m_3, 2e+H})$									
$(\mathbf{m_1+k, m_2+e+H, m_3+J, 2e})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+J, 2e+H})$				v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+J, m_2+2e+H, m_3+e, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+J, m_2+2e+H, m_3+k, e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+J, m_2+2e, m_3+e+H, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+J, m_2+2e, m_3+k, e+H})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+J, m_2+e+k, m_3+e+H, e})$		v^2+v^4	v^2+v^4		v^2+v^4	v^2+v^4		v^2+v^4	v^2+v^4
$(\mathbf{m_1+J, m_2+e+k, m_3+e, e+H})$		v^3+v^5	v^3+v^5		v^3+v^5	v^3+v^5		v^3+v^5	v^3+v^5
$(\mathbf{m_1+J, m_2+e+H, m_3+k, 2e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+k, 2e+H})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1+H, m_2+2e, m_3+e+J, k})$			v^2			v^2			v^2
$(\mathbf{m_1+H, m_2+2e, m_3+k, e+J})$			v^2			v^2			v^2
$(\mathbf{m_1+H, m_2+e+k, m_3+e+J, e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+e+k, m_3+e, e+J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+e+J, m_3+e, e+k})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+e+J, m_3+k, 2e})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+2e+H, m_3+e+J, k})$			v^3			v^3			v^3
$(\mathbf{m_1, m_2+2e+H, m_3+k, e+J})$			v^3			v^3			v^3
$(\mathbf{m_1, m_2+e+k, m_3+e+J, e+H})$		v^4	v^4		v^4	v^4			v^4
$(\mathbf{m_1, m_2+e+k, m_3+e+H, e+J})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+H, e+k})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+k, 2e+H})$					v^5	v^5		v^5	v^5

Wall $c : a = b = 0$

G

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
b	**1**	**1**	**2**	**2**	$> \mathbf{2}$	$> \mathbf{2}$
$(\mathbf{m_1+k, m_2+k, m_3+k})$	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2-e+k, m_3+e, k})$			v	v	v	v
$(\mathbf{m_1, m_2-e+k, m_3+e+k, k})$					v	v
$(\mathbf{m_1-e+k, m_2+k, m_3+e, k})$		v		v		v
$(\mathbf{m_1-e+k, m_2, m_3+e+k, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1-e+k, m_2-e+k, m_3+e+k, e})$					v^2	v^2

H

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
b	**1**	**1**	**2**	**2**	$> \mathbf{2}$	$> \mathbf{2}$
$(\mathbf{m_1+i+k, m_2+k, m_3+k})$	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2+i+k, m_3+k})$	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+k, m_3+i+k})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+i+k, m_2-e+k, m_3+e, k})$			v	v	v	v
$(\mathbf{m_1+k, m_2-e+i+k, m_3+e, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2-e+k, m_3+e, i+k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2-e+k, m_3+e+i+k, k})$					v^2	v^2
$(\mathbf{m_1, m_2-e+k, m_3+e+k, i+k})$					v^3	v^3
$(\mathbf{m_1-e+i+k, m_2+k, m_3+e, k})$		v		v		v
$(\mathbf{m_1-e+i+k, m_2, m_3+e+k, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1-e+i+k, m_2-e+k, m_3+e+k, e})$					v^2	v^2
$(\mathbf{m_1-e+k, m_2+i+k, m_3+e, k})$		v^2		v^2		v^2
$(\mathbf{m_1-e+k, m_2+k, m_3+e, i+k})$		v^3		v^3		v^3
$(\mathbf{m_1-e+k, m_2, m_3+e+i+k, k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1-e+k, m_2, m_3+e+k, i+k})$			v^4	v^4	v^4	v^4
$(\mathbf{m_1-e+k, m_2-e+i+k, m_3+e+k, e})$					v^3	v^3
$(\mathbf{m_1-e+k, m_2-e+k, m_3+e+i+k, e})$					v^4	v^4
$(\mathbf{m_1, m_2-e+i+k, m_3+e+k, k})$					v	v

I

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
b	**1**	**1**	**2**	**2**	$> \mathbf{2}$	$> \mathbf{2}$
$(\mathbf{m_1+e, m_2+k, m_3+k})$	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2+e, m_3+k})$	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+k, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2-e+k, m_3+e, k})$			v	v	v	v
$(\mathbf{m_1+k, m_2, m_3+e, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+k, m_3+e, k})$		v		v		v
$(\mathbf{m_1, m_2, m_3+e+k, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2-e+k, m_3+e+k, e})$					v^2	v^2
$(\mathbf{m_1-e+k, m_2+e, m_3+e, k})$		v^2		v^2		v^2
$(\mathbf{m_1-e+k, m_2, m_3+e+k, e})$			v^3	v^3	v^3	v^3

J

a	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$	**1**	$> \mathbf{1}$
b	**1**	**1**	**2**	**2**	$> \mathbf{2}$	$> \mathbf{2}$
$(\mathbf{m_1+e+I, m_2+k, m_3+k})$	1	1	1	1	1	1
$(\mathbf{m_1+k, m_2+e+I, m_3+k})$	v	v	v	v	v	v
$(\mathbf{m_1+k, m_2+k, m_3+e+I})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+I, m_2-e+k, m_3+e, k})$			v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3+k, I})$	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2-e+k, m_3+e+I, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+k, m_3+e, I})$	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+I, m_3+e, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2, m_3+e+I, k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+k, m_3+e, k})$		v		v		v
$(\mathbf{m_1+I, m_2, m_3+e+k, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2-e+k, m_3+e+k, e})$					v^2	v^2
$(\mathbf{m_1, m_2+k, m_3+e+I, k})$		v^2		v^2		v^2
$(\mathbf{m_1, m_2+I, m_3+e+k, k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2-e+k, m_3+e+k, e+I})$					v^3	v^3
$(\mathbf{m_1-e+k, m_2+e+I, m_3+e, k})$		v^2		v^2		v^2
$(\mathbf{m_1-e+k, m_2+e, m_3+e+I, k})$		v^3		v^3		v^3
$(\mathbf{m_1-e+k, m_2+I, m_3+e+k, e})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1-e+k, m_2, m_3+e+k, e+I})$			v^4	v^4	v^4	v^4

K

$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+j+k, m_2+j+k, m_3+k)}$	1	1	1	1	1	1
$\mathbf{(m_1+j+k, m_2+k, m_3+j+k)}$	v	v	v	v	v	v
$\mathbf{(m_1+k, m_2+j+k, m_3+j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2-e+j+k, m_3+e, k)}$			v	v	v	v
$\mathbf{(m_1+j+k, m_2-e+k, m_3+e, j+k)}$			v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2-e+j+k, m_3+e, j+k)}$			v^3	v^3	v^3	v^3
$\mathbf{(m_1, m_2-e+j+k, m_3+e+j+k, k)}$					v	v
$\mathbf{(m_1, m_2-e+j+k, m_3+e+k, j+k)}$					v^2	v^2
$\mathbf{(m_1, m_2-e+k, m_3+e+j+k, j+k)}$					v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+j+k, m_3+e, k)}$		v		v		v
$\mathbf{(m_1-e+j+k, m_2+k, m_3+e, j+k)}$		v^2		v^2		v^2
$\mathbf{(m_1-e+j+k, m_2, m_3+e+j+k, k)}$			v^2	v^2	v^2	v^2
$\mathbf{(m_1-e+j+k, m_2, m_3+e+k, j+k)}$			v^3	v^3	v^3	v^3
$\mathbf{(m_1-e+j+k, m_2-e+j+k, m_3+e+k, e)}$					v^2	v^2
$\mathbf{(m_1-e+j+k, m_2-e+k, m_3+e+j+k, e)}$					v^3	v^3
$\mathbf{(m_1-e+k, m_2+j+k, m_3+e, j+k)}$		v^3		v^3		v^3
$\mathbf{(m_1-e+k, m_2, m_3+e+j+k, j+k)}$			v^4	v^4	v^4	v^4
$\mathbf{(m_1-e+k, m_2-e+j+k, m_3+e+j+k, e)}$					v^4	v^4

L

$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e, m_2+e, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e, m_2+k, m_3+e)}$	v	v	v	v
$\mathbf{(m_1+k, m_2+e, m_3+e)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2, m_3+e, k)}$			v	v
$\mathbf{(m_1, m_2+e, m_3+e, k)}$		v		v
$\mathbf{(m_1, m_2, m_3+e+k, e)}$			v^2	v^2

M

$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+I, m_2+e+I, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+e+I)}$	v	v	v	v
$\mathbf{(m_1+k, m_2+e+I, m_3+e+I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+e, m_3+k, I)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+k, m_3+e, I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+I, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+I, m_2, m_3+e+I, k)}$			v^2	v^2
$\mathbf{(m_1+e, m_2+e+I, m_3+k, I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+k, m_3+e+I, I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e, m_2+I, m_3+e+I, k)}$			v^3	v^3
$\mathbf{(m_1+k, m_2+e+I, m_3+e, I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+e+I, I)}$	v^4	v^4	v^4	v^4
$\mathbf{(m_1+I, m_2+e+I, m_3+e, k)}$		v		v
$\mathbf{(m_1+I, m_2+e, m_3+e+I, k)}$		v^2		v^2
$\mathbf{(m_1+I, m_2+I, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1+I, m_2, m_3+e+k, e+I)}$			v^3	v^3
$\mathbf{(m_1, m_2+e+I, m_3+e+I, k)}$		v^3		v^3
$\mathbf{(m_1, m_2+I, m_3+e+k, e+I)}$			v^4	v^4

N

$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+i+j+k, m_2+j+k, m_3+k)}$	1	1	1	1	1	1
$\mathbf{(m_1+i+j+k, m_2+k, m_3+j+k)}$	v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+i+j+k, m_3+k)}$	v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+k, m_3+i+j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+i+j+k, m_3+j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3+i+j+k)}$	v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+i+j+k, m_2-e+j+k, m_3+e, k)}$			v	v	v	v
$\mathbf{(m_1+i+j+k, m_2-e+k, m_3+e, j+k)}$			v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2-e+i+j+k, m_3+e, k)}$			v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2-e+k, m_3+e, i+j+k)}$			v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2-e+i+j+k, m_3+e, j+k)}$			v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2-e+j+k, m_3+e, i+j+k)}$			v^4	v^4	v^4	v^4
$\mathbf{(m_1, m_2-e+i+j+k, m_3+e+j+k, k)}$					v	v
$\mathbf{(m_1, m_2-e+i+j+k, m_3+e+k, j+k)}$					v^2	v^2
$\mathbf{(m_1, m_2-e+j+k, m_3+e+i+j+k, k)}$					v^2	v^2

$(m_1, m_2 - e + j + k, m_3 + e + k, i + j + k)$					v^3	v^3
$(m_1, m_2 - e + k, m_3 + e + i + j + k, j + k)$					v^3	v^3
$(m_1, m_2 - e + k, m_3 + e + j + k, i + j + k)$					v^4	v^4
$(m_1 - e + i + j + k, m_2 + j + k, m_3 + e, k)$		v		v		v
$(m_1 - e + i + j + k, m_2 + k, m_3 + e, j + k)$		v^2		v^2		v^2
$(m_1 - e + i + j + k, m_2, m_3 + e + j + k, k)$			v^2	v^2	v^2	v^2
$(m_1 - e + i + j + k, m_2, m_3 + e + k, j + k)$			v^3	v^3	v^3	v^3
$(m_1 - e + i + j + k, m_2 - e + j + k, m_3 + e + k, e)$					v^2	v^2
$(m_1 - e + i + j + k, m_2 - e + k, m_3 + e + j + k, e)$					v^3	v^3
$(m_1 - e + j + k, m_2 + i + j + k, m_3 + e, k)$		v^2		v^2		v^2
$(m_1 - e + j + k, m_2 + k, m_3 + e, i + j + k)$		v^3		v^3		v^3
$(m_1 - e + j + k, m_2, m_3 + e + i + j + k, k)$			v^3	v^3	v^3	v^3
$(m_1 - e + j + k, m_2, m_3 + e + k, i + j + k)$			v^4	v^4	v^4	v^4
$(m_1 - e + j + k, m_2 - e + i + j + k, m_3 + e + k, e)$					v^3	v^3
$(m_1 - e + j + k, m_2 - e + k, m_3 + e + i + j + k, e)$					v^4	v^4
$(m_1 - e + k, m_2 + i + j + k, m_3 + e, j + k)$		v^3		v^3		v^3
$(m_1 - e + k, m_2 + j + k, m_3 + e, i + j + k)$		v^4		v^4		v^4
$(m_1 - e + k, m_2, m_3 + e + i + j + k, j + k)$			v^4	v^4	v^4	v^4
$(m_1 - e + k, m_2, m_3 + e + j + k, i + j + k)$			v^5	v^5	v^5	v^5
$(m_1 - e + k, m_2 - e + i + j + k, m_3 + e + j + k, e)$					v^4	v^4
$(m_1 - e + k, m_2 - e + j + k, m_3 + e + i + j + k, e)$					v^5	v^5

O

a	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
$(m_1 + e, m_2 + j + k, m_3 + k)$	1	1	1	1	1	1
$(m_1 + e, m_2 + k, m_3 + j + k)$	v	v	v	v	v	v
$(m_1 + j + k, m_2 + e, m_3 + k)$	v	v	v	v	v	v
$(m_1 + j + k, m_2 + k, m_3 + e)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + k, m_2 + e, m_3 + j + k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + k, m_2 + j + k, m_3 + e)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1 + e, m_2 - e + j + k, m_3 + e, k)$			v	v	v	v
$(m_1 + e, m_2 - e + k, m_3 + e, j + k)$			v^2	v^2	v^2	v^2
$(m_1 + j + k, m_2, m_3 + e, k)$			v^2	v^2	v^2	v^2
$(m_1 + k, m_2, m_3 + e, j + k)$			v^3	v^3	v^3	v^3
$(m_1, m_2 + j + k, m_3 + e, k)$		v		v		v
$(m_1, m_2 + k, m_3 + e, j + k)$		v^2		v^2		v^2
$(m_1, m_2, m_3 + e + j + k, k)$			v^2	v^2	v^2	v^2
$(m_1, m_2, m_3 + e + k, j + k)$			v^3	v^3	v^3	v^3
$(m_1, m_2 - e + j + k, m_3 + e + k, e)$					v^2	v^2
$(m_1, m_2 - e + k, m_3 + e + j + k, e)$					v^3	v^3
$(m_1 - e + j + k, m_2 + e, m_3 + e, k)$		v^2		v^2		v^2
$(m_1 - e + j + k, m_2, m_3 + e + k, e)$			v^3	v^3	v^3	v^3
$(m_1 - e + k, m_2 + e, m_3 + e, j + k)$		v^3		v^3		v^3
$(m_1 - e + k, m_2, m_3 + e + j + k, e)$			v^4	v^4	v^4	v^4

P

a	**1**	**> 1**	**1**	**> 1**	**1**	**> 1**
b	**1**	**1**	**2**	**2**	**> 2**	**> 2**
$(m_1 + e + I, m_2 + j + k, m_3 + k)$	1	1	1	1	1	1
$(m_1 + e + I, m_2 + k, m_3 + j + k)$	v	v	v	v	v	v
$(m_1 + j + k, m_2 + e + I, m_3 + k)$	v	v	v	v	v	v
$(m_1 + j + k, m_2 + k, m_3 + e + I)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + k, m_2 + e + I, m_3 + j + k)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + k, m_2 + j + k, m_3 + e + I)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1 + e + I, m_2 - e + j + k, m_3 + e, k)$			v	v	v	v
$(m_1 + e + I, m_2 - e + k, m_3 + e, j + k)$			v^2	v^2	v^2	v^2
$(m_1 + e, m_2 + j + k, m_3 + k, I)$	v	v	v	v	v	v
$(m_1 + e, m_2 + k, m_3 + j + k, I)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + e, m_2 - e + j + k, m_3 + e + I, k)$			v^2	v^2	v^2	v^2
$(m_1 + e, m_2 - e + k, m_3 + e + I, j + k)$			v^3	v^3	v^3	v^3
$(m_1 + j + k, m_2 + e, m_3 + k, I)$	v^2	v^2	v^2	v^2	v^2	v^2
$(m_1 + j + k, m_2 + k, m_3 + e, I)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1 + j + k, m_2 + I, m_3 + e, k)$			v^2	v^2	v^2	v^2
$(m_1 + j + k, m_2, m_3 + e + I, k)$			v^2	v^2	v^2	v^2
$(m_1 + k, m_2 + e, m_3 + j + k, I)$	v^3	v^3	v^3	v^3	v^3	v^3
$(m_1 + k, m_2 + j + k, m_3 + e, I)$	v^4	v^4	v^4	v^4	v^4	v^4
$(m_1 + k, m_2 + I, m_3 + e, j + k)$			v^3	v^3	v^3	v^3
$(m_1 + k, m_2, m_3 + e + I, j + k)$			v^4	v^4	v^4	v^4
$(m_1 + I, m_2 + j + k, m_3 + e, k)$		v		v		v
$(m_1 + I, m_2 + k, m_3 + e, j + k)$		v^2		v^2		v^2
$(m_1 + I, m_2, m_3 + e + j + k, k)$			v^2	v^2	v^2	v^2

$\mathbf{(m_1+I, m_2, m_3+e+k, j+k)}$			v^3	v^3	v^3	v^3			
$\mathbf{(m_1+I, m_2-e+j+k, m_3+e+k, e)}$					v^2	v^2			
$\mathbf{(m_1+I, m_2-e+k, m_3+e+j+k, e)}$					v^3	v^3			
$\mathbf{(m_1, m_2+j+k, m_3+e+I, k)}$		v^2		v^2		v^2			
$\mathbf{(m_1, m_2+k, m_3+e+I, j+k)}$		v^3		v^3		v^3			
$\mathbf{(m_1, m_2+I, m_3+e+j+k, k)}$			v^3	v^3	v^3	v^3			
$\mathbf{(m_1, m_2+I, m_3+e+k, j+k)}$			v^4	v^4	v^4	v^4			
$\mathbf{(m_1, m_2-e+j+k, m_3+e+k, e+I)}$					v^3	v^3			
$\mathbf{(m_1, m_2-e+k, m_3+e+j+k, e+I)}$					v^4	v^4			
$\mathbf{(m_1-e+j+k, m_2+e+I, m_3+e, k)}$		v^2		v^2		v^2			
$\mathbf{(m_1-e+j+k, m_2+e, m_3+e+I, k)}$		v^3		v^3		v^3			
$\mathbf{(m_1-e+j+k, m_2+I, m_3+e+k, e)}$			v^3	v^3	v^3	v^3			
$\mathbf{(m_1-e+j+k, m_2, m_3+e+k, e+I)}$			v^4	v^4	v^4	v^4			
$\mathbf{(m_1-e+k, m_2+e+I, m_3+e, j+k)}$		v^3		v^3		v^3			
$\mathbf{(m_1-e+k, m_2+e, m_3+e+I, j+k)}$		v^4		v^4		v^4			
$\mathbf{(m_1-e+k, m_2+I, m_3+e+j+k, e)}$			v^4	v^4	v^4	v^4			
$\mathbf{(m_1-e+k, m_2, m_3+e+j+k, e+I)}$			v^5	v^5	v^5	v^5			

Q

a	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$
b	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+e+k, m_2+j+k, m_3+k)}$	1	1	1	1	1	1	1	1	1
$\mathbf{(m_1+e+k, m_2+k, m_3+j+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+e+k, m_3+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+k, m_3+e+k)}$				v	v	v	v	v	v
$\mathbf{(m_1+k, m_2+e+I, m_3+j+k)}$		v	v		v	v		v	v
$\mathbf{(m_1+k, m_2+j+k, m_3+e+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2-e+j+k, m_3+e, k)}$				v	v	v	v	v	v
$\mathbf{(m_1+e+k, m_2-e+k, m_3+e, j+k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+k, m_3+j+k, k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e, m_2-e+j+k, m_3+e+k, k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2-e+k, m_3+e+k, j+k)}$							v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+k, m_3+e, k)}$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$\mathbf{(m_1+j+k, m_2, m_3+e+k, k)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+j+k, k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3+e, k)}$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$\mathbf{(m_1+k, m_2+k, m_3+e, j+k)}$		v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+k, m_2, m_3+e+j+k, k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2, m_3+e+k, j+k)}$				$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$\mathbf{(m_1+k, m_2-e+j+k, m_3+e+k, e)}$							v^2	v^2	v^2
$\mathbf{(m_1+k, m_2-e+k, m_3+e+j+k, e)}$							v^3	v^3	v^3
$\mathbf{(m_1, m_2+j+k, m_3+e+k, k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1, m_2+k, m_3+e+j+k, k)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1, m_2+k, m_3+e+k, j+k)}$				v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$\mathbf{(m_1, m_2-e+k, m_3+e+j+k, e+k)}$							v^3	v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+e+k, m_3+e, k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1-e+j+k, m_2+e, m_3+e+k, k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+j+k, m_2+k, m_3+e+k, e)}$				v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1-e+k, m_2+e+k, m_3+e, j+k)}$			v^2			v^2			v^2
$\mathbf{(m_1-e+k, m_2+e, m_3+e+k, j+k)}$		v^3	v^3		v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+k, m_2+k, m_3+e+j+k, e)}$					v^3	v^3		v^3	v^3
$\mathbf{(m_1-e+k, m_2, m_3+e+j+k, e+k)}$				v^4	v^4	v^4	v^4	v^4	v^4

R

a	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$
b	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$\mathbf{(m_1+e+I, m_2+j+k, m_3+k)}$	1	1	1	1	1	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+j+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e+k, m_2+j+k, m_3+I)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+e+k, m_2+I, m_3+j+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+e+I, m_3+k)}$	v	v	v	v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+e+k, m_3+I)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+j+k, m_2+I, m_3+e+k)}$				v	v	v	v	v	v
$\mathbf{(m_1+j+k, m_2+k, m_3+e+I)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+I, m_2+e+k, m_3+j+k)}$		v	v		v	v		v	v
$\mathbf{(m_1+I, m_2+j+k, m_3+e+k)}$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+I, m_3+j+k)}$		v^2	v^2		v^2	v^2		v^2	v^2
$\mathbf{(m_1+k, m_2+j+k, m_3+e+I)}$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+I, m_2-e+j+k, m_3+e, k)}$				v	v	v	v	v	v
$\mathbf{(m_1+e+I, m_2-e+k, m_3+e, j+k)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2-e+j+k, m_3+e, I)}$				v^2	v^2	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2-e+I, m_3+e, j+k)}$				v^3	v^3	v^3	v^3	v^3	v^3

$(\mathbf{m_1+e, m_2+I, m_3+j+k, k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3+j+k, I})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2-e+j+k, m_3+e+I, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2-e+j+k, m_3+e+k, I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2-e+I, m_3+e+k, j+k})$							v^2	v^2	v^2
$(\mathbf{m_1+e, m_2-e+k, m_3+e+I, j+k})$							v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2+I, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+j+k, m_2+k, m_3+e, I})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+j+k, m_2, m_3+e+I, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+j+k, m_2, m_3+e+k, I})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+I, m_2+e, m_3+j+k, k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2+j+k, m_3+e, k})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+I, m_2+k, m_3+e, j+k})$		v^2	v^2	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+I, m_2, m_3+e+j+k, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+I, m_2, m_3+e+k, j+k})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+I, m_2-e+j+k, m_3+e+k, e})$							v^2	v^2	v^2
$(\mathbf{m_1+I, m_2-e+k, m_3+e+j+k, e})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+j+k, I})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+j+k, m_3+e, I})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+I, m_3+e, j+k})$	v^3	v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+k, m_2, m_3+e+j+k, I})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+I, j+k})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2-e+j+k, m_3+e+I, e})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2-e+I, m_3+e+j+k, e})$							v^4	v^4	v^4
$(\mathbf{m_1, m_2+j+k, m_3+e+I, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1, m_2+j+k, m_3+e+k, I})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+I, m_3+e+k, j+k})$				v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1, m_2+I, m_3+e+j+k, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2+k, m_3+e+I, j+k})$				v^5	v^2+v^4	v^2+v^4	v^5	v^2+v^4	v^2+v^4
$(\mathbf{m_1, m_2+k, m_3+e+j+k, I})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2-e+j+k, m_3+e+I, e+k})$							v^2	v^2	v^2
$(\mathbf{m_1, m_2-e+j+k, m_3+e+k, e+I})$							v^3	v^3	v^3
$(\mathbf{m_1, m_2-e+k, m_3+e+j+k, e+k})$							v^3	v^3	v^3
$(\mathbf{m_1, m_2-e+I, m_3+e+j+k, e+k})$							v^4	v^4	v^4
$(\mathbf{m_1-e+j+k, m_2+e+I, m_3+e, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1-e+j+k, m_2+e+k, m_3+e, I})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+j+k, m_2+e, m_3+e+I, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+j+k, m_2+e, m_3+e+k, I})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1-e+j+k, m_2+I, m_3+e+k, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1-e+j+k, m_2+k, m_3+e+I, e})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1-e+I, m_2+e+k, m_3+e, j+k})$			v^2			v^2			v^2
$(\mathbf{m_1-e+I, m_2+e, m_3+e+k, j+k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+I, m_2+k, m_3+e+j+k, e})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1-e+I, m_2, m_3+e+j+k, e+k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1-e+k, m_2+e+I, m_3+e, j+k})$			v^3			v^3			v^3
$(\mathbf{m_1-e+k, m_2+e, m_3+e+I, j+k})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1-e+k, m_2+I, m_3+e+j+k, e})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1-e+k, m_2, m_3+e+j+k, e+I})$				v^5	v^5	v^5	v^5	v^5	v^5

S						
a	**1**	$>$ **1**	**1**	$>$ **1**	**1**	$>$ **1**
b	**1**	**1**	**2**	**2**	$>$ **2**	$>$ **2**
$(\mathbf{m_1+e+i, m_2+e, m_3+k})$	1	1	1	1	1	1
$(\mathbf{m_1+e+i, m_2+k, m_3+e})$	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+e+i, m_3+k})$	v	v	v	v	v	v
$(\mathbf{m_1+e, m_2+k, m_3+e+i})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e+i, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+e, m_3+e+i})$	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+i, m_2, m_3+e, k})$			v	v	v	v
$(\mathbf{m_1+e, m_2+e, m_3+k, i})$	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+e, i})$	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+i, m_3+e, k})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2, m_3+e+i, k})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+e, i})$	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+i, m_2+e, m_3+e, k})$		v		v		v
$(\mathbf{m_1+i, m_2, m_3+e+k, e})$			v^2	v^2	v^2	v^2
$(\mathbf{m_1, m_2+e+i, m_3+e, k})$		v^2		v^2		v^2
$(\mathbf{m_1, m_2+e, m_3+e+i, k})$		v^3		v^3		v^3
$(\mathbf{m_1, m_2+i, m_3+e+k, e})$			v^3	v^3	v^3	v^3
$(\mathbf{m_1, m_2, m_3+e+k, e+i})$			v^4	v^4	v^4	v^4

$\mathbf{T}$				
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+k, m_2+e, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+k, m_2+k, m_3+e)}$	v	v	v	v
$\mathbf{(m_1+e, m_2+e+k, m_3+k)}$	v	v	v	v
$\mathbf{(m_1+e, m_2+k, m_3+e+k)}$			v	v
$\mathbf{(m_1+k, m_2+e+k, m_3+e)}$		v		v
$\mathbf{(m_1+k, m_2+e, m_3+e+k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2, m_3+e, k)}$			v	v
$\mathbf{(m_1+e, m_2+k, m_3+e, k)}$	v^2	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e, m_2, m_3+e+k, k)}$			v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+e, k)}$	v^3	$v+v^3$	v^3	$v+v^3$
$\mathbf{(m_1+k, m_2, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1, m_2+e+k, m_3+e, k)}$		v^2		v^2
$\mathbf{(m_1, m_2+e, m_3+e+k, k)}$		v^3		v^3
$\mathbf{(m_1, m_2+k, m_3+e+k, e)}$			v^3	v^3

$\mathbf{U}$				
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+I, m_2+e, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+e)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+e, m_3+I)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+I, m_3+e)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+e+I, m_3+k)}$	v	v	v	v
$\mathbf{(m_1+e, m_2+e+k, m_3+I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+I, m_3+e+k)}$			v	v
$\mathbf{(m_1+e, m_2+k, m_3+e+I)}$			v^2	v^2
$\mathbf{(m_1+I, m_2+e+k, m_3+e)}$		v		v
$\mathbf{(m_1+I, m_2+e, m_3+e+k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+I, m_3+e)}$		v^2		v^2
$\mathbf{(m_1+k, m_2+e, m_3+e+I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+I, m_2, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+k, m_2, m_3+e, I)}$			v^2	v^2
$\mathbf{(m_1+e, m_2+I, m_3+e, k)}$	v^2	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e, m_2+k, m_3+e, I)}$	v^3	v^3	$2v^3$	$2v^3$
$\mathbf{(m_1+e, m_2, m_3+e+I, k)}$			v^3	v^3
$\mathbf{(m_1+e, m_2, m_3+e+k, I)}$			v^4	v^4
$\mathbf{(m_1+I, m_2+e, m_3+e, k)}$	v^3	$v+v^3$	v^3	$v+v^3$
$\mathbf{(m_1+I, m_2, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1+k, m_2+e, m_3+e, I)}$	v^4	v^2+v^4	v^4	v^2+v^4
$\mathbf{(m_1+k, m_2, m_3+e+I, e)}$			v^3	v^3
$\mathbf{(m_1, m_2+e+I, m_3+e, k)}$		v^2		v^2
$\mathbf{(m_1, m_2+e+k, m_3+e, I)}$		v^3		v^3
$\mathbf{(m_1, m_2+e, m_3+e+I, k)}$		v^3		v^3
$\mathbf{(m_1, m_2+e, m_3+e+k, I)}$		v^4		v^4
$\mathbf{(m_1, m_2+I, m_3+e+k, e)}$			v^3	v^3
$\mathbf{(m_1, m_2+k, m_3+e+I, e)}$			v^4	v^4

$\mathbf{V}$				
$\mathbf{a}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{1}$	$\mathbf{>1}$
$\mathbf{b}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{>1}$	$\mathbf{>1}$
$\mathbf{(m_1+e+I, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+e+J)}$	v	v	v	v
$\mathbf{(m_1+e+J, m_2+e+I, m_3+k)}$	v	v	v	v
$\mathbf{(m_1+e+J, m_2+k, m_3+e+I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+I, m_3+e+J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+J, m_3+e+I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+I, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+k, m_3+e, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+J, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+I, m_2, m_3+e+J, k)}$			v^2	v^2
$\mathbf{(m_1+e+J, m_2+e, m_3+k, I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+I, m_3+e, k)}$			v^2	v^2
$\mathbf{(m_1+e+J, m_2+k, m_3+e, I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+J, m_2, m_3+e+I, k)}$			v^3	v^3
$\mathbf{(m_1+e, m_2+e+I, m_3+k, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+e+J, m_3+k, I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e, m_2+k, m_3+e+I, J)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e, m_2+k, m_3+e+J, I)}$	v^4	v^4	v^4	v^4

$\mathbf{(m_1+e, m_2+I, m_3+e+J, k)}$			v^3	v^3
$\mathbf{(m_1+e, m_2+J, m_3+e+I, k)}$			v^4	v^4
$\mathbf{(m_1+k, m_2+e+I, m_3+e, J)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+e+J, m_3+e, I)}$	v^4	v^4	v^4	v^4
$\mathbf{(m_1+k, m_2+e, m_3+e+I, J)}$	v^4	v^4	v^4	v^4
$\mathbf{(m_1+k, m_2+e, m_3+e+J, I)}$	v^5	v^5	v^5	v^5
$\mathbf{(m_1+I, m_2+e+J, m_3+e, k)}$		v		v
$\mathbf{(m_1+I, m_2+e, m_3+e+J, k)}$		v^2		v^2
$\mathbf{(m_1+I, m_2+J, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1+I, m_2, m_3+e+k, e+J)}$			v^3	v^3
$\mathbf{(m_1+J, m_2+e+I, m_3+e, k)}$		v^2		v^2
$\mathbf{(m_1+J, m_2+e, m_3+e+I, k)}$		v^3		v^3
$\mathbf{(m_1+J, m_2+I, m_3+e+k, e)}$			v^3	v^3
$\mathbf{(m_1+J, m_2, m_3+e+k, e+I)}$			v^4	v^4
$\mathbf{(m_1, m_2+e+I, m_3+e+J, k)}$		v^3		v^3
$\mathbf{(m_1, m_2+e+J, m_3+e+I, k)}$		v^4		v^4
$\mathbf{(m_1, m_2+I, m_3+e+k, e+J)}$			v^4	v^4
$\mathbf{(m_1, m_2+J, m_3+e+k, e+I)}$			v^5	v^5

W

a	**1**	$>$ **1**	**1**	$>$ **1**
b	**1**	**1**	$>$ **1**	$>$ **1**
$\mathbf{(m_1+e+k, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+k, m_2+k, m_3+e+J)}$	v	v	v	v
$\mathbf{(m_1+e+J, m_2+e+k, m_3+k)}$	v	v	v	v
$\mathbf{(m_1+e+J, m_2+k, m_3+e+k)}$			v	v
$\mathbf{(m_1+k, m_2+e+k, m_3+e+J)}$		v		v
$\mathbf{(m_1+k, m_2+e+J, m_3+e+k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+k, m_3+e, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+J, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+k, m_2, m_3+e+J, k)}$			v^2	v^2
$\mathbf{(m_1+e+J, m_2+k, m_3+e, k)}$	v^2	v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e+J, m_2, m_3+e+k, k)}$			v^3	v^3
$\mathbf{(m_1+e, m_2+e+k, m_3+k, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e, m_2+k, m_3+e+k, J)}$			v^2	v^2
$\mathbf{(m_1+e, m_2+k, m_3+e+J, k)}$	v^3	v^3	$2v^3$	$2v^3$
$\mathbf{(m_1+e, m_2+J, m_3+e+k, k)}$			v^4	v^4
$\mathbf{(m_1+k, m_2+e+k, m_3+e, J)}$		v^2		v^2
$\mathbf{(m_1+k, m_2+e+J, m_3+e, k)}$	v^3	$v+v^3$	v^3	$v+v^3$
$\mathbf{(m_1+k, m_2+e, m_3+e+k, J)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+k, m_2+e, m_3+e+J, k)}$	v^4	v^2+v^4	v^4	v^2+v^4
$\mathbf{(m_1+k, m_2+J, m_3+e+k, e)}$			v^2	v^2
$\mathbf{(m_1+k, m_2, m_3+e+k, e+J)}$			v^3	v^3
$\mathbf{(m_1+J, m_2+e+k, m_3+e, k)}$		v^2		v^2
$\mathbf{(m_1+J, m_2+e, m_3+e+k, k)}$		v^3		v^3
$\mathbf{(m_1+J, m_2+k, m_3+e+k, e)}$			v^3	v^3
$\mathbf{(m_1, m_2+e+k, m_3+e+J, k)}$		v^3		v^3
$\mathbf{(m_1, m_2+e+J, m_3+e+k, k)}$		v^4		v^4
$\mathbf{(m_1, m_2+k, m_3+e+k, e+J)}$			v^4	v^4

X

a	**1**	$>$ **1**	**1**	$>$ **1**
b	**1**	**1**	$>$ **1**	$>$ **1**
$\mathbf{(m_1+e+I, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+I, m_2+k, m_3+e+J)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+e+J, m_3+I)}$	v	v	v	v
$\mathbf{(m_1+e+k, m_2+I, m_3+e+J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+e+I, m_3+k)}$	v	v	v	v
$\mathbf{(m_1+e+J, m_2+e+k, m_3+I)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+J, m_2+I, m_3+e+k)}$			v	v
$\mathbf{(m_1+e+J, m_2+k, m_3+e+I)}$			v^2	v^2
$\mathbf{(m_1+I, m_2+e+k, m_3+e+J)}$		v		v
$\mathbf{(m_1+I, m_2+e+J, m_3+e+k)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+k, m_2+e+I, m_3+e+J)}$		v^2		v^2
$\mathbf{(m_1+k, m_2+e+J, m_3+e+I)}$	v^3	v^3	v^3	v^3
$\mathbf{(m_1+e+I, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+e+I, m_2+k, m_3+e, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+I, m_2+J, m_3+e, k)}$			v	v
$\mathbf{(m_1+e+I, m_2, m_3+e+J, k)}$			v^2	v^2

$(\mathbf{m_1+e+k, m_2+e, m_3+I, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3+e, J})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+J, m_3+e, I})$			v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+e+J, I})$			v^3	v^3
$(\mathbf{m_1+e+J, m_2+I, m_3+e, k})$	v^2	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+k, m_3+e, I})$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e+J, m_2, m_3+e+I, k})$			v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+k, I})$			v^4	v^4
$(\mathbf{m_1+e, m_2+e+I, m_3+k, J})$	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+I, J})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3+e+k, J})$			v^2	v^2
$(\mathbf{m_1+e, m_2+I, m_3+e+J, k})$	v^3	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+k, m_3+e+I, J})$			v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, I})$	v^4	v^4	$2v^4$	$2v^4$
$(\mathbf{m_1+e, m_2+J, m_3+e+I, k})$			v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+k, I})$			v^5	v^5
$(\mathbf{m_1+I, m_2+e+k, m_3+e, J})$		v^2		v^2
$(\mathbf{m_1+I, m_2+e+J, m_3+e, k})$	v^3	$v+v^3$	v^3	$v+v^3$
$(\mathbf{m_1+I, m_2+e, m_3+e+k, J})$	v^3	v^3	v^3	v^3
$(\mathbf{m_1+I, m_2+e, m_3+e+J, k})$	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+I, m_2+J, m_3+e+k, e})$			v^2	v^2
$(\mathbf{m_1+I, m_2, m_3+e+k, e+J})$			v^3	v^3
$(\mathbf{m_1+k, m_2+e+I, m_3+e, J})$		v^3		v^3
$(\mathbf{m_1+k, m_2+e+J, m_3+e, I})$	v^4	v^2+v^4	v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+e, m_3+e+I, J})$	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+e+J, I})$	v^5	v^2+v^4	v^5	v^2+v^4
$(\mathbf{m_1+k, m_2+J, m_3+e+I, e})$			v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+e+I, e+J})$			v^4	v^4
$(\mathbf{m_1+J, m_2+e+I, m_3+e, k})$		v^2		v^2
$(\mathbf{m_1+J, m_2+e+k, m_3+e, I})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+I, k})$		v^3		v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+k, I})$		v^4		v^4
$(\mathbf{m_1+J, m_2+I, m_3+e+k, e})$			v^3	v^3
$(\mathbf{m_1+J, m_2+k, m_3+e+I, e})$			v^4	v^4
$(\mathbf{m_1, m_2+e+I, m_3+e+J, k})$		v^3		v^3
$(\mathbf{m_1, m_2+e+k, m_3+e+J, I})$		v^4		v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+I, k})$		v^4		v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+k, I})$		v^5		v^5
$(\mathbf{m_1, m_2+I, m_3+e+k, e+J})$			v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+e+I, e+J})$			v^5	v^5

Y									
a	**1**	**2**	**> 2**	**1**	**2**	**> 2**	**1**	**2**	**> 2**
b	**1**	**1**	**1**	**2**	**2**	**2**	**> 2**	**> 2**	**> 2**
$(\mathbf{m_1+2e, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+2e, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+2e, m_3+k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e+k})$				v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+2e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+e+J})$		v	v		v	v		v	v
$(\mathbf{m_1+e, m_2+e+J, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+2e, m_3+e+J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+2e})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+J, m_3+e, k})$				v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2, m_3+e+J, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e, m_3+e, J})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2, m_3+e+J, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2, m_3+2e, k})$							v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2, m_3+e+k, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+2e, m_3+k, J})$		v	v		v	v		v	v
$(\mathbf{m_1+e, m_2+e+k, m_3+e, J})$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e, m_2+e+J, m_3+e, k})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+e, m_3+e+k, J})$	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e, m_2+e, m_3+e+J, k})$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(\mathbf{m_1+e, m_2+k, m_3+2e, J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3+e+J, e})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$

$(\mathbf{m_1+e, m_2+J, m_3+2e, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+J, m_3+e+k, e})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2, m_3+e+k, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+e, J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+2e, J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e, m_3+e+J, e})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+J, m_3+2e, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+2e, e+J})$							v^3	v^3	v^3
$(\mathbf{m_1+J, m_2+2e, m_3+e, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+J, m_2+e, m_3+2e, k})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+J, m_2+e, m_3+e+k, e})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+J, m_2+k, m_3+2e, e})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+2e, m_3+e+J, k})$			v^2			v^2			v^2
$(\mathbf{m_1, m_2+e+k, m_3+e+J, e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+e+J, m_3+2e, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+e+J, m_3+e+k, e})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+e, m_3+e+k, e+J})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1, m_2+k, m_3+2e, e+J})$				v^4	v^4	v^4	v^4	v^4	v^4

Z									
a	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$	**1**	**2**	$\mathbf{>2}$
b	**1**	**1**	**1**	**2**	**2**	**2**	$\mathbf{>2}$	$\mathbf{>2}$	$\mathbf{>2}$
$(\mathbf{m_1+2e+H, m_2+e+J, m_3+k})$	1	1	1	1	1	1	1	1	1
$(\mathbf{m_1+2e+H, m_2+k, m_3+e+J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e+H})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e+J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+2e+H, m_3+k})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e+H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e+k})$				v	v	v	v	v	v
$(\mathbf{m_1+e+J, m_2+k, m_3+2e+H})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+H, m_2+e+k, m_3+e+J})$		v	v		v	v		v	v
$(\mathbf{m_1+e+H, m_2+e+J, m_3+e+k})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+k, m_2+2e+H, m_3+e+J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+k, m_2+e+J, m_3+2e+H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e+H, m_2+e, m_3+k, J})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e+H, m_2+k, m_3+e, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e+H, m_2+J, m_3+e, k})$				v	v	v	v	v	v
$(\mathbf{m_1+2e+H, m_2, m_3+e+J, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+e+J, m_3+k, H})$	v	v	v	v	v	v	v	v	v
$(\mathbf{m_1+2e, m_2+e+H, m_3+k, J})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+e+J, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+k, m_3+e+H, J})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+2e, m_2+J, m_3+e+H, k})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+2e, m_2+H, m_3+e+J, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e+J, m_3+e, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2+e+H, m_3+e, J})$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+k, m_2+e, m_3+e+J, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+k, m_2+e, m_3+e+H, J})$	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+k, m_2+H, m_3+e+J, e})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+k, m_2, m_3+e+J, e+H})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+2e, m_3+k, H})$	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e+k, m_3+e, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+e+H, m_3+e, k})$	v^2	v^2	v^2	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$	$2v^2$
$(\mathbf{m_1+e+J, m_2+e, m_3+e+k, H})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+e, m_3+e+H, k})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e+J, m_2+k, m_3+2e, H})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2+H, m_3+2e, k})$							v^2	v^2	v^2
$(\mathbf{m_1+e+J, m_2+H, m_3+e+k, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+2e+H, k})$							v^3	v^3	v^3
$(\mathbf{m_1+e+J, m_2, m_3+e+k, e+H})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e+H, m_2+2e, m_3+k, J})$		v	v		v	v		v	v
$(\mathbf{m_1+e+H, m_2+e+k, m_3+e, J})$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1+e+H, m_2+e+J, m_3+e, k})$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$	v^3	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+H, m_2+e, m_3+e+k, J})$	v^3	v^3	v^3	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$	$v+v^3$
$(\mathbf{m_1+e+H, m_2+e, m_3+e+J, k})$	v^2+v^4	$2v^2+v^4$	$2v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$	$2v^2+v^4$	$3v^2+v^4$	$3v^2+v^4$
$(\mathbf{m_1+e+H, m_2+k, m_3+2e, J})$				v^2	v^2	v^2	v^2	v^2	v^2
$(\mathbf{m_1+e+H, m_2+k, m_3+e+J, e})$	v^3	v^3	v^3	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$	$2v^3$
$(\mathbf{m_1+e+H, m_2+J, m_3+2e, k})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e+H, m_2+J, m_3+e+k, e})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e+H, m_2, m_3+e+k, e+J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+2e+H, m_3+k, J})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+e, m_2+e+k, m_3+e+J, H})$		v^2	v^2		v^2	v^2		v^2	v^2

$(\mathbf{m_1+e, m_2+e+k, m_3+e+H, J})$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+e, m_2+e+J, m_3+e+k, H})$	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+e+J, m_3+e+H, k})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+H, m_3+e+k, J})$	v^4	v^4	v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+e+H, m_3+e+J, k})$	v^2+v^4	$2v^3+v^5$	$2v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$	$2v^3+v^5$	$3v^3+v^5$	$3v^3+v^5$
$(\mathbf{m_1+e, m_2+k, m_3+2e+H, J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+e, m_2+k, m_3+e+J, e+H})$	v^4	v^4	v^4	$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$	$2v^4$
$(\mathbf{m_1+e, m_2+J, m_3+2e+H, k})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+e, m_2+J, m_3+e+k, e+H})$				v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+e, m_2+H, m_3+e+k, e+J})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+2e+H, m_3+e, J})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+e+J, H})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+k, m_2+2e, m_3+e+H, J})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+k, m_2+e+J, m_3+2e, H})$	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e+H, m_3+2e, J})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+e+H, m_3+e+J, e})$	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4	v^4	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+e, m_3+2e+H, J})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+e, m_3+e+J, e+H})$	v^5	v^2+v^4	v^2+v^4	v^5	v^2+v^4	v^2+v^4	v^5	v^2+v^4	v^2+v^4
$(\mathbf{m_1+k, m_2+J, m_3+2e+H, e})$				v^3	v^3	v^3	v^3	v^3	v^3
$(\mathbf{m_1+k, m_2+J, m_3+2e, e+H})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+k, m_2+H, m_3+2e, e+J})$							v^3	v^3	v^3
$(\mathbf{m_1+k, m_2, m_3+2e+H, e+J})$							v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+2e+H, m_3+e, k})$		v^2	v^2		v^2	v^2		v^2	v^2
$(\mathbf{m_1+J, m_2+2e, m_3+e+H, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+J, m_2+e+H, m_3+2e, k})$					v^2	v^2		v^2	v^2
$(\mathbf{m_1+J, m_2+e+H, m_3+e+k, e})$		v^3	v^3	v^3	$2v^3$	$2v^3$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1+J, m_2+e, m_3+2e+H, k})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+J, m_2+e, m_3+e+k, e+H})$		v^4	v^4	v^4	$2v^4$	$2v^4$	v^4	$2v^4$	$2v^4$
$(\mathbf{m_1+J, m_2+k, m_3+2e+H, e})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1+J, m_2+k, m_3+2e, e+H})$				v^5	v^5	v^5	v^5	v^5	v^5
$(\mathbf{m_1+H, m_2+2e, m_3+e+J, k})$			v^2			v^2			v^2
$(\mathbf{m_1+H, m_2+e+k, m_3+e+J, e})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+e+J, m_3+2e, k})$		v^3	v^3		v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+e+J, m_3+e+k, e})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1+H, m_2+e, m_3+e+k, e+J})$					v^3	v^3		v^3	v^3
$(\mathbf{m_1+H, m_2+k, m_3+2e, e+J})$				v^4	v^4	v^4	v^4	v^4	v^4
$(\mathbf{m_1, m_2+2e+H, m_3+e+J, k})$			v^3			v^3			v^3
$(\mathbf{m_1, m_2+e+k, m_3+e+J, e+H})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+2e+H, k})$		v^4	v^4		v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+e+J, m_3+e+k, e+H})$		v^5	v^5		v^5	v^5		v^5	v^5
$(\mathbf{m_1, m_2+e+H, m_3+e+k, e+J})$					v^4	v^4		v^4	v^4
$(\mathbf{m_1, m_2+k, m_3+2e+H, e+J})$				v^5	v^5	v^5	v^5	v^5	v^5

Spoke $a : b = c = 0$

K

$\mathbf{a = 4x + y + 2}$ where $\mathbf{0 \le y < 4}$	$\mathbf{a=1}$	$\mathbf{y=0}$	$\mathbf{y=1}$	$\mathbf{y=2}$	$\mathbf{y=3}$	
$(m_1+j+k, m_2+j+k, m_3+k)$	1	1	1	1	1	
$(m_1-e+j+k, m_2+e+k, m_3+j+k)$		v	v	v	v	
$(m_1-(3z-1)e, m_2+(z+1)e+j+k, m_3+ze+j+k, ze+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-2)e+j+k, m_2+(z+1)e+k, m_3+(z+1)e, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-2)e+j+k, m_2+(z+1e+j+k, m_3+(z+1)e, ze+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(2z-2)e+k, m_2+(z+1)e+j+k, m_3+(z+1)e, ze+j+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-2)e, m_2+(z+1)e+j+k, m_3+(z+1)e+k, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-3)e+j+k, m_2+(z+2)e, m_3+(z+1)e+k, ze+j+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-3)e+k, m_2+(z+2)e, m_3+(z+1)e+j+k, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-3)e+j+k, m_2+(z+1)e+j+k, m_3+(z+1)e+k, (z+1)e)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-4)e+j+k, m_2+(z+2)e+k, m_3+(z+1)e+j+k, (z+1)e)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-4)e+j+k, m_2+(z+2)e, m_3+(z+1)e+j+k, (z+1)e+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3x-1)e, m_2+(x+1)e+j+k, m_3+xe+j+k, xe+k)$			v	v	v	
$(m_1-(3x-2)e+j+k, m_2+(x+1)e+k, m_3+(x+1)e, xe+j+k)$			v^2	v^2	v^2	
$(m_1-(3x-2)e+j+k, m_2+(x+1e+j+k, m_3+(x+1)e, xe+k)$				v	v	
$(m_1-(2x-2)e+k, m_2+(x+1)e+j+k, m_3+(x+1)e, xe+j+k)$				v	v	
$(m_1-(3x-2)e, m_2+(x+1)e+j+k, m_3+(x+1)e+k, xe+j+k)$				v^2	v^2	
$(m_1-(3x-3)e+j+k, m_2+(x+2)e, m_3+(x+1)e+k, xe+j+k)$					v	
$(m_1-(3x-3)e+k, m_2+(x+2)e, m_3+(x+1)e+j+k, xe+j+k)$					v^2	
$(m_1-(3x-3)e+j+k, m_2+(x+1)e+j+k, m_3+(x+1)e+k, (x+1)e)$					v^2	

N

$\mathbf{a = 4x + y + 2}$ where $\mathbf{0 \le y < 4}$	$\mathbf{a=1}$	$\mathbf{y=0}$	$\mathbf{y=1}$	$\mathbf{y=2}$	$\mathbf{y=3}$	
$(m_1+i+j+k, m_2+j+k, m_3+k)$	1	1	1	1	1	
$(m_1+j+k, m_2+i+j+k, m_3+k)$	v	v	v	v	v	
$(m_1-e+i+j+k, m_2+e+k, m_3+j+k)$		v	v	v	v	
$(m_1-e+j+k, m_2+e+k, m_3+i+j+k)$		v^2	v^2	v^2	v^2	
$(m_1-(3z-1)e, m_2+(z+1)e+i+j+k, m_3+ze+j+k, ze+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-1)e, m_2+(z+1)e+j+k, m_3+ze+i+j+k, ze+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-2)e+i+j+k, m_2+(z+1)e+k, m_3+(z+1)e, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-2)e+j+k, m_2+(z+1)e+k, m_3+(z+1)e, ze+i+j+k)$		v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(3z-2)e+i+j+k, m_2+(z+1e+j+k, m_3+(z+1)e, ze+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-2)e+j+k, m_2+(z+1e+i+j+k, m_3+(z+1)e, ze+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(2z-2)e+k, m_2+(z+1)e+i+j+k, m_3+(z+1)e, ze+j+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(2z-2)e+k, m_2+(z+1)e+j+k, m_3+(z+1)e, ze+i+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-2)e, m_2+(z+1)e+i+j+k, m_3+(z+1)e+k, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-2)e, m_2+(z+1)e+j+k, m_3+(z+1)e+k, ze+i+j+k)$		v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(3z-3)e+i+j+k, m_2+(z+2)e, m_3+(z+1)e+k, ze+j+k)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-3)e+j+k, m_2+(z+2)e, m_3+(z+1)e+k, ze+i+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-3)e+k, m_2+(z+2)e, m_3+(z+1)e+i+j+k, ze+j+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-3)e+k, m_2+(z+2)e, m_3+(z+1)e+j+k, ze+i+j+k)$		v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(3z-3)e+i+j+k, m_2+(z+1)e+j+k, m_3+(z+1)e+k, (z+1)e)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-3)e+j+k, m_2+(z+1)e+i+j+k, m_3+(z+1)e+k, (z+1)e)$		v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(3z-4)e+i+j+k, m_2+(z+2)e+k, m_3+(z+1)e+j+k, (z+1)e)$		v	v	v	v	$0 \le z < x$
$(m_1-(3z-4)e+j+k, m_2+(z+2)e+k, m_3+(z+1)e+i+j+k, (z+1)e)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-4)e+i+j+k, m_2+(z+2)e, m_3+(z+1)e+j+k, (z+1)e+k)$		v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(3z-4)e+j+k, m_2+(z+2)e, m_3+(z+1)e+i+j+k, (z+1)e+k)$		v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(3x-1)e, m_2+(x+1)e+i+j+k, m_3+xe+j+k, xe+k)$			v	v	v	
$(m_1-(3x-1)e, m_2+(x+1)e+j+k, m_3+xe+i+j+k, xe+k)$			v^2	v^2	v^2	
$(m_1-(3x-2)e+i+j+k, m_2+(x+1)e+k, m_3+(x+1)e, xe+j+k)$			v^2	v^2	v^2	
$(m_1-(3x-2)e+j+k, m_2+(x+1)e+k, m_3+(x+1)e, xe+i+j+k)$			v^3	v^3	v^3	
$(m_1-(3x-2)e+i+j+k, m_2+(x+1e+j+k, m_3+(x+1)e, xe+k)$				v	v	
$(m_1-(3x-2)e+j+k, m_2+(x+1)e++j+k, m_3+(x+1)e, xe+k)$				v^2	v^2	
$(m_1-(2x-2)e+k, m_2+(x+1)e+i+j+k, m_3+(x+1)e, xe+j+k)$				v	v	
$(m_1-(2x-2)e+k, m_2+(x+1)e+j+k, m_3+(x+1)e, xe+i+j+k)$				v^2	v^2	
$(m_1-(3x-2)e, m_2+(x+1)e+i+j+k, m_3+(x+1)e+k, xe+j+k)$				v^2	v^2	
$(m_1-(3x-2)e, m_2+(x+1)e+j+k, m_3+(x+1)e+k, xe+i+j+k)$				v^3	v^3	
$(m_1-(3x-3)e+i+j+k, m_2+(x+2)e, m_3+(x+1)e+k, xe+j+k)$					v	
$(m_1-(3x-3)e+j+k, m_2+(x+2)e, m_3+(x+1)e+k, xe+i+j+k)$					v^2	
$(m_1-(3x-3)e+k, m_2+(x+2)e, m_3+(x+1)e+i+j+k, xe+j+k)$					v^2	
$(m_1-(3x-3)e+k, m_2+(x+2)e, m_3+(x+1)e+j+k, xe+i+j+k)$					v^3	
$(m_1-(3x-3)e+i+j+k, m_2+(x+1)e+j+k, m_3+(x+1)e+k, (x+1)e)$					v^2	
$(m_1-(3x-3)e+j+k, m_2+(x+1)e+i+j+k, m_3+(x+1)e+k, (x+1)e)$					v^3	

O

$\mathbf{a = 4x + y + 2}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{a = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$\mathbf{(m_1 + e, m_2 + j + k, m_3 + k)}$	1	1	1	0	1	
$\mathbf{(m_1 + j + k, m_2 + e, m_3 + k)}$	v	v	v	v	v	
$\mathbf{(m_1, m_2 + e + k, m_3 + j + k)}$		v	v	v	v	
$\mathbf{(m_1 - e + j + k, m_2 + e + k, m_3 + e)}$		v^2	v^2	v^2	v^2	
$\mathbf{(m_1 - (3z + 1)e, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e, ze + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 1)e, m_2 + (z + 1)e + k, m_3 + (z + 1)e, ze + j + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e + k, m_2 + (z + 2)e, m_3 + (z + 1)e, ze + j + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e + k, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e, m_2 + (z + 2)e, m_3 + (z + 1)e + k, ze + j + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + j + k, m_2 + (z + 2)e, m_3 + (z + 1)e + k, (z + 1)e)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + k, m_2 + (z + 2)e, m_3 + (z + 1)e + j + k, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (2z + 3)e, m_2 + (z + 2)e + k, m_3 + (z + 1)e + j + k, (z + 1)e)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z - 4)e + j + k, m_2 + (z + 2)e + k, m_3 + (z + 2)e, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e, m_2 + (z + 2)e, m_3 + (z + 1)e + j + k, (z + 1)e + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3x + 1)e, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e, xe + k)}$			v	v	v	
$\mathbf{(m_1 - (3x + 1)e, m_2 + (x + 1)e + k, m_3 + (x + 1)e, xe + j + k)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (3x + 2)e + k, m_2 + (x + 2)e, m_3 + (x + 1)e, xe + j + k)}$				v	v	
$\mathbf{(m_1 - (3x + 2)e, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e + k, (x + 1)e)}$				v^2	v^2	
$\mathbf{(m_1 - (3x + 2)e, m_2 + (x + 2)e, m_3 + (x + 1)e + k, xe + j + k)}$					v	
$\mathbf{(m_1 - (3x + 3)e + j + k, m_2 + (x + 2)e, m_3 + (x + 1)e + k, (x + 1)e)}$					v	
$\mathbf{(m_1 - (3x + 3)e + k, m_2 + (x + 2)e, m_3 + (x + 1)e + j + k, (x + 1)e)}$					v^2	

P

$\mathbf{a = 4x + y + 2}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{a = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$\mathbf{(m_1 + e + I, m_2 + j + k, m_3 + k)}$	1	1	1	1	1	
$\mathbf{(m_1 + e, m_2 + j + k, m_3 + k, I)}$	v	v	v	v	v	
$\mathbf{(m_1 + j + k, m_2 + e + I, m_3 + k)}$	v	v	v	v	v	
$\mathbf{(m_1 + j + k, m_2 + e, m_3 + k, I)}$	v^2	v^2	v^2	v^2	v^2	
$\mathbf{(m_1 + I, m_2 + e + k, m_3 + j + k)}$		v	v	v	v	
$\mathbf{(m_1, m_2 + e + k, m_3 + j + k, I)}$		v^2	v^2	v^2	v^2	
$\mathbf{(m_1 - e + j + k, m_2 + e + k, m_3 + e + I)}$		v^2	v^2	v^2	v^2	
$\mathbf{(m_1 - e + j + k, m_2 + e + k, m_3 + e, I)}$		v^3	v^3	v^3	v^3	
$\mathbf{(m_1 - (3z + 1)e + I, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e, ze + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 1)e, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e + I, ze + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 1)e + I, m_2 + (z + 1)e + k, m_3 + (z + 1)e, ze + j + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 1)e, m_2 + (z + 1)e + k, m_3 + (z + 1)e + I, ze + j + k)}$		v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e + k, m_2 + (z + 2)e + I, m_3 + (z + 1)e, ze + j + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e + k, m_2 + (z + 2)e, m_3 + (z + 1)e + I, ze + j + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e + I, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e + k, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e, m_2 + (z + 1)e + j + k, m_3 + (z + 1)e + k, (z + 1)e + I)}$		v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e + I, m_2 + (z + 2)e, m_3 + (z + 1)e + k, ze + j + k)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 2)e, m_2 + (z + 2)e + I, m_3 + (z + 1)e + k, ze + j + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + j + k, m_2 + (z + 2)e + I, m_3 + (z + 1)e + k, (z + 1)e)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + j + k, m_2 + (z + 2)e, m_3 + (z + 1)e + k, (z + 1)e + I)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + k, m_2 + (z + 2)e + I, m_3 + (z + 1)e + j + k, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + k, m_2 + (z + 2)e, m_3 + (z + 1)e + j + k, (z + 1)e + I)}$		v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (2z + 3)e + I, m_2 + (z + 2)e + k, m_3 + (z + 1)e + j + k, (z + 1)e)}$		v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (2z + 3)e, m_2 + (z + 2)e + k, m_3 + (z + 1)e + j + k, (z + 1)e + I)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z - 4)e + j + k, m_2 + (z + 2)e + k, m_3 + (z + 2)e + I, (z + 1)e)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z - 4)e + j + k, m_2 + (z + 2)e + k, m_3 + (z + 2)e, (z + 1)e + I)}$		v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e + I, m_2 + (z + 2)e, m_3 + (z + 1)e + j + k, (z + 1)e + k)}$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z + 3)e, m_2 + (z + 2)e + I, m_3 + (z + 1)e + j + k, (z + 1)e + k)}$		v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3x + 1)e + I, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e, xe + k)}$			v	v	v	
$\mathbf{(m_1 - (3x + 1)e, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e + I, xe + k)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (3x + 1)e + I, m_2 + (x + 1)e + k, m_3 + (x + 1)e, xe + j + k)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (3x + 1)e, m_2 + (x + 1)e + k, m_3 + (x + 1)e + I, xe + j + k)}$			v^3	v^3	v^3	
$\mathbf{(m_1 - (3x + 2)e + k, m_2 + (x + 2)e + I, m_3 + (x + 1)e, xe + j + k)}$				v	v	
$\mathbf{(m_1 - (3x + 2)e + k, m_2 + (x + 2)e, m_3 + (x + 1)e + I, xe + j + k)}$				v^2	v^2	
$\mathbf{(m_1 - (3x + 2)e + I, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e + k, (x + 1)e)}$				v^2	v^2	
$\mathbf{(m_1 - (3x + 2)e, m_2 + (x + 1)e + j + k, m_3 + (x + 1)e + k, (x + 1)e + I)}$				v^3	v^3	
$\mathbf{(m_1 - (3x + 2)e + I, m_2 + (x + 2)e, m_3 + (x + 1)e + k, xe + j + k)}$					v	
$\mathbf{(m_1 - (3x + 2)e, m_2 + (x + 2)e + I, m_3 + (x + 1)e + k, xe + j + k)}$					v^2	
$\mathbf{(m_1 - (3x + 3)e + j + k, m_2 + (x + 2)e + I, m_3 + (x + 1)e + k, (x + 1)e)}$					v	
$\mathbf{(m_1 - (3x + 3)e + j + k, m_2 + (x + 2)e, m_3 + (x + 1)e + k, (x + 1)e + I)}$					v^2	
$\mathbf{(m_1 - (3x + 3)e + k, m_2 + (x + 2)e + I, m_3 + (x + 1)e + j + k, (x + 1)e)}$					v^2	
$\mathbf{(m_1 - (3x + 3)e + k, m_2 + (x + 2)e, m_3 + (x + 1)e + j + k, (x + 1)e + I)}$					v^3	

Q

$\mathbf{a = 4x + y + 1}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$\mathbf{(m_1 + e + k, m_2 + j + k, m_3 + k)}$	1	1	1	1	
$\mathbf{(m_1 + j + k, m_2 + e + k, m_3 + k)}$	v	v	v	v	
$\mathbf{(m_1 - 3ze + k, m_2 + (z+1)e + k, m_3 + ze + j + k, ze)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - 3ze, m_2 + (z+1)e + k, m_3 + ze + j + k, ze + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + j + k, m_2 + (z+1)e + k, m_3 + (z+1)e, ze + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + k, m_2 + (z+1)e + j + k, m_3 + (z+1)e, ze + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e, m_2 + (z+1)e + j + k, m_3 + (z+1)e + k, ze + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + k, m_2 + (z+1)e + k, m_3 + (z+1)e, ze + j + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + k, m_2 + (z+2)e, m_3 + (z+1)e + k, ze + j + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + k, m_2 + (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+3)e + j + k, m_2 + (z+2)e + k, m_3 + (z+1)e + k, (z+1)e)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z-3)e + k, m_2 + (z+2)e, m_3 + (z+1)e + j + k, (z+1)e + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - 3xe + k, m_2 + (x+1)e + k, m_3 + xe + j + k, xe)}$		v	v	v	
$\mathbf{(m_1 - 3xe, m_2 + (x+1)e + k, m_3 + xe + j + k, xe + k)}$		v	v	v	
$\mathbf{(m_1 - (3x+1)e + j + k, m_2 + (x+1)e + k, m_3 + (x+1)e, xe + k)}$		v^2	v^2	v^2	
$\mathbf{(m_1 - (3x+1)e + k, m_2 + (x+1)e + j + k, m_3 + (x+1)e, xe + k)}$			v	v	
$\mathbf{(m_1 - (3x+1)e, m_2 + (x+1)e + j + k, m_3 + (x+1)e + k, xe + k)}$			v^2	v^2	
$\mathbf{(m_1 - (3x+1)e + k, m_2 + (x+1)e + k, m_3 + (x+1)e, xe + j + k)}$			v^2	v^2	
$\mathbf{(m_1 - (3x+2)e + k, m_2 + (x+2)e, m_3 + (x+1)e + k, xe + j + k)}$				v	
$\mathbf{(m_1 - (3x+2)e + k, m_2 + (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e)}$				v^2	

R

$\mathbf{a = 4x + y + 1}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	$y = 4$	
$\mathbf{(m_1 + e + I, m_2 + j + k, m_3 + k)}$	1	1	1	1	
$\mathbf{(m_1 + e + k, m_2 + j + k, m_3 + I)}$	v	v	v	v	
$\mathbf{(m_1 + j + k, m_2 + e + I, m_3 + k)}$	v	v	v	v	
$\mathbf{(m_1 + j + k, m_2 + e + k, m_3 + I)}$	v^2	v^2	v^2	v^2	
$\mathbf{(m_1 - 3ze + I, m_2 + (z+1)e + k, m_3 + ze + j + k, ze)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - 3ze + k, m_2 + (z+1)e + I, m_3 + ze + j + k, ze)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - 3ze, m_2 + (z+1)e + I, m_3 + ze + j + k, ze + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - 3ze, m_2 + (z+1)e + k, m_3 + ze + j + k, ze + I)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + j + k, m_2 + (z+1)e + I, m_3 + (z+1)e, ze + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + j + k, m_2 + (z+1)e + k, m_3 + (z+1)e, ze + I)}$	v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + I, m_2 + (z+1)e + j + k, m_3 + (z+1)e, ze + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + k, m_2 + (z+1)e + j + k, m_3 + (z+1)e, ze + I)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e, m_2 + (z+1)e + j + k, m_3 + (z+1)e + I, ze + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e, m_2 + (z+1)e + j + k, m_3 + (z+1)e + k, ze + I)}$	v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + I, m_2 + (z+1)e + k, m_3 + (z+1)e, ze + j + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+1)e + k, m_2 + (z+1)e + I, m_3 + (z+1)e, ze + j + k)}$	v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + I, m_2 + (z+2)e, m_3 + (z+1)e + k, ze + j + k)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + k, m_2 + (z+2)e, m_3 + (z+1)e + I, ze + j + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + I, m_2 + (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z+2)e + k, m_2 + (z+1)e + j + k, m_3 + (z+1)e + I, (z+1)e)}$	v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - (3z+3)e + j + k, m_2 + (z+2)e + I, m_3 + (z+1)e + k, (z+1)e)}$	v	v	v	v	$0 \leq z < x$
$\mathbf{(m_1 - (3z+3)e + j + k, m_2 + (z+2)e + k, m_3 + (z+1)e + I, (z+1)e)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z-3)e + I, m_2 + (z+2)e, m_3 + (z+1)e + j + k, (z+1)e + k)}$	v^2	v^2	v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (3z-3)e + k, m_2 + (z+2)e, m_3 + (z+1)e + j + k, (z+1)e + I)}$	v^3	v^3	v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - 3xe + I, m_2 + (x+1)e + k, m_3 + xe + j + k, xe)}$		v	v	v	
$\mathbf{(m_1 - 3xe + k, m_2 + (x+1)e + I, m_3 + xe + j + k, xe)}$		v^2	v^2	v^2	
$\mathbf{(m_1 - 3xe, m_2 + (x+1)e + I, m_3 + xe + j + k, xe + k)}$		v	v	v	
$\mathbf{(m_1 - 3xe, m_2 + (x+1)e + k, m_3 + xe + j + k, xe + I)}$		v^2	v^2	v^2	
$\mathbf{(m_1 - (3x+1)e + j + k, m_2 + (x+1)e + I, m_2 + (x+1)e, xe + k)}$		v^2	v^2	v^2	
$\mathbf{(m_1 - (3x+1)e + j + k, m_2 + (x+1)e + k, m_3 + (x+1)e, xe + I)}$		v^3	v^3	v^3	
$\mathbf{(m_1 - (3x+1)e + I, m_2 + (x+1)e + j + k, m_3 + (x+1)e, xe + k)}$			v	v	
$\mathbf{(m_1 - (3x+1)e + k, m_2 + (x+1)e + j + k, m_3 + (x+1)e, xe + I)}$			v^2	v^2	
$\mathbf{(m_1 - (3x+1)e, m_2 + (x+1)e + j + k, m_3 + (x+1)e + I, xe + k)}$			v^2	v^2	
$\mathbf{(m_1 - (3x+1)e, m_2 + (x+1)e + j + k, m_3 + (x+1)e + k, xe + I)}$			v^3	v^3	
$\mathbf{(m_1 - (3x+1)e + I, m_2 + (x+1)e + k, m_3 + (x+1)e, xe + j + k)}$			v^2	v^2	
$\mathbf{(m_1 - (3x+1)e + k, m_2 + (x+1)e + I, m_3 + (x+1)e, xe + j + k)}$			v^3	v^3	
$\mathbf{(m_1 - (3x+2)e + I, m_2 + (x+2)e, m_3 + (x+1)e + k, xe + j + k)}$				v	
$\mathbf{(m_1 - (3x+2)e + k, m_2 + (x+2)e, m_3 + (x+1)e + I, xe + j + k)}$				v^2	
$\mathbf{(m_1 - (3x+2)e + I, m_2 + (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e)}$				v^2	
$\mathbf{(m_1 - (3x+2)e + k, m_2 + (x+1)e + j + k, m_3 + (x+1)e + I, (x+1)e)}$				v^3	

L

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e,m_2+e,m_3+k})$	1	1
$(\mathbf{m_1,m_2+e+k,m_3+e})$		v

M

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I,m_2+e+I,m_3+k})$	1	1
$(\mathbf{m_1+e+I,m_2+e,m_3+k,I})$	v	v
$(\mathbf{m_1+e,m_2+e+I,m_3+k,I})$	v^2	v^2
$(\mathbf{m_1+I,m_2+e+k,m_3+e+I})$		v
$(\mathbf{m_1+I,m_2+e+k,m_3+e,I})$		v^2
$(\mathbf{m_1,m_2+e+k,m_3+e+I,I})$		v^3

S

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+i,m_2+e,m_3+k})$	1	1
$(\mathbf{m_1+e,m_2+e+i,m_3+k})$	v	v
$(\mathbf{m_1+e,m_2+e,m_3+k,I})$	v^2	v^2
$(\mathbf{m_1+i,m_2+e+k,m_3+e})$		v
$(\mathbf{m_1,m_2+e+k,m_3+e+I})$		v^2
$(\mathbf{m_1,m_2+e+k,m_3+e,I})$		v^3

T

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+k,m_2+e,m_3+k})$	1	1
$(\mathbf{m_1+e,m_2+e+k,m_3+k})$	v	v
$(\mathbf{m_1+k,m_2+e+k,m_3+e})$		v
$(\mathbf{m_1,m_2+e+k,m_3+e,k})$		v^2

U

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I,m_2+e,m_3+k})$	1	1
$(\mathbf{m_1+e+k,m_2+e,m_3+I})$	v	v
$(\mathbf{m_1+e,m_2+e+I,m_3+k})$	v	v
$(\mathbf{m_1+e,m_2+e+k,m_3+I})$	v^2	v^2
$(\mathbf{m_1+I,m_2+e+k,m_3+e})$		v
$(\mathbf{m_1+k,m_2+e+I,m_3+e})$		v^2
$(\mathbf{m_1,m_2+e+I,m_3+e,k})$		v^2
$(\mathbf{m_1,m_2+e+k,m_3+e,I})$		v^3

V

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+I,m_2+e+J,m_3+k})$	1	1
$(\mathbf{m_1+e+J,m_2+e+I,m_3+k})$	v	v
$(\mathbf{m_1+I,m_2+e+k,m_3+e+J})$		v
$(\mathbf{m_1+J,m_2+e+k,m_3+e+I})$		v^2
$(\mathbf{m_1+e+I,m_2+e,m_3+k,J})$	v	v
$(\mathbf{m_1+e+J,m_2+e,m_3+k,I})$	v^2	v^2
$(\mathbf{m_1+e,m_2+e+I,m_3+k,J})$	v^2	v^2
$(\mathbf{m_1+e,m_2+e+J,m_3+k,I})$	v^3	v^3
$(\mathbf{m_1+I,m_2+e+k,m_3+e,J})$		v^2
$(\mathbf{m_1+J,m_2+e+k,m_3+e,I})$		v^3
$(\mathbf{m_1,m_2+e+k,m_3+e+I,J})$		v^3
$(\mathbf{m_1,m_2+e+k,m_3+e+J,I})$		v^4

W

$\mathbf{a}$	$\mathbf{1}$	$> \mathbf{1}$
$(\mathbf{m_1+e+k,m_2+e+J,m_3+k})$	1	1
$(\mathbf{m_1+e+J,m_2+e+k,m_3+k})$	v	v
$(\mathbf{m_1+k,m_2+e+k,m_3+e+J})$		v
$(\mathbf{m_1+e+k,m_2+e,m_3+k,J})$	v	v
$(\mathbf{m_1+e,m_2+e+k,m_3+k,J})$	v^2	v^2
$(\mathbf{m_1+k,m_2+e+k,m_3+e,J})$		v^2
$(\mathbf{m_1+J,m_2+e+k,m_3+e,k})$		v^2
$(\mathbf{m_1,m_2+e+k,m_3+e+J,k})$		v^3

X

a	**1**	$> \mathbf{1}$
$(\mathbf{m_1 + e + I, m_2 + e + J, m_3 + k})$	1	1
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + I})$	v	v
$(\mathbf{m_1 + e + J, m_2 + e + I, m_3 + k})$	v	v
$(\mathbf{m_1 + e + J, m_2 + e + k, m_3 + I})$	v^2	v^2
$(\mathbf{m_1 + k, m_2 + e + I, m_3 + e + J})$		v^2
$(\mathbf{m_1 + I, m_2 + e + k, m_3 + e, J})$		v^2
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + k, J})$	v	v
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + I, J})$	v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + I, m_3 + k, J})$	v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + k, m_3 + I, J})$	v^3	v^3
$(\mathbf{m_1 + I, m_2 + e + k, m_3 + e + J})$		v
$(\mathbf{m_1 + k, m_2 + e + I, m_3 + e, J})$		v^3
$(\mathbf{m_1 + J, m_2 + e + I, m_3 + e, k})$		v^2
$(\mathbf{m_1 + J, m_2 + e + k, m_3 + e, I})$		v^3
$(\mathbf{m_1, m_2 + e + I, m_3 + e + J, k})$		v^3
$(\mathbf{m_1, m_2 + e + k, m_3 + e + J, I})$		v^4

Y

a	**1**	**2**	$> \mathbf{2}$
$(\mathbf{m_1 + 2e, m_2 + e + J, m_3 + k})$	1	1	1
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + e})$	v	v	v
$(\mathbf{m_1 + e + J, m_2 + 2e, m_3 + k})$	v	v	v
$(\mathbf{m_1 + e + J, m_2 + e + k, m_3 + e})$	v^2	v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + k, m_3 + e + J})$		v	v
$(\mathbf{m_1 + k, m_2 + 2e, m_3 + e + J})$		v^2	v^2
$(\mathbf{m_1 + 2e, m_2 + e, m_3 + k, J})$	v	v	v
$(\mathbf{m_1 + e, m_2 + 2e, m_3 + k, J})$		v	v
$(\mathbf{m_1 + e, m_2 + e + k, m_3 + e, J})$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1 + k, m_2 + 2e, m_3 + e, J})$		v^3	v^3
$(\mathbf{m_1 + J, m_2 + 2e, m_3 + e, k})$		v^2	v^2
$(\mathbf{m_1, m_2 + 2e, m_3 + e + J, k})$			v^2
$(\mathbf{m_1, m_2 + e + k, m_3 + e + J, e})$		v^3	v^3

Z

a	**1**	**2**	$> \mathbf{2}$
$(\mathbf{m_1 + 2e + H, m_2 + e + J, m_3 + k})$	1	1	1
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + e + H})$	v	v	v
$(\mathbf{m_1 + e + J, m_2 + 2e + H, m_3 + k})$	v	v	v
$(\mathbf{m_1 + e + J, m_2 + e + k, m_3 + e + H})$	v^2	v^2	v^2
$(\mathbf{m_1 + e + H, m_2 + e + k, m_3 + e + J})$		v	v
$(\mathbf{m_1 + k, m_2 + 2e + H, m_3 + e + J})$		v^2	v^2
$(\mathbf{m_1 + 2e + H, m_2 + e, m_3 + k, J})$	v	v	v
$(\mathbf{m_1 + 2e, m_2 + e + J, m_3 + k, H})$	v	v	v
$(\mathbf{m_1 + 2e, m_2 + e + H, m_3 + k, J})$	v^2	v^2	v^2
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + e, H})$	v^2	v^2	v^2
$(\mathbf{m_1 + e + J, m_2 + 2e, m_3 + k, H})$	v^2	v^2	v^2
$(\mathbf{m_1 + e + J, m_2 + e + k, m_3 + e, H})$	v^3	v^3	v^3
$(\mathbf{m_1 + e + H, m_2 + 2e, m_3 + k, J})$		v	v
$(\mathbf{m_1 + e + H, m_2 + e + k, m_3 + e, J})$	v^2	$2v^2$	$2v^2$
$(\mathbf{m_1 + e, m_2 + 2e + H, m_3 + k, J})$		v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + k, m_3 + e + J, H})$		v^2	v^2
$(\mathbf{m_1 + e, m_2 + e + k, m_3 + e + H, J})$	v^3	$2v^3$	$2v^3$
$(\mathbf{m_1 + k, m_2 + 2e + H, m_3 + e, J})$		v^3	v^3
$(\mathbf{m_1 + k, m_2 + 2e, m_3 + e + J, H})$		v^3	v^3
$(\mathbf{m_1 + k, m_2 + 2e, m_3 + e + H, J})$		v^4	v^4
$(\mathbf{m_1 + J, m_2 + 2e + H, m_3 + e, k})$		v^2	v^2
$(\mathbf{m_1 + J, m_2 + 2e, m_3 + e + H, k})$		v^3	v^3
$(\mathbf{m_1 + H, m_2 + 2e, m_3 + e + J, k})$			v^2
$(\mathbf{m_1 + H, m_2 + e + k, m_3 + e + J, e})$		v^3	v^3
$(\mathbf{m_1, m_2 + 2e + H, m_3 + e + J, k})$			v^3
$(\mathbf{m_1, m_2 + e + k, m_3 + e + J, e + H})$		v^4	v^4

Spoke $b : a = c = 0$

H

$\mathbf{b = 2x + y + 3}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b=1}$	$\mathbf{b=2}$	$\mathbf{y=0}$	$\mathbf{y=1}$	
$(m_1 + i + k, m_2 + k, m_3 + k)$	1	1	1	1	
$(m_1 + k, m_2, m_3 + i + k, k)$	v	v	v	v	
$(m_1 + k, m_2 - e + i + k, m_3 + e + k)$		v	v	v	
$(m_1 + i + k, m_2 - e + k, m_3 + e, k)$		v	v	v	
$(m_1 + k, m_2 - e + i + k, m_3 + e, k)$		v^2	v^2	v^2	
$(m_1 + k, m_2 - e + k, m_3 + e, i + k)$			v	v	
$(m_1 + k, m_2 - e, m_3 + e + i + k, k)$			v	v	
$(m_1 + k, m_2 - e, m_3 + e + k, i + k)$			$2v^2$	$2v^2$	
$(m_1, m_2 - e + i + k, m_3 + e + k, k)$			v	v	
$(m_1, m_2 - e + k, m_3 + e + i + k, k)$			$2v^2$	$2v^2$	
$(m_1, m_2 - e + k, m_3 + e + k, i + k)$			$2v^3$	$2v^3$	
$(m_1 - e + i + k, m_2 - e + k, m_3 + e + k, e)$			v^2	v^2	
$(m_1 - e + k, m_2 - e, m_3 + e + i + k, e + k)$			v^3	v^3	
$(m_1 - ze + k, m_2 - (z+2)e + i + k, m_3 + (z+1)e + k, (z+1)e)$			v	v	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+2)e + k, m_3 + (z+1)e + i + k, (z+1)e)$			v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+2)2e + k, m_3 + (z+1)e + i + k, (z+1)e + k)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + k, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + k)$			v	v	$0 \leq z < x$
$(m_1 - (z+1)e + i + k, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + k, m_2 - (z+1)e, m_3 + (z+2)e + k, ze + k)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + k)$			$v + v^3$	$v + v^3$	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + k, ze + i + k)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+1)e + i + k, (z+1)e + k)$			v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e + i + k, m_3 + (z+2)e + k, (z+1)e)$			$v + v^3$	$v + v^3$	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e + i + k, m_3 + (z+2)e, (z+1)e + k)$			$2v^2 + v^4$	$2v^2 + v^4$	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + i + k)$			v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e, m_2 - (z+2)e + i + k, m_3 + (z+2)e + k, (z+1)e + k)$			v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + i + k, m_2 - (z+2)e + k, m_3 + (z+2)e + k, (z+1)e)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + i + k)$			v	v	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e, m_3 + (z+2)e + i + k, (z+1)e + k)$			v	v	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+2)e, m_3 + (z+2)e + k, (z+1)e + i + k)$			$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - (z+1)e, m_2 - (z+2)e + i + k, m_3 + (z+2)e + k, (z+1)e + k)$			v	v	$0 \leq z < x$
$(m_1 - (z+1)e, m_2 - (z+2)e + k, m_3 + (z+2)e + i + k, (z+1)e + k)$			$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - (z+1)e, m_2 - (z+2)e + k, m_3 + (z+2)e + k, (z+1)e + i + k)$			$3v^3$	$3v^3$	$0 \leq z < x$
$(m_1 - (z+2)e + i + k, m_2 - (z+2)e + k, m_3 + (z+2)e + k, (z+2)e)$			v^2	v^2	$0 \leq z < x$
$(m_1 - (z+2)e + k, m_2 - (z+2)e, m_3 + (z+2)e + i + k, (z+2)e + k)$			v^3	v^3	$0 \leq z < x$
$(m_1 - xe + k, m_2 - (x+2)e + i + k, m_3 + (x+1)e + k, (x+1)e)$				v	
$(m_1 - xe + k, m_2 - (x+2)e + k, m_3 + (x+1)e + i + k, (x+1)e)$				v^2	
$(m_1 - xe, m_2 - (x+2)2e + k, m_3 + (x+1)e + i + k, (x+1)e + k)$				v^2	
$(m_1 - (x+1)e + i + k, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + k)$				v	
$(m_1 - (x+1)e + i + k, m_2 - (x+1)e + k, m_3 + (x+1)e + k, (x+1)e)$				v^2	
$(m_1 - (x+1)e + i + k, m_2 - (x+1)e, m_3 + (x+2)e + k, xe + k)$				v^2	
$(m_1 - (x+1)e + i + k, m_2 - (x+2)e + k, m_3 + (x+2)e, (x+1)e + k)$				$v + v^3$	
$(m_1 - (x+1)e + k, m_2 - (x+1)e, m_3 + (x+2)e + k, xe + i + k)$				v^2	
$(m_1 - (x+1)e + k, m_2 - (x+1)e, m_3 + (x+1)e + i + k, (x+1)e + k)$				v^3	
$(m_1 - (x+1)e + k, m_2 - (x+2)e + i + k, m_3 + (x+2)e + k, (x+1)e)$				$v + v^3$	
$(m_1 - (x+1)e + k, m_2 - (x+2)e + i + k, m_3 + (x+2)e, (x+1)e + k)$				$2v^2 + v^4$	
$(m_1 - (x+1)e + k, m_2 - (x+2)e + k, m_3 + (x+2)e, (x+1)e + i + k)$				v^3	
$(m_1 - (x+1)e, m_2 - (x+2)e + i + k, m_3 + (x+2)e + k, (x+1)e + k)$				v^3	

I

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b=1}$	$\mathbf{y=0}$	$\mathbf{y=1}$	
$(m_1 + e, m_2 + k, m_3 + k)$	1	1	1	
$(m_1 + k, m_2, m_3 + e + k)$		v	v	
$(m_1 + e, m_2 - e + k, m_3 + e, k)$		v	v	
$(m_1 + k, m_2, m_3 + e, k)$		v^2	v^2	
$(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e)$		v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + k)$		v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e)$		$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + k)$		v	v	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e)$		v	v	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + k)$		$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e)$			v	
$(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + k)$			v	
$(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + k, (x+1)e)$			$2v^2$	

J

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \le y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$\mathbf{(m_1 + e + I, m_2 + k, m_3 + k)}$	1	1	1	
$\mathbf{(m_1 + e, m_2 + k, m_3 + k, I)}$	v	v	v	
$\mathbf{(m_1 + k, m_2 + I, m_3 + e + k)}$		v	v	
$\mathbf{(m_1 + k, m_2, m_3 + e + k, I)}$		v^2	v^2	
$\mathbf{(m_1 + e + I, m_2 - e + k, m_3 + e, k)}$		v	v	
$\mathbf{(m_1 + e, m_2 - e + k, m_3 + e + I, k)}$		v^2	v^2	
$\mathbf{(m_1 + k, m_2 + I, m_3 + e, k)}$		v^2	v^2	
$\mathbf{(m_1 + k, m_2, m_3 + e + I, k)}$		v^3	v^3	
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e + I)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + k)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + I, ze + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e)}$		$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e + I)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + k)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e + I, (z+1)e + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + I)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k)}$		$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + k)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e)}$			v	
$\mathbf{(m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + k)}$			v	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + k)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + k, (x+1)e)}$			$2v^2$	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + k, (x+1)e)}$			$2v^3$	

N

$\mathbf{b = 2x + y + 3}$ where $\mathbf{0 \le y < 2}$	$\mathbf{b = 1}$	$\mathbf{b = 2}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$\mathbf{(m_1 + i + j + k, m_2 + j + k, m_3 + k)}$	1	1	1	1	
$\mathbf{(m_1 + i + j + k, m_2 + k, m_3 + j + k)}$	v	v	v	v	
$\mathbf{(m_1 + j + k, m_2, m_3 + i + j + k, k)}$	v	v	v	v	
$\mathbf{(m_1 + k, m_2, m_3 + i + j + k, j + k)}$	v^2	v^2	v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 - e + i + j + k, m_3 + e + k)}$		v	v	v	
$\mathbf{(m_1 + k, m_2 - e + i + j + k, m_3 + e + j + k)}$		v^2	v^2	v^2	
$\mathbf{(m_1 + i + j + k, m_2 - e + j + k, m_3 + e, k)}$		v	v	v	
$\mathbf{(m_1 + i + j + k, m_2 - e + k, m_3 + e, j + k)}$		v^2	v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 - e + i + j + k, m_3 + e, k)}$		v^2	v^2	v^2	
$\mathbf{(m_1 + k, m_2 - e + i + j + k, m_3 + e, j + k)}$		v^3	v^3	v^3	
$\mathbf{(m_1 + j + k, m_2 - e + k, m_3 + e, i + j + k)}$			v	v	
$\mathbf{(m_1 + k, m_2 - e + j + k, m_3 + e, i + j + k)}$			v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 - e, m_3 + e + i + j + k, k)}$			v	v	
$\mathbf{(m_1 + k, m_2 - e, m_3 + e + i + j + k, j + k)}$			v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 - e, m_3 + e + k, i + j + k)}$			$2v^2$	$2v^2$	
$\mathbf{(m_1 + k, m_2 - e, m_3 + e + j + k, i + j + k)}$			$2v^3$	$2v^3$	
$\mathbf{(m_1, m_2 - e + i + j + k, m_3 + e + j + k, k)}$			v	v	
$\mathbf{(m_1, m_2 - e + i + j + k, m_3 + e + k, j + k)}$			v^2	v^2	
$\mathbf{(m_1, m_2 - e + j + k, m_3 + e + i + j + k, k)}$			$2v^2$	$2v^2$	
$\mathbf{(m_1, m_2 - e + k, m_3 + e + i + j + k, j + k)}$			$2v^3$	$2v^3$	
$\mathbf{(m_1, m_2 - e + j + k, m_3 + e + k, i + j + k)}$			$2v^3$	$2v^3$	
$\mathbf{(m_1, m_2 - e + k, m_3 + e + j + k, i + j + k)}$			$2v^4$	$2v^4$	
$\mathbf{(m_1 - e + i + j + k, m_2 - e + j + k, m_3 + e + k, e)}$			v^2	v^2	
$\mathbf{(m_1 - e + i + j + k, m_2 - e + k, m_3 + e + j + k, e)}$			v^3	v^3	
$\mathbf{(m_1 - e + j + k, m_2 - e, m_3 + e + i + j + k, e + k)}$			v^3	v^3	
$\mathbf{(m_1 - e + k, m_2 - e, m_3 + e + i + j + k, e + j + k)}$			v^4	v^4	
$\mathbf{(m_1 - ze + j + k, m_2 - (z+2)e + i + j + k, m_3 + (z+1)e + k, (z+1)e)}$			v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+2)e + i + j + k, m_3 + (z+1)e + j + k, (z+1)e)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + j + k, m_2 - (z+2)e + k, m_3 + (z+1)e + i + j + k, (z+1)e)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+2)e + j + k, m_3 + (z+1)e + i + j + k, (z+1)e)}$			v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)2e + j + k, m_3 + (z+1)e + i + j + k, (z+1)e + k)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)2e + k, m_3 + (z+1)e + i + j + k, (z+1)e + j + k)}$			v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + j + k, m_3 + (z+2)e, ze + k)}$			v	v	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + j + k)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e)}$			v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e, m_3 + (z+2)e + j + k, ze + k)}$			v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e, m_3 + (z+2)e + k, ze + j + k)}$			v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + k)}$			$v + v^3$	$v + v^3$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + i + j + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + j + k)}$			$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + k, ze + i + j + k)}$			v^2	v^2	$0 \le z < x$

$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + j + k, ze + i + j + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+1)e + i + j + k, (z+1)e + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+1)e + i + j + k, (z+1)e + j + k})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + k, (z+1)e})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + j + k, (z+1)e})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e, (z+1)e + k})$	$2v^2 + v^4$	$2v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e, (z+1)e + j + k})$	$2v^3 + v^5$	$2v^3 + v^5$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + i + j + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + i + j + k})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + j + k, (z+1)e + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + k, (z+1)e + j + k})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + i + j + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e + k, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + i + j + k, m_2 - (z+2)e + k, m_3 + (z+2)e + j + k, (z+1)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + i + j + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + i + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e, m_3 + (z+2)e + i + j + k, (z+1)e + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e, m_3 + (z+2)e + i + j + k, (z+1)e + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+2)e, m_3 + (z+2)e + k, (z+1)e + i + j + k})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+2)e, m_3 + (z+2)e + j + k, (z+1)e + i + j + k})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + j + k, (z+1)e + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + i + j + k, m_3 + (z+2)e + k, (z+1)e + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + j + k, m_3 + (z+2)e + i + j + k, (z+1)e + k})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + k, m_3 + (z+2)e + i + j + k, (z+1)e + j + k})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + j + k, m_3 + (z+2)e + k, (z+1)e + i + j + k})$	$3v^3$	$3v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+2)e + k, m_3 + (z+2)e + j + k, (z+1)e + i + j + k})$	$3v^4$	$3v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z+2)e + i + j + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e + k, (z+2)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+2)e + i + j + k, m_2 - (z+2)e + k, m_3 + (z+2)e + j + k, (z+2)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+2)e + j + k, m_2 - (z+2)e, m_3 + (z+2)e + i + j + k, (z+2)e + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+2)e + k, m_2 - (z+2)e, m_3 + (z+2)e + i + j + k, (z+2)e + j + k})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - xe + j + k, m_2 - (x+2)e + i + j + k, m_3 + (x+1)e + k, (x+1)e})$		v	
$(\mathbf{m_1 - xe + k, m_2 - (x+2)e + i + j + k, m_3 + (x+1)e + j + k, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe + j + k, m_2 - (x+2)e + k, m_3 + (x+1)e + i + j + k, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - (x+2)e + j + k, m_3 + (x+1)e + i + j + k, (x+1)e})$		v^3	
$(\mathbf{m_1 - xe, m_2 - (x+2)2e + j + k, m_3 + (x+1)e + i + j + k, (x+1)e + k})$		v^2	
$(\mathbf{m_1 - xe, m_2 - (x+2)2e + k, m_3 + (x+1)e + i + j + k, (x+1)e + j + k})$		v^3	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + j + k, m_3 + (x+2)e, xe + k})$		v	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + j + k})$		v^2	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e})$		v^2	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e})$		v^3	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e, m_3 + (x+2)e + j + k, xe + k})$		v^2	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+1)e, m_3 + (x+2)e + k, xe + j + k})$		v^3	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+2)e + j + k, m_3 + (x+2)e, (x+1)e + k})$		$v + v^3$	
$(\mathbf{m_1 - (x+1)e + i + j + k, m_2 - (x+2)e + k, m_3 + (x+2)e, (x+1)e + j + k})$		$v^2 + v^4$	
$(\mathbf{m_1 - (x+1)e + j + k, m_2 - (x+1)e, m_3 + (x+2)e + k, xe + i + j + k})$		v^2	
$(\mathbf{m_1 - (x+1)e + k, m_2 - (x+1)e, m_3 + (x+2)e + j + k, xe + i + j + k})$		v^3	
$(\mathbf{m_1 - (x+1)e + j + k, m_2 - (x+1)e, m_3 + (x+1)e + i + j + k, (x+1)e + k})$		v^3	
$(\mathbf{m_1 - (x+1)e + k, m_2 - (x+1)e, m_3 + (x+1)e + i + j + k, (x+1)e + j + k})$		v^4	
$(\mathbf{m_1 - (x+1)e + j + k, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e + k, (x+1)e})$		$v + v^3$	
$(\mathbf{m_1 - (x+1)e + k, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e + j + k, (x+1)e})$		$v^2 + v^4$	
$(\mathbf{m_1 - (x+1)e + j + k, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e, (x+1)e + k})$		$2v^2 + v^4$	
$(\mathbf{m_1 - (x+1)e + k, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e, (x+1)e + j + k})$		$2v^3 + v^5$	
$(\mathbf{m_1 - (x+1)e + j + k, m_2 - (x+2)e + k, m_3 + (x+2)e, (x+1)e + i + j + k})$		v^3	
$(\mathbf{m_1 - (x+1)e + k, m_2 - (x+2)e + j + k, m_3 + (x+2)e, (x+1)e + i + j + k})$		v^4	
$(\mathbf{m_1 - (x+1)e, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e + j + k, (x+1)e + k})$		v^3	
$(\mathbf{m_1 - (x+1)e, m_2 - (x+2)e + i + j + k, m_3 + (x+2)e + k, (x+1)e + j + k})$		v^4	

O

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$(\mathbf{m_1 + e, m_2 + j + k, m_3 + k})$	1	1	1	
$(\mathbf{m_1 + e, m_2 + k, m_3 + j + k})$	v	v	v	
$(\mathbf{m_1 + e, m_2 - e + j + k, m_3 + e, k})$		v	v	
$(\mathbf{m_1 + e, m_2 - e + k, m_3 + e, j + k})$		v^2	v^2	
$(\mathbf{m_1 + j + k, m_2, m_3 + e + k})$		v	v	
$(\mathbf{m_1 + j + k, m_2, m_3 + e, k})$		v^2	v^2	
$(\mathbf{m_1 + k, m_2, m_3 + e + j + k})$		v^2	v^2	
$(\mathbf{m_1 + k, m_2, m_3 + e, j + k})$		v^3	v^3	
$(\mathbf{m_1 - ze + j + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e})$		v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + j + k, (z+1)e})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+2)e, ze + k})$		v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e})$		$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + j + k})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e})$		$2v^3$	$2v^3$	$0 \leq z < x$

$\mathbf{(m_1 - ze, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + k)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + j + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + k)}$		$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + j + k, (z+1)e)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + j + k)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - xe + j + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e)}$			v	
$\mathbf{(m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + j + k, (x+1)e)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+2)e, xe + k)}$			v	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e)}$			$2v^2$	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + j + k)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e)}$			$2v^3$	

P

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \le y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$\mathbf{(m_1 + e + I, m_2 + j + k, m_3 + k)}$	1	1	1	
$\mathbf{(m_1 + e, m_2 + j + k, m_3 + k, I)}$	v	v	v	
$\mathbf{(m_1 + e + I, m_2 + k, m_3 + j + k)}$	v	v	v	
$\mathbf{(m_1 + e, m_2 + k, m_3 + j + k, I)}$	v^2	v^2	v^2	
$\mathbf{(m_1 + e + I, m_2 - e + j + k, m_3 + e, k)}$		v	v	
$\mathbf{(m_1 + e, m_2 - e + j + k, m_3 + e + I, k)}$		v^2	v^2	
$\mathbf{(m_1 + e + I, m_2 - e + k, m_3 + e, j + k)}$		v^2	v^2	
$\mathbf{(m_1 + e, m_2 - e + k, m_3 + e + I, j + k)}$		v^3	v^3	
$\mathbf{(m_1 + j + k, m_2 + I, m_3 + e + k)}$		v	v	
$\mathbf{(m_1 + j + k, m_2, m_3 + e + k, I)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 + I, m_3 + e, k)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2, m_3 + e + I, k)}$		v^3	v^3	
$\mathbf{(m_1 + k, m_2 + I, m_3 + e + j + k)}$		v^2	v^2	
$\mathbf{(m_1 + k, m_2, m_3 + e + j + k, I)}$		v^3	v^3	
$\mathbf{(m_1 + k, m_2 + I, m_3 + e, j + k)}$		v^3	v^3	
$\mathbf{(m_1 + k, m_2, m_3 + e + I, j + k)}$		v^4	v^4	
$\mathbf{(m_1 - ze + j + k, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + j + k, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1) + I)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+1)e + j + k, (z+1)e)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + j + k, (z+1)e + I)}$		v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + j + k, m_3 + (z+2)e, ze + k)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+2)e + I, ze + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e)}$		$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+1)e + k, (z+1)e + I)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + j + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + I, ze + j + k)}$		v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e + I)}$		$2v^4$	$2v^4$	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + k)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)e + j + k, m_3 + (z+2)e + I, (z+1)e + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + j + k)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e + I, (z+1)e + j + k)}$		v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e)}$		v	v	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + I)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k)}$		$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + k)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+2)e + j + k, (z+1)e)}$		v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + j + k, (z+1)e + I)}$		v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + j + k)}$		$2v^3$	$2v^3$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + j + k)}$		$2v^4$	$2v^4$	$0 \le z < x$
$\mathbf{(m_1 - xe + j + k, m_2 - (x+1)e + I, m_3 + (x+1)e + k, (x+1)e)}$			v	
$\mathbf{(m_1 - xe + j + k, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e + I)}$			v^2	
$\mathbf{(m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+1)e + j + k, (x+1)e)}$			v^2	
$\mathbf{(m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + j + k, (x+1)e + I)}$			v^3	
$\mathbf{(m_1 - xe + I, m_2 - (x+1)e + j + k, m_3 + (x+2)e, xe + k)}$			v	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+2)e + I, xe + k)}$			v^2	
$\mathbf{(m_1 - xe + I, m_2 - (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e)}$			$2v^2$	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e + I)}$			$2v^3$	
$\mathbf{(m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + j + k)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e + I, xe + j + k)}$			v^3	
$\mathbf{(m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e)}$			$2v^3$	
$\mathbf{(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e + I)}$			$2v^4$	

Q

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$\mathbf{(m_1 - +e + k, m_2 + j + k, m_3 + k)}$	1	1	1	
$\mathbf{(m_1 + e + k, m_2 + k, m_3 + j + k)}$	v	v	v	
$\mathbf{(m_1 + e, m_2 + k, m_3 + j + k, k)}$	v	v	v	
$\mathbf{(m_1 + e + k, m_2 - e + j + k, m_3 + e, k)}$		v	v	
$\mathbf{(m_1 + e + k, m_2 - e + k, m_3 + e, j + k)}$		v^2	v^2	
$\mathbf{(m_1 + e, m_2 - e + j + k, m_3 + e + k, k)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 + k, m_3 + e + k)}$		v	v	
$\mathbf{(m_1 + j + k, m_2 + k, m_3 + e, k)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2, m_3 + e + k, k)}$		$v + v^3$	$v + v^3$	
$\mathbf{(m_1 + k, m_2, m_3 + e + j + k, k)}$		v^2	v^2	
$\mathbf{(m_1 + k, m_2, m_3 + e + k, j + k)}$		v^3	v^3	
$\mathbf{(m_1 - (z - 1)e, m_2 + k - (z + 1)e, m_3 + (z + 1)e + k, ze + j + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze + j + k, m_2 - (z + 1)e + k, m_3 + (z + 1)e + k, (z + 1)e)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e + j + k, m_3 + (z + 2)e, ze + k)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e + j + k, m_3 + (z + 1)e + k, (z + 1)e)}$		$2v^2$	$2v^2$	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e + k, m_3 + (z + 2)e, ze + j + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e + k, m_3 + (z + 1)e + j + k, (z + 1)e)}$		$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e, m_3 + (z + 1)e + j + k, (z + 1)e + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze, m_2 - (z + 1)e + j + k, m_3 + (z + 2)e + k, ze + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze, m_2 - (z + 1)e + k, m_3 + (z + 1)e + j + k, (z + 1)e + k)}$		$2v^3$	$2v^3$	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 2)e + j + k, m_3 + (z + 2)e, (z + 1)e + k)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 2)e + k, m_3 + (z + 2)e, (z + 1)e + j + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze, m_2 - (z + 2)e + j + k, m_3 + (z + 2)e + k, (z + 1)e + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze, m_2 - (z + 1)e + k, m_3 + (z + 2)e + k, ze + j + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (z + 1)e + j + k, m_2 - (z + 1)e + k, m_3 + (z + 2)e + k, (z + 1)e)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - (z + 1)e + j + k, m_2 - (z + 1)e + k, m_3 + (z + 2)e, (z + 1)e + k)}$		$2v^2$	$2v^2$	$0 \leq z < x$
$\mathbf{(m_1 - (z + 1)e + j + k, m_2 - (z + 1)e, m_3 + (z + 2)e + k, (z + 1)e + k)}$		$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$\mathbf{(m_1 - (z + 1)e + k, m_2 - (z + 1)e, m_3 + (z + 2)e + j + k, (z + 1)e + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (z + 1)e + k, m_2 - (z + 1)e, m_3 + (z + 2)e + k, (z + 1)e + j + k)}$		$2v^3$	$2v^3$	$0 \leq z < x$
$\mathbf{(m_1 - (x - 1)e, m_2 + k - (x + 1)e, m_3 + (x + 1)e + k, xe + j + k)}$			v^2	
$\mathbf{(m_1 - xe + j + k, m_2 - (x + 1)e + k, m_3 + (x + 1)e + k, (x + 1)e)}$			v	
$\mathbf{(m_1 - xe + k, m_2 - (x + 1)e + j + k, m_3 + (x + 2)e, xe + k)}$			v	
$\mathbf{(m_1 - xe + k, m_2 - (x + 1)e + j + k, m_3 + (x + 1)e + k, (x + 1)e)}$			$2v^2$	
$\mathbf{(m_1 - xe + k, m_2 - (x + 1)e + k, m_3 + (x + 2)e, xe + j + k)}$			v^2	
$\mathbf{(m_1 - xe + k, m_2 - (x + 1)e + k, m_3 + (x + 1)e + j + k, (x + 1)e)}$			$v + v^3$	
$\mathbf{(m_1 - xe + k, m_2 - (x + 1)e, m_3 + (x + 1)e + j + k, x + 1)e + k)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x + 1)e + j + k, m_3 + (x + 2)e + k, xe + k)}$			v^2	
$\mathbf{(m_1 - xe, m_2 - (x + 1)e + k, m_3 + (x + 1)e + j + k, (x + 1)e + k)}$			$2v^3$	

R

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$\mathbf{(m_1 + e + I, m_2 + j + k, m_3 + k)}$	1	1	1	
$\mathbf{(m_1 + e + k, m_2 + j + k, m_3 + I)}$	v	v	v	
$\mathbf{(m_1 + e + I, m_2 + k, m_3 + j + k)}$	v	v	v	
$\mathbf{(m_1 + e + k, m_2 + I, m_3 + j + k)}$	v^2	v^2	v^2	
$\mathbf{(m_1 + e, m_2 + I, m_3 + j + k, k)}$	v	v	v	
$\mathbf{(m_1 + e, m_2 + k, m_3 + j + k, I)}$	v^2	v^2	v^2	
$\mathbf{(m_1 + e + I, m_2 - e + j + k, m_3 + e, k)}$		v	v	
$\mathbf{(m_1 + e + k, m_2 - e + j + k, m_3 + e, I)}$		v^2	v^2	
$\mathbf{(m_1 + e + I, m_2 - e + k, m_3 + e, j + k)}$		v^2	v^2	
$\mathbf{(m_1 + e + k, m_2 - e + I, m_3 + e, j + k)}$		v^3	v^3	
$\mathbf{(m_1 + e, m_2 - e + j + k, m_3 + e + I, k)}$		v^2	v^2	
$\mathbf{(m_1 + e, m_2 - e + j + k, m_3 + e + k, I)}$		v^3	v^3	
$\mathbf{(m_1 + j + k, m_2 + I, m_3 + e + k)}$		v	v	
$\mathbf{(m_1 + j + k, m_2 + k, m_3 + e + I)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 + I, m_3 + e, k)}$		v^2	v^2	
$\mathbf{(m_1 + j + k, m_2 + k, m_3 + e, I)}$		v^3	v^3	
$\mathbf{(m_1 + j + k, m_2, m_3 + e + I, k)}$		$v + v^3$	$v + v^3$	
$\mathbf{(m_1 + j + k, m_2, m_3 + e + k, I)}$		$v^2 + v^4$	$v^2 + v^4$	
$\mathbf{(m_1 + I, m_2, m_3 + e + j + k, k)}$		v^2	v^2	
$\mathbf{(m_1 + k, m_2, m_3 + e + j + k, I)}$		v^3	v^3	
$\mathbf{(m_1 + I, m_2, m_3 + e + k, j + k)}$		v^3	v^3	
$\mathbf{(m_1 + k, m_2, m_3 + eI, j + k)}$		v^4	v^4	
$\mathbf{(m_1 - (z - 1)e, m_2 - (z + 1)e + I, m_3 + (z + 1)e + k, ze + j + k)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - (z - 1)e, m_2 - (z + 1)e + k, m_3 + (z + 1)e + I, ze + j + k)}$		v^3	v^3	$0 \leq z < x$
$\mathbf{(m_1 - ze + j + k, m_2 - (z + 1)e + I, m_3 + (z + 1)e + k, (z + 1)e)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - ze + j + k, m_2 - (z + 1)e + k, m_3 + (z + 1)e + I, (z + 1)e)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z + 1)e + j + k, m_3 + (z + 2)e, ze + k)}$		v	v	$0 \leq z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z + 1)e + j + k, m_3 + (z + 2)e, ze + I)}$		v^2	v^2	$0 \leq z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z + 1)e + j + k, m_3 + (z + 1)e + k, (z + 1)e)}$		$2v^2$	$2v^2$	$0 \leq z < x$

$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + j + k, m_3 + (z+1)e + I, (z+1)e})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, ze + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e, ze + j + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e})$	$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+1)e + j + k, (z+1)e})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e, m_3 + (z+1)e + j + k, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+1)e + j + k, (z+1)e + I})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+2)e + I, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + j + k, m_3 + (z+2)e + k, ze + I})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+1)e + j + k, (z+1)e + k})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+1)e + j + k, (z+1)e + I})$	$2v^4$	$2v^4$	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+2)e + j + k, m_3 + (z+2)e, (z+1)e + I})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+2)e + I, m_3 + (z+2)e, (z+1)e + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+2)e + k, m_3 + (z+2)e, (z+1)e + j + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+2)e + j + k, m_3 + (z+2)e + k, (z+1)e + I})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+2)e + j + k, m_3 + (z+2)e + I, (z+1)e + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e + k, ze + j + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + I, ze + j + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e + k, m_3 + (z+2)e + I, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + I})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + k})$	$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + I})$	$v^2 + 2v^4$	$v^2 + 2v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + I, m_2 - (z+1)e, m_3 + (z+2)e + j + k, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + j + k, (z+1)e + I})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + I, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + j + k})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + j + k})$	$2v^4$	$2v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (x-1)e, m_2 - (x+1)e + I, m_3 + (x+1)e + k, xe + j + k})$		v^2	
$(\mathbf{m_1 - (x-1)e, m_2 - (x+1)e + k, m_3 + (x+1)e + I, xe + j + k})$		v^3	
$(\mathbf{m_1 - xe + j + k, m_2 - (x+1)e + I, m_3 + (x+1)e + k, (x+1)e})$		v	
$(\mathbf{m_1 - xe + j + k, m_2 - (x+1)e + k, m_3 + (x+1)e + I, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + j + k, m_3 + (x+2)e, xe + k})$		v	
$(\mathbf{m_1 - xe + k, m_2 - (x+1)e + j + k, m_3 + (x+2)e, xe + I})$		v^2	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + j + k, m_3 + (x+1)e + I, (x+1)e})$		$2v^2$	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + j + k, m_3 + (x+1)e + k, (x+1)e})$		$2v^3$	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+2)e, xe + j + k})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+2)e, xe + j + k})$		v^3	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e})$		$v + 2v^3$	
$(\mathbf{m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+1)e + j + k, (x+1)e})$		$v^2 + 2v^4$	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e, m_3 + (x+1)e + j + k, (x+1)e + k})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - (x+1)e, m_3 + (x+1)e + j + k, (x+1)e + I})$		v^3	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+2)e + I, xe + k})$		v^2	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + j + k, m_3 + (x+2)e + k, xe + I})$		v^3	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + I, m_3 + (x+1)e + j + k, (x+1)e + k})$		$2v^3$	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+1)e + j + k, (x+1)e + I})$		$2v^4$	

S

$\mathbf{b = 2x + y + 2}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{b = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + k})$	1	1	1	
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + e})$	v	v	v	
$(\mathbf{m_1 + e, m_2 + k, m_3 + e, I})$	v	v	v	
$(\mathbf{m_1 + e + I, m_2, m_3 + e, k})$		v	v	
$(\mathbf{m_1 + e, m_2 + I, m_3 + e + k})$		v	v	
$(\mathbf{m_1 + e, m_2 + I, m_3 + e, k})$		v^2	v^2	
$(\mathbf{m_1 + e, m_2, m_3 + e + k, I})$		v^2	v^2	
$(\mathbf{m_1 + e, m_2, m_3 + e + I, k})$		$v + v^3$	$v + v^3$	
$(\mathbf{m_1 + e, m_2 - e + k, m_3 + e + I, e})$		v^2	v^2	
$(\mathbf{m_1 + k, m_2 + I, m_3 + 2e})$		v^2	v^2	
$(\mathbf{m_1 + k, m_2, m_3 + e + I, e})$		v^3	v^3	
$(\mathbf{m_1 - (z-1)e, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e})$		v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e + I})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e, ze + I})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - ze, m_3 + (z+2)e, ze + k})$		v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e})$		$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e})$		$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + I, (z+1)e})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + I})$		$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + I})$		v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e, m_3 + (z+2)e, (z+1)e + k})$		v	v	$0 \leq z < x$

$(\mathbf{m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + I})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + k})$	$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e + I, (z+2)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+3)e, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+2)e})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (x-1)e, m_2 - (x+1)e + I, m_3 + (x+1)e + k, (x+1)e})$		v	
$(\mathbf{m_1 - (x-1)e, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e + I})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+2)e, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - xe, m_3 + (x+2)e, xe + I})$		v^2	
$(\mathbf{m_1 - xe + I, m_2 - xe, m_3 + (x+2)e, xe + k})$		v	
$(\mathbf{m_1 - xe + I, m_2 - xe, m_3 + (x+1)e + k, (x+1)e})$		$2v^2$	
$(\mathbf{m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+2)e, x + 1)e})$		$v + v^3$	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e + I, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e, (x+1)e + I})$		$2v^3$	

T

$\mathbf{b = 2x + y + 1}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + k})$	1	1	
$(\mathbf{m_1 + e + k, m_2 + k, m_3 + e})$	v	v	
$(\mathbf{m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e, ze + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + k, ze})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - ze + k, m_3 + (z+1)e, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze, m_3 + (z+1)e + k, ze + k})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + k, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e, ze + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + k, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e, xe + k})$		v	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + k, xe})$		v	
$(\mathbf{m_1 - (x+1)e, m_2 - xe + k, m_3 + (x+1)e, xe + k})$		v^2	
$(\mathbf{m_1 - (x-1)e, m_2 - xe, m_3 + (x+1)e + k, xe + k})$		$v + v^3$	
$(\mathbf{m_1 - xe + k, m_2 - xe, m_3 + (x+1)e + k, (x+1)e})$		v^2	

U

$\mathbf{b = 2x + y + 1}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	
$(\mathbf{m_1 + e + I, m_2 + e, m_3 + k})$	1	1	
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + I})$	v	v	
$(\mathbf{m_1 + e + I, m_2 + k, m_3 + e})$	v	v	
$(\mathbf{m_1 + e + k, m_2 + I, m_3 + e})$	v^2	v^2	
$(\mathbf{m_1 - (z-1)e + I, m_2 - ze, m_3 + (z+1)e, ze + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e, ze + I})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + I, m_3 + (z+1)e + k, ze})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + I, ze})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - ze + I, m_3 + (z+1)e, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - ze + k, m_3 + (z+1)e, ze + I})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze, m_3 + (z+1)e + I, ze + k})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze, m_3 + (z+1)e + k, ze + I})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + I, (z+1)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - (z+1)e + k, m_3 + (z+1)e + I, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - ze, m_3 + (z+2)e, ze + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e, ze + I})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + I, (z+1)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + I})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (x-1)e + I, m_2 - xe, m_3 + (x+1)e, xe + k})$		v	
$(\mathbf{m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e, xe + I})$		v^2	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + I, m_3 + (x+1)e + k, xe})$		v	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + I, xe})$		v^2	
$(\mathbf{m_1 - (x+1)e, m_2 - xe + I, m_3 + (x+1)e, xe + k})$		v^2	
$(\mathbf{m_1 - (x+1)e, m_2 - xe + k, m_3 + (x+1)e, xe + I})$		v^3	
$(\mathbf{m_1 - (x-1)e, m_2 - xe, m_3 + (x+1)e + I, xe + k})$		$v + v^3$	
$(\mathbf{m_1 - (x-1)e, m_2 - xe, m_3 + (x+1)e + k, xe + I})$		$v^2 + v^4$	
$(\mathbf{m_1 - xe + I, m_2 - xe, m_3 + (x+1)e + k, (x+1)e})$		v^2	
$(\mathbf{m_1 - xe + k, m_2 - xe, m_3 + (x+1)e + I, (x+1)e})$		v^3	

V

$b = 2x + y + 2$ where $0 \leq y < 2$	$b = 1$	$y = 0$	$y = 1$	
$(m_1 + e + I, m_2 + e + J, m_3 + k)$	1	1	1	
$(m_1 + e + I, m_2 + e, m_3 + k, J)$	v	v	v	
$(m_1 + e + I, m_2 + k, m_3 + e + J)$	v	v	v	
$(m_1 + e + I, m_2 + k, m_3 + e, J)$	v^2	v^2	v^2	
$(m_1 + e + J, m_2 + k, m_3 + e, I)$	v	v	v	
$(m_1 + e, m_2 + k, m_3 + e + J, I)$	v^2	v^2	v^2	
$(m_1 + e + I, m_2 + J, m_3 + e, k)$		v	v	
$(m_1 + e + I, m_2, m_3 + e + J, k)$		v^2	v^2	
$(m_1 + e + J, m_2 + I, m_3 + e + k)$		v	v	
$(m_1 + e, m_2 + I, m_3 + e + k, J)$		v^2	v^2	
$(m_1 + e + J, m_2 + I, m_3 + e, k)$		v^2	v^2	
$(m_1 + e, m_2 + I, m_3 + e + J, k)$		v^3	v^3	
$(m_1 + e + J, m_2, m_3 + e + k, I)$		v^2	v^2	
$(m_1 + e, m_2 + J, m_3 + e + k, I)$		v^3	v^3	
$(m_1 + e + J, m_2, m_3 + e + I, k)$		$v + v^3$	$v + v^3$	
$(m_1 + e, m_2 + J, m_3 + e + I, k)$		$v^2 + v^4$	$v^2 + v^4$	
$(m_1 + e + J, m_2 - e + k, m_3 + e + I, e)$		v^2	v^2	
$(m_1 + e, m_2 - e + k, m_3 + e + I, e + J)$		v^3	v^3	
$(m_1 + k, m_2 + I, m_3 + 2e + J)$		v^2	v^2	
$(m_1 + k, m_2 + I, m_3 + 2e, J)$		v^3	v^3	
$(m_1 + k, m_2 + J, m_3 + e + I, e)$		v^3	v^3	
$(m_1 + k, m_2, m_3 + e + I, e + J)$		v^4	v^4	
$(m_1 - (z-1)e + J, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e)$		v	v	$0 \leq z < x$
$(m_1 - (z-1)e, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e + J)$		v^2	v^2	$0 \leq z < x$
$(m_1 - (z-1)e + J, m_2 - (z+1)e, m_3 + (z+1)e + k, (z+1)e + I)$		v^2	v^2	$0 \leq z < x$
$(m_1 - (z-1)e, m_2 - (z+1)e + J, m_3 + (z+1)e + k, (z+1)e + I)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e + J, (z+1)e)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + J)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, ze + I)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + J, ze + I)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + I, m_2 - ze + J, m_3 + (z+2)e, ze + k)$		v	v	$0 \leq z < x$
$(m_1 - ze + I, m_2 - ze, m_3 + (z+2)e + J, ze + k)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + I, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)$		$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)$		$2v^3$	$2v^3$	$0 \leq z < x$
$(m_1 - ze + I, m_2 - (z+1)e + k, m_2 + (z+2)e + J, (z+1)e)$		$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + J)$		$v^2 + 2v^4$	$v^2 + 2v^4$	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e + k, m_3 + (z+2)e + I, (z+1)e)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + I, (z+1)e + J)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + I)$		$2v^3$	$2v^3$	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e + I)$		$2v^4$	$2v^4$	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + J, m_3 + (z+2)e, (z+1)e + I)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e, m_3 + (z+2)e + J, (z+1)e + I)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + I, m_2 - (z+1)e + J, m_3 + (z+2)e, (z+1)e + k)$		v	v	$0 \leq z < x$
$(m_1 - ze + I, m_2 - (z+1)e, m_3 + (z+2)e + J, (z+1)e + k)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e)$		v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e + k, (z+1)e + J)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k)$		$2v^2$	$2v^2$	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e + J, (z+1)e + k)$		$2v^3$	$2v^3$	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e, m_3 + (z+2)e + k, (z+1)e + I)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + J, m_3 + (z+2)e + k, (z+1)e + I)$		v^3	v^3	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+1)e + k)$		$v + 2v^3$	$v + 2v^3$	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + J, m_3 + (z+2)e + I, (z+1)e + k)$		$v^2 + 2v^4$	$v^2 + 2v^4$	$0 \leq z < x$
$(m_1 - ze + J, m_2 - (z+2)e + k, m_3 + (z+2)e + I, (z+2)e)$		v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+2)e + k, m_3 + (z+2)e + I, (z+2)e + J)$		v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+3)e + J, (z+1)e)$		v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e + I, m_3 + (z+3)e, (z+1)e + J)$		v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e + J, m_3 + (z+2)e + I, (z+2)e)$		$2v^3$	$2v^3$	$0 \leq z < x$
$(m_1 - (z+1)e + k, m_2 - (z+1)e, m_3 + (z+2)e + I, (z+2)e + J)$		$2v^4$	$2v^4$	$0 \leq z < x$
$(m_1 - (x-1)e + J, m_2 - (x+1)e + I, m_3 + (x+1)e + k, (x+1)e)$			v	
$(m_1 - (x-1)e, m_2 - (x+1)e + I, m_3 + (x+1)e + k, (x+1)e + J)$			v^2	
$(m_1 - (x-1)e + J, m_2 - (x+1)e, m_3 + (x+1)e + k, (x+1)e + I)$			v^2	
$(m_1 - (x-1)e, m_2 - (x+1)e + J, m_3 + (x+1)e + k, (x+1)e + I)$			v^3	
$(m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+2)e + J, (x+1)e)$			v^2	
$(m_1 - xe + k, m_2 - (x+1)e + I, m_3 + (x+2)e, (x+1)e + J)$			v^3	
$(m_1 - xe + k, m_2 - xe + J, m_3 + (x+2)e, xe + I)$			v^2	
$(m_1 - xe + k, m_2 - xe, m_3 + (x+2)e + J, xe + I)$			v^3	
$(m_1 - xe + I, m_2 - xe + J, m_3 + (x+2)e, xe + k)$			v	
$(m_1 - xe + I, m_2 - xe, m_3 + (x+2)e + J, xe + k)$			v^2	
$(m_1 - xe + I, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e)$			$2v^2$	

$(m_1 - xe + I, m_2 - xe, m_3 + (x+1)e + k, (x+1)e + J)$	$2v^3$
$(m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+2)e + J, x+1)e)$	$v + 2v^3$
$(m_1 - xe + I, m_2 - (x+1)e + k, m_3 + (x+2)e, x+1)e + J)$	$v^2 + 2v^4$
$(m_1 - xe + J, m_2 - (x+1)e + k, m_3 + (x+2)e + I, (x+1)e)$	v^2
$(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e + I, (x+1)e + J)$	v^3
$(m_1 - xe + J, m_2 - (x+1)e + k, m_3 + (x+2)e, (x+1)e + I)$	$2v^3$
$(m_1 - xe, m_2 - (x+1)e + k, m_3 + (x+2)e + J, (x+1)e + I)$	$2v^4$

W

$b = 2x + y + 1$ where $0 \le y < 2$	$y = 0$	$y = 1$	
$(m_1 + e + k, m_2 + e + J, m_3 + k)$	1	1	
$(m_1 + e + k, m_2 + e, m_3 + k, J)$	v	v	
$(m_1 + e + k, m_2 + k, m_3 + e + J)$	v	v	
$(m_1 + e + k, m_2 + k, m_3 + e, J)$	v^2	v^2	
$(m_1 - (z-1)e + k, m_2 - ze + J, m_3 + (z+1)e, ze + k)$	v	v	$0 \le z < x$
$(m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e + J, ze + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+1)e + k, ze)$	v	v	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+1)e, ze + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+1)e + k, ze + k)$	$v + v^3$	$v + v^3$	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + k, ze + J)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + J, ze + k)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+1)e + k, ze + k)$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)$	v^2	v^2	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e)$	v	v	$0 \le z < x$
$(m_1 - (z+1)e, m_2 - (z+1)e + k, m_3 + (z+1)e + k, (z+1)e + J)$	v^2	v^2	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, ze + k)$	v	v	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)$	v^2	v^2	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + J, ze + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)$	v^3	v^3	$0 \le z < x$
$(m_1 - ze + k, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e)$	$v + v^3$	$v + v^3$	$0 \le z < x$
$(m_1 - ze + k, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + J)$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$(m_1 - ze + J, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e + k)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e + k, m_2 - ze + J, m_3 + (z+1)e, ze + k)$		v	
$(m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e + J, xe + k)$		v^2	
$(m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+1)e + k, xe)$		v	
$(m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+1)e, xe + k)$		v^2	
$(m_1 - (x-1)e + J, m_2 - xe, m_3 + (x+1)e + k, xe + k)$		$v + v^3$	
$(m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + k, xe + J)$		v^2	
$(m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + J, xe + k)$		v^3	
$(m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+1)e + k, xe + k)$		$v^2 + v^4$	
$(m_1 - xe + k, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e)$		v^2	
$(m_1 - xe + k, m_2 - xe, m_3 + (x+1)e + k, (x+1)e + J)$		v^3	

X

$b = 2x + y + 1$ where $0 \le y < 2$	$y = 0$	$y = 1$	
$(m_1 + e + I, m_2 + e + J, m_3 + k)$	1	1	
$(m_1 + e + k, m_2 + e + J, m_3 + I)$	v	v	
$(m_1 + e + I, m_2 + e, m_3 + k, J)$	v	v	
$(m_1 + e + k, m_2 + e, m_3 + I, J)$	v^2	v^2	
$(m_1 + e + I, m_2 + k, m_3 + e + J)$	v	v	
$(m_1 + e + k, m_2 + I, m_3 + e + J)$	v^2	v^2	
$(m_1 + e + I, m_2 + k, m_3 + e, J)$	v^2	v^2	
$(m_1 + e + k, m_2 + I, m_3 + e, J)$	v^3	v^3	
$(m_1 - (z-1)e + I, m_2 - ze + J, m_3 + (z+1)e, ze + k)$	v	v	$0 \le z < x$
$(m_1 - (z-1)e + k, m_2 - ze + J, m_3 + (z+1)e, ze + I)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + I, m_2 - ze, m_3 + (z+1)e + J, ze + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e + J, ze + I)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + I, m_3 + (z+1)e + k, ze)$	v	v	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+1)e + I, ze)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + I, m_3 + (z+1)e, ze + k)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+1)e, ze + I)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+1)e + I, ze + k)$	$v + v^3$	$v + v^3$	$0 \le z < x$
$(m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+1)e + k, ze + I)$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + I, m_3 + (z+1)e + k, ze + J)$	v^2	v^2	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + I, ze + J)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + I, m_3 + (z+1)e + J, ze + k)$	v^3	v^3	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + J, ze + I)$	v^4	v^4	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+1)e + I, ze + k)$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$(m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+1)e + k, ze + I)$	$v^3 + v^5$	$v^3 + v^5$	$0 \le z < x$
$(m_1 - ze + I, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)$	v^2	v^2	$0 \le z < x$

$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 + (z+1)e + I, (z+1)e)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + I, (z+1)e + J)}$	v^4	v^4	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - (z+1)e + k, m_3 + (z+1)e + I, (z+1)e)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e, m_2 - (z+1)e + I, m_3 + (z+1)e + k, (z+1)e + J)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e, m_2 - (z+1)e + k, m_3 + (z+1)e + I, (z+1)e + J)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze + J, m_3 + (z+2)e, ze + k)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, ze + I)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 + (z+1)e + I, (z+1)e)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 + (z+2)e + J, ze + k)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + J, ze + I)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 + (z+1)e + I, (z+1)e + J)}$	v^4	v^4	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e)}$	$v + v^3$	$v + v^3$	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e + J, (z+1)e)}$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + J)}$	$v^2 + v^4$	$v^2 + v^4$	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + J)}$	$v^3 + v^5$	$v^3 + v^5$	$0 \le z < x$
$\mathbf{(m_1 - ze + J, m_2 - (z+1)e + I, m_3 + (z+2)e, (z+1)e + k)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + J, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + I)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + I, m_3 + (z+2)e + J, (z+1)e + k)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e + I)}$	v^4	v^4	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + I, m_2 - ze + J, m_3 + (z+1)e, ze + k)}$		v	
$\mathbf{(m_1 - (z-1)e + k, m_2 - ze + J, m_3 + (z+1)e, ze + I)}$		v^2	
$\mathbf{(m_1 - (x-1)e + I, m_2 - xe, m_3 + (x+1)e + J, xe + k)}$		v^2	
$\mathbf{(m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e + J, xe + I)}$		v^3	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + I, m_3 + (x+1)e + k, xe)}$		v	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+1)e + I, xe)}$		v^2	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + I, m_3 + (x+1)e, xe + k)}$		v^2	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+1)e, xe + I)}$		v^3	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe, m_3 + (x+1)e + I, xe + k)}$		$v + v^3$	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe, m_3 + (x+1)e + k, xe + I)}$		$v^2 + v^4$	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 + (x+1)e + k, xe + J)}$		v^2	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + I, xe + J)}$		v^3	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 + (x+1)e + J, xe + k)}$		v^3	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + J, xe + I)}$		v^4	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+1)e + I, xe + k)}$		$v^2 + v^4$	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+1)e + k, xe + I)}$		$v^3 + v^5$	
$\mathbf{(m_1 - xe + I, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e)}$		v^2	
$\mathbf{(m_1 - xe + k, m_2 - xe + J, m_3 + (x+1)e + I, (x+1)e)}$		v^3	
$\mathbf{(m_1 - xe + I, m_2 - xe, m_3 + (x+1)e + k, (x+1)e + J)}$		v^3	
$\mathbf{(m_1 - xe + k, m_2 - xe, m_3 + (x+1)e + I, (x+1)e + J)}$		v^4	

Y

$\mathbf{b = 2x + y + 1}$ where $\mathbf{0 \le y < 2}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	
$\mathbf{(m_1 + 2e, m_2 + e + J, m_3 + k)}$	1	1	
$\mathbf{(m_1 + e + k, m_2 + e + J, m_3 + e)}$	v	v	
$\mathbf{(m_1 + 2e, m_2 + e, m_3 + k, J)}$	v	v	
$\mathbf{(m_1 + 2e, m_2 + k, m_3 + e + J)}$	v	v	
$\mathbf{(m_1 + 2e, m_2 + k, m_3 + e, J)}$	v^2	v^2	
$\mathbf{(m_1 + e + k, m_2 + e, m_3 + e + J)}$	v^2	v^2	
$\mathbf{(m_1 + e + k, m_2 + e, m_3 + e, J)}$	$v + v^3$	$v + v^3$	
$\mathbf{(m_1 - (z-2)e, m_2 - ze + J, m_3 + (z+1)e, ze + k)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-2)e, m_2 - ze, m_3 + (z+1)e + J, ze + k)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e + J, (z+1)e)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - (z-1)e, m_3 + (z+1)e + k, ze)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - (z-1)e, m_3 + (z-1)e, ze + k)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+2)e, ze)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+1)e + k, (z+1)e)}$	$v + v^3$	$v + v^3$	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e, m_2 - ze + k, m_3 + (z+2)e, ze + J)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + J, (z+1)e)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+2)e, ze + k)}$	$v + v^3$	$v + v^3$	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e)}$	$2v^2 + v^4$	$2v^2 + v^4$	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, (z+1)e)}$	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+2)e, ze + k)}$	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + J)}$	$2v^2$	$2v^2$	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, (z+1)e)}$	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + J, (z+1)e)}$	$2v^2$	$2v^2$	$0 \le z < x$

$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e, (z+1)e + J})$	$3v^3$	$3v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + J, m_2 - ze, m_3 + (z+2)e, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+2)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (x-2)e, m_2 - xe + J, m_3 + (x+1)e, xe + k})$		v	
$(\mathbf{m_1 - (x-2)e, m_2 - xe, m_3 + (x+1)e + J, xe + k})$		v^2	
$(\mathbf{m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e + J, (x+1)e})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e, m_3 + (x+1)e + k, xe})$		v	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e, m_3 + (x-1)e, xe + k})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+2)e, xe})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe, m_3 + (x+1)e + k, x + 1)e})$		$v + v^3$	
$(\mathbf{m_1 - (x+1)e, m_2 - xe + k, m_3 + (x+2)e, xe + J})$		v^2	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + J, (x+1)e})$		v^3	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+2)e, xe + k})$		$v + v^3$	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e})$		$2v^2 + v^4$	
$(\mathbf{m_1 - (x-1)e, m_2 - xe, m_3 + (x+1)e + k, (x+1)e + J})$		v^3	
$(\mathbf{m_1 - xe + k, m_2 - xe + J, m_3 + (x+2)e, (x+1)e})$		v^3	

Z

$\mathbf{b = 2x + y + 1}$ where $\mathbf{0 \leq y < 2}$	$\mathbf{y = 1}$	$\mathbf{y = 1}$	
$(\mathbf{m_1 + 2e + H, m_2 + e + J, m_3 + k})$	1	1	
$(\mathbf{m_1 + 2e, m_2 + e + J, m_3 + k, H})$	v	v	
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + e + H})$	v	v	
$(\mathbf{m_1 + e + k, m_2 + e + J, m_3 + e, H})$	v^2	v^2	
$(\mathbf{m_1 + 2e + H, m_2 + e, m_3 + k, J})$	v	v	
$(\mathbf{m_1 + 2e, m_2 + e + H, m_3 + k, J})$	v^2	v^2	
$(\mathbf{m_1 + 2e + H, m_2 + k, m_3 + e + J})$	v	v	
$(\mathbf{m_1 + 2e, m_2 + k, m_3 + e + J, H})$	v^2	v^2	
$(\mathbf{m_1 + 2e + H, m_2 + k, m_3 + e, J})$	v^2	v^2	
$(\mathbf{m_1 + 2e, m_2 + k, m_3 + e + H, J})$	v^3	v^3	
$(\mathbf{m_1 + e + k, m_2 + e + H, m_3 + e + J})$	v^2	v^2	
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + e + J, H})$	v^3	v^3	
$(\mathbf{m_1 + e + k, m_2 + e + H, m_3 + e, J})$	$v + v^3$	$v + v^3$	
$(\mathbf{m_1 + e + k, m_2 + e, m_3 + e + H, J})$	$v^2 + v^4$	$v^2 + v^4$	
$(\mathbf{m_1 - (z-2)e + H, m_2 - ze + J, m_3 + (z+1)e, ze + k})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-2)e, m_2 - ze + J, m_3 + (z+1)e + H, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-2)e + H, m_2 - ze, m_3 + (z+1)e + J, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-2)e, m_2 - ze + H, m_3 + (z+1)e + J, ze + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + k, m_2 - ze + H, m_3 + (z+1)e + J, (z+1)e})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + k, m_2 - ze, m_3 + (z+1)e + J, (z+1)e + H})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - (z-1)e + H, m_3 + (z+1)e + k, ze})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - (z-1)e, m_3 + (z+1)e + k, ze + H})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - (z-1)e + H, m_3 + (z-1)e, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - (z-1)e, m_3 + (z-1)e + H, ze + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+2)e + H, ze})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze + k, m_3 + (z+2)e, ze + H})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze + H, m_3 + (z+1)e + k, (z+1)e})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + H})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e + H, m_2 - ze + k, m_3 + (z+2)e, ze + J})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z+1)e, m_2 - ze + k, m_3 + (z+2)e + H, ze + J})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - ze + k, m_3 + (z+1)e + J, (z+1)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + k, m_3 + (z+1)e + J, (z+1)e + H})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - ze + J, m_3 + (z+2)e, ze + k})$	$v + v^3$	$v + v^3$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+2)e + H, ze + k})$	$v^2 + v^4$	$v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e})$	$2v^2 + v^4$	$2v^2 + v^4$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + J, m_3 + (z+1)e + k, (z+1)e + H})$	$2v^3 + v^5$	$2v^3 + v^5$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + H, m_3 + (z+1)e + k, (z+1)e + J})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e + H, (z+1)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, (z+1)e + H})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze + H, m_3 + (z+2)e, ze + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + J, m_2 - ze, m_3 + (z+2)e + H, ze + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - ze, m_3 + (z+1)e + k, (z+1)e + J})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - ze + H, m_3 + (z+1)e + k, (z+1)e + J})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+1)e + H})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e + H, m_2 - (z+1)e + k, m_3 + (z+2)e, (z+1)e + J})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - (z-1)e, m_2 - (z+1)e + k, m_3 + (z+2)e + H, (z+1)e + J})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e + H, (z+1)e})$	v	v	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + J, m_3 + (z+2)e, (z+1)e + H})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + H, m_3 + (z+2)e + J, (z+1)e})$	$2v^2$	$2v^2$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + J, (z+1)e + H})$	$2v^3$	$2v^3$	$0 \leq z < x$
$(\mathbf{m_1 - ze + k, m_2 - ze + H, m_3 + (z+2)e, (z+1)e + J})$	$3v^3$	$3v^3$	$0 \leq z < x$

$(\mathbf{m_1 - ze + k, m_2 - ze, m_3 + (z+2)e + H, (z+1)e + J})$	$3v^4$	$3v^4$	$0 \leq z < x$
$(\mathbf{m_1 - ze + J, m_2 - ze + H, m_3 + (z+2)e, (z+1)e + k})$	v^2	v^2	$0 \leq z < x$
$(\mathbf{m_1 - ze + J, m_2 - ze, m_3 + (z+2)e + H, (z+1)e + k})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze + H, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+2)e})$	v^3	v^3	$0 \leq z < x$
$(\mathbf{m_1 - ze, m_2 - (z+1)e + k, m_3 + (z+2)e + J, (z+2)e + H})$	v^4	v^4	$0 \leq z < x$
$(\mathbf{m_1 - (x-2)e + H, m_2 - xe + J, m_3 + (x+1)e, xe + k})$		v	
$(\mathbf{m_1 - (x-2)e, m_2 - xe + J, m_3 + (x+1)e + H, xe + k})$		v^2	
$(\mathbf{m_1 - (x-2)e + H, m_2 - xe, m_3 + (x+1)e + J, xe + k})$		v^2	
$(\mathbf{m_1 - (x-2)e, m_2 - xe + H, m_3 + (x+1)e + J, xe + k})$		v^3	
$(\mathbf{m_1 - (x-1)e + k, m_2 - xe + H, m_3 + (x+1)e + J, (x+1)e})$		v^2	
$(\mathbf{m_1 - (x-1)e + k, m_2 - xe, m_3 + (x+1)e + J, (x+1)e + H})$		v^3	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e + H, m_3 + (x+1)e + k, xe})$		v	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e, m_3 + (x+1)e + k, xe + H})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e + H, m_3 + (x-1)e, xe + k})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - (x-1)e, m_3 + (x-1)e + H, xe + k})$		v^3	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+2)e + H, xe})$		v^2	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe + k, m_3 + (x+2)e, xe + H})$		v^3	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe + H, m_3 + (x+1)e + k, x+1)e})$		$v + v^3$	
$(\mathbf{m_1 - (x-1)e + J, m_2 - xe, m_3 + (x+1)e + k, x+1)e + H})$		$v^2 + v^4$	
$(\mathbf{m_1 - (x+1)e + H, m_2 - xe + k, m_3 + (x+2)e, xe + J})$		v^2	
$(\mathbf{m_1 - (x+1)e, m_2 - xe + k, m_3 + (x+2)e + H, xe + J})$		v^3	
$(\mathbf{m_1 - (x-1)e + H, m_2 - xe + k, m_3 + (x+1)e + J, (x+1)e})$		v^3	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + k, m_3 + (x+1)e + J, (x+1)e + H})$		v^4	
$(\mathbf{m_1 - (x-1)e + H, m_2 - xe + J, m_3 + (x+2)e, xe + k})$		$v + v^3$	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+2)e + H, xe + k})$		$v^2 + v^4$	
$(\mathbf{m_1 - (x-1)e + H, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e})$		$2v^2 + v^4$	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + J, m_3 + (x+1)e + k, (x+1)e + H})$		$2v^3 + v^5$	
$(\mathbf{m_1 - (x-1)e + H, m_2 - xe, m_3 + (x+1)e + k, (x+1)e + J})$		v^3	
$(\mathbf{m_1 - (x-1)e, m_2 - xe + H, m_3 + (x+1)e + k, (x+1)e + J})$		v^4	
$(\mathbf{m_1 - xe + k, m_2 - xe + J, m_3 + (x+2)e + H, (x+1)e})$		v^3	
$(\mathbf{m_1 - xe + k, m_2 - xe + J, m_3 + (x+2)e, (x+1)e + H})$		v^4	

Spoke $c : a = b = 0$

D

$\mathbf{c = 4x + y + 2}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{c = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$(m_1 + i + j, m_2 + j, m_3)$	1	1	1	1	1	
$(m_1 + i + j, m_2, m_3 - e + j, e)$		v	v	v	v	
$(m_1 - ze + j, m_2 - ze, m_3 - (z+1)e, (3z+1)e + i + j)$		v	v	v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + i + j, m_3 - (z+1)e + j, (3z+2)e)$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze + j, m_2 - (z+1)e + i + j, m_3 - (z+1)e, (3z+2)e)$		v	v	v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + i + j, m_3 - (z+1)e, (3z+2)e + j)$		v	v	v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + j, m_3 - (z+1)e, (3z+2)e + i + j)$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + j, m_3 - (z+2)e + i + j, (3z+3)e)$		v	v	v	v	$0 \leq z < x$
$(m_1 - (z+1)e, m_2 - (z+1)e, m_3 - (z+2)e + i + j, (3z+3)e + j)$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + j, m_2 - (z+1)e + j, m_3 - (z+1)e, (3z+3)e)$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + j, m_2 - (z+1)e, m_3 - (z+2)e + j, (3z+4)e)$		v	v	v	v	$0 \leq z < x$
$(m_1 - (z+1)e + j, m_2 - (z+1)e, m_3 - (z+2)e + i + j, (3z+4)e)$		v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - xe + j, m_2 - xe, m_3 - (x+1)e, (3x+1)e + i + j)$			v	v	v	
$(m_1 - xe, m_2 - (x+1)e + i + j, m_3 - (x+1)e + j, (3x+2)e)$			v^2	v^2	v^2	
$(m_1 - xe + j, m_2 - (x+1)e + i + j, m_3 - (x+1)e, (3x+2)e)$				v	v	
$(m_1 - xe, m_2 - (x+1)e + i + j, m_3 - (x+1)e, (3x+2)e + j)$				v	v	
$(m_1 - xe, m_2 - (x+1)e + j, m_3 - (x+1)e, (3x+2)2e + i + j)$				v^2	v^2	
$(m_1 - xe, m_2 - (x+1)e + j, m_3 - (x+2)e + i + j, (3x+3)e)$					v	
$(m_1 - (x+1)e, m_2 - (x+1)e, m_3 - (x+2)e + i + j, (3x+3)3e + j)$					v^2	
$(m_1 - (x+1)e + i + j, m_2 - (x+1)e + j, m_3 - (x+1)e, (3x+3)e)$					v^2	

N

$\mathbf{c = 4x + y + 2}$ where $\mathbf{0 \leq y < 4}$	$\mathbf{c = 0}$	$\mathbf{c = 1}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$(m_1 + i + j + k, m_2 + j + k, m_3 + k)$	1	1	1	1	1	1	
$(m_1 + i + j + k, m_2 + j + k, m_3, k)$		v	v	v	v	v	
$(m_1 + i + j + k, m_2 + k, m_3 - e + j + k, e)$			v	v	v	v	
$(m_1 + i + j + k, m_2, m_3 - e + j + k, e + k)$			v^2	v^2	v^2	v^2	
$(m_1 - ze + j + k, m_2 - ze + k, m_3 - (z+1)e, (3z+1)e + i + j + k)$			v	v	v	v	$0 \leq z < x$
$(m_1 - ze + j + k, m_2 - ze, m_3 - (z+1)e + k, (3z+1)e + i + j + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e + j + k, (3z+2)e)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e + j + k, (z+2)e + k)$			v^3	v^3	v^3	v^3	$0 \leq z < x$
$(m_1 - ze + j + k, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e + k, (3z+2)e)$			v	v	v	v	$0 \leq z < x$
$(m_1 - ze + j + k, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e, (3z+2)e + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e, (3z+2)e + j + k)$			v	v	v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + i + j + k, m_3 - (z+1)e + k, (3z+2)e + j + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + j + k, m_3 - (z+1)e, (3z+2)e + i + j + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + j + k, m_3 - (z+1)e + k, (3z+2)e + i + j + k)$			v^3	v^3	v^3	v^3	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e + j + k, m_3 - (z+2)e + i + j + k, (3z+3)e)$			v	v	v	v	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + j + k, m_3 - (z-2)e + i + j + k, (3z+3)e + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze + k, m_2 - (z+1)e, m_3 - (z+2)e + i + j + k, (3z+3)e + j + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - ze, m_2 - (z+1)e + k, m_3 - (z+2)e + i + j + k, (3z+3)e + j + k)$			v^3	v^3	v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + j + k, m_3 - (z+1)e + k, (3z+3)e)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + j + k, m_3 - (z+1)e, (3z+3)e + k)$			v^3	v^3	v^3	v^3	$0 \leq z < x$
$(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e + k, m_3 - (z-2)e + j + k, (3z+4)e)$			v	v	v	v	$0 \leq z < x$
$(m_1 - (z+1)e + i + j + k, m_2 - (z+1)e, m_3 - (z+2)e + j + k, (3z+4)e + k)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + j + k, m_2 - (z+1)e + k, m_3 - (z+2)e + i + j + k, (3z+4)e)$			v^2	v^2	v^2	v^2	$0 \leq z < x$
$(m_1 - (z+1)e + j + k, m_2 - (z+1)e, m_3 - (z+2)e + i + j + k, (3z+4)e + k)$			v^3	v^3	v^3	v^3	$0 \leq z < x$
$(m_1 - xe + j + k, m_2 - xe + k, m_3 - (x+1)e, (3x+1)e + i + j + k)$				v	v	v	
$(m_1 - xe + j + k, m_2 - xe, m_3 - (x+1)e + k, (3x+1)e + i + j + k)$				v^2	v^2	v^2	
$(m_1 - xe + k, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e + j + k, (3x+2)e)$				v^2	v^2	v^2	
$(m_1 - xe, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e + j + k, (x+2)e + k)$				v^3	v^3	v^3	
$(m_1 - xe + j + k, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e + k, (3x+2)e)$					v	v	
$(m_1 - xe + j + k, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e, (3x+2)e + k)$					v^2	v^2	
$(m_1 - xe + k, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e, (3x+2)e + j + k)$					v	v	
$(m_1 - xe, m_2 - (x+1)e + i + j + k, m_3 - (x+1)e + k, (3x+2)e + j + k)$					v^2	v^2	
$(m_1 - xe + k, m_2 - (x+1)e + j + k, m_3 - (x+1)e, (3x+2)e + i + j + k)$					v^2	v^2	
$(m_1 - xe, m_2 - (x+1)e + j + k, m_3 - (x+1)e + k, (3x+2)e + i + j + k)$					v^3	v^3	
$(m_1 - xe + k, m_2 - (x+1)e + j + k, m_3 - (x+2)e + i + j + k, (3x+3)e)$						v	
$(m_1 - xe, m_2 - (x+1)e + j + k, m_3 - (x-2)e + i + j + k, (3x+3)e + k)$						v^2	
$(m_1 - xe + k, m_2 - (x+1)e, m_3 - (x+2)e + i + j + k, (3x+3)e + j + k)$						v^2	
$(m_1 - xe, m_2 - (x+1)e + k, m_3 - (x+2)e + i + j + k, (3x+3)e + j + k)$						v^3	
$(m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + j + k, m_3 - (x+1)e + k, (3x+3)e)$						v^2	
$(m_1 - (x+1)e + i + j + k, m_2 - (x+1)e + j + k, m_3 - (x+1)e, (3x+3)e + k)$						v^3	

O

$c = 4x + y + 2$ where $0 \le y < 4$	$c=0$	$c=1$	$y=0$	$y=1$	$y=2$	$y=3$	
(m_1+e, m_2+j+k, m_3+k)	1	1	1	1	1	1	
(m_1+e, m_2+j+k, m_3, k)		v	v	v	v	v	
$(m_1+e, m_2+k, m_3-e+j+k, e)$			v	v	v	v	
$(m_1+e, m_2, m_3-e+j+k, e+k)$			v^2	v^2	v^2	v^2	
$(m_1-ze+j+k, m_2-ze, m_3-(z+1)e+k, (3z+2)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze+k, m_2-ze, m_3-(z+1)e+j+k, (3z+2)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+k, m_2-ze, m_3-(z+1)e, (3z+2)e+j+k)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+j+k, m_3-(z+1)e+k, (3z+3)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+k, m_2-(z+1)e+j+k, m_3-(z+1)e, (3z+3)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+j+k, m_3-(z+1)e, (3z+3)e+k)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+k, m_3-(z+1)e, (3z+3)e+j+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-3z, m_2-(z+1)e+k, m_3-(z+2)e+j+k, (3z+4)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e, m_3-(z+2)e+j+k, (3z+4)e+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(z+1)e+j+k, m_2-(z+1)e+k, m_3-(z+1)e, (3z+4)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-xe+j+k, m_2-xe, m_3-(x+1)e+k, (3x+2)e)$				v	v	v	
$(m_1-xe+k, m_2-xe, m_3-(x+1)e+j+k, (3x+2)e)$				v^2	v^2	v^2	
$(m_1-xe+k, m_2-xe, m_3-(x+1)e, (3x+2)e+j+k)$					v	v	
$(m_1-xe, m_2-(x+1)e+j+k, m_3-(x+1)e+k, (3x+3)e)$					v^2	v^2	
$(m_1-xe+k, m_2-(x+1)e+j+k, m_3-(x+1)e, (3x+3)e)$						v	
$(m_1-xe, m_2-(x+1)e+j+k, m_3-(x+1)e, (3x+3)e+k)$						v	
$(m_1-xe, m_2-(x+1)e+k, m_3-(x+1)e, (3x+3)e+j+k)$						v^2	

P

$c = 4x + y + 2$ where $0 \le y < 4$	$c=0$	$c=1$	$y=0$	$y=1$	$y=2$	$y=3$	
$(m_1+e+I, m_2+j+k, m_3+k)$	1	1	1	1	1	1	
$(m_1+e, m_2+j+k, m_3+k, I)$	v	v	v	v	v	v	
$(m_1+e+I, m_2+j+k, m_3, k)$		v	v	v	v	v	
$(m_1+e, m_2+j+k, m_3+I, k)$		v^2	v^2	v^2	v^2	v^2	
$(m_1+e+I, m_2+k, m_3-e+j+k, e)$			v	v	v	v	
$(m_1+e, m_2+k, m_3-e+j+k, e+I)$			v^2	v^2	v^2	v^2	
$(m_1+e+I, m_2, m_3-e+j+k, e+k)$			v^2	v^2	v^2	v^2	
$(m_1+e, m_2+I, m_3-e+j+k, e+k)$			v^3	v^3	v^3	v^3	
$(m_1-ze+j+k, m_2-ze+I, m_3-(z+1)e+k, (3z+2)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze+j+k, m_2-ze, m_3-(z+1)e+k, (3z+2)e+I)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+k, m_2-ze+I, m_3-(z+1)e+j+k, (3z+2)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+k, m_2-ze, m_3-(z+1)e+j+k, (3z+2)e+I)$			v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-ze+k, m_2-ze+I, m_3-(z+1)e, (3z+2)e+j+k)$			v	v	v	v	$0 \le z < x$
$(m_1-ze+k, m_2-ze, m_3-(z+1)e+I, (3z+2)e+j+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+I, m_2-(z+1)e+j+k, m_3-(z+1)e+k, (3z+3)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-3z, m_2-(z+1)e+j+k, m_3-(z+1)e+k, (3z+3)e+I)$			v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-ze+k, m_2-(z+1)e+j+k, m_3-(z+1)e+I, (3z+3)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze+k, m_2-(z+1)e+j+k, m_3-(z+1)e, (3z+3)e+I)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+I, m_2-(z+1)e+j+k, m_3-(z+1)e, (3z-3)e+k)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+j+k, m_3-(z+1)e+I, (3z+3)e+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+I, m_2-(z+1)e+k, m_3-(z+1)e, (3z+3)e+j+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+k, m_3-(z+1)e+I, (3z+3)e+j+k)$			v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-ze+I, m_2-(z+1)e+k, m_3-(z+2)e+j+k, (3z+4)e)$			v	v	v	v	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+k, m_3-(z+2)e+j+k, (3z+4)e+I)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze+I, m_2-(z+1)e, m_3-(z+2)e+j+k, (3z+4)e+k)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-ze, m_2-(z+1)e+I, m_3-(z+2)e+j+k, (3z+4)e+k)$			v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-(z+1)e+j+k, m_2-(z+1)e+k, m_3-(z+1)e+I, (3z+4)e)$			v^2	v^2	v^2	v^2	$0 \le z < x$
$(m_1-(z+1)e+j+k, m_2-(z+1)e+k, m_3-(z+1)e, (3z+4)e+I)$			v^3	v^3	v^3	v^3	$0 \le z < x$
$(m_1-xe+j+k, m_2-xe+I, m_3-(x+1)e+k, (3x+2)e)$				v	v	v	
$(m_1-xe+j+k, m_2-xe, m_3-(x+1)e+k, (3x+2)e+I)$				v^2	v^2	v^2	
$(m_1-xe+k, m_2-xe+I, m_3-(x+1)e+j+k, (3x+2)e)$				v^2	v^2	v^2	
$(m_1-xe+k, m_2-xe, m_3-(x+1)e+j+k, (3x+2)e+I)$				v^3	v^3	v^3	
$(m_1-xe+k, m_2-xe+I, m_3-(x+1)e, (3x+2)e+j+k)$					v	v	
$(m_1-xe+k, m_2-xe, m_3-(x+1)e+I, (3x+2)e+j+k)$					v^2	v^2	
$(m_1-xe+I, m_2-(x+1)e+j+k, m_3-(x+1)e+k, (3x+3)e)$					v^2	v^2	
$(m_1-3x, m_2-(x+1)e+j+k, m_3-(x+1)e+k, (3x+3)e+I)$					v^3	v^3	
$(m_1-xe+k, m_2-(x+1)e+j+k, m_3-(x+1)e+I, (3x+3)e)$						v	
$(m_1-xe+k, m_2-(x+1)e+j+k, m_3-(x+1)e, (3x+3)e+I)$						v^2	
$(m_1-xe+I, m_2-(x+1)e+j+k, m_3-(x+1)e, (3x-3)e+k)$						v	
$(m_1-xe, m_2-(x+1)e+j+k, m_3-(x+1)e+I, (3x+3)e+k)$						v^2	
$(m_1-xe+I, m_2-(x+1)e+k, m_3-(x+1)e, (3x+3)e+j+k)$						v^2	
$(m_1-xe, m_2-(x+1)e+k, m_3-(x+1)e+I, (3x+3)e+j+k)$						v^3	

S

$\mathbf{c = 4x + y + 1}$ where $\mathbf{0 \le y < 4}$	$\mathbf{c = 0}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$\mathbf{(m_1 + e + I, m_2 + e, m_3 + k)}$	1	1	1	1	1	
$\mathbf{(m_1 + e + I, m_2 + e, m_3, k)}$		v	v	v	v	
$\mathbf{(m_1 - (z-1)e + I, m_2 - ze + k, m_3 - ze, (3z+1)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + k, m_3 - ze, (3z+1)e + I)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + I, m_3 - ze, (3z+1)e + k)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + I, m_3 - (z+1)e + k, (3z+2)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze, m_3 - (z+1)e + k, (3z+2)e + I)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + I, m_3 - ze, (3z+2)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 - (z+1)e + I, (3z+3)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 - (z+1)e + k, (3z+3)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 - (z+1)e, (3z+3)e + k)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 - (z+1)e + I, (3z+4)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (x-1)e + I, m_2 - xe + k, m_3 - xe, (3x+1)e)}$			v	v	v	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + k, m_3 - xe, (3x+1)e + I)}$			v	v	v	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 - xe, (3x+1)e + k)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 - (x+1)e + k, (3x+2)e)}$				v	v	
$\mathbf{(m_1 - (x-1)e, m_2 - xe, m_3 - (x+1)e + k, (3x+2)e + I)}$				v^2	v^2	
$\mathbf{(m_1 - xe + k, m_2 - xe + I, m_3 - xe, (3x+2)e)}$				v^2	v^2	
$\mathbf{(m_1 - xe + k, m_2 - xe, m_3 - (x+1)e + I, (3x+3)e)}$					v	
$\mathbf{(m_1 - xe + I, m_2 - xe, m_3 - (x+1)e + k, (3x+3)e)}$					v^2	

V

$\mathbf{c = 4x + y + 1}$ where $\mathbf{0 \le y < 4}$	$\mathbf{c = 0}$	$\mathbf{y = 0}$	$\mathbf{y = 1}$	$\mathbf{y = 2}$	$\mathbf{y = 3}$	
$\mathbf{(m_1 + e + I, m_2 + e + J, m_3 + k)}$	1	1	1	1	1	
$\mathbf{(m_1 + e + I, m_2 + e, m_3 + k, J)}$	v	v	v	v	v	
$\mathbf{(m_1 + e + I, m_2 + e + J, m_3, k)}$		v	v	v	v	
$\mathbf{(m_1 + e + I, m_2 + e, m_3 + J, k)}$		v^2	v^2	v^2	v^2	
$\mathbf{(m_1 - (z-1)e + I, m_2 - ze + k, m_3 - ze + J, (3z+1)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1) + e + I, m_2 - ze + k, m_3 - ze, (3z+1)e + J)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze + k, m_3 - ze, (z+1)e + I)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + k, m_3 - ze + J, (3z+1)e + I)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze + I, m_3 - ze, (3z+1)e + k)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + I, m_3 - ze + J, (3z+1)e + k)}$		v^3	v^3	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e + J, m_2 - ze + I, m_3 - (z+1)e + k, (3z+2)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - (z-1)e, m_2 - ze + I, m_3 - (z+1)e + k, (3z+2)e + J)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e + J, m_2 - ze, m_3 - (z+1)e + k, (3z+2)e + I)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - (z+1)e, m_2 - ze + J, m_3 - (z+1)e + k, (3z+2)e + I)}$		v^3	v^3	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + I, m_3 - ze + J, (3z+2)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + I, m_3 - ze, (3z+2)e + J)}$		v^3	v^3	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze + J, m_3 - (z+1)e + I, (3x+3)e)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + k, m_2 - ze, m_3 - (z+1)e + I, (3z+3)e + J)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze + J, m_3 - (z+1)e + k, (3z+3)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 - (z+1)e + k, (3z+3)e + J)}$		v^3	v^3	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze + J, m_3 - (z+1)e, (3z+3)e + k)}$		v	v	v	v	$0 \le z < x$
$\mathbf{(m_1 - ze + I, m_2 - ze, m_3 - (z+1)e + J, (3z+3)e + k)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze + J, m_2 - (z+1)e + k, m_3 - (z+1)e + I, (3z+4)e)}$		v^2	v^2	v^2	v^2	$0 \le z < x$
$\mathbf{(m_1 - ze, m_2 - (z+1)e + k, m_3 - (z+1)e + I, (3z+4)e + J)}$		v^3	v^3	v^3	v^3	$0 \le z < x$
$\mathbf{(m_1 - (x-1)e + I, m_2 - xe + k, m_3 - xe + J, (3x+1)e)}$			v	v	v	
$\mathbf{(m_1 - (x-1) + e + I, m_2 - xe + k, m_3 - xe, (3x+1)e + J)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + k, m_3 - xe, (x+1)e + I)}$			v	v	v	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + k, m_3 - xe + J, (3x+1)e + I)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + I, m_3 - xe, (3x+1)e + k)}$			v^2	v^2	v^2	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 - xe + J, (3x+1)e + k)}$			v^3	v^3	v^3	
$\mathbf{(m_1 - (x-1)e + J, m_2 - xe + I, m_3 - (x+1)e + k, (3x+2)e)}$				v	v	
$\mathbf{(m_1 - (x-1)e, m_2 - xe + I, m_3 - (x+1)e + k, (3x+2)e + J)}$				v^2	v^2	
$\mathbf{(m_1 - (x+1)e + J, m_2 - xe, m_3 - (x+1)e + k, (3x+2)e + I)}$				v^2	v^2	
$\mathbf{(m_1 - (x+1)e, m_2 - xe + J, m_3 - (x+1)e + k, (3x+2)e + I)}$				v^3	v^3	
$\mathbf{(m_1 - xe + k, m_2 - xe + I, m_3 - xe + J, (3x+2)e)}$				v^2	v^2	
$\mathbf{(m_1 - xe + k, m_2 - xe + I, m_3 - xe, (3x+2)e + J)}$				v^3	v^3	
$\mathbf{(m_1 - xe + k, m_2 - xe + J, m_3 - (x+1)e + I, (3x+3)e)}$					v	
$\mathbf{(m_1 - xe + k, m_2 - xe, m_3 - (x+1)e + I, (3x+3)e + J)}$					v^2	
$\mathbf{(m_1 - xe + I, m_2 - xe + J, m_3 - (x+1)e + k, (3x+3)e)}$					v^2	
$\mathbf{(m_1 - xe + I, m_2 - xe, m_3 - (x+1)e + k, (3x+3)e + J)}$					v^3	

E

c	**1**	**> 1**
$(\mathbf{m_1+e, m_2+j, m_3})$	1	1
$(\mathbf{m_1+e, m_2, m_3-e+j, e})$		v

F

c	**1**	**> 1**
$(\mathbf{m_1+e+I, m_2+j, m_3})$	1	1
$(\mathbf{m_1+e+I, m_2, m_3-e+j, e})$		v
$(\mathbf{m_1+e, m_2+j, m_3, I})$	v	v
$(\mathbf{m_1+e, m_2, m_3-e+j, e+I})$		v^2

Q

c	**0**	**1**	**> 1**
$(\mathbf{m_1+e+k, m_2+j+k, m_3+k})$	1	1	1
$(\mathbf{m_1+e+k, m_2+j+k, m_3, k})$		v	v
$(\mathbf{m_1+e+k, m_2+k, m_3-e+j+k, e})$			v
$(\mathbf{m_1+e+k, m_2, m_3-e+j+k, e+k})$			v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, k})$		v^2	v^2
$(\mathbf{m_1+e, m_2+k, m_3-e+j+k, e+k})$			v^3

R

c	**0**	**1**	**> 1**
$(\mathbf{m_1+e+I, m_2+j+k, m_3+k})$	1	1	1
$(\mathbf{m_1+e+k, m_2+j+k, m_3+I})$	v	v	v
$(\mathbf{m_1+e+I, m_2+j+k, m_3, k})$		v	v
$(\mathbf{m_1+e+I, m_2+k, m_3-e+j+k, e})$			v
$(\mathbf{m_1+e+I, m_2, m_3-e+j+k, e+k})$			v^2
$(\mathbf{m_1+e+k, m_2+j+k, m_3, I})$		v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3-e+j+k, e})$			v^2
$(\mathbf{m_1+e+k, m_2, m_3-e+j+k, e+I})$			v^3
$(\mathbf{m_1+e, m_2+j+k, m_3+I, k})$		v^2	v^2
$(\mathbf{m_1+e, m_2+j+k, m_3+k, I})$		v^3	v^3
$(\mathbf{m_1+e, m_2+I, m_3-e+j+k, e+k})$			v^3
$(\mathbf{m_1+e, m_2+k, m_3-e+j+k, e+I})$			v^4

T

c	**0**	**1**	**> 1**
$(\mathbf{m_1+e+k, m_2+e, m_3+k})$	1	1	1
$(\mathbf{m_1+e+k, m_2+e, m_3, k})$		v	v
$(\mathbf{m_1+e+k, m_2+k, m_3, e})$			v
$(\mathbf{m_1+e, m_2+k, m_3, e+k})$			v^2

B

c	**0**	**1**	**> 1**
$(\mathbf{m_1+e+I, m_2+e, m_3+k})$	1	1	1
$(\mathbf{m_1+e+k, m_2+e, m_3+I})$	v	v	v
$(\mathbf{m_1+e+I, m_2+e, m_3, k})$		v	v
$(\mathbf{m_1+e+I, m_2+k, m_3, e})$			v
$(\mathbf{m_1+e+k, m_2+e, m_3, I})$		v^2	v^2
$(\mathbf{m_1+e+k, m_2+I, m_3, e})$			v^2
$(\mathbf{m_1+e, m_2+I, m_3, e+k})$			v^2
$(\mathbf{m_1+e, m_2+k, m_3, e+I})$			v^3

W

c	**0**	**1**	**> 1**
$(\mathbf{m_1+e+k, m_2+e+J, m_3+k})$	1	1	1
$(\mathbf{m_1+e+k, m_2+e+J, m_3, k})$		v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+k, J})$	v	v	v
$(\mathbf{m_1+e+k, m_2+e, m_3+J, k})$		v^2	v^2
$(\mathbf{m_1+e+k, m_2+k, m_3+J, e})$			v
$(\mathbf{m_1+e+k, m_2+k, m_3, e+J})$			v^2
$(\mathbf{m_1+e+J, m_2+k, m_3, e+k})$			v^2
$(\mathbf{m_1+e, m_2+k, m_3+J, e+k})$			v^3

X

c	**0**	**1**	$> \mathbf{1}$
$\mathbf{(m_1+e+I, m_2+e+J, m_3+k)}$	1	1	1
$\mathbf{(m_1+e+k, m_2+e+J, m_3+I)}$	v	v	v
$\mathbf{(m_1+e+I, m_2+e+J, m_3, k)}$		v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+k, J)}$	v	v	v
$\mathbf{(m_1+e+I, m_2+e, m_3+J, k)}$		v^2	v^2
$\mathbf{(m_1+e+I, m_2+k, m_3+J, e)}$			v
$\mathbf{(m_1+e+I, m_2+k, m_3, e+J)}$			v^2
$\mathbf{(m_1+e+k, m_2+e+J, m_3, I)}$		v^2	v^2
$\mathbf{(m_1+e+k, m_2+e, m_3+I, J)}$	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+e, m_3+J, I)}$		v^3	v^3
$\mathbf{(m_1+e+k, m_2+I, m_3+J, e)}$			v^2
$\mathbf{(m_1+e+k, m_2+I, m_3, e+J)}$			v^3
$\mathbf{(m_1+e+J, m_2+I, m_3, e+k)}$			v^2
$\mathbf{(m_1+e+J, m_2+k, m_3, e+I)}$			v^3
$\mathbf{(m_1+e, m_2+I, m_3+J, e+k)}$			v^3
$\mathbf{(m_1+e, m_2+k, m_3+J, e+I)}$			v^4

Y

c	**0**	**1**	**2**	$> \mathbf{2}$
$\mathbf{(m_1+2e, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+k, m_2+e+J, m_3+e)}$	v	v	v	v
$\mathbf{(m_1+2e, m_2+e+J, m_3, k)}$		v	v	v
$\mathbf{(m_1+2e, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+2e, m_2+e, m_3+J, k)}$		v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2+k, m_3+J, e)}$			v	v
$\mathbf{(m_1+2e, m_2+k, m_3, e+J)}$			v^2	v^2
$\mathbf{(m_1+e+k, m_2+e+J, m_3, e)}$			v	v
$\mathbf{(m_1+e+k, m_2+e, m_3+J, e)}$		v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e+k, m_2+e, m_3, e+J)}$			v^3	v^3
$\mathbf{(m_1+e+J, m_2+e, m_3, e+k)}$			v^2	v^2
$\mathbf{(m_1+e+J, m_2+k, m_3, 2e)}$				v^2
$\mathbf{(m_1+e, m_2+k, m_3+J, 2e)}$			v^3	v^3

Z

c	**0**	**1**	**2**	$> \mathbf{2}$
$\mathbf{(m_1+2e+H, m_2+e+J, m_3+k)}$	1	1	1	1
$\mathbf{(m_1+e+k, m_2+e+J, m_3+e+H)}$	v	v	v	v
$\mathbf{(m_1+2e+H, m_2+e+J, m_3, k)}$		v	v	v
$\mathbf{(m_1+2e+H, m_2+e, m_3+k, J)}$	v	v	v	v
$\mathbf{(m_1+2e+H, m_2+e, m_3+J, k)}$		v^2	v^2	v^2
$\mathbf{(m_1+2e+H, m_2+k, m_3+J, e)}$			v	v
$\mathbf{(m_1+2e+H, m_2+k, m_3, e+J)}$			v^2	v^2
$\mathbf{(m_1+2e, m_2+e+J, m_3+k, H)}$	v	v	v	v
$\mathbf{(m_1+2e, m_2+e+J, m_3+H, k)}$		v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2+e+H, m_3+k, J)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+2e, m_2+e+H, m_3+J, k)}$		v^3	v^3	v^3
$\mathbf{(m_1+2e, m_2+k, m_3+J, e+H)}$			v^2	v^2
$\mathbf{(m_1+2e, m_2+k, m_3+H, e+J)}$			v^3	v^3
$\mathbf{(m_1+e+k, m_2+e+J, m_3+e, H)}$	v^2	v^2	v^2	v^2
$\mathbf{(m_1+e+k, m_2+e+J, m_3+H, e)}$			v	v
$\mathbf{(m_1+e+k, m_2+e+J, m_3, e+H)}$			v^2	v^2
$\mathbf{(m_1+e+k, m_2+e+H, m_3+J, e)}$		v^2	$2v^2$	$2v^2$
$\mathbf{(m_1+e+k, m_2+e+H, m_3, e+J)}$			v^3	v^3
$\mathbf{(m_1+e+k, m_2+e, m_3+J, e+H)}$		v^3	$2v^3$	$2v^3$
$\mathbf{(m_1+e+k, m_2+e, m_3+H, e+J)}$			v^4	v^4
$\mathbf{(m_1+e+J, m_2+e+H, m_3, e+k)}$			v^2	v^2
$\mathbf{(m_1+e+J, m_2+e, m_3+H, e+k)}$			v^3	v^3
$\mathbf{(m_1+e+J, m_2+k, m_3+H, 2e)}$				v^2
$\mathbf{(m_1+e+J, m_2+k, m_3, 2e+H)}$				v^3
$\mathbf{(m_1+e+H, m_2+k, m_3+J, 2e)}$			v^3	v^3
$\mathbf{(m_1+e, m_2+k, m_3+J, 2e+H)}$			v^4	v^4

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