EXPANSIONS OF FIELDS BY ANGULAR FUNCTIONS

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ABSTRACT. The notion of an angular function has been introduced by Zilber as one possible way of connecting non-commutative geometry with two 'counterexamples' from model theory: the non-classical Zariski curves of Hrushovski and Zilber, and Poizat's field with green points. This article discusses some questions of Zilber relating to existentially closed structures in the class of algebraically closed fields with an angular function.
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1. INTRODUCTION; ANGULAR FUNCTIONS

This article is motivated by some questions posed by Boris Zilber in [13]. In that paper, Zilber makes connections between non-commutative geometry and two 'counterexamples' in model theory: the non-classical Zariski curve of Hrushovski and Zilber from 10.1 of [4] (see Example 1.3 below), and Poizat's 'field with green points' from [9] (see Section 3.2 here). We will not attempt to describe these connections here: the reader should consult [13]. Instead we treat Zilber's questions purely in terms of their model-theoretic content.

We begin by saying what is meant by an angular function. For the moment, fix a natural number N, an algebraically closed field F of characteristic zero and multiplicatively independent $\alpha, \beta \in F^* = F \setminus \{0\}$. Let Γ be the group of N-th roots of unity in F, and ϵ a primitive N-th root of unity. An *angular function* (with these data) is a function ang $: F^* \to \Gamma$ satisfying, for all $t \in F^*$:

$$\text{(1)} \qquad \text{ang}(\epsilon t) = \text{ang}(t)$$
$$\text{(2)} \qquad \text{ang}(\beta t) = \text{ang}(t)$$
$$\text{(3)} \qquad \text{ang}(\alpha t) = \epsilon \text{ang}(t)$$

In Question 2.4 of [13], Zilber asks:

Question 1.1. Consider a structure which is existentially closed in the class of structures $(F, +, \cdot, \alpha, \beta, \text{ang})$ satisfying these equations. What is its model-theoretic status? Is it supersimple?

Before making more precise what our answer to this question is, we give some definitions and the examples which motivate the terminology and the question.

Note that associated to any angular function ang there are two definable subgroups of the multiplicative group of the field. The group of *periods* of ang is

$$G = \{g \in F^* : \mathrm{ang}(gt) = \mathrm{ang}(t) \; \forall t \in F^*\};$$

and the group of *quasiperiods* of ang is

$$G^+ = \{h \in F^* : \exists \gamma \in \Gamma \; \forall t \in F^* \; \mathrm{ang}(ht) = \gamma \mathrm{ang}(t)\}.$$

So $\Gamma \leq G \leq G^+$ and there is a definable homomorphism $\chi : G^+ \to \Gamma$ with kernel G (given by $\mathrm{ang}(ht) = \chi(h)\mathrm{ang}(t)$ for $h \in G^+$ and $t \in F^*$). Note that as defined above, χ is surjective, so the induced map $\bar{\chi} : G^+/G \to \Gamma$ is an isomorphism.

Example 1.2. The following example from 2.3 of [13] provides the motivation for the terminology 'angular function.' Let $F = \mathbb{C}$, the complex numbers and $\epsilon = \exp(2\pi i/N)$. For $k = 0, \ldots, N-1$, let P_k be the sector of the complex plane consisting of non-zero complex numbers z with an argument $\arg z$ in the range $2\pi k/N \leq \arg z < 2\pi(k+1)/N$. We define $\mathrm{ang} : \mathbb{C}^* \to \Gamma$ by, for $t \in \mathbb{C}^*$:

$$\mathrm{ang}(t) = \epsilon^k \Leftrightarrow t^N \in P_k.$$

The group of periods of ang is $G = \mathbb{R}^{>0}\Gamma$ and the group of quasiperiods is $G^+ = \mathbb{R}^{>0}\langle \epsilon_1 \rangle$ where $\epsilon_1 = \exp(2\pi i/N^2)$.

It is clear that $(\mathbb{C}, +, \cdot, \mathrm{ang})$ has the strict order property (consider additive translates of the definable subset P_0).

Example 1.3. We describe briefly, following Section 1 of [13], how to obtain an example of a non-classical Zariski curve from a suitable angular function. Given $(F; +, \cdot, \alpha, \beta, \mathrm{ang})$ satisfying the above equations, (and where $\langle \alpha, \beta \rangle$ is free abelian of rank 2) define $U, V : F^* \to F^*$ by

$$U(t) = \alpha t$$
$$V(t) = \beta \mathrm{ang}(t)t.$$

These are definable permutations of F^* and

$$VU(t) = \epsilon UV(t).$$

Let T denote the set F with only the structure given by the definable permutations U, V and $p : T \to F$ given by $p(t) = t^N$. Then the structure $((T; U, V), (F; +, \cdot), p : T \to F)$ is a finite cover of $(F; +, \cdot)$ which is interpretable in $(F; +, \cdot, \mathrm{ang})$, but not in $(F; +, \cdot)$. More detatils can be found in Section 1 of [13] and Section 10 of [4].

We now return to Question 1.1. The intention is to add a generic angular function, having prescribed in advance the groups of periods and

quasiperiods (rather than also having an existential closure condition on, say, the group of periods). Thus we make the following definitions.

Definition 1.4. Let L_0 be a first-order language which contains the language of rings $(+, -, \cdot, 0, 1)$ and additional unary predicates $\mathbf{F}, \mathbf{\Gamma}, \mathcal{G}$ and $\mathcal{G}^+$, and a unary function symbol χ. Let F be an algebraically closed field of characteristic 0, and $\Gamma \leq G \leq G^+ \leq F^*$ subgroups of the multiplicative group of F. Suppose $\chi : G^+ \to \Gamma$ is a surjective homomorphism with kernel G. We consider this as an L_0-structure, where $\mathbf{F}$ is interpreted as F, $\mathbf{\Gamma}$ as Γ, $\mathcal{G}$ as G, $\mathcal{G}^+$ as G^+ and χ as χ. Let $T_0 = Th(F; +, -, \cdot, 0, 1, \Gamma, G, G^+, \chi, \dots)$ be the L_0-theory of this structure. Without loss, we shall assume that L_0 has been extended so that T_0 is model-complete. Note that if Γ is finite, then G^+ and χ are definable (with parameters) from G in the multiplicative group F^*.

Now let L_A be the expansion of L_0 by an extra unary function symbol A. We define T_A to be the theory axiomatized by T_0 and axioms:

(i) $(A(0) = 0) \wedge (\forall t)((t \neq 0) \to \mathbf{\Gamma}(A(t)))$;

(ii) $(\forall t)(\forall g)(\mathcal{G}(g) \to A(g \cdot t) = A(t))$;

(iii) $(\forall t)(\forall h)(\mathcal{G}^+(h) \wedge (t \neq 0) \to A(h \cdot t) = \chi(h) \cdot A(t))$

Thus in a model M of T_A, the subgroups $\mathcal{G}(M)$ and $\mathcal{G}^+(M)$ are contained in the periods and quasiperiods of the angular function A^M and it is easy to show that if M is an existentially closed model of T_A, then they are exactly the periods and quasiperiods (*a priori* it is not immediately obvious that T_A is even consistent, but this will become clear from Lemma 2.1). Question 1.1 can then be seen as asking whether the class of existentially closed models of T_A is axiomatizable, and if so whether completions of its theory are supersimple. In Section 2 we prove:

Theorem 1.5. (1) *If T_0 eliminates the quantifier $\exists^\infty$ in the sorts $\mathbf{F}, \mathbf{F}/\mathcal{G}$ and $\mathbf{F}/\mathcal{G}^+$ then T_A has a model companion T_A^*.*

(2) *If additionally T_0 is simple and Γ is finite then all completions of T_A^* are simple (and in the same simplicity class as T_0).*

By the parenthetic remark at the end of (2), we mean, for example, that if T_0 is supersimple of SU-rank κ, then so is any completion of T_A^*. If Γ is non-trivial, then T_A^* will not be stable.

Essentially both parts of the theorem follow quickly from known results. As we shall see, the first follows from an old theorem of Winkler [10], and the second follows from results of Nübling in [5]. Of course, for the result to be of any relevance to Zilber's question we need to identify suitable T_0 which satisfy its hypotheses.

We consider two situations where the model theory of an algebraically closed field of characteristic zero expanded by a predicate for a subgroup of the multiplicative group is understood. In the first case, the

subgroup is in the divisible hull of a finitely generated group (– structures of *Lang type*). In the second case, we consider structures constructed via a Hrushovski predimension as in Poizat's 'field with green points' from [9] (– structures of *Poizat type*). The first case is relevant to Section 2 in Zilber's paper [13] and the second case is relevant to Sections 3 and 4 there.

In both cases we are dealing with superstable structures (of infinite U-rank). Both cases have weak nfcp, that is, they eliminate $\exists^\infty$ in all (real and imaginary) sorts. For structures of Lang type, this is folk-lore; for structures of Poizat type we give a proof using belles paires.

In Section 4 we show how to put a probability measure on definable sets in models of T_A^*, giving one possible answer to the problem at the end of Section 2 of [13].

2. Winkler's Theorem and the proof of Theorem 1.5

Throughout this section we use the notation of Definition 1.4. Suppose we have a model M of T_A. If the model is understood from the context, we will abuse notation and write $\mathcal{G}$ instead of $\mathcal{G}(M)$ etc.

Rather than considering the angular function $A : \mathbf{F} \to \mathbf{\Gamma}$ we consider a *section* $s : \mathbf{F}^*/\mathcal{G}^+ \to \mathbf{F}^*/\mathcal{G}$ of the natural map $\nu : \mathbf{F}^*/\mathcal{G} \to \mathbf{F}^*/\mathcal{G}^+$ (given by $\nu(x\mathcal{G}) = x\mathcal{G}^+$) which is interdefinable with A. More formally, suppose L_0 has been expanded to include the part of L_0^{eq} involving the sorts $\mathbf{F}^*/\mathcal{G}^+$ and $\mathbf{F}^*/\mathcal{G}$. Expand L_0 further to a language L_s which has an extra unary function symbol s between these sorts. Let T_s be the L_s-theory axiomatized by T_0 and the axiom:

$$(\forall y \in \mathbf{F}^*/\mathcal{G}^+)(s(y)\mathcal{G}^+ = y)$$

saying that s is a section of the L_0-definable map ν.

Lemma 2.1. *There is a definable correspondence between the models of T_A and the models of T_s which preserves the property of existential closure. Thus T_A has a model companion (respectively, model completion) if and only if T_s does.*

Proof: The correspondence is given by the equation:

$$A(t) = \bar{\chi}(s(t\mathcal{G}^+)^{-1}t\mathcal{G}).$$

Indeed, given a section s of ν, then $s(t\mathcal{G}^+)^{-1}t\mathcal{G} \in \mathcal{G}^+/\mathcal{G}$ and $\bar{\chi}$ maps this into $\mathbf{\Gamma}$. So A given by this equation is certainly a map from $\mathbf{F}^*$ to $\mathbf{\Gamma}$ and it is easy to check that it satisfies the axioms in Definition 1.4. Conversely given an angular function A, the above equation determines a function $s : \mathbf{F}^*/\mathcal{G}^+ \to \mathbf{F}^*/\mathcal{G}$ given by:

$$s(t\mathcal{G}^+) = t\mathcal{G}(\bar{\chi}^{-1}(A(t)))^{-1},$$

because $\bar{\chi}$ is bijective, and one checks easily that this is a section of ν.

The above correspondence is given by a quantifier-free formula, so it translates existentially closed structures in one class into existentially

closed structures in the other. This proves the statement about the existence of model companions. For the statement about model completions, observe that T_A has the amalgamation property iff T_s has the amalgamation property. □

By the above lemma it will suffice to prove Theorem 1.5 for the theory T_s. The advantage of this is that s only has a simple axiom to satisfy: that of being a section of a certain definable map. This is exactly the situation in which we can apply a result of Winkler from the 1970's. In the following, L is any first-order language and T any L-theory.

Definition 2.2. We say that T *eliminates the quantifier 'there exist infinitely many'* (or *eliminates* $\exists^\infty$, or is *algebraically bounded*) if for all L-formulas $\phi(x, \bar{y})$ there is a natural number N_ϕ with the property that for all models M of T and $\bar{a}$ in M, if $\phi[M, \bar{a}]$ has more than N_ϕ elements, then it is infinite.

Definition 2.3. Suppose $\phi(x, \bar{y})$ is an L-formula, where $\bar{y}$ is an n-tuple of variables. Let L^+ be the expansion of L by a new n-ary function symbol σ. The L^+ theory T^+ is axiomatized by T together with the new axiom:
$$(\forall \bar{y})((\exists x)\phi(x, \bar{y}) \rightarrow \phi(\sigma(\bar{y}), \bar{y})).$$
We refer to T^+ as a *Skolem expansion* of T.

So of course this says that σ is a Skolem function for the formula $\phi(x, \bar{y})$. Strictly speaking, σ should be defined for all n-tuples, even if they do not satisfy $(\exists x)\phi(x, \bar{y})$. We can avoid this issue by including a sort for the definable set $(\exists x)\phi(x, \bar{y})$ and only defining σ on this sort. Alternatively we can define $\sigma(\bar{y})$ to be some fixed $\emptyset$-definable element.

Theorem 2.4 (P. Winkler, Theorem 2 of [10])**.** *Suppose T is a model-complete L-theory which eliminates the quantifier $\exists^\infty$. Then any Skolem expansion T^+ of T has a model completion $(T^+)^*$.* □

Proof of Theorem 1.5: (1) The theory T_s is the Skolem expansion of T_0 with respect to the formula $\phi(x, y)$:
$$(x \in \mathbf{F}/\mathcal{G}) \wedge (y \in \mathbf{F}/\mathcal{G}^+) \wedge (x\mathcal{G}^+ = y).$$
The assumptions on eliminating $\exists^\infty$ in the indicated sorts means that we can apply Winkler's Theorem to deduce that T_s has a model completion T_s^*. So by Lemma 2.1 there is a model completion T_A^* of T_A.

(2) If Γ is finite, then $\mathbf{\Gamma}$ is finite in all models of T_0 and the map ν is $|\Gamma|$-to-1. So the Skolem expansion T_s is an *algebraic* Skolem expansion (in the terminology of [5]) and the results in Section 1 of [5] give immediately that if T_0 is simple then so is any completion of T_s^* (and in the same simplicity class). By interdefinability of T_A and T_s, the same is true of T_A^*. □

Remarks 2.5. The result in [10] is more general than is stated in Theorem 2.4: one has a model completion for the expansion by an arbitrary number of Skolem functions. By Theorem 4 of [10] there is also a converse to Theorem 2.4: if there is a model companion for T^+ then T^+ eliminates $\exists^\infty$. By combining these results, one obtains ([10], Corollary 1) that the model completion $(T^+)^*$ in Theorem 2.4 eliminates $\exists^\infty$.

The proof of Theorem 2.4 gives an axiomatization of T_s^*. As well as T_s we have axioms of the form:

$$\forall \bar{x}((\exists^\infty \bar{y})\alpha(\bar{x}, \bar{y}) \to (\exists \bar{y})(\alpha(\bar{x}, \bar{y}) \wedge \beta(\bar{x}, \bar{y})))$$

where

  (i) α is an L_0-formula and β is a conjunction of formulas of the form '$s(v_1) = v_2$' where v_1, v_2 are amongst the variables in $\bar{x}, \bar{y}$, not both in $\bar{x}$ (and no variable appearing as v_1 is in more than one such conjunct).

  (ii) If $i \neq j$ then $y_i \neq y_j$ is a conjunct of α and if '$s(v_1) = v_2$' is a conjunct of β, then $\phi(v_2, v_1)$ is a conjunct of α.

It should be noted that '$\exists^\infty \bar{y} \ldots$' here means 'there exist infinitely many $\bar{y}$ *which are different in each coordinate*' (see the definition in Section 1 of [10]).

Remarks 2.6. We note that if, in Theorem 1.5 the group Γ is not finite (and $\mathbf{F}^*/\mathcal{G}^+$ is infinite), then T_s^* and T_A^* cannot be simple. The argument is essentially that of Lemma 3.1 of [5]. In fact, the formula $\theta(x; y, z)$ given by $A(x \cdot y) = z$ has the tree property (with respect to T_s^*). In some model of T_s^* take $(a_i : i < \omega)$ in $\mathbf{F}^*$ lying in different $\mathcal{G}^+$-cosets, and distinct elements $(g_j : j < \omega)$ of Γ. For $\eta \in \omega^{<\omega}$ with domain $n > 0$ consider the parameters $c_\eta = (a_n, g_{\eta(n-1)})$. Clearly $\{\theta(x; c_{(\eta^\frown j)}) : j < \omega\}$ is pairwise inconsistent for each $\eta \in \omega^\omega$, and using the above axioms for T_s^* one shows that for $\zeta \in \omega^\omega$ the set $\{\theta(x; c_{\zeta|n}) : n < \omega\}$ is consistent.

3. Weak nfcp

Suppose L is a first-order language and T a complete L-theory. Then by a theorem of Shelah, T has nfcp ('does not have the finite cover property') if and only if it is stable and eliminates the quantifier $\exists^\infty$ in all (real and imaginary) sorts. The latter property (for arbitrary T) is sometimes referred to as *weak nfcp*. In this section we discuss two types of stable theories T_0 to which we would like to apply Theorem 1.5, and show that they have weak nfcp.

3.1. **Structures of Lang type.** Recall that an abelian group is said to be of *finite rank* if it is in the divisible hull of a finitely generated subgroup. By a structure of *Lang type* we mean an algebraically closed

field of characteristic zero expanded by a predicate $\mathcal{G}$ for a finite rank subgroup G of some semiabelian variety (see [7], where the use of this terminology is different, but consistent with ours). We shall be interested in the particular case where G is a finite rank subgroup of the multiplicative group (though it should be noted that the following also holds in the more general context).

Theorem 3.1. *Suppose F is an algebraically closed field of characteristic 0 and infinite transcendence rank and G is a finite rank subgroup of the multiplicative group F^*. Consider this as an L-structure where L is the language of rings expanded by a unary predicate $\mathcal{G}$ for the subgroup G. Then:*

   (i) *The induced structure on $\mathcal{G}$ is that of a stable one-based group and if G is κ-stable, then so is $T = Th(F; +, \cdot, 0, 1, \mathcal{G})$.*

   (ii) *T has nfcp.*

Proof: Part (i) is from [7], particularly Proposition 2.6.

For (ii) one can easily adapt the proof of Proposition 2.6 of [7] to show that T is non-multidimensional. It follows (for example by Remark 8.2.13 of [6]) that T has nfcp. Alternatively, by (i) the induced structure on $\mathcal{G}$ is that of a one-based group and so has nfcp. Moreover, G is *small* in F (in the terminology of [1]), so one can apply Proposition 5.7 of [1] to obtain that T has nfcp. $\qquad\qquad\square$

We summarise our answer to Zilber's question (Question 1.1) as follows, using the notation (L_0, T_0, L_A, T_A) from Definition 1.4.

Theorem 3.2. *Suppose F is an algebraically closed field of characteristic zero of infinite transcendence rank and $\alpha, \beta \in F$ are multiplicatively independent. Let N be a natural number, ϵ a primitive N-th root of 1 and $\Gamma = \langle \epsilon \rangle$; $G = \langle \alpha^N, \beta, \epsilon \rangle$; and $G^+ = \langle \alpha, \beta, \epsilon \rangle$. Define $\chi : G^+ \to \Gamma$ to have kernel G and $\chi(\alpha) = \epsilon$. Then:*

   (i) *$T_0 = Th(F; +, -, \cdot, 0, 1, \Gamma, G, G^+, \chi)$ is superstable of Lascar rank ω and has nfcp.*

   (ii) *T_A has a model completion T_A^* and all completions of this are supersimple of SU-rank ω.*

Proof: (i) All of the structure is interpretable in $(F, G, \alpha, \beta, \epsilon)$ and the required properties of this are given by Theorem 3.1 and the proof of Proposition 2.6 in [7].

(ii) This follows from part (i) and Theorem 1.5. $\qquad\qquad\square$

Of course, in a model of T_A as above the function A is an angular function, that is, it satisfies the properties (1)-(3) in the Introduction.

By Theorem 10 of [10], any one-cardinal model of T_0 can be expanded to a model of T_A^*. In particular, there is a countable model of T_A^* in which $\mathcal{G} = G$. It is natural to ask whether there is a model of T_A^* in which the field is the complex numbers and the group of periods is

precisely G. To do this it would be enough to construct an e.c. model of T_A (or of T_s) of cardinality continuum in which $\mathcal{G} = G$. This cannot be an entirely straightforward union-of-a-chain argument, as the following example shows.

Let M_0 be a model of T_0, let $c \in \mathbf{F}(M_0)$ be algebraically independent from $\mathcal{G}(M_0)$ and $\psi(x, y, c)$ be the q.f. formula $x \in \mathcal{G} \wedge y = x + c$. Then M_0 can be expanded to a model M of T_s in which $s(\nu(y\mathcal{G})) \neq y\mathcal{G}$ for all $y \in c + \mathcal{G}$. On the other hand, $c + \mathcal{G}(M_0)$ meets infinitely many (multiplicative) cosets modulo $\mathcal{G}$, so there is an elementary extension N_0 of M_0 and $b \in \mathcal{G}(N_0)$ such that $(b+c)\mathcal{G}(N_0) \notin \mathbf{F}(M_0)/\mathcal{G}(N_0)$. Thus we can expand N_0 to a model N of T_s so that M is a submodel of N and $\psi(x, y, c) \wedge s(\nu(y\mathcal{G})) = y\mathcal{G}$ has a solution $(b, b + c)$ in N. It follows that any e.c. model M_1 of T_s which contains M has to have $\mathcal{G}(M_1) > \mathcal{G}(M)$.

3.2. **Poizat's field with green points.** We wish to consider Poizat's 'field with green points' $(F, G(F))$ from [9]. This is a structure in a language L for fields with an extra unary predicate G. The field F is algebraically closed of characteristic zero and the subset $G(F)$ (– the 'green points') is a torsion free divisible subgroup of the multiplicative group $F^{\times}$. For every algebraically closed subfield A of finite transcendence degree we have the predimension inequality

$$\delta(A) = 2\mathrm{trdeg}(A) - \mathrm{rk}_{\mathbb{Q}}G(A) \geq 0.$$

Moreover $(F, G(F))$ is 'rich' : the familiar existential closure condition with respect to self-sufficient embeddings. The first-order axiomatization of $T_G = Th(F, G(F))$ is described in Section 3 of [9].

The main result of this Section is:

Theorem 3.3. T_G *has nfcp.*

It should be noted that we will *not* use T_G when applying Theorem 1.5. Instead, we will use variations on Poizat's original construction (see Section 3.3). The axiomatisation of these variations and the proofs of nfcp are similar to those for T_G, and it therefore seems reasonable to present the proof of nfcp for the original 'field with green points.'

The proof we give of Theorem 3.3 uses Poizat's technology of belles paires from [8] and we begin by briefly recalling this.

Let P be a new unary predicate symbol and L^P the language obtained by adjoining P to L. We consider a pair $M \preceq N$ of models of T_G as an L^P-structure by interpreting P as the subset M. We refer to this as a *belle paire* if M is ω^+-saturated (in the L-sense) and for every finite subset of N, every 1-type (in the L-sense) over $M \cup A$ is realised in N. Let T_G^P denote the L^P-theory of all belles paires. As

T_G is stable, T_G^P is complete ([8], Théorème 4) and T_G has nfcp iff every ω^+-saturated (in the L^P-sense) model of T_G^P is a belle paire ([8], Théorème 6).

Henceforth, let $M \preceq N$ be an ω^+-saturated model of T_G^P. We need to show that this is a belle paire, and of course, the issue is the relative ω-saturation of N over M. Let cl denote self-sufficient closure in N, and acl_f denote field-theoretic algebraic closure. Let d be the dimension function associated with the predimension δ.

Let A be a finite subset of N and $p(x)$ a 1-type (in the L-sense) over $M \cup A$. Let $M_1 = \mathrm{cl}_N(M \cup A)$. We want to show that p is realised in N. Suppose it is realised by c in the elementary extension N_1 of N. We may assume that $M \preceq N_1$ is a belle paire, so $(M, N) \preceq_{L^P} (M, N_1)$. Now, $M_2 = \mathrm{cl}_{N_1}(M \cup A \cup \{c\})$ is of finite transcendence degree over M_1 ([9], Corollaire 1.4), so we may assume that $M_1 \leq M_2$ is a *minimal* extension: for every algebraically closed C with $M_1 \subset C \subseteq M_2$, we have $\mathrm{cl}(C) = M_2$. There are three possible cases to consider, depending on $d(c/M_1)$ (computed in N_1).

Case 1. (White generic) $d(c/M_1) = 2$. So $c \notin M_1$; $G(M_1) = G(M_2)$ and $M_2 = \mathrm{acl}_f(M_1 c)$.

Case 2. (Green generic) $d(c/M_1) = 1$. So $c \notin M_1$; c is green; $G(M_1)$ is in the divisible hull of the subgroup generated by $G(M)$ and c; $\mathrm{acl}_f(M_1 c) = M_2$.

Case 3. $d(c/M_1) = 0$.

Claim 1. For any natural number n there exist $g_1, \ldots, g_n \in G(N)$, algebraically independent over M, with $d(g_1, \ldots, g_n/M) = n$.

So the claim is that N contains a Morley sequence $g_1, \ldots, g_n$ of length n for the green generic over M: the g_i should be green points, algebraically independent over M, and $\mathrm{acl}_f(Mg_1 \ldots g_n)$ should self-sufficient in N, with its G-part $\mathbb{Q}$-dependent on $G(M)$ and $g_1, \ldots, g_n$. We show that there is a set $\Phi(x_1, \ldots, x_n)$ of L^P-formulas (without parameters) which is equivalent to the $L(M)$-type of such a sequence. The claim then follows, because of the ω^+-saturation of the pair (M, N).

The formulas should express that:

- the g_i are in G and algebraically independent over the P-part M
- for any $s \geq 0$, if $e_1, \ldots, e_{2s+1}$ are green and have transcendence degree at most s over $M\bar{g}$, then they are multiplicatively dependent over $G(M)\bar{g}$

Note that the case $s = 0$ is equivalent to the G-part of $\mathrm{acl}_f(M\bar{g})$ being $\mathbb{Q}$-dependent on $G(M)\bar{g}$; the remaining cases say that $\mathrm{acl}_f(M\bar{g})$ is self-sufficiently embedded.

Clearly we can ensure that Φ contains L^P-formulas expressing the first group of conditions. For the second, we use Proposition 3.2 of [9]. Suppose $V(\bar{y}, \bar{z})$ is a variety over $\mathbb{Q}$, with the length of $\bar{y}$ being

$k = n + 2s + 1$. For an appropriate tuple of parameters b, we denote by $V(b)$ the variety in k-space defined by $V(\bar{y}, b)$. There is a finite set of proper basic tori $T_1, \ldots, T_t$, depending only on V, with the property that for any torus T and any b, any irreducible component of $V(b) \cap T$ of dimension greater than $\dim V(b) + \dim T - k$ (an *atypical* component) is contained in a coset of one of the T_i.

Then in Φ we also take, for every such variety V over $\mathbb{Q}$ a formula (with appropriate r, t, T_i):

$$(\forall \bar{e})(\forall b \in P^r)$$

('$V(b)$ irreducible of dimension $\leq n + s$' $\wedge$ $((\bar{x}, \bar{e}) \in V(b)) \wedge G(\bar{e})$

$$\longrightarrow (\bar{x}, \bar{e}) \in \bigcup_{i \leq t}(P^\times)^{n+2s+1} T_i).$$

To prove the claim we show that for any pair $M_1 \preceq N_1$ of models of T_G and n-tuple $\bar{g}$ in N_1, we have $(M_1, N_1) \models \Phi(\bar{g})$ iff $\bar{g}$ is a Morley n-sequence of the green generic over $P(N_1) = M_1$.

First, suppose $(M_1, N_1) \models \Phi(\bar{g})$. So the g_i are algebraically independent over M_1 and we need to show that the second bullet point above holds. Take $e_1, \ldots, e_{2s+1}$ as there and let $V(b)$ be an irreducible variety over M_1 with $\bar{g}\bar{e}$ as generic point. So V is a variety over $\mathbb{Q}$ and b is a tuple of parameters in M_1, and $V(b)$ is of dimension at most $n + s$. By the appropriate element of Φ, the elements of $\bar{g}\bar{e}$ satisfy $g_1^{a_1} \ldots g_n^{a_n} e_1^{b_1} \ldots e_{2s+1}^{b_{2s+1}} = c \in M_1$, for some integers a_i, b_j not all zero. Clearly $c \in G(M_1)$, so this gives what we want.

Conversely suppose $\bar{g}$ is a Morley n-sequence of the green generic over $P(N_1) = M_1$. We need to show that it satisfies the formulas in Φ. So take V, b and $\bar{e}$ as in the displayed formula with $\bar{g}\bar{e} \in V(b)$. Let T be the minimum torus of $\bar{g}\bar{e}$ over M_1 and W the connected component of $V(b) \cap T$ containing $\bar{g}\bar{e}$. We claim this is atypical. Indeed, $\dim W = n + \operatorname{trdeg}(\bar{e}/M_1\bar{g})$ and $\dim T = n + \operatorname{rk}_{\mathbb{Q}}(\bar{e}/M_1\bar{g})$ (where $\operatorname{rk}_{\mathbb{Q}}$ denotes multiplicative rank). As $\operatorname{acl}_f(M_1\bar{g}) \leq N_1$ we have

$$2\operatorname{trdeg}(\bar{e}/M_1\bar{g}) \geq \operatorname{rk}_{\mathbb{Q}}(\bar{e}/M_1\bar{g}).$$

Now, using these equations and the fact that $\dim V(b) \leq n + s$, we have that $\dim V(b) + \dim T - (n + 2s + 1) < \dim W$ if $2\operatorname{trdeg}(\bar{e}/M_1\bar{g}) - \operatorname{rk}_{\mathbb{Q}}(\bar{e}/M_1\bar{g}) > \operatorname{trdeg}(\bar{e}/M_1\bar{g}) - s - 1$. But the left hand side here is ≥ 0, whereas the right hand side is negative, as we are supposing $\operatorname{trdeg}(\bar{e}/M_1\bar{g}) \leq s$. So W is an atypical component of the intersection $V(b) \cap T$, and is therefore contained in a coset of one of the basic tori T_i. Both $V(b)$ and T are defined over M_1, so W is defined over M_1. Therefore the coset representative can be taken in M_1: hence $\bar{g}$ satisfies the required formula in Φ.

($\square$ Claim 1)

Claim 2. In Case 2, we may take c in N.

Indeed, take $n > d(A/M)$, and $g_1, \ldots, g_n \in N$ as in Claim 1. Then for some $i \leq n$ we have $d(g_i/AM) = 1$. But then $\mathrm{tp}(g_i/M_1) = \mathrm{tp}(c/M_1)$, as required. ($\square$ Claim 2)

Claim 3. In Case 1, we may take $c \in N$.

Again, take $n > d(A/M)$ but this time, consider $g_1, \ldots, g_{2n} \in N$ as in Claim 1. Let $b_i = g_i + g_{2i}$, for $i = 1, \ldots, n$. Then one checks that these are white generics, independent over M. The same argument as in Claim 1 then shows that we can assume that $d(b_1/MA) = 2$, so we can take $c = b_1$. ($\square$ Claim 3)

We now deal with Case 3. There exists a tuple $\bar{c} = (c_1, \ldots, c_{2n})$ of elements of M_2 such that $c_1, \ldots, c_n$ are a transcendence basis of M_2 over M_1, and $c_1, \ldots, c_{2n}$ is a $\mathbb{Q}$-basis for $G(M_2)$ over $G(M_1)$. It is enough to show that the type of $\bar{c}$ over M_1 is realised in N.

Let $V(b, \bar{y})$ be the variety over M_1 which has $\bar{c}$ as generic point. So $V(\bar{x}, \bar{y})$ is over $\mathbb{Q}$, and b is a tuple of parameters in M_1.

Claim 4. There is a formula $\psi(\bar{x})$ in the field language such that $N \models \psi(b)$ and if $\psi(f)$ holds (in an algebraically closed field of characteristic zero), then:

- $V(f, \bar{y})$ is irreducible and $\dim V(f, \bar{y}) = n$
- the generic of $V(f, \bar{y})$ is multiplicatively independent over $\mathrm{acl}_f(f)$
- if $\bar{g}$ is a generic point of $V(f, \bar{y})$ and we colour its coordinates (and a torsion-free divisible multiplicative group generated by them) green, then $\mathrm{acl}_f(f) \leq \mathrm{acl}_f(f\bar{g})$.

The proof of this is in Poizat's paper ([9], from bottom of page 1674). ($\square$ Claim 4)

There is a set $\Psi(\bar{x}, \bar{y})$ of L^P-formulas which says:

- $V(\bar{x}, \bar{y})$ and $G(\bar{y})$
- $\mathrm{trdeg}(\bar{y}/P\bar{x}) = n$
- $\bar{y}$ is multiplicatively independent over $P\bar{x}$

Claim 5. For each $\phi(\bar{x}, \bar{y}) \in \Psi$, the closed formula

$$\forall \bar{x}(\psi(\bar{x}) \rightarrow (\exists \bar{y})\phi(\bar{x}, \bar{y}))$$

is in T^P (the theory of belles paires).

Let $M' \preceq N'$ be a belle paire and f a tuple in N' satisfying ψ. Let $A = \mathrm{acl}_f(f)$. Take $B = \mathrm{acl}_f(f\bar{g})$ as in the third bullet point in Claim 4. By the relative ω-saturation in the belle paire, the free amalgam of B and $\mathrm{cl}_{N'}(M'f)$ embeds into N over $\mathrm{cl}_{N'}(M'f)$. Let $\bar{g}'$ be the image of $\bar{g}$ under this embedding. Then $(M', N') \models \Psi(f, \bar{g}')$. ($\square$ Claim 5)

We can now finish off the proof in Case 3. Recall that (M, N) is an ω^+-saturated model of T_G^P. Because of the tuple $\bar{c}$, we have $N_1 \models \psi(b)$. By Claims 4 and 5 and the ω^+-saturation, there is therefore a tuple $\bar{c}'$ in N with $(M, N) \models \Psi(b, \bar{c}')$. As $M_1 \leq N_1$ and $\delta(\mathrm{acl}_f(M_1\bar{c}')/M_1) = 0$, we have $\mathrm{acl}_f(M_1\bar{c}') \leq N_1$. From this and Ψ, we obtain $\mathrm{tp}_{N_1}(\bar{c}'/M_1) = \mathrm{tp}_{N_1}(\bar{c}/M_1)$, as required. ($\square$ Theorem)

We remark that Martin Hils pointed out to us that Theorem 3.3 follows from a result of his [2]: the theory of T_G (in a language expanded by definitions so that it is model complete) together with an automorphism has a model companion.

3.3. **Variations on green points.** In [11], Zilber proves that, under the assumption of Schanuel's conjecture, the field of complex numbers expanded by a predicate for the multiplicative subgroup

$$\exp((1 + \mathrm{i})\mathbb{R} + \mathbb{Q})$$

is a model of Poizat's T_G. For the purposes of [13], it is more natural to consider $\mathbb{C}$ expanded by predicates for slightly different multiplicative subgroups.

Example 3.4. (See Section 3.5 of [13].) Consider $\alpha, \beta \in \mathbb{C}^\times$ linearly independent over $\mathbb{R}$, and let $i_a, i_b \in \mathbb{R}$ be such that $\mathrm{i} = i_a \alpha + i_b \beta$. Let $h \in \mathbb{R}$ be such that $1, 2\pi i_a, 2\pi i_a h$ are linearly independent over $\mathbb{Q}$ and let $N \in \mathbb{N}$. Define

$$G_0 = \exp(\frac{2\pi\mathrm{i}}{hN}\mathbb{Z} + \frac{\alpha}{h}\mathbb{Z} + \beta\mathbb{R}).$$

There is a theory $\tilde{T}_G$ axiomatised in the same way as Poizat's T_G, but where the subgroup of 'green' points is elementarily equivalent to $\mathbb{Z}^2$, rather than $\mathbb{Q}$. The theory $\tilde{T}_G$ is superstable of U-rank $\omega.2$ and assuming Schanuel's conjecture, $\tilde{T}_G = Th(\mathbb{C}, G_0)$ (see Proposition 3.5 of [13]). (One caveat here is that the class of coloured algebraically closed fields $(A, G(A))$ of characteristic zero with $G(A) \equiv \mathbb{Z}^2$ satisfies the amalgamation property but not the joint embedding property. So one should really work with the subclass of coloured fields containing a fixed $(A_0, G(A_0))$, say with $\delta(A_0) = 0$, and include constants for generators of $G(A_0)$ in the language.)

In the notation of Definition 1.4, we expand $(\mathbb{C}, G_0)$ to an L_0-structure by setting $\Gamma = \langle \epsilon \rangle$; $G = G_0.\Gamma$; $G^+ = \exp((2\pi\mathrm{i}/hN)\mathbb{Z} + (\alpha/hN)\mathbb{Z} + \beta\mathbb{R}).\Gamma$ and $\chi(a) = \epsilon$, where ϵ is a primitive N-th root of 1, and $a = \exp(\alpha/hN)$. Note that G is of index N in G^+ and so all of these are definable in $(\mathbb{C}, G_0, a)$. Thus (assuming Schanuel's conjecture) we obtain a structure whose theory T_0 is superstable of rank $\omega.2$. The proof of nfcp for T_G given in Section 3.2 also works for $\tilde{T}_G$ with little modification. So Theorem 1.5 applies, and as in Theorem 3.2 we have:

Theorem 3.5. *With the above notation, and that of Definition 1.4:*
  (i) *T_0 is superstable of Lascar rank $\omega.2$ and has nfcp.*
  (ii) *T_A has a model completion T_A^* and all completions of this are supersimple of SU-rank $\omega.2$.*

In Section 3.5 of [13] Zilber gives an explicit construction of an angular function ang_N with the above data and asks whether the theory of the complex numbers expanded by ang_N is supersimple (assuming

Schanuel's conjecture). In the light of the above result, it appears reasonable to ask whether it is a model of T_A^*.

Example 3.6. In Section 3.3 of [13] Zilber works with an angular function ang having the following data: $\Gamma = \exp(2\pi\mathrm{i}h\mathbb{Z})$; $G = \exp(2\pi\mathrm{i}h\mathbb{Z} + \beta\mathbb{R})$; $G^+ = \exp(2\pi\mathrm{i}h\mathbb{Z} + \alpha\mathbb{Z} + \beta\mathbb{R})$ and $\chi(\exp(\alpha)) = \exp(2\pi\mathrm{i}h)$, where α, β and h are as in Example 3.4. Again, Schanuel's conjecture implies that $Th(\mathbb{C}; G, \Gamma)$ can be axiomatized in a similar way to Poizat's T_G (the appropriate predimension to use is $\delta(A) = 2.\mathrm{trdeg}(A) - \mathrm{rk}_{\mathbb{Q}}(\mathcal{G}(A)) - \mathrm{rk}_{\mathbb{Q}}(\Gamma(A)))$, and is superstable of Lascar rank $\omega.2$ (see Proposition 2 of Section 3.3 in [13]). However, as Γ is infinite, the group G^+ and the homomorphism χ are not definable in this, so we require a slightly different approach.

Let L_0' be a language consisting of the language of rings together with unary predicates $\mathcal{G}, \Gamma, \Gamma^+$ and unary functions χ, χ^{-1}. Consider the class of L_0'-structures consisting of an algebraically closed field A of characteristic zero, and multiplicative subgroups $\mathcal{G}(A), \Gamma(A), \Gamma^+(A)$ with the properties that

- $\Gamma(A)$ is a pure subgroup of $\mathcal{G}(A)$ and $\mathcal{G}(A) \cap \Gamma^+(A) = 1$;
- the groups $\mathcal{G}(A)$ and $\Gamma(A)$ are elementarily equivalent to $\mathbb{Z}$;
- $\chi : \Gamma^+(A) \to \Gamma(A)$ and $\chi^{-1} : \Gamma(A) \to \Gamma^+(A)$ are mutually inverse group isomorphisms;
- the predimension inequality $\delta \geq 0$ holds,

where $\delta(A) = 2.\mathrm{trdeg}(A) - \mathrm{rk}_{\mathbb{Q}}(\mathcal{G}(A)) - 3.\mathrm{rk}_{\mathbb{Q}}(\Gamma(A))$. Note that the intention is that we can take (definably) $\mathcal{G}^+$ to be the direct product $\mathcal{G}.\Gamma^+$ and χ extends to $\mathcal{G}^+$ by projection to the second factor. As with Poizat's T_G one can construct a complete L_0'-theory $\hat{T}_G$ whose ω-saturated models are the 'rich' structures in the class; moreover $\hat{T}_G$ is superstable of Lascar rank $\omega.2$, $\mathcal{G}$ is of rank ω and Γ of rank 1. (As in 3.4 we should really work over a fixed coloured field.) This is similar to the construction in [12] and as there, it is expected that $(\mathbb{C}; \Gamma, G, G^+, \chi)$ as given above is a model of $\hat{T}_G$ (as always, assuming Schanuel's conjecture), though we have not checked the details.

The proof of Theorem 3.3 can easily be adapted to show that $\hat{T}_G$ has nfcp. Thus, as in Theorem 1.5, the theory $(\hat{T}_G)_A$ of models of $\hat{T}_G$ (with the language expanded to get model completemenss) with an angular function has a model companion $(\hat{T}_G)_A^*$. Note that by Remark 2.6, this is not simple.

The above provides one possible approach to Problems 1 and 2 of Section 3.3 of [13] and it is of course of interest to check whether Zilber's explicit angular function ang with the above data gives a model of $(\hat{T}_G)_A^*$.

4. A PROBABILITY MEASURE

In this section we describe (– under certain hypotheses on T_0) how to put a probability measure on the definable subsets of F in models of T_A^*. In particular, this gives one possible answer to the Problem in 2.4 of [13].

We continue to use the notation of Definition 1.4 and assume that T_0 is a superstable L_0-theory with nfcp and Γ is finite; we also assume that T_0 has quantifier elimination (as usual, this can be achieved by expanding the language). As we are working with superstable fields, there is a unique 1-type of maximal rank (– the generic type) over any algebraically closed subset of a model of T_0. We will assume that whenever C is algebraically closed (in a model of T_0) and e is generic over C, then $\mathcal{G}^+(\mathrm{acl}(Ce)) = \mathcal{G}^+(C)$. This is the case in the examples of interest in Section 3.

Theorem 1.5 applies, and we fix a completion of T_A^*, which we will also denote by T_A^*. Results in [5] imply that this is determined by the values of the angular function on $\mathrm{acl}(\emptyset)$; it is supersimple; algebraic closure in the T_A^* and T_0 senses are the same, and in a model $(F; \mathrm{ang})$ of T_A^* a generic type $\mathrm{tp}(e/C)$ is determined by the restriction of the angular function ang to $\mathrm{acl}(Ce)$. (Of course, the results in [5] apply to T_s^*, and we obtain the corresponding results about T_A^* as in the proof of Theorem 1.5.)

Work in a big model $(F; \mathrm{ang})$ of T_A^* and let $C \subset F$ be algebraically closed. Let $\mathcal{X}$ be the set of generic types over C, equipped with the Stone space topology. Note that $\mathcal{X}$ is closed in the Stone space of all T_A^*-types over C, as it consists of those types which extend the T_0 generic type over C. In particular, $\mathcal{X}$ is compact.

Fix some $e \in F$ which is generic over C and let $B = \mathrm{acl}(Ce)$. Consider B just as an L_0-structure and let $\mathcal{Y}$ denote the set of all possible angular functions from B to Γ which extend the given $\mathrm{ang}|C$. By the above remarks each element of $\mathcal{Y}$ determines a generic type over C, and two elements of $\mathcal{Y}$ give rise to the same generic type iff there is an L_0-automorphism of B over Ce which sends one to the other. In summary:

Lemma 4.1. *There is a surjective map* $\Theta : \mathcal{Y} \to \mathcal{X}$ *whose fibres are the* $\mathrm{Aut}_{L_0}(B/Ce)$*-orbits on* $\mathcal{Y}$.

Note that the quotient of any two elements of $\mathcal{Y}$ (as functions $B \to \Gamma$) is constant on $\mathcal{G}^+(B)$-cosets, and so is a function from $B/\mathcal{G}^+(B)$ to Γ which is the identity on $C/\mathcal{G}^+(B)$. Conversely if we multiply any element of $\mathcal{Y}$ by such a function, we obtain another element of $\mathcal{Y}$. Thus we have a regular action of the group

$$P = \Gamma^{(B/\mathcal{G}^+(B))\backslash(C/\mathcal{G}^+(B))}$$

of such functions on $\mathcal{Y}$.

Now, P is a product of copies of the finite set Γ, so we can give it the product topology. We can regard Γ as a uniform probablility space, and give P the product measure κ (of course, this is the Haar measure on P). By choosing an element y_0 of $\mathcal{Y}$ we obtain a bijection $\eta : P \to \mathcal{Y}$ (with $\eta(g) = gy_0$), and thereby can transfer the topology and measure on P to $\mathcal{Y}$. The resulting topology and measure λ on $\mathcal{Y}$ are independent of the choice of y_0 here.

Claim: the map Θ in the above lemma is continuous.

For $s \in \mathbb{N}$ and $\bar{a} = (a_1, \ldots, a_s) \in B^s$, let $\psi(x_1, \ldots, x_s, y)$ be an $L_0(C)$-formula such that $\psi(\bar{x}, e)$ isolates the L_0-type of $\bar{a}$ over Ce. An $L_A(C)$-formula $\theta(y)$ of the form

$$(\exists \bar{x})(\psi(\bar{x}, y) \wedge \bigwedge_{i=1}^{s} (\mathrm{ang}(x_i) = \gamma_i))$$

for $\gamma_1, \ldots, \gamma_s \in \Gamma$ will temporarily be called a *basic* formula. Then $p, q \in \mathcal{X}$ are equal iff they contain the same basic formulas, so the clopen sets determined by the basic formulas are a basis of the topology on $\mathcal{X}$.

Now, for any such basic $\theta(y)$ there is a finite $A \subseteq B$ such that, for $f, g \in \mathcal{Y}$, if f, g have the same restriction to A, then $\theta(y) \in \Theta(f)$ iff $\theta(y) \in \Theta(g)$ (simply take A to contain all $\mathrm{Aut}(B/Ce)$-translates of the a_i). Continuity of Θ then follows.

Definition 4.2. If D is a C-definable subset of F, let $[D]$ be the closed subset $\{p \in \mathcal{X} : D \in p\}$ of $\mathcal{X}$, and define

$$\mu(D) = \lambda(\Theta^{-1}([D])).$$

This is a finitely-additive measure on the collection of $L_A(C)$-definable subsets of F. Of course, as λ is a measure on the Borel sets of $\mathcal{Y}$, μ extends to a σ-additive measure on the collection of countable boolean combinations of C-definable sets.

Lemma 4.3. *The measure μ does not depend on the choice of C.*

Proof. Temporarily augment the notation by including a subscript to indicate dependency on C. We need to show that if $C' \supseteq C$ is algebraically closed and D is C-definable, then $\mu_C(D) = \mu_{C'}(D)$.

Let $e \in F$ be generic over C' and $B' = \mathrm{acl}(C'e)$. Then e is also generic over C, and we let $B = \mathrm{acl}(Ce)$. Obviously $\mathcal{G}^+(B') \cap B = \mathcal{G}^+(B)$ so there is a natural injective function $\iota : B/\mathcal{G}^+(B) \to B'/\mathcal{G}^+(B')$ (given by $x\mathcal{G}^+(B) \mapsto xG^+(B')$). By the assumption that $\mathcal{G}^+(B') = \mathcal{G}^+(C')$, it follows that $\iota^{-1}(C'/\mathcal{G}^+(B')) = C/\mathcal{G}^+(B)$. Thus ι induces a surjective map $\pi : P_{C'} \to P_C$ (essentially by restriction), and of course this is a continuous group homomorphism.

Let $\eta_{C'} : P_{C'} \to \mathcal{Y}_{C'}$ and $\eta_C : P_C \to \mathcal{Y}_C$ be the bijections obtained by taking $\eta_{C'}(1) = \mathrm{ang}|C'$ and $\eta_C(1) = \mathrm{ang}|C$. Restriction gives a continuous map $\rho : \mathcal{Y}_{C'} \to \mathcal{Y}_C$ and the square

$$P_{C'} \xrightarrow{\ \pi\ } P_C$$

$$\eta_{C'} \downarrow \qquad\qquad \downarrow \eta_C$$

$$\mathcal{Y}_{C'} \xrightarrow{\ \rho\ } \mathcal{Y}_C$$

is commutative. In particular, ρ is surjective.

Restriction also gives a continuous surjection $\tau : \mathcal{X}_{C'} \to \mathcal{X}_C$, and the square:

$$\mathcal{Y}_{C'} \xrightarrow{\ \rho\ } \mathcal{Y}_C$$

$$\Theta_{C'} \downarrow \qquad\qquad \downarrow \Theta_C$$

$$\mathcal{X}_{C'} \xrightarrow{\ \tau\ } \mathcal{X}_C$$

is commutative.

Note that if D is C-definable, then $[D]_{C'} = \tau^{-1}[D]_C$ (i.e. any generic type over C containing D extends to one over C'). So we need to verify that if $Z \subseteq \mathcal{X}_C$ is closed then

$$\lambda_{C'}(\Theta_{C'}^{-1}\tau^{-1}(Z)) = \lambda_C(\Theta_C^{-1}(Z)).$$

By commutativity of the square, this follows if we show that whenever $Y \subseteq \mathcal{Y}_C$ is closed (or even Borel), then

$$\lambda_{C'}(\rho^{-1}(Y)) = \lambda_C(Y).$$

By the first square and definition of λ, this follows if $\kappa_{C'}(\pi^{-1}(X)) = \kappa_C(X)$ for all closed (or Borel) $X \subseteq P_C$. But each side of this equation is a Haar measure on P_C: hence the equality. $\square$

Example 4.4. The following example shows that the measure μ is not preserved under definable bijections.

Let T_0 be as in Example 3.4, $N \in \mathbb{N}$ not divisible by 2 and Γ the N-th roots of unity, with ϵ a primitive N-th root. Let T_A^* be as in Theorem 3.5. Consider the $L_A(\Gamma)$-definable sets:

$$X = \{e \in F^* : \mathrm{ang}(e) = 1,\ \mathrm{ang}(\{g : g^2 = e\}) = \{\epsilon, \epsilon^2\}\};$$

$$Y = \{g \in F^* : \mathrm{ang}(g) = \epsilon,\ \mathrm{ang}(g^2) = 1, \mathrm{ang}(-g) = \epsilon^2\}.$$

There is a definable bijection between these sets as the different values of the angular function allow one to select in a definable way a square root of each $e \in X$. On the other hand, $\mu(X) = 2/N^3$ whereas $\mu(Y) = 1/N^3$.

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