The Geometries of the Hrushovski Constructions

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Abstract

In 1984 Zilber conjectured that any strongly minimal structure is geometrically equivalent to one of the following types of strongly minimal structures, in the appropriate language: Pure sets, Vector Spaces over a fixed Division Ring and Algebraically Closed Fields.

In 1993, in his article 'A new strongly minimal set' Hrushovski produced a family of counterexamples to Zilber's conjecture. His method consists in two steps. Firstly he builds a 'limit' structure from a suitable class of finite structures in a language consisting only of a ternary relational symbol. Secondly, in a step called the collapse, he defines a continuum of subclasses such that the corresponding 'limit' structures are new strongly minimal structures. These new strongly minimal structures are non isomorphic but Hrushovski then asks if they are geometrically equivalent.

We first analyze the pregeometries arising from different variations of the construction before the collapse. In particular we prove that if we repeat the construction starting with an n-ary relational symbol instead of a 3-ary relational symbol, then the pregeometries associated to the corresponding 'limit' structures are not locally isomorphic when we vary the arity.

Second we prove that these new strongly minimal structures are geometrically equivalent. In fact we prove that their geometries are isomorphic to the geometry of the 'limit' structure obtained before the collapse.

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Chapter 1

Introduction

The algebraic closure of a strongly minimal structure gives a pregeometry, thus a geometry. We say that two strongly minimal structures are geometrically equivalent if their associated geometries are locally isomorphic. In 1984 Zilber conjectured that any strongly minimal structure is geometrically equivalent to one the following types of strongly minimal structures, in the appropriate language: Pure sets, Vector Spaces over a fixed Division Ring and Algebraically Closed Fields.

In 1993, Hrushovski in his paper [4], introduced a method in order to produce a counterexample to Zilber's Conjecture. Basically this method builds a limit structure, called the generic model, from a suitable class of finite structures. Then he studies the pregeometries associated to this limit structure to show that the strongly minimal set obtained is in fact new, in the sense of Zilber's conjecture. We give a brief description of Hrushovski's example.

Consider the language $L_3 = \{R_3\}$, where R_3 is a ternary relational symbol. Then consider a function δ that assigns to each finite L_3-structure A an integer number $\delta(A) = |A| - |R_3^A|$, where R_3^A is the set of triples in R_3 with coordinates in A. Next we restrict δ to the class $\mathcal{C}_3$ of finite L_3-structures such that $\delta(A)$ is nonnegative for every $A \in \mathcal{C}_3$ and such that $\mathcal{C}_3$ is still closed under substructures. So we define the class

$$\mathcal{C}_3 = \{A : A \text{ is a finite } L_3 \text{ structure and } \delta(A') \geq 0 \, \forall A' \subseteq A\}.$$

We call this restriction $\delta : \mathcal{C}_3 \to \mathbb{R}_0^+$ a predimension. Now we define a binary relation $\leq$ on $\mathcal{C}_3$ by putting $A \leq B$ if and only if $A \subseteq B$ and $\delta(A) \leq \delta(A')$ for all A' such that $A \subseteq A' \subseteq B$.

The key step to build the limit structure from $(\mathcal{C}_3, \leq)$ is to prove an amalgamation lemma. More precisely: 'If $A, B_1, B_2 \in \mathcal{C}_3$, $A \leq B_1$ and $A \leq B_2$ then there is $C \in \mathcal{C}_3$ and embeddings $f_i : B_i \to C$ such that these embeddings agree in A and such that $f_i(B_i) \leq C$'. Using this amalgamation lemma we can build a limit structure from the class $(\mathcal{C}_3, \leq)$ such that its theory is in this case ω-stable. To produce a strongly minimal structure we need to restrict further the class $\mathcal{C}_3$ to a class $(\mathcal{C}_\mu, \leq)$ and to prove another amalgamation lemma for this restricted class. This step is called 'the collapse' and the generic model arising from this amalgamation class is strongly minimal and it is a counterexample to Zilber's conjecture.

Now we give a brief description of the content of the chapters. Chapters 2, 3 and 4 are largely expository. The results presented in this chapters are known but we include full proofs of various results for the sake of completeness. Also sometimes the results are stated in a slightly different way than what is found in the literature. The reader can safely skip Chapters 2, 3 and 4 and move on to the main chapters of this thesis: Chapter 5 and Chapter 6, which consist mainly of new results.

In Chapter 2 we define the notions of pregeometry and geometry. We introduce here basic notions about pregeometries like independent sets, basis of a pregeometry, dimension function of a pregeometry and localization of a pregeometry. Moreover we dedicate a substantial part of the chapter to explain how to obtain a pregeometry from a predimension function satisfying submodularity, motivating in this way the notion of predimension and submodularity. The main results of the chapter are Corollary 2.2.8 and Proposition 2.2.9. The results achieved in this chapter are already known, but we present here a self-contained approach.

In Chapter 3 we define the notion of a strongly minimal set and we prove that the algebraic closure in a strongly minimal structure induces a pregeometry, see 3.2.4. The results in this chapter are classical and we follow the approach given by David Marker in his book [5].

In Chapter 4 we talk about predimension and amalgamation. In his paper [7], Frank Wagner presents an axiomatic approach to Hrushovski's method and uses it to obtain stability and ω-stability. We will discuss here this axiomatic approach. There is another axiomatic approach due to John T. Baldwin and Niandong Shi, in their paper [1], where they allow the closure of finite sets to be possibly infinite. This chapter is basically a reorganization of Frank Wagner's paper [7], but there is some degree of originality in the presentation. In contrast with Wagner's approach, rather than start with an amalgamation class and a dimension function, we start immediately with a predimension function and then define the amalgamation class and the dimension function from it. This approach might lose some generality but allows a more concise approach to the subject. For example, rather than just defining the pregeometry of the generic model, this way of presenting the subject allow us to define a pregeometry in each structure such that every substructure is in the amalgamation class, basically using Proposition 2.2.9 from Chapter 2.

We give an almost self-contained presentation of the subject (for relational structures) towards two versions of the same theorem that summarizes most of the main results of the chapter. In Theorem 4.3.20 we isolate testable conditions that guarantee the saturation of the generic model and the stability/ω-

stability of its theory. In Theorem 4.3.22 we use weaker conditions, but somewhat more difficult to be tested in practical contexts. We finalize this chapter applying the results developed in the chapter to obtain the uncollapsed version of Hrushovski original construction, see [4] for the original work.

Chapter 5 and Chapter 6 are the main chapters of the thesis, consisting mostly of original results.

In Chapter 5 we deal with variations of the original Hrushovski constructions before the collapse. We consider variations where we allow the language to be more general than just a ternary relational symbol and compare the pregeometries arising from the corresponding generic model when we change the language.

We consider as our more general case the construction arising from the predimension

$$\delta_f(A) = |A| - \sum_{i \in I} \alpha_i |R_i^A|$$

where each R_i is a relational symbol and each α_i is a nonnegative integer, see Proposition 5.1.14 for definition and set up. Now we emphasize the main results of the chapter.

If we restrict our language in our general construction we obtain a restricted amalgamation class. In Theorem 5.2.1 we prove that the pregeometry of the generic model of this restricted amalgamation class is still isomorphic to the pregeometry of the generic model produced before the restriction. In Theorem 5.2.8 we build a chain of embeddings of pregeometries arising from the construction corresponding to different languages. In particular if we consider $\mathcal{M}_3$ to be the generic model of the original construction and $\mathcal{M}_n$ the variation where we work with a n-ary relational symbol ($n \geq 3$) instead of a ternary relational symbol, then we get that the corresponding pregeometries, $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$, embed in each other for $m, n \geq 3$.

This result made us ask if actually $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$ were isomorphic for $m, n \geq 3$. However we prove in Theorem 5.3.3 that for $m \neq n$ we have that $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$ are not isomorphic. The key step to prove this and other results are the Changing Lemmas. For this particular result we use the first two Changing Lemmas.

The First Changing Lemma 5.3.1 says that if we replace a closed substructure of a structure in our amalgamation class by another structure in the class with the same underlying set, then the resulting structure is still in the class. The Second Changing Lemma 5.3.2 adds that if we do this replacement without changing the pregeometry of the closed substructure, then the resulting structure has the same pregeometry as the original one.

The Third Changing Lemma 5.4.1 deals with changes in the generic model. Basically it says that if we replace a finite closed substructure of the generic model by another structure in the class with the same underlying set, then the resulting structure is still isomorphic to the generic model. The Fourth Changing Lemma 5.4.3 says that if two structures in the class with the same underlying set differ only on a closed substructure then the localized pregeometry over this common subset is the same for both structures.

These Changing Lemmas are similar to lemmas used in some of David Evans' articles, namely [2] and [3].

With these four Changing Lemmas we prove in Theorem 5.4.5 that localizing the pregeometry of the generic model over a finite subset does not change the isomorphism type of the pregeometry, that is, the localized pregeometry is isomorphic to the pregeometry before the localization. Using this and Theorem 5.3.3 we get that for $m \neq n$ the pregeometries $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$ are not even locally isomorphic.

We end the chapter by considering an amalgamation class of pregeometries in an appropriate language. We define the class $(P_n, \preceq_n)$ of pregeometries by applying the forgetful functor to the class $(\mathcal{C}_n, \leq_n)$. Then using the first two Changing Lemmas we prove that $\preceq_n$ is transitive and that $(P_n, \preceq_n)$ is an amalgamation class. This set up does not match exactly the framework of Chapter 4, but we still can define a reasonable notion of generic model of $(P_n, \preceq_n)$ and prove the existence and uniqueness of such a generic model $\mathcal{P}_n$ of this class of pregeometries. In the last theorem of the chapter, Theorem 5.5.9, we prove that $\mathcal{P}_n$ is isomorphic to $PG(\mathcal{M}_n)$: for this result we use the first three Changing Lemmas. This amalgamation class of pregeometries and in particular this last theorem will prove useful in the proof our main result in Chapter 6.

Chapter 6 is entirely dedicated to the proof of our main result, dealing with the pregeometries arising after the collapse. Following Hrushovski, we start by defining the notion of minimally simply algebraic extensions. Then we consider amalgamation classes $\mathcal{C}_\mu$ where μ is a function from the isomorphism types of minimally simply algebraic extensions to the natural numbers. Basically, the structures in $\mathcal{C}_\mu$ are the structures in $\mathcal{C}_3$ such that if we see minimally simply algebraic extensions $A \leq B_i$ for $1 \leq i \leq k$ and we have isomorphisms $f_i : B_1 \to B_i$ over A with $B_i \cap B_j = A$ for $1 \leq i, j \leq k$ and $i \neq j$, then $k \leq \mu(A \leq B_1)$. So the μ function restricts the number of realizations of minimally simply algebraic extensions. We have that $(\mathcal{C}_\mu, \leq)$ is an amalgamation class and that the corresponding generic model $\mathcal{M}_\mu$ is strongly minimal. We describe this construction in the beginning of the chapter, but for a more detailed exposition you can see Hrushovski's article [4] or Wagner's article [7].

Since Hrushovski's article [4] in 1993, there was the open question if the pregeometries arising from different μ functions were locally isomorphic. We answer this question affirmatively proving that they are in fact isomorphic. More precisely we prove in Theorem 6.2.7 that for $\mu \geq 1$ we have $PG(\mathcal{M}_\mu) \simeq PG(\mathcal{M}_3)$, that is collapsing does not affect the isomorphism type of the pregeometry. The idea of the proof is by back and forth between $PG(\mathcal{M}_\mu)$

and $PG(\mathcal{M}_3)$. On the $PG(\mathcal{M}_3)$ side we have the tools from Chapter 5, specifically the first three Changing Lemmas and Theorem 5.5.9. However the Changing Lemmas fail for the $PG(\mathcal{M}_\mu)$ side, so we prove a battery of lemmas in order to develop another tool, the Hard Changing Lemma.

The Hard Changing Lemma is somewhat different from the other ones, because it combines the notion $\leq$ of self-sufficiency of structures with the notion $\preceq$ of self-sufficiency of pregeometries. Basically it says that if we have $A \leq B$ with $A \in \mathcal{C}_\mu$ and $B \in \mathcal{C}_3$ then we can find $B' \in \mathcal{C}_\mu$ such that $A \leq B'$ and $PG(B) \preceq PG(B')$, however B is not a substructure of B'. With this technical tool on the $PG(\mathcal{M}_\mu)$ side we can use the back and forth argument to build an isomorphism between $PG(\mathcal{M}_\mu)$ and $PG(\mathcal{M}_3)$.

We end this work with a small Chapter 7 where we give two open problems.

In the first problem we discuss an alternative and more natural definition for the generic model of the class of pregeometries $(P_n, \preceq)$, we call it the $\sqsubseteq$-*generic* and $\preceq$-*generic* stands for the old version of the generic, that is $\mathcal{P}_n$. The question is if the $\sqsubseteq$-generic exists. We notice however that if it exists then it is isomorphic to the $\preceq$-generic.

The second problem deals with a variation of this construction by Hrushovski from Proposition 18 in the article [4]. In this variation the amalgamation class is chosen in a way that we can recover the structures from the pregeometries. This means in particular that in this case the pregeometries $PG(\mathcal{M}_\mu)$ will not be isomorphic for different choices of the function μ. The question is then if they are locally isomorphic. We believe that this is the case, we hope that by localizing to a well chosen finite set we will get the pregeometry of the unordered version of $\mathcal{M}_3$.

Chapter 2

Pregeometries

2.1 Basic definitions

Let $\mathcal{P}(S)$ denote the set of subsets of S and $\mathcal{P}_F(S)$ denote the set of finite subsets of S.

Definition 2.1.1. Let D be a set and $\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ be a function. We say that (D, cl) is a *closure* if and only if for all subsets A and B of D we have,

1. $A \subseteq \mathrm{cl}(A)$.
2. $A \subseteq B \implies \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.
3. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

We say that a closure (D, cl) is a *good closure* if for all subsets A of D it satisfies,

$$\mathrm{cl}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} \mathrm{cl}(A_0).$$

We say that a good closure is a *pregeometry* if it satisfies the exchange principle, that is, for all $A \subseteq D$ and $b, c \in D$ we have,

$$c \in \mathrm{cl}(A \cup \{b\}) - \mathrm{cl}(A) \implies b \in \mathrm{cl}(A \cup \{c\}).$$

We say that a pregeometry is a *geometry* if for all $x \in D$ we have,

$$\mathrm{cl}(\{x\}) = \{x\}.$$

Given a closure (D, cl) we can define new closures from it in two different ways, either by inducing closures on subsets or localizing closures at subsets. See the next definition.

Definition 2.1.2. Let (D, cl) be a closure and $D', X \subseteq D$. We define,

- $(D', \mathrm{cl}^{D'})$ by $\mathrm{cl}^{D'}(A) = \mathrm{cl}(A) \cap D'$. (induced closure on D')
- (D, cl_X) by $\mathrm{cl}_X(A) = \mathrm{cl}(A \cup X)$. (localization at X)
- $(D', \mathrm{cl}_X^{D'})$ by $\mathrm{cl}_X^{D'}(A) = \mathrm{cl}(A \cup X) \cap D'$.

Note that we have $\mathrm{cl}_X^{D'} = (\mathrm{cl}_X)^{D'}$ and if $X \subseteq D'$ we also have $\mathrm{cl}_X^{D'} = (\mathrm{cl}^{D'})_X$.

The next proposition is easy to prove and is left to the reader.

Proposition 2.1.3. *Let* (D, cl) *be a closure/good closure/pregeometry and* $D', X \subseteq D$. *Then* $(D', \mathrm{cl}^{D'})$ *and* (D, cl_X) *are also closures/good closures/pregeometries.*

It is not true that a localization of a geometry is still a geometry because in the localization of a geometry the closure of the emptyset is not the emptyset. However every pregeometry has an associated geometry. We show how to do it in the next definition.

Definition 2.1.4. Let (D, cl) be a pregeometry. We define the corresponding geometry $D^{\sim} := (D^{\sim}, \mathrm{cl}^{\sim})$ as follows. We start by defining an equivalence relation $\sim$ on $D \setminus \mathrm{cl}(\emptyset)$ by saying that $x \sim y$ if and only if $\mathrm{cl}(x) = \mathrm{cl}(y)$. Then we define the set $D^{\sim} := (D \setminus \mathrm{cl}(\emptyset))/\sim$, that is, the set of equivalence classes of $\sim$. Let $x \in D \setminus \mathrm{cl}(\emptyset)$, we denote the equivalence class of x by $[x]_{\sim}$. Let $A \subseteq D^{\sim}$ then we define $\mathrm{cl}^{\sim}(A)$ by saying that $[x]_{\sim} \in \mathrm{cl}^{\sim}(A)$ if and only if $x \in \mathrm{cl}(\bigcup A)$. Then $D^{\sim} := (D^{\sim}, \mathrm{cl}^{\sim})$ is a geometry.

This procedure allow us to define the localization of a geometry in a such way that the localization is a geometry.

Definition 2.1.5. Let (D, cl) be a geometry and $X \subseteq D$. We define the localization of (D, cl) at X as $(D_X)^{\sim}$ where D_X is the localization as pregeometry.

We will need later the following definition.

Definition 2.1.6. Two pregeometries D^1 and D^2 are said to be locally isomorphic if there are finite sets $X_1 \subseteq D^1$ and $X_2 \subseteq D^2$ such that $D^1_{X_1} \simeq D^2_{X_2}$. Two geometries D^1 and D^2 are said to be locally isomorphic if there are finite sets $X_1 \subseteq D^1$ and $X_2 \subseteq D^2$ such that $(D^1_{X_1})^{\sim} \simeq (D^2_{X_2})^{\sim}$.

In a pregeometry we can define the notions of independent sets and basis in a similar way to vector spaces.

Definition 2.1.7. Let (D, cl) be a pregeometry. We say that $A \subseteq D$ is *independent* if for all $a \in A$ we have $a \notin \mathrm{cl}(A \setminus \{a\})$. If $X \subseteq D$ we say that $A \subseteq D$ is *independent over* X if for all $a \in A$ we have $a \notin \mathrm{cl}_X(A \setminus \{a\})$. Given $Y \subseteq D$ we say that $A \subseteq Y$ is a *basis* for Y if A is independent and $\mathrm{cl}(A) = \mathrm{cl}(Y)$.

Proposition 2.1.8. *In a pregeometry a set is independent if and only if every finite subset is independent.*

Proof. Let A_0 be a finite subset of A and $a \in A_0$ be such that $a \in \mathrm{cl}(A_0 \setminus \{a\})$. Then $a \in \mathrm{cl}(A \setminus \{a\})$ so if A_0 is dependent then so is A. In the other direction let $a \in A$ be such that $a \in \mathrm{cl}(A \setminus \{a\})$. Then $a \in \mathrm{cl}(A_0 \setminus \{a\})$ for some finite subset A_0 of A. That is, if A is dependent so is some finite subset of A. $\square$

We shall prove now that in a pregeometry every subset has a basis but first we need the following result.

Proposition 2.1.9. *Let (D, cl) be a pregeometry and $Y \subseteq D$. The bases of Y are exactly the maximal independent subsets of Y.*

Proof. ($\Rightarrow$) Let A be a maximal independent subset of Y. We want to prove that $Y \subseteq \mathrm{cl}(A)$. For that we will prove that if there exists $b \in Y \setminus \mathrm{cl}(A)$ then $A \cup \{b\}$ would be an independent subset of Y, contradicting the maximality of A. In fact $b \notin \mathrm{cl}(A)$ and given $a \in A$ we would also get $a \notin \mathrm{cl}(A \cup \{b\} - \{a\})$. This is true because if we have $a \in \mathrm{cl}(A \cup \{b\} - \{a\})$ then by independence of A we would have $a \in \mathrm{cl}(A \setminus \{a\} \cup \{b\}) - \mathrm{cl}(A \setminus \{a\})$ and by exchange we would have $b \in \mathrm{cl}(A \setminus \{a\} \cup \{a\}) = \mathrm{cl}(A)$, which is false. Thus $a \notin \mathrm{cl}(A \cup \{b\} - \{a\})$ and $A \cup \{b\}$ would be independent. Finally $Y \subseteq \mathrm{cl}(A)$ implies $\mathrm{cl}(Y) \subseteq \mathrm{cl}(A)$ and $A \subseteq Y$ implies $\mathrm{cl}(A) \subseteq \mathrm{cl}(Y)$. Thus $\mathrm{cl}(A) = \mathrm{cl}(Y)$ and as A is independent then A is a basis of Y.

($\Leftarrow$) Assume now that A is a basis of Y. If A was not a maximal independent subset of Y then there would exist $b \in Y - A$ such that $A \cup \{b\}$ was independent. In particular $b \notin \mathrm{cl}(A)$ so $\mathrm{cl}(Y) \neq \mathrm{cl}(A)$ contradicting the fact that A is a basis of Y. $\square$

Corollary 2.1.10. *In a pregeometry every independent subset of a set Y can be extended to a basis of Y.*

Proof. Let Y be a subset of the pregeometry and $A \subseteq Y$ an independent set. Because of the last proposition we just need to prove that there exists a maximal independent subset of Y containing A. But unions of chains of independent subsets are independent because every finite subset of them is independent, thus by Zorn's Lemma, there exists a maximal independent subset of Y containing A. $\square$

Now we want to define the notion of dimension of a subset of a pregeometry as the cardinality of a basis. But we first need to prove that every two bases of a subset of a pregeometry have the same cardinality.

Notation 2.1.11. From now on we may write AB instead of $A \cup B$ and Ac instead of $A \cup \{c\}$.

Proposition 2.1.12. *Let (D, cl) be a pregeometry and $Y \subseteq D$. If $A \subseteq Y$ is independent and B is a basis of Y then $|A| \leq |B|$ and if A is also a basis of Y then $|A| = |B|$.*

Proof. If $A \subseteq B$ then $|A| \leq |B|$. If not let $(a_\epsilon)_{\epsilon \in \alpha}$ be an enumeration of $A \setminus (A \cap B)$. Let $A_0 = A \cap B = B_0$. Note that $B_0 \subseteq B$, $|A_0| = |B_0|$ and $A \setminus A_0 \cup B_0$ is independent. Let δ be an ordinal and suppose that we had built for each $\epsilon \in \delta$, A_ϵ and B_ϵ such that $B_\epsilon \subseteq B$, $|A_\epsilon| = |B_\epsilon|$ and $A \setminus A_\epsilon \cup B_\epsilon$ is independent. If $\delta = \epsilon + 1$ then we put $A_{\epsilon+1} = A_\epsilon \cup \{a_\epsilon\}$ and $B_{\epsilon+1} = B_\epsilon \cup \{b_\epsilon\}$ where b_ϵ is some element of $B \setminus \mathrm{cl}(A \setminus A_\epsilon \cup B_\epsilon - \{a_\epsilon\})$, in particular we will have $b_\epsilon \notin B_\epsilon$. Such a b_ϵ exists otherwise we would have $B \subseteq \mathrm{cl}(A \setminus A_\epsilon \cup B_\epsilon - \{a_\epsilon\})$ thus $\mathrm{cl}(B) \subseteq \mathrm{cl}(A \setminus A_\epsilon \cup B_\epsilon - \{a_\epsilon\})$ and then $a_\epsilon \in A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(Y) = \mathrm{cl}(B) \subseteq \mathrm{cl}(A \setminus A_\epsilon \cup B_\epsilon - \{a_\epsilon\})$ contradicting the independence of $A \setminus A_\epsilon \cup B_\epsilon$.

Now we prove that $A \setminus A_{\epsilon+1} \cup B_{\epsilon+1} = A \setminus (A_\epsilon a_\epsilon) \cup B_\epsilon b_\epsilon$ is independent. In fact $b_\epsilon \notin \mathrm{cl}(A \setminus (A_\epsilon a_\epsilon) \cup B_\epsilon)$ by choice of b_ϵ and for an arbitrary $c \in A \setminus (A_\epsilon a_\epsilon) \cup B_\epsilon b_\epsilon$, if $c \in \mathrm{cl}(A \setminus (A_\epsilon a_\epsilon) \cup B_\epsilon b_\epsilon - \{c\})$ then by exchange we would get again $b_\epsilon \in \mathrm{cl}(A \setminus (A_\epsilon a_\epsilon) \cup B_\epsilon)$, thus $A \setminus A_{\epsilon+1} \cup B_{\epsilon+1}$ is independent. Clearly we still have $B_{\epsilon+1} \subseteq B$ and $|A_{\epsilon+1}| = |B_{\epsilon+1}|$.

Now if δ is a limit ordinal we put $A_\delta = \bigcup_{\epsilon\in\delta} A_\epsilon$ and $B_\delta = \bigcup_{\epsilon\in\delta} B_\epsilon$, thus $B_\delta \subseteq B$ and $|A_\delta| = |B_\delta|$. It remains to prove that $A \setminus A_\delta \cup B_\delta$ is independent.

Suppose that $A\setminus A_\delta \cup B_\delta$ is dependent. Then there would exist $c \in A\setminus A_\delta \cup B_\delta$ such that $c \in \mathrm{cl}(A \setminus A_\delta \cup B_\delta - \{c\})$. In particular c would belong to the closure of some finite subset F of $A \setminus A_\delta \cup B_\delta - \{c\}$. Let $\epsilon \in \delta$ be such that $F \cup \{c\} \subseteq A \setminus A_\epsilon \cup B_\epsilon$. Such an ϵ exists because $F \cup \{c\}$ is a finite subset of $A \setminus A_\delta \cup B_\delta$ and δ is a limit ordinal. We have $c \in \mathrm{cl}(F) \subseteq \mathrm{cl}(A \setminus A_\epsilon \cup B_\epsilon - \{c\})$ contradicting the independence of $A\setminus A_\epsilon \cup B_\epsilon$. Thus $A\setminus A_\delta \cup B_\delta$ is independent.

Finally, we just need to observe that we will get $A = A_\delta$ and $B_\delta \subseteq B$ for some ordinal δ. In particular $|A| \leq |B|$ and $B_\delta = A \setminus A_\delta \cup B_\delta$ is independent.

If A is also a basis of Y then by the same argument, because B is independent, we will get $|B| \leq |A|$, thus $|A| = |B|$. □

We can now define the notion of dimension in a pregeometry.

Definition 2.1.13. Let (D, cl) be a pregeometry and $Y \subseteq D$. We define $d(Y)$, the *dimension of* Y, as the cardinality of a basis of Y.

2.2 Pregeometries and Predimensions

We can ask the question whether a pregeometry can be recovered from its dimension function d. Clearly $\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ can be recovered by its restriction to finite subsets of D, thus the next proposition gives an affirmative answer to the question above.

Proposition 2.2.1. *Let* (D, cl) *be a pregeometry and* $A \subseteq D$ *finite. We have*

$$\mathrm{cl}(A) = \{c \in D : d(Ac) = d(A)\}.$$

More precisely we have,

- $c \in \mathrm{cl}(A) \Leftrightarrow d(Ac) = d(A)$
- $c \notin \mathrm{cl}(A) \Leftrightarrow d(Ac) = d(A) + 1$

Proof. It suffices to prove that $c \notin \mathrm{cl}(A) \Rightarrow d(Ac) = d(A) + 1$ and $c \in \mathrm{cl}(A) \Rightarrow d(Ac) = d(A)$. First we assume that $c \notin \mathrm{cl}(A)$. It is enough to show that whenever A_0 is a basis of A then A_0c is a basis of Ac. Let A_0 be a basis of A. First we prove that A_0c is independent.

Notice that $c \in \mathrm{cl}(A_0)$ would imply $c \in \mathrm{cl}(A)$ which contradict our assumption that $c \notin \mathrm{cl}(A)$. For an arbitrary $a \in A_0$ if $a \in \mathrm{cl}(A_0c \setminus \{a\})$ then $a \in \mathrm{cl}(A_0 \setminus \{a\} \cup \{c\}) \setminus \mathrm{cl}(A_0 \setminus \{a\})$ by independence of A_0. But then by exchange we would get again $c \in \mathrm{cl}(A_0 \setminus \{a\} \cup \{a\}) = \mathrm{cl}(A_0)$. Thus A_oc is independent.

Now let us see that $\mathrm{cl}(A_0c) = \mathrm{cl}(Ac)$. Trivially $\mathrm{cl}(A_0c) \subseteq \mathrm{cl}(Ac)$ and because $\mathrm{cl}(A_0) = \mathrm{cl}(A)$ we get also $\mathrm{cl}(Ac) \subseteq \mathrm{cl}(\mathrm{cl}(A)c) = \mathrm{cl}(\mathrm{cl}(A_0)c) \subseteq \mathrm{cl}(\mathrm{cl}(A_0c)) = \mathrm{cl}(A_0c)$. Thus A_0c is a basis of Ac and $d(Ac) = d(A) + 1$.

Now we assume that $c \in \mathrm{cl}(A)$. Let A_0 be a basis of A. Then $\mathrm{cl}(Ac) \subseteq \mathrm{cl}(\mathrm{cl}(A)c) = \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ so we get $\mathrm{cl}(A_0) = \mathrm{cl}(A) = \mathrm{cl}(Ac)$. Thus A_0 is a basis of Ac and this shows that $d(A) = d(Ac)$. □

Given a set D and a function $f : \mathcal{P}_F(D) \to \mathbb{R}$ it is natural to attempt building a pregeometry on D by putting $\mathrm{cl}(A) = \{c \in D : f(Ac) = f(A)\}$ for every finite subset A of D. Then we could extend cl to $\mathcal{P}(D)$ in the natural way. So first we should deal with the question: When does a given function $C : \mathcal{P}_F(D) \to \mathcal{P}(D)$ extend (in the only possible way) to a pregeometry? The next lemma will give precise conditions.

Lemma 2.2.2. *Let D be a set and $C : \mathcal{P}_F(D) \to \mathcal{P}(D)$ be a function. For* *every set $A \subseteq D$ define*

$$\mathrm{cl}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} C(A_0).$$

Suppose that $C : \mathcal{P}_F(D) \to \mathcal{P}(D)$ satisfies the following conditions for all finite subsets A and B of D:

1. $A \subseteq C(A)$
2. $A \subseteq B \Rightarrow C(A) \subseteq C(B)$
3. $A_0 \subseteq C(A)$ *and* A_0 *finite* $\Rightarrow C(A_0) \subseteq C(A)$

Then $\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ *is an extension of* $C : \mathcal{P}_F(D) \to \mathcal{P}(D)$ *and* (D, cl) *is a good closure.*
Moreover if C satisfies

4. $c \in C(Ab) \setminus C(A) \Rightarrow b \in C(Ac)$

then (D, cl) *is a pregeometry.*

Proof. First we prove that $A \subseteq \mathrm{cl}(A)$. We just need to observe that $\mathrm{cl}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} C(A_0)$ so by 1) this contains $\bigcup_{A_0 \in \mathcal{P}_F(A)} A_0 = A$.

Now we prove that $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. Just observe that $\mathrm{cl}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} C(A_0)$ and because $A \subseteq B$ this is contained in $\bigcup_{A_0 \in \mathcal{P}_F(B)} C(A_0) = \mathrm{cl}(B)$.

Now we need to prove that $\mathrm{cl}(\cdot)$ is an extension of $C(\cdot)$, that is, for a finite A we have $\mathrm{cl}(A) = C(A)$. Let $A \subseteq D$ be finite. First we observe that $\mathrm{cl}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} C(A_0)$ contains $C(A)$ because A is finite and $A \subseteq A$. In the other direction we need to prove that $\mathrm{cl}(A) \subseteq C(A)$. It suffices to show that for every finite subset A_0 of A we have $C(A_0) \subseteq C(A)$, but this follows from 2) since $A_0 \subseteq A$.

Finally, in order to obtain a good closure we shall prove $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all $A \subseteq D$. Observe that we already have $A \subseteq \mathrm{cl}(A)$ and we already prove the monotonicity of cl so $\mathrm{cl}(A) \subseteq \mathrm{cl}(\mathrm{cl}(A))$. It remains to prove that $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$. Suppose that $c \in \mathrm{cl}(\mathrm{cl}(A))$. Then $c \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq \mathrm{cl}(A) = \bigcup_{B_0 \in \mathcal{P}_F(A)} C(B_0)$. But then, as A_0 is finite, by 2) we have $B_1, \cdots, B_n \subseteq A$ finite such that $A_0 \subseteq \bigcup_i C(B_i) \subseteq C(\bigcup_i B_i)$. Thus as $\bigcup_i B_i$ is finite, then by 3) we get $C(A_0) \subseteq C(\bigcup_i B_i)$ and $c \in \mathrm{cl}(A_0) = C(A_0) \subseteq C(\bigcup_i B_i) = \mathrm{cl}(\bigcup_i B_i) \subseteq \mathrm{cl}(A)$. Thus $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$.

We proved that (D, cl) is a good closure, in order to prove that is a pregeometry we need to assume 4). Suppose $c \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then we have $c \in C(A_0 b) \setminus C(A_0)$ for some finite subset A_0 of A. But then, by 4) we have $b \in C(A_0 c) \subseteq \mathrm{cl}(Ac)$. Thus (D, cl) satisfies the exchange principle and is a pregeometry. $\square$

We will now analyze when a function $f : \mathcal{P}_F(D) \to \mathbb{R}$ induces a pregeometry by putting $\mathrm{cl}(A) = \{c \in D : f(Ac) = f(A)\}$ for finite $A \subseteq D$.

Theorem 2.2.3. *Let D be a set and $f : \mathcal{P}_F(D) \to \mathbb{R}$ be a function. For finite $A \subseteq D$ put $C(A) = \{c \in D : f(Ac) = f(A)\}$. Then $C : \mathcal{P}_F(D) \to \mathcal{P}(D)$ extends to a good closure (in the only possible way) if and only if for every finite $A \subseteq D$ and $b, c \in D$ we have,*

- *(A1) $f(Ab) = f(A) \Rightarrow f(Abc) = f(Ac)$.*

Moreover C extends to a pregeometry if and only if for every finite $A \subseteq D$ and $b, c \in D$ we have (A_1) and

- *(A2) $f(Ab) \neq f(A) \wedge f(Abc) = f(Ac) \Rightarrow f(Abc) = f(Ab)$.*

Proof. Assume that C extends to a good closure. Observe that $A1$ is consequence of monotonicity. In fact $f(Ab) = f(A) \Leftrightarrow b \in C(A) \Rightarrow b \in C(Ac) \Leftrightarrow f(Abc) = f(Ac)$. Assume further that C extend to a pregeometry. Observe that $A2$ is a consequence of the exchange principle. In fact $f(Ab) \neq f(A) \wedge f(Abc) = f(Ac) \Leftrightarrow b \notin C(A) \wedge b \in C(Ac) \Leftrightarrow b \in C(Ac) \setminus C(A) \Rightarrow c \in C(Ab) \Leftrightarrow f(Abc) = f(Ab)$.

We shall prove now that if $A1$ holds then C extend to a good closure and if $A2$ also holds then C extend to a pregeometry. For this we will use the last lemma.

First we prove that $A \subseteq C(A)$. It is trivial, just observe that $a \in A \Rightarrow f(Aa) = f(A) \Rightarrow a \in C(A)$.

Now we prove that $A \subseteq B \Rightarrow C(A) \subseteq C(B)$. Let $B \setminus A = \{b_1 \cdots b_n\}$. Let $c \in C(A)$ so $f(Ac) = f(A)$. Then by applying $A1$ recursively we get $f(Ac) = f(A) \Rightarrow f(Ab_1c) = f(Ab_1) \Rightarrow f(Ab_1b_2c) = f(Ab_1b_2) \Rightarrow \cdots \Rightarrow f(Ab_1 \cdots b_nc) = f(Ab_1 \cdots b_n) \Leftrightarrow f(Bc) = f(B) \Leftrightarrow c \in C(B)$, thus $C(A) \subseteq C(B)$.

Now we prove that if $A_0 \subseteq C(A)$ and A_0 finite then $C(A_0) \subseteq C(A)$. Let $A = \{a'_1, \cdots, a'_m\}$, $A_0 = \{a_1, \cdots, a_n\}$ and $c \in D$. We want to prove that $f(A_0c) = f(A_0) \Rightarrow f(Ac) = f(A)$. We will prove,

$$f(A_0c) = f(A_0) \underset{(i)}{\Rightarrow} f(A_0Ac) = f(A_0A) \underset{(ii)}{\Rightarrow} f(Ac) = f(A).$$

(Proof of (i)) We just need to apply A_1 recursively. That is $f(A_0c) = f(A_0) \Rightarrow f(A_0a'_1c) = f(A_0a'_1) \Rightarrow \cdots \Rightarrow f(A_0a'_1 \cdots a'_mc) = f(A_0a'_1 \cdots a'_m) \Leftrightarrow f(A_0Ac) = f(A_0A)$.

(Proof of (ii)) First we observe that $f(A) = f(Aa_1) = f(Aa_1a_2) = \cdots = f(Aa_1 \cdots a_n) = f(AA_0)$. This is true because we have $a_k \in A_0 \subseteq C(A) \subseteq C(Aa_1 \cdots a_{k-1})$ implies $f(Aa_1 \cdots a_k) = f(Aa_1 \cdots a_{k-1})$ for all $1 \leq k \leq$

n. By hypothesis we have $f(AA_0) = f(AA_0c)$, that is $f(Aa_1 \cdots a_n) = f(Aa_1 \cdots a_n c)$. We also have $f(Aa_1 \cdots a_n) = f(Aa_1 \cdots a_{n-1})$ so by $A1$ we get $f(Aa_1 \cdots a_n c) = f(Aa_1 \cdots a_{n-1}c)$. Thus we have that $f(Aa_1 \cdots a_{n-1}) = f(Aa_1 \cdots a_{n-1}c)$. Now as $f(Aa_1 \cdots a_{n-i}) = f(Aa_1 \cdots a_{n-(i+1)})$ we can repeat the process. In the end we obtain $f(Aa_1) = f(Aa_1c)$ and finally $f(A) = f(Ac)$ as desired. Thus C extends to a good closure by the previous lemma.

Finally we want to prove that if $A2$ holds then C extend to a pregeometry. By the previous lemma it suffices to show that $c \in C(Ab) \setminus C(A) \Rightarrow b \in C(Ac)$. But $c \in C(Ab) \setminus C(A) \Leftrightarrow f(Abc) = f(Ab) \wedge f(Ac) \neq f(A)$ so by $A2$ we get $f(Abc) = f(Ac)$, that is $b \in C(Ac)$. Now we apply the previous lemma to conclude that C extend to a pregeometry. □

Suppose that $f : \mathcal{P}_F(D) \to \mathcal{P}(D)$ induces a pregeometry in the usual way. We can ask how to recover the dimension function. The dimension function can be recovered by putting

- $d(\emptyset) = 0$
- $d(Ac) = d(A)$ if $f(Ac) = f(A)$
- $d(Ac) = d(A) + 1$ if $f(Ac) \neq f(A)$

for all finite $A \subseteq D$ and $c \in D$. This is true because by 2.2.1 we have $f(Ac) = f(A)$ if and only if $c \in \mathrm{cl}(A)$ if and only if $d(Ac) = d(A)$ and $f(Ac) \neq f(A)$ if and only if $c \notin \mathrm{cl}(A)$ if and only if $d(Ac) = d(A) + 1$. In particular we get $f = d$ if and only if $f(\emptyset) = 0$ and $f(Ac) - f(A) \in \{0, 1\}$, for all finite $A \subseteq D$ and $c \in D$. We obtain the following corollary of this observation.

Corollary 2.2.4. *Pregeometries on a set D are in correspondence with dimension functions $f : \mathcal{P}_F(D) \to \mathbb{N}$ satisfying $A1$ and*

- $f(\emptyset) = 0$
- $f(A) \leq f(Ac) \leq f(A) + 1$

for all finite $A \subseteq D$ *and* $c \in D$.

Proof. Because of the observations above, the only thing that remains to prove is that $A2$ holds under these assumptions. Assume that $f(Ab) \neq f(A)$ (that is $f(Ab) = f(A) + 1$) and $f(Abc) = f(Ac)$. Assume for purpose of contradiction that $f(Abc) \neq f(Ab)$, that is $f(Abc) = f(Ab) + 1$. Then we get $f(Abc) = f(Ab) + 1 = f(A) + 1 + 1 = f(A) + 2$, but on the other side $f(Abc) = f(Ac) \leq f(A) + 1$ so we get the desired contradiction. □

Now we explain a way of obtaining a dimension function $f : \mathcal{P}_F(D) \to \mathbb{R}_0^+$ satisfying the axioms $A1$ and $A2$ beginning with another function (predimension) $f_0 : \mathcal{P}_F(D) \to \mathbb{R}_0^+$.

Proposition 2.2.5. *Let* D *be a set and* $f_0 : \mathcal{P}_F(D) \to \mathbb{R}_0^+$ *be a function. Define* $f : \mathcal{P}_F(D) \to \mathbb{R}_0^+$ *by*

$$f(A) = \inf\{f_0(B) : A \subseteq B \in \mathcal{P}_F(D)\}$$

Then we have $f(A) \leq f(Ac)$ *for all finite* $A \subseteq D$ *and* $c \in D$.

Proof. Trivial, follows from the definition of f. □

Proposition 2.2.6. *Let* D *be a set and* $f_0 : \mathcal{P}_F(D) \to \mathbb{N}$ *be a function and let* $f : \mathcal{P}(D) \to \mathbb{R}_0^+$ *be defined as in Proposition 2.2.5. If* f_0 *satisfies* $f_0(Ac) \leq f_0(A) + 1$ *for all finite* $A \subseteq D$ *and* $c \in D$ *then* f *satisfies*

$$f(A) \leq f(Ac) \leq f(A) + 1$$

for all finite $A \subseteq D$ *and* $c \in D$.

Proof. It was already seen in the last proposition that $f(A) \leq f(Ac)$. Note that because f_0 has its image contained in the natural numbers then $f(A) = \inf\{f_0(B) : A \subseteq B \in \mathcal{P}_F(D)\}$ is achieved for some B. Thus we have $f_0(B) = f(A)$ for some finite $A \subseteq B \subseteq D$. Now By definition of f we have $f(Ac) \leq f_0(Bc)$ and by hypothesis we have $f_0(Bc) \leq f_0(B)+1$. We also have $f_0(B) = f(A)$ therefore $f(Ac) \leq f(A) + 1$. $\square$

Consider the following important axiom (submodularity) for a function $f_0 : \mathcal{P}_F(D) \to \mathbb{R}_0^+$,

(SM) $f_0(A \cup B) \leq f_0(A) + f_0(B) - f_0(A \cap B)$, for all finite $A, B \subseteq D$.

Proposition 2.2.7. *Let D be a set and $f_0 : \mathcal{P}_F(D) \to \mathbb{R}_0^+$ be a function. If f_0 satisfies SM then f satisfies $A1$.*

Proof. We want to prove that if $f(Ab) = f(A)$ then $f(Abc) = f(Ac)$. By the monotonicity of f proved above we have $f(Ac) \leq f(Abc)$. Let $\gamma > 0$. Then by definition of f we can find a finite $E_1 \supseteq Ac$ such that $f_0(E_1)-f(Ac) \leq \gamma/2$ and a finite $E_2 \supseteq Ab$ such that $f_0(E_2)-f(Ab) \leq \gamma/2$. Note that $E_1 \cap E_2 \supseteq A$ so $f_0(E_1 \cap E_2) \geq f(A) = f(Ab) \geq f(E_2) - \gamma/2$, thus $f_0(E_2) - f_0(E_1 \cap E_2) \leq \gamma/2$. Note also that we have $f_0(E_1) \leq f(Ac) + \gamma/2$.

Now as $E_1 \cup E_2 \supseteq Abc$ we have $f(Abc) \leq f_0(E_1 \cup E_2)$ and by submodularity we have $f_0(E_1 \cup E_2) \leq f_0(E_1)+f_0(E_2)-f_0(E_1 \cap E_2) \leq (f(Ac)+\gamma/2)+\gamma/2 = f(Ac) + \gamma$. Thus we have $f(Abc) \leq f(Ac) + \gamma$, for every $\gamma > 0$. So we must have $f(Abc) \leq f(Ac)$. $\square$

We obtain now without pain the following corollary.

Corollary 2.2.8. *Let D be a set and $f_0 : \mathcal{P}_F(D) \to \mathbb{R}_0^+$ be a function. If f_0 satisfies SM then f induces a good closure on D by putting* $\mathrm{cl}(A) = \{c \in D : f(Ac) = f(A)\}$ *for every finite $A \subseteq D$.*

If we impose some more restrictions on f_0 we obtain a pregeometry.

Proposition 2.2.9. *Let D be a set and $f_0 : \mathcal{P}_F(D) \to \mathbb{N}$ be a function. Suppose that f_0 satisfies the following conditions:*

1. $f_0(\emptyset) = 0$
2. $f_0(\{c\}) \leq 1$, *for all* $c \in D$
3. *(SM)* $f_0(A \cup B) \leq f_0(A) + f_0(B) - f_0(A \cap B)$, *for all finite* $A, B \subseteq D$.

Let $f : \mathcal{P}_F(D) \to \mathbb{N}$ be defined by $f(A) = \inf\{f_0(B) : A \subseteq B \in \mathcal{P}_F(D)\}$. Then f induces a pregeometry on D by putting $\mathrm{cl}(A) = \{c \in D : f(Ac) = f(A)\}$ for every finite $A \subseteq D$ and the respective dimension function coincides with f on finite sets.

Proof. By Proposition 2.2.7, the submodularity of f_0 implies that f satisfies A_1, so by Corollary 2.2.4 we only need to show that $f(\emptyset) = 0$ and $f(A) \leq f(Ac) \leq f(A) + 1$ for all finite $A \subseteq D$ and $c \in D$. By 1) we have $0 \leq f(\emptyset) \leq f_0(\emptyset) = 0$ so $f(\emptyset) = 0$. For proving that $f(A) \leq f(Ac) \leq f(A) + 1$ for all finite $A \subseteq D$ and $c \in D$ it suffices to prove (because of Proposition 2.2.6) that f_0 satisfies $f_0(Ac) \leq f_0(A) + 1$, for all finite $A \subseteq D$ and $c \in D$.

But by submodularity we have $f_0(Ac) \underset{3)}{\leq} f_0(A) + f_0(\{c\}) - f_0(A \cap \{c\}) = f_0(A) + f_0(\{c\}) - f_0(\emptyset) \underset{1)}{=} f_0(A) + f_0(\{c\}) \underset{2)}{\leq} f_0(A) + 1$. This concludes the proof. $\square$

Chapter 3

Strongly Minimal Structures

3.1 Minimality and Strong Minimality

Notation 3.1.1. Let $\mathcal{M}$ be a structure. We write M for the underlying set of $\mathcal{M}$. When $\bar{a} \in M^n$ for some natural number n we instead write $\bar{a} \in M$ when no confusion arises. Normally $\bar{a}, \bar{b}, \bar{c} \cdots$ stand for tuples of elements in the structure and $\bar{x}, \bar{y}, \bar{z} \cdots$ for tuples of variables.

Definition 3.1.2. Let $\mathcal{M}$ be an L-structure. Let $\psi(\bar{x}, \bar{z})$ be an L-formula and $\bar{a} \in M$. Put $D = \psi(M, \bar{a}) = \{\bar{c} \in M : \mathcal{M} \models \psi(\bar{c}, \bar{a})\}$.
We say that $\psi(\bar{x}, \bar{a})$ is *minimal in* $\mathcal{M}$ if D is infinite and if for every L-formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b} \in M$ we have,

$$\text{either } \varphi(M, \bar{b}) \cap D \text{ or } D \setminus \varphi(M, \bar{b}) \text{ is finite.}$$

We say that $\psi(\bar{x}, \bar{a})$ is *strongly minimal in* $\mathcal{M}$ if D is infinite and if for every L-formula $\varphi(\bar{x}, \bar{y})$ there exists $d \in \mathbb{N}$ such that for every $\bar{b} \in M$ we have

$$\text{either } \varphi(M, \bar{b}) \cap D \text{ or } D \setminus \varphi(M, \bar{b}) \text{ has no more than } d \text{ elements.}$$

If $\phi(\bar{x}, \bar{a})$ is minimal/strongly minimal in $\mathcal{M}$ we also say that D is minimal/strongly minimal in $\mathcal{M}$.

Remark 3.1.3. a) If a definable set $D \subseteq M^n$ is strongly minimal in $\mathcal{M}$ then it is minimal in $\mathcal{M}$.

b) Let $(\mathcal{M}, (m_i)_{i \in I})$ be an expansion of $\mathcal{M}$ by constants and $D \subseteq M^n$. Then D is definable/minimal/strongly minimal in $(\mathcal{M}, (m_i)_{i \in I})$ if and only if it is in $\mathcal{M}$.

The next proposition shows that strong minimality is a stronger version of minimality in a sense that strong minimality of a formula $\psi(\bar{x}, \bar{a})$ does not depend on the model $\mathcal{M}$ but only on its theory $Th(\mathcal{M}, \bar{a})$. So strong minimality of a formula can be defined relatively to a complete theory, in a suitable language. Remember that an L-structure is *weakly saturated* if every n-type over $\emptyset$ consistent with the theory of the structure is realized in the structure.

Proposition 3.1.4. *Let $\mathcal{M}$ be an L-structure, $\psi(\bar{x}, \bar{z})$ an L-formula and $\bar{a} \in M$. The following are equivalent:*

1. *$\psi(\bar{x}, \bar{a})$ is strongly minimal in $\mathcal{M}$.*
2. *$\psi(\bar{x}, \bar{a})$ is minimal in $\mathcal{N}$ for every $\mathcal{N} \succeq \mathcal{M}$.*
3. *$\psi(\bar{x}, \bar{a})$ is minimal in $\mathcal{N}$ for a fixed weakly saturated $(\mathcal{N}, \bar{a}) \vDash Th(\mathcal{M}, \bar{a})$.*
4. *$\psi(\bar{x}, \bar{a})$ is strongly minimal in $\mathcal{N}$ for every $(\mathcal{N}, \bar{a}) \vDash Th(\mathcal{M}, \bar{a})$.*

Proof. We will prove $1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$ and $4 \Rightarrow 3 \Rightarrow 4$.

($1 \Rightarrow 4$) For every L-formula $\varphi(\bar{x}, \bar{y})$ and $d \in \mathbb{N}$ consider the following formula $F_{\varphi,d}(\bar{y})$:

$$\forall \bar{x}_1 \cdots \bar{x}_{d+1} ((\bigwedge_{i \neq j} \bar{x}_i \neq \bar{x}_j \wedge \bigwedge_i \psi(\bar{x}_i, \bar{a})) \to \bigvee_i \neg\varphi(\bar{x}_i, \bar{y})).$$

Note that for every choice of $\bar{b} \in M$, $\mathcal{M} \vDash F_{\varphi,d}(\bar{b})$ means 'we cannot have $d+1$ distinct elements of $\psi(M,\bar{a})$ in $\varphi(M,\bar{b})$'. Thus if $\psi(\bar{x},\bar{a})$ is strongly minimal in $\mathcal{M}$ then for every choice of $\varphi(\bar{x},\bar{y})$ there will exist $d \in \mathbb{N}$ such that $(\mathcal{M},\bar{a}) \vDash \forall\bar{y}(F_{\varphi,d}(\bar{y}) \vee F_{\neg\varphi,d}(\bar{y}))$. Now we have $\forall\bar{y}(F_{\varphi,d}(\bar{y}) \vee F_{\neg\varphi,d}(\bar{y})) \in Th(\mathcal{M},\bar{a})$ and as $(\mathcal{N},\bar{a}) \vDash Th(\mathcal{M},\bar{a})$ we have $(\mathcal{N},\bar{a}) \vDash \forall\bar{y}(F_{\varphi,d}(\bar{y}) \vee F_{\neg\varphi,d}(\bar{y}))$. In particular this means that $\psi(\bar{x},\bar{a})$ is strongly minimal in $\mathcal{N}$, we are just using the same d as in $\mathcal{M}$, for every formula $\varphi(\bar{x},\bar{y})$.

($4 \Rightarrow 2$) Trivial, just observe that strongly minimality implies minimality and that $\mathcal{N} \succeq \mathcal{M}$ implies $(\mathcal{N},\bar{a}) \vDash Th(\mathcal{M},\bar{a})$.

($2 \Rightarrow 1$) Assume that $\psi(\bar{x},\bar{a})$ is minimal in every $\mathcal{N} \succeq \mathcal{M}$. Suppose for the purpose of contradiction that $\psi(\bar{x},\bar{a})$ is not strongly minimal in $\mathcal{M}$, so there exists $\varphi(\bar{x},\bar{y})$ such that for every $d \in \mathbb{N}$ exist $\bar{b}_d \in M$ such that,

$$|\varphi(M,\bar{b}_d) \cap \psi(M,\bar{a})| > d \text{ and } |\neg\varphi(M,\bar{b}_d) \cap \psi(M,\bar{a})| > d.$$

Let $\sigma_d(\bar{y})$ be the $L_{\bar{a}}$-formula stating

$$|\varphi(M,\bar{y}) \cap \psi(M,\bar{a})| > d \text{ and } |\neg\varphi(M,\bar{y}) \cap \psi(M,\bar{a})| > d,$$

so $\mathcal{M}$ realizes every finite subset of $\{\sigma_d(\bar{y}) : d \in \mathbb{N}\}$. Now let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$ such that $(\mathcal{N},\bar{a})$ is weakly saturated. Thus $(\mathcal{N},\bar{a}) \vDash Th(\mathcal{M},\bar{a})$ and $\mathcal{N}$ realizes $\{\sigma_d(\bar{y}) : d \in \mathbb{N}\}$, let $\bar{b}$ be such a realization. Then we get that in $\mathcal{N}$ both $\varphi(N,\bar{b}) \cap \psi(N,\bar{a})$ and $\neg\varphi(N,\bar{b}) \cap \psi(N,\bar{a})$ are infinite, contradicting the minimality of $\psi(\bar{x},\bar{a})$ in $\mathcal{N}$. Thus $\psi(\bar{x},\bar{a})$ must be strongly minimal in $\mathcal{M}$.

($4 \Rightarrow 3$) Trivial, just observe that we can always find weakly saturated models of $Th(\mathcal{M},\bar{a})$.

($3 \Rightarrow 4$) Let $\mathcal{M}'$ be a weakly saturated (in the language $L_{\bar{a}}$) model of $Th(\mathcal{M},\bar{a})$, so by hypothesis $\psi(\bar{x},\bar{a})$ is minimal in $\mathcal{M}'$. Using the same argument as in $2 \Rightarrow 1$ we can prove that $\psi(\bar{x}, \bar{a})$ is strongly minimal in $\mathcal{M}'$ (here we just do not need an elementary extension of $\mathcal{M}'$ because $(\mathcal{M}', \bar{a})$ is already weakly saturated). Now using $1 \Rightarrow 4$) we get that $\psi(\bar{x}, \bar{a})$ is strongly minimal in $\mathcal{N}$ for every $(\mathcal{N}, \bar{a}) \models Th(\mathcal{M}', \bar{a}) = Th(\mathcal{M}, \bar{a})$. □

Corollary 3.1.5. *Let $\mathcal{M}$ be an L-structure and $\mathcal{N}$ be an ω-saturated elementary extension of $\mathcal{M}$. Then $\psi(\bar{x}, \bar{a})$ is strongly minimal in $\mathcal{M}$ if and only if $\psi(\bar{x}, \bar{a})$ is minimal in $\mathcal{N}$.*

Proof. We have $(\mathcal{N}, \bar{a}) \models Th(\mathcal{M}, \bar{a})$ because $\mathcal{N} \succeq \mathcal{M}$. Also $(\mathcal{N}, \bar{a})$ is ω-saturated because $\mathcal{N}$ is ω-saturated, in particular $(\mathcal{N}, \bar{a})$ is weakly saturated. So by the last proposition, minimality in $\mathcal{N}$ implies strongly minimality in $\mathcal{M}$. Also by last proposition, strongly minimality in $\mathcal{M}$ implies minimality in every elementary extension, in particular in $\mathcal{N}$. □

3.2 Pregeometries of Strongly Minimal Sets

Next we attach a closure relation to every L-structure and we prove this is in fact a good closure.

Definition 3.2.1. Let $\mathcal{M}$ be a an L-structure. Given a subset A of M we define the *algebraic closure of* A, which we denote by $\operatorname{acl}(A)$, as:

$$\{c \in M : \text{there exists } \phi(x, \bar{y}) \in L \text{ and } \bar{b} \in A \text{ such that } c \in \phi(M, \bar{b}) \text{ and } \phi(M, \bar{b}) \text{ is finite}\}.$$

Proposition 3.2.2. *Let $\mathcal{M}$ be an L-structure. Then* $\operatorname{acl} : \mathcal{P}(M) \to \mathcal{P}(M)$ *is a good closure and consequently, for every $X, D \subseteq M$ so are* $\operatorname{acl}^D, \operatorname{acl}_X$ *and* acl_X^D.

Proof. It suffices to show that acl is a good closure.

First we prove that $A \subseteq \operatorname{acl}(A)$. If $a \in A$, we want to prove that there exists an L-formula $\phi(x, \bar{y})$ and $\bar{b} \in A$ such that $\phi(M, \bar{b})$ is finite and $a \in \phi(M, \bar{b})$.

Just let $\phi(x,\bar{y})$ be $x = y$ and $\bar{b} = a$, so $\phi(M,a) = \{a\}$ is finite and $a \in \phi(M,a)$.

Now we prove that $A \subseteq B \Rightarrow \operatorname{acl}(A) \subseteq \operatorname{acl}(B)$. Let $c \in \operatorname{acl}(A)$, so there exists an L-formula $\phi(x,\bar{y})$ and $\bar{b} \in A$ such that $c \in \phi(M,\bar{b})$ finite. In order to prove that $c \in \operatorname{acl}(B)$ just note that we can use the same formula $\phi(x,\bar{y})$ and the same parameters $\bar{b}$, because $\bar{b} \in A \subseteq B$.

To prove that acl is a closure we need to prove that $\operatorname{acl}(\operatorname{acl}(A)) \subseteq \operatorname{acl}(A)$. Let $c \in \operatorname{acl}(\operatorname{acl}(A))$ so there exists an L-formula $\phi(x,\bar{y})$ and $\bar{b} = (b_1, \cdots b_m) \in \operatorname{acl}(A)$ such that $c \in \phi(M,\bar{b})$ and $\phi(M,\bar{b})$ is finite. For each b_i there exists an L_A formula $\varphi_i(y_i)$ (that is, a formula with parameters in A) such that $b_i \in \varphi_i(M)$ and $\varphi_i(M)$ is finite. Let $\varphi(\bar{y})$ be $\bigwedge_i \varphi_i(y_i)$. Then $\bar{b} \in \varphi(M)$ and $\varphi(M)$ is finite because for each y_i there are only finitely many possibilities. Now let $n = |\phi(M,\bar{b})|$ and let $\psi(x,\bar{y})$ be

$$\phi(x,\bar{y}) \wedge \varphi(\bar{y}) \wedge \exists x_1 \cdots x_n \forall z(\phi(z,\bar{y}) \to \bigvee_{1 \leq i \leq n} z = x_i)$$

Note that $\mathcal{M} \models \psi(c,\bar{b})$ and that $\psi(x,\bar{y})$ has finitely many realizations in $\mathcal{M}$. In fact, for each $\bar{b}'$ realizing $\varphi(\bar{y}) \wedge \exists x_1 \cdots x_n \forall z(\phi(z,\bar{y}) \to \bigvee_{1 \leq i \leq n} z = x_i)$ there are at most n values of x satisfying $\phi(x,\bar{b}')$, also there are only finitely many realizations of $\varphi(\bar{y})$ in $\mathcal{M}$, so $\psi(x,\bar{y})$ has finitely many realizations in $\mathcal{M}$. In particular we have $c \in \exists\bar{y}\psi(M,\bar{y})$ and $\exists\bar{y}\psi(M,\bar{y})$ finite and also $\exists\psi(x,\bar{y}) \in L_A$, thus $c \in \operatorname{acl}(A)$.

Finally, in order to prove that acl is a good closure we need to prove that $c \in \operatorname{acl}(A) \Rightarrow c \in \operatorname{acl}(A_0)$ for some finite $A_0 \subseteq A$. This is trivial, the L_A-formula used to show that $c \in \operatorname{acl}(A)$ uses only finitely many parameters from A, thus we just need to consider a finite subset A_0 of A containing the set of parameters used. $\square$

Remark 3.2.3. Let $\mathcal{M}$ be an L-structure.

a) Let D be a 0-definable subset of M and $A \subseteq D$. Remember the notion of induced closure given by:

$$\mathrm{acl}^D(A) = \mathrm{acl}(A) \cap D$$

Then $\mathrm{acl}^D(A)$ equals:

$\{c \in D :$ there exists $\phi(x, \bar{y}) \in L$ and $\bar{b} \in A$ such that $c \in \phi(D, \bar{b})$ and $\phi(D, \bar{b})$ is finite$\}$.

b) If $X \subseteq M$ then acl_X is precisely the algebraic closure in the expansion $(\mathcal{M}, (x)_{x \in X})$.

We have seen that in a structure $\mathcal{M}$ the algebraic closure is a good closure, the next proposition shows in particular that if M is minimal then (M, acl) is a pregeometry.

Proposition 3.2.4. *Let $\mathcal{M}$ be an L-structure and $D \subseteq M$. If D is 0-definable and minimal in $\mathcal{M}$ then (D, acl^D) is a pregeometry.*

Proof. We have seen before that acl^D is a good closure so we only have to prove the exchange principle, that is, $c \in \mathrm{acl}^D(Ab) \setminus \mathrm{acl}^D(A) \Rightarrow b \in \mathrm{acl}^D(Ac)$. Let us first say that we are going to use without mentioning it again that, since D is 0-definable then $\mathrm{acl}^D(A) = \mathrm{acl}(A) \cap D = \{c \in D :$ there exists $\phi(x, \bar{y}) \in L$ and $\bar{b} \in A$ such that $c \in \phi(D, \bar{b})$ finite$\}$.

Assume that $c \in \mathrm{acl}^D(Ab) \setminus \mathrm{acl}^D(A)$, thus there exists $\phi(x, y) \in L_A$ and $n \in \mathbb{N}^+$ such that $c \in \phi(D, b)$ and $|\phi(D, b)| = n$. Let $\psi(w)$ be the formula asserting that $|\phi(D, w)| = n$, so in particular we have $b \in \psi(D)$, also note that since D is 0-definable then $\psi(w) \in L_A$. Now by minimality of D, either $\psi(D)$ or $D \setminus \psi(D)$ is finite. But $\psi(D)$ cannot be finite, otherwise $\psi(w) \in L_A$ and $b \in \psi(D)$ finite would imply $b \in \mathrm{acl}^D(A)$ which is incompatible with $c \in \mathrm{acl}^D(Ab) \setminus \mathrm{acl}^D(A)$. Thus $\psi(D)$ is cofinite (and infinite) in D. Observe now that $\phi(c, y) \wedge \psi(y)$ is an L_{Ac}-formula and that $\mathcal{M} \vDash \phi(c, b) \wedge \psi(b)$, so if $\{y \in D : \phi(c, y) \wedge \psi(y)\}$ is finite then $b \in \mathrm{acl}^D(Ac)$ and we are done. We shall prove now that this is always the case.

Assume for purpose of contradiction that $\{y \in D : \phi(c, y) \wedge \psi(y)\}$ is infinite. Then by minimality of D it must be cofinite in D, so there exists $l \in \mathbb{N}$ such that $|D - \{y \in D : \phi(c, y) \wedge \psi(y)\}| = l$. Let $\chi(x)$ be the L_A-formula (again D is 0-definable) asserting that $|D - \{y \in D : \phi(x, y) \wedge \psi(y)\}| = l$.

If $\chi(D)$ was finite then we would have $c \in \chi(D)$ finite and $\chi(x) \in L_A$ so $c \in \mathrm{acl}^D(A)$, which is false. Thus $\chi(D)$ is infinite. Because $\chi(D)$ is infinite we can find distinct $a_1, \cdots, a_{n+1} \in D$ such that $\mathcal{M} \models \chi(a_i)$ and observe that $B_i := \{y \in D : \phi(a_i, y) \wedge \psi(y)\}$ is cofinite in D because $\mathcal{M} \models \chi(a_i)$ means that $|D - \{y \in D : \phi(a_i, y) \wedge \psi(y)\}| = l$. Of course each B_i is also infinite because it is cofinite in an infinite set. Now we can choose $b' \in \bigcap_i B_i$ because a finite intersection of infinite cofinite sets is nonempty.

Finally we get that $\mathcal{M} \models \phi(a_i, b') \wedge \psi(b')$ for all $1 \leq i \leq n + 1$. In particular this shows that $|\phi(D, b')| \geq n + 1$. But we also have $\mathcal{M} \models \psi(b')$ which means that $|\phi(D, b')| = n$: here is the desired contradiction that concludes the proof. $\square$

The next corollary shows what we get if we only have definability instead of 0-definability.

Corollary 3.2.5. *Let $\mathcal{M}$ be an L-structure and $D \subseteq M$. If D is minimal in $\mathcal{M}$ and definable using parameters $\bar{a}$ then $(D, \mathrm{acl}^D_{\bar{a}})$ is a pregeometry.*

Proof. First expand $\mathcal{M}$ to $L_{\bar{a}}$. Then D is still minimal in the expansion and is now 0-definable in the expanded language. Now remember that the algebraic closure in the expansion is precisely the localization $\mathrm{acl}_{\bar{a}}$. Finally we apply the last proposition which says that $(\mathrm{acl}_{\bar{a}})^D = \mathrm{acl}^D_{\bar{a}}$ is a pregeometry. $\square$

Chapter 4

Predimension and Amalgamation

4.1 Predimension and The Generic Model

We will now describe the axiomatic approach of Wagner, see [7]. All the results in this chapter are essentially in the Wagner's article [7], but not necessarily stated in the same way. We start by refining the more general notion of *predimention function* given in chapter 2. From now on when we talk about a predimention function it is in the sense of the following definition.

Definition 4.1.1. Let L be a countable relational language and $\mathcal{C}$ be a class of finite L-structures closed under isomorphisms and substructures and with at most countably many isomorphism types. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a function. We say that δ is a *predimension* if the following axioms are satisfied:

P1 There is no infinite chain $A_0 \subseteq A_1 \subseteq \cdots$ in $\mathcal{C}$ with $\delta(A_i) > \delta(A_{i+1})$ for every $i \in \mathbb{N}$.

P2 (Submodularity) If $AB \in \mathcal{C}$ is a structure with underlying set $A \cup B$ then:

$$\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B).$$

P3 $A \simeq B$ (A isomorphic to B) implies $\delta(A) = \delta(B)$ for all $A, B \in \mathcal{C}$.

P4 $\delta(\emptyset) = 0$.

A typical example of a predimension is $\delta(A) = |A| - \sum_{i \in I} \alpha_i |R_i[A]|$ where I is any countable set, where each α_i is greater or equal than zero and where each R_i is a relational symbol and $R_i[A]$ is the set of tuples in R_i with coordinates in A. For purpose of convergence we shall also assume that for each A there are only finitely many $i \in I$ such that $R_i[A]$ is nonempty. In Chapter 5 we study a particular example of this predimension.

Given a predimension $\delta : \mathcal{C} \to \mathbb{R}_0^+$ we attach now a binary relation $\leq$ to $\mathcal{C}$.

Definition 4.1.2. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $A \subseteq B \in \mathcal{C}$, we say that $A \leq B$ (A is *closed in* B) if $\delta(A) \leq \delta(A')$ for all A' such that $A \subseteq A' \subseteq B$. We say that $A \not\leq B$ if $A \subseteq B$ but not $A \leq B$. We say that $A \not\leq_{min} B$ if $A \not\leq B$ and $A \subseteq B' \subset B \Rightarrow A \leq B'$.

We can find a proof of the following two propositions in [7], Lemma 4.3.

Proposition 4.1.3. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Then $(\mathcal{C}, \leq)$ satisfies the following properties:*

C1 $(A \leq B \wedge A \subseteq B' \subseteq B) \Rightarrow A \leq B'$, *for all* $A, B, B' \in \mathcal{C}$.

C2 There is no infinite chain $A_1 \not\leq_{min} A_2 \not\leq_{min} \cdots$ *in* $\mathcal{C}$.

C3 $\emptyset \leq A$, *for all* $A \in \mathcal{C}$.

C4 $A \leq B \Rightarrow A \cap X \leq X$, *for all* $A, B \in \mathcal{C}$ *and* $X \subseteq B$.

Proposition 4.1.4. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Then,*

a) $\leq$ *is reflexive and transitive.*

b) $\leq$ *is invariant under isomorphism, that is, if $f : B \to B'$ is an embedding then $A \leq B$ if and only if $f(A) \leq f(B)$.*

Now we want to consider an extension of $\mathcal{C}$ containing possibly infinite structures.

Definition 4.1.5. Let $\mathcal{M}$ be an L-structure. We define:

$age(\mathcal{M}) =$ Class of the finite structures isomorphic to substructures of $\mathcal{M}$.

Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. We define the class

$$\bar{\mathcal{C}} = \{\mathcal{M} : age(\mathcal{M}) \subseteq \mathcal{C}\}.$$

Remark 4.1.6. Observe that we can extend $\leq$ from $\mathcal{C} \times \mathcal{C}$ to $\bar{\mathcal{C}} \times \bar{\mathcal{C}}$ in a way that we still get reflexivity, transitivity and invariance under isomorphism.

First we extend to $\mathcal{C} \times \bar{\mathcal{C}}$ by putting $A \leq M$ if an only if

$$A \subseteq M \text{ and } \delta(A) \leq \delta(A') \text{ for all } A' \text{ such that } A \subseteq A' \in \mathcal{P}_F(M).$$

Note that this is the same as putting $A \leq M$ if and only if $A \subseteq M$ and $A \leq B$ for all B such that $A \subseteq B \in \mathcal{P}_F(M)$.

Now we can extend to $\bar{\mathcal{C}} \times \bar{\mathcal{C}}$ by putting $M_1 \leq M_2$ if and only if

$$M_1 \subseteq M_2 \text{ and } \forall A \in \mathcal{P}_F(M_1)(A \leq M_1 \to A \leq M_2).$$

Note that we should check that the definition for $\bar{\mathcal{C}} \times \bar{\mathcal{C}}$ agree with the definition for $\mathcal{C} \times \bar{\mathcal{C}}$. In fact, this holds because of $C1$.

We can now attach a good closure to each element of $\bar{\mathcal{C}}$.

Lemma 4.1.7. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. If A is a finite subset of $\mathcal{M}$ then there exists a finite smallest closed subset of M containing A. So we define,*

$$\mathrm{cl}_{\mathcal{M}}(A) = \text{ \textit{Smallest closed subset of} } M \text{ \textit{containing} } A.$$

Proof. By $C2$ there is a maximal chain $A \nleq_{min} A_1 \nleq_{min} A_2 \nleq_{min} \cdots \nleq_{min} A_n$ with A_n finite. Now we prove that $A_n \leq \mathcal{M}$. In fact if $A_n \nleq \mathcal{M}$ then we would have $A_n \nleq_{min} B$ for some finite $B \subseteq \mathcal{M}$ contradicting the maximality of our chain. Thus there is a finite closed subset of $\mathcal{M}$ containing A.

Now we prove that there is a unique smallest closed subset of $\mathcal{M}$ containing A. First we observe that the intersection of two finite closed subsets of $\mathcal{M}$ is closed in $\mathcal{M}$. In fact let B_1 and B_2 be two finite closed subsets of $\mathcal{M}$. Let D be an arbitrary subset of $\mathcal{M}$ containing $B_1 \cap B_2$ and let C be some subset of $\mathcal{M}$ containing $B_1 \cup B_2 \cup D$. Firstly we observe that by definition $B_1 \leq \mathcal{M}$ implies $B_1 \leq C$. But then by the axiom $C4$ we get $B_1 \cap B_2 = B_1 \cap (B_2 \cap D) \leq B_2 \cap D$. Secondly we observe that $B_2 \leq \mathcal{M}$ implies that $B_2 \leq C$ thus by axiom $C4$ we have $B_2 \cap D \leq D$. But now by transitivity we have $B_1 \cap B_2 \leq D$. Finally as D was arbitrary this proves that $B_1 \cap B_2 \leq \mathcal{M}$. Now assume that B_1 and B_2 are two minimal finite closed subsets of $\mathcal{M}$ containing A. But then as $B_1 \cap B_2 \leq \mathcal{M}$ we would have by minimality of B_1 and B_2 that $B_1 = B_1 \cap B_2 = B_2$. Thus there exists the smallest closed subset of $\mathcal{M}$ containing A and it is finite.

$\square$

Proposition 4.1.8. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Then $\mathrm{cl}_{\mathcal{M}} : \mathcal{P}_F(M) \to \mathcal{P}_F(M)$ extends to a good closure $\mathrm{cl}_{\mathcal{M}} : \mathcal{P}(M) \to \mathcal{P}(M)$ by putting,*

$$\mathrm{cl}_{\mathcal{M}}(A) = \bigcup_{A_0 \in \mathcal{P}_F(A)} \mathrm{cl}_{\mathcal{M}}(A_0).$$

Consequently, the closure of a finite set is finite and infinite sets have the same cardinality as their closures.

Proposition 4.1.9. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M}_1 \subseteq \mathcal{M}_2 \in \bar{\mathcal{C}}$. Then we have*

$$\mathcal{M}_1 \leq \mathcal{M}_2 \textit{ if and only if } \mathrm{cl}_{\mathcal{M}_2}(M_1) = M_1.$$

As in Hrushovski's example we want to construct a limit structure of $(\mathcal{C}, \leq)$ and study the properties of its theory. The next definition makes precise what we mean by 'limit structure' of $(\mathcal{C}, \leq)$.

Definition 4.1.10. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M}$ an L-structure. We say that $\mathcal{M}$ is a *generic model of* $(\mathcal{C}, \leq)$ if it is countable and if satisfies $F1$ and $F2$, where,

- F1 $\mathcal{M} \in \bar{\mathcal{C}}$
- F2 (extension property) If $A \leq \mathcal{M}$ and $A \leq B \in \mathcal{C}$ then there exists an embedding $f : B \to \mathcal{M}$ such that $f_{|A} = Id_{|A}$ (where $Id_{|A}$ is the identity map) and such that $f(B) \leq \mathcal{M}$.

However, in order to construct our generic model, we need $(\mathcal{C}, \leq)$ to satisfy an extra requirement that we call $\leq$-amalgamation property.

Definition 4.1.11. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. We say that $(\mathcal{C}, \leq)$ has the $\leq$-*amalgamation property* if whenever we have $A_0 \leq A_1 \in \mathcal{C}$ and $A_0 \leq A_2 \in \mathcal{C}$, then there exists $D \in \mathcal{C}$ and embeddings $f_1 : A_1 \to D$, $f_2 : A_2 \to D$ such that $f_{1|A_0} = f_{2|A_0}$ and $f_1(A_1) \leq D$ and $f_2(A_2) \leq D$.

We can now construct the generic model and prove its uniqueness up to isomorphism in the next two theorems. These results are standard and can be found together in Proposition 2.3 in [7] but we include the proofs here.

Theorem 4.1.12. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and suppose that $(\mathcal{C}, \leq)$ has the $\leq$-amalgamation property. Then there exists a generic model for $(\mathcal{C}, \leq)$.*

Proof. Let $A_0, A_1, A_2, \cdots$ be representatives of the isomorphism types of structures in $\mathcal{C}$. We are going to construct the generic model $\mathcal{M}$ as the union of a countable chain of finite structures.

Let $\mathcal{M}_0 = \emptyset$. We want to construct $\mathcal{M}_{i+1}$ from $\mathcal{M}_i$. For each $A \leq \mathcal{M}_i$ consider the set of all possible extensions $A \leq B$ of A with $B \in \mathcal{C}$ isomorphic to some element of $\{A_0, \cdots, A_i\}$ and consider an equivalence relation on this set given by $B' \sim B''$ if and only if there is an isomorphism from B' to B'' fixing the elements of A. For each A, this equivalence relation has finitely many equivalence classes, also there are only finitely many possibilities for A inside $\mathcal{M}_i$. Let $A_1 \leq B_1, A_2 \leq B_2, \cdots, A_r \leq B_r$ be representatives of the equivalence classes (without fixing A).

We are going to construct a finite chain $\mathcal{M}_i = D_0 \leq D_1 \leq D_2 \leq \cdots \leq D_r = \mathcal{M}_{i+1}$ such that D_t contains a closed copy of B_t. For this we use the $\leq$-amalgamation property. In fact we have $A_t \leq B_t \in \mathcal{C}$ and $A_t \leq \mathcal{M}_i \leq D_{t-1} \in \mathcal{C}$ so by the $\leq$-amalgamation property we get $D_t \in \mathcal{C}$ and embeddings $f : B_t \to D_t$ and $g : D_{t-1} \to D_t$ such that $f_{|A_t} = g_{|A_t}$ and $f(B_t) \leq D_t$ and $g(D_{t-1}) \leq D_t$. We can also assume that g is the inclusion $D_{t-1} \to D_t$. Thus we have a closed copy of B_t inside D_t.

Finally put $\mathcal{M} = \bigcup_{i \in \omega} \mathcal{M}_i$. But we still need to check that this construction works for our purpose. In fact, given $A \leq \mathcal{M}$, then we have $A \leq \mathcal{M}_i$ for i large enough, also if $A \leq B \in \mathcal{C}$ then for i large enough we have B_t from the construction and an isomorphism $h : B \to B_t$ fixing the elements of A. By construction of D_t we also have an isomorphism $f : B_t \to D_t$ fixing the elements of A and such that $f(B_t) \leq D_t \leq \mathcal{M}_{i+1} \leq \mathcal{M}$. Thus $f \circ h : B \to \mathcal{M}$ is an embedding such that $f \circ h_{|A} = Id_{|A}$ and $f \circ h(B) \leq \mathcal{M}$. This proves that our model $\mathcal{M}$ satisfies $F2$.

Note that $\mathcal{M}$ is countable and that $\mathcal{M}$ satisfies $F1$ because every finite substructure of $\mathcal{M}$ is inside some $\mathcal{M}_i \in \mathcal{C}$ and $\mathcal{C}$ is closed under substructures. Thus $\mathcal{M}$ is a generic model for $(\mathcal{C}, \leq)$. $\square$

The next theorem shows in particular the uniqueness of the generic model up to isomorphism.

Theorem 4.1.13. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M}$ and $\mathcal{M}'$ be generic models of $(\mathcal{C}, \leq)$. Then $\mathcal{M} \simeq \mathcal{M}'$ and every isomorphism $f_0 : A \to A'$ between finite closed substructures $A \leq \mathcal{M}$ and $A' \leq \mathcal{M}'$ extends to an isomorphism $f : \mathcal{M} \to \mathcal{M}'$.*

Proof. Let $f_0 : A \to A'$ be an isomorphism as above, we want to extend it to an isomorphism $f : \mathcal{M} \to \mathcal{M}'$. By countability of $\mathcal{M}$ and $\mathcal{M}'$ we can assume that $\mathcal{M}$ and $\mathcal{M}'$ are the union of increasing chains of finite sets $\mathcal{M} = \bigcup_{i\in\omega} A_i$ and $\mathcal{M}' = \bigcup_{i\in\omega} A'_i$ with $A_0 = A$ and $A'_0 = A'$. Let $\mathcal{M}_0 = A$ and $\mathcal{M}'_0 = A'$, so we have $f_0 : \mathcal{M}_0 \to \mathcal{M}'_0$ such that $A_0 \subseteq \mathcal{M}_0 \leq \mathcal{M}$ and $A'_0 \subseteq \mathcal{M}'_0 \leq \mathcal{M}'$. Assume that we have constructed $f_n : \mathcal{M}_n \to \mathcal{M}'_n$ such that $A_n \subseteq \mathcal{M}_n \leq \mathcal{M}$ and $A'_n \subseteq \mathcal{M}'_n \leq \mathcal{M}$. We want to extend f_n to some $f_{n+1} : \mathcal{M}_{n+1} \to \mathcal{M}'_{n+1}$ such that $A_{n+1} \subseteq \mathcal{M}_{n+1} \leq \mathcal{M}$ and $A'_{n+1} \subseteq \mathcal{M}'_{n+1} \leq \mathcal{M}'$.

Now we want to increase the domain of f_n. Consider a finite set B such that $\mathcal{M}_n \cup A_{n+1} \subseteq B \leq \mathcal{M}$, there exists such B, namely the closure of $\mathcal{M}_n \cup A_{n+1}$ in $\mathcal{M}$ (remember that the closure of a finite set is finite). We have by hypothesis $\mathcal{M}_n \leq \mathcal{M}$ so we have also $\mathcal{M}_n \leq B$ and if we extend $f_n : \mathcal{M}_n \to \mathcal{M}'_n$ to some isomorphism $\widetilde{f_n} : B \to B''$ then we get $\mathcal{M}'_n = \widetilde{f_n}(\mathcal{M}_n) \leq \widetilde{f_n}(B) = B''$ and $B'' \in \mathcal{C}$ because $B'' \simeq B$. But we have $\mathcal{M}'_n \leq \mathcal{M}'$ by hypothesis and $\mathcal{M}'_n \leq B'' \in \mathcal{C}$ so by $F2$ there exists $B' \leq \mathcal{M}'$ and an isomorphism $h : B'' \to B'$ extending the identity in $\mathcal{M}'_n$. Now if we consider $g = h \circ \widetilde{f_n}$ then we have that $g : B \to B'$ is an isomorphism extending $f_n : \mathcal{M}_n \to \mathcal{M}'_n$ with $B' \leq \mathcal{M}'$ and the domain of g containing A_{n+1}.

Now we would like to increase the image of g, or by other words, the domain of g^{-1}, we will proceed in the same way as in the previous step. Consider a finite set $\mathcal{M}'_{n+1}$ such that $B' \cup A'_{n+1} \subseteq \mathcal{M}'_{n+1} \leq \mathcal{M}'$, there exists such $\mathcal{M}'_{n+1}$, namely the closure of $B' \cup A'_{n+1}$ in $\mathcal{M}'$ (again remember that the closure of a finite set is finite). We have $B' \leq \mathcal{M}'$ so we have also $B' \leq \mathcal{M}'_{n+1}$ and if we extend $g^{-1} : B' \to B$ to some isomorphism $\widetilde{g^{-1}} : \mathcal{M}'_{n+1} \to \mathcal{M}''_{n+1}$ then we get $B = \widetilde{g^{-1}}(B') \leq \widetilde{g^{-1}}(\mathcal{M}'_{n+1}) = \mathcal{M}''_{n+1}$ and $\mathcal{M}''_{n+1} \in \mathcal{C}$ because $\mathcal{M}''_{n+1} \simeq \mathcal{M}'_{n+1}$. But we have $B \leq \mathcal{M}$ and $B \leq \mathcal{M}''_{n+1} \in \mathcal{C}$ so by $F2$ there exists $\mathcal{M}_{n+1} \leq \mathcal{M}$ and an isomorphism $h' : \mathcal{M}''_{n+1} \to \mathcal{M}_{n+1}$ extending the identity in B. Now we if we consider $f^{-1}_{n+1} = h' \circ \widetilde{g^{-1}}$ then we have $f_{n+1} : \mathcal{M}_{n+1} \to \mathcal{M}'_{n+1}$ extending $g : B \to B'$ and consequently extending $f_n : \mathcal{M}_n \to \mathcal{M}'_n$ with $A_{n+1} \subseteq \mathcal{M}_{n+1} \leq \mathcal{M}$ and $A'_{n+1} \subseteq \mathcal{M}'_{n+1} \leq \mathcal{M}$ as desired.

Finally we put $f = \bigcup_{n\in\omega} f_n$ and $f : \mathcal{M} \to \mathcal{M}'$ is the desired isomorphism. $\square$

4.2 Dimension and Independence

We called our δ a predimension, this is because δ can be used to define a dimension function in each $\mathcal{M} \in \bar{\mathcal{C}}$.

Definition 4.2.1. Let $\delta : \mathcal{C} \to \mathbb{R}^+_0$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. We define the *dimension function* $d_{\mathcal{M}} : \mathcal{P}_F(M) \to \mathbb{R}^+_0$ as follows,

$$d_{\mathcal{M}}(A) = \inf\{\delta(A') : A \subseteq A' \in \mathcal{P}_F(M)\}, \text{ for all } A \in \mathcal{P}_F(M).$$

We also define for $A, B \in \mathcal{P}_F(M)$,

$$d_{\mathcal{M}}(A/B) = d_{\mathcal{M}}(A \cup B) - d_{\mathcal{M}}(B).$$

Moreover if $A \in \mathcal{P}_F(M)$ and if $B \subseteq M$ we can also define,

$$d_{\mathcal{M}}(A/B) = \inf\{d_{\mathcal{M}}(A/B') : B' \in \mathcal{P}_F(B)\}.$$

We will write d instead of $d_{\mathcal{M}}$ and cl instead of $\mathrm{cl}_{\mathcal{M}}$ when no confusion arises.

Proposition 4.2.2. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. The following properties hold for A and B finite closed subsets of $\mathcal{M}$:*

D1 $A \subseteq B \Rightarrow d(A) \leq d(B)$,

D2 $d(\mathrm{cl}(A \cup B)) + d(A \cap B) \leq d(A) + d(B)$,

D3 $A \simeq B \Rightarrow d(A) = d(B)$.

The following proposition shows that the infimum in the definition of d is always achieved. We can find a proof of the next proposition in [7], Lemma 4.3.

Proposition 4.2.3. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Let $A \in \mathcal{P}_F(M)$, then:*

a) $d(A) = \delta(\mathrm{cl}(A))$, *consequently* $d(A) = d(\mathrm{cl}(A))$.

b) $A \leq \mathcal{M}$ *if and only if* $d(A) = \delta(A)$.

Lemma 4.2.4. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Let $A, A_1, A_2 \in \mathcal{P}_F(M)$ and $B \subseteq M$, then the following hold:*

1. $d(A/B) \geq 0$
2. $B \subseteq B' \Rightarrow d(A/B) \geq d(A/B')$
3. $d(A/B) = d(\mathrm{cl}(A)/B) = d(A/\mathrm{cl}(B)) = d(\mathrm{cl}(A)/\mathrm{cl}(B))$
4. $d(A_1A_2/B) = d(A_1/A_2B) + d(A_2/B)$

Proof. See [7], Lemma 3.2. □

Definition 4.2.5. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Let $B \subseteq M$ and $A_1, A_2 \in \mathcal{P}_F(M)$. We say that $A_1 \mathop{\downarrow}^d_B A_2$ (A_1 and A_2 are *independent over* B) if

- $d(A_1/A_2B) = d(A_1/B)$
- $\mathrm{cl}(A_1B) \cap \mathrm{cl}(A_2B) = \mathrm{cl}(B)$.

We can extend this definition to the case where A_1 and A_2 are possibly infinite subsets of $\mathcal{M}$ by putting,

$$A_1 \mathop{\downarrow}^{d}_{B} A_2 \text{ if and only if } A_1' \mathop{\downarrow}^{d}_{B} A_2' \text{ for all } A_1' \in \mathcal{P}_F(A_1) \text{ and for all } A_2' \in \mathcal{P}_F(A_2).$$

Remark 4.2.6. Note that $d(A_1/A_2B) = d(A_1/B) \Leftrightarrow d(A_1A_2/B) = d(A_1/B) + d(A_2/B)$, thus independence over B is a symmetric relation.

4.3 The theory of the generic model

In this section we will prove the stability of the theory of the generic model under certain assumptions. Let us first recall the notions of λ-stability and stability.

Definition 4.3.1. Let T be a complete theory in a countable language. Let λ be an infinite cardinal. We say that T is λ-*stable* if whenever $\mathcal{M} \models T$, $A \subseteq M$ and $|A| = \lambda$ then the number of complete n-types over A is less or equal to λ. We say that T is *stable* if it is λ-stable for some infinite cardinal λ.

The following proposition plays a crucial role in counting types. Remember that $diag(A)$ (the diagram of a substructure A) is the set of quantifier free formulas realized by A, where we see A as a tuple.

Proposition 4.3.2. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and let $\mathcal{M}$ satisfy $F1$ and $F2$. Let $A \leq \mathcal{M}$ and $B \leq \mathcal{M}$, then*

$$diag(A/\emptyset) = diag(B/\emptyset) \Rightarrow tp_{\mathcal{M}}(A/\emptyset) = tp_{\mathcal{M}}(B/\emptyset).$$

Proof. See corollary 2.4 of Wagner's article [7]. □

We need one more lemma to allow us to count types, but for proving this lemma we need δ to satisfy some further axioms.

Definition 4.3.3. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Let $A_0 \leq A_i \in \mathcal{C}\,(i = 1, 2)$ and $A_1 \cap A_2 = A_0$. The *free amalgam of* A_1 and A_2 over A_0 is the structure with underlying set $A_1 \cup A_2$, whose only relations are those induced from A_1 and A_2. We denote it by $A_1 \amalg_{A_0} A_2$.

In the next definition, we consider AB to be the substructure of $\mathcal{M}$ with underlying set $A \cup B$.

Definition 4.3.4. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Consider the following axioms:

$PS(\mathcal{M})$ For all $A, B \leq \mathcal{M}$ one of the following holds:

1) $AB = A \amalg_{A \cap B} B$
2) $\exists \gamma > 0\ \exists A_0 \in \mathcal{P}_F(A)\ \exists B_0 \in \mathcal{P}_F(B)$ such that if $A_0 \subseteq A' \in \mathcal{P}_F(A)$, $B_0 \subseteq B' \in \mathcal{P}_F(B)$, $A' \leq A$ and $B' \leq B$ then we have

$$\delta(A') + \delta(B') \geq \delta(A'B') + \delta(A' \cap B') + \gamma$$

.

$PW(\mathcal{M})$ The following statements hold:

- $\forall a \in M\ \forall X \subseteq M\ \exists X_0 \in \mathcal{P}_F(X)$ such that $d(a/X) = d(a/X_0)$.

- For all finite closed subsets A, B of $\mathcal{M}$ we have

$$A \underset{A\cap B}{\overset{d}{\bigcup}} B \Rightarrow AB = A \amalg_{A\cap B} B.$$

We say that δ satisfies PS if δ satisfies $PS(\mathcal{M})$ for every $\mathcal{M} \in \bar{\mathcal{C}}$ and we say that δ satisfies PW if δ satisfies $PW(\mathcal{M})$ for every $\mathcal{M} \in \bar{\mathcal{C}}$.

Proposition 4.3.5. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. If $\{\delta(X) - \delta(Y) : X, Y \in \mathcal{C}\}$ is a discrete subset of $\mathbb{R}$ then $PS(\mathcal{M})$ implies $PW(\mathcal{M})$.*

We can now state our lemma.

Lemma 4.3.6. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and $\mathcal{M} \in \bar{\mathcal{C}}$. Assume that δ satisfies $PS(\mathcal{M})$ or $PW(\mathcal{M})$. If $A, B \leq \mathcal{M}$ and $A \bigcup^d_{A\cap B} B$ then $AB = A \amalg_{A\cap B} B \leq \mathcal{M}$.*

Proof. In [7] Wagner define the axioms DW and DS for a dimension function. Then he shows that if a predimension function δ satisfies PW (respectively PS) then the corresponding dimension function satisfies DW (respectively DS). The proof of this lemma follows directly from Lemma 3.3 from Wagner's article [7]. $\square$

We can now prove the next theorem.

Theorem 4.3.7. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension and let $\mathcal{M}$ be the generic model of $(\mathcal{C}, \leq)$. Assume that $\mathcal{N}$ satisfies $F1$ and $F2$ (we write $\mathcal{N}$ satisfies $F1 \wedge F2$) for every ω-saturated $\mathcal{N} \models Th(\mathcal{M})$. Then:*

a) If δ satisfies PS then $Th(\mathcal{M})$ is stable

b) If δ satisfies PW then $Th(\mathcal{M})$ is ω-stable.

Proof. First we prove a). Let λ be a cardinal such that $\lambda^\omega = \lambda$ and $\lambda \geq 2^\omega$, such cardinals exist, for example $\lambda = 2^\omega$. Let $\mathcal{M}' \models Th(\mathcal{M})$ and $\mathcal{N} \succ \mathcal{M}'$ such that $\mathcal{N}$ is λ^+-saturated. Let $X' \subseteq M'$ be a set such that $|X'| = \lambda$ and put $X = \mathrm{cl}_{\mathcal{N}}(X')$, thus $|X| = |X'| = \lambda$. Every n-type over X' consistent with $Th_{X'}(\mathcal{M}')$ is realized in $\mathcal{N}$ by λ^+-saturation. We will prove that we have at most λ n-types over X realized in $\mathcal{N}$, so in particular we will have at most λ n-types over X' consistent with $Th_{X'}(\mathcal{M}')$, proving in this way the λ-stability of $Th(\mathcal{M})$.

Let $\bar{a}$ be a tuple in $\mathcal{N}$, we want to count the possibilities for $tp(\bar{a}/X)$. Note that we can find a countable subset X_0 of X such that $d_{\mathcal{N}}(\bar{a}/X) = d_{\mathcal{N}}(\bar{a}/X_0)$. In fact, there exist $X_{0,1}, X_{0,2}, X_{0,3}, \cdots$ finite subsets of X with $d(\bar{a}/X) \leq d(\bar{a}/X_{0,n})$ and $\lim_{n\to\infty} d(\bar{a}/X_{0,n}) = d(\bar{a}/X)$, so we put $X_0 = \bigcup_{n\in\omega} X_{0,n}$. We have then $d(\bar{a}/X) \leq d(\bar{a}/X_0) \leq d(\bar{a}/X_{0,n})$ and taking this to the limit we get $d(\bar{a}/X) = d(\bar{a}/X_0)$.

Let $A = \mathrm{cl}_{\mathcal{N}}(\bar{a}X_0)$ and $X_1 = A \cap X$, then A, X_1 are both countable and closed in $\mathcal{N}$. We want to prove that $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{X_1} X$. First note that $\mathrm{cl}(AX_1) \cap \mathrm{cl}(XX_1) = \mathrm{cl}(X_1) \Leftrightarrow \mathrm{cl}(A) \cap \mathrm{cl}(X) = \mathrm{cl}(X_1) \Leftrightarrow A \cap X = X_1$, which is true. Now observe that $d(A/XX_1) = d(A/X) \leq d(A/X_1) = d(\mathrm{cl}(\bar{a}X_0)/X_1) = d(\bar{a}X_0/X_1) = d(\bar{a}/X_1) \leq d(\bar{a}/X_0) = d(\bar{a}/X) = d(\bar{a}X_0/X) = d(\mathrm{cl}(\bar{a}X_0)/X) = d(A/X)$. Thus $d(A/XX_1) = d(A/X_1)$ and we get $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{X_1} X$. Now we have $A, X \leq \mathcal{N}$ and $A \mathop{\smile\!\!\!\!\!\mid}^{d}_{A\cap X} X$ so as $\mathcal{N}$ satisfies $F1$ and as δ satisfies PS we can apply our lemma and we get $AX = A \amalg_{A\cap X} X \leq \mathcal{N}$.

Now we want to count the possible diagrams for AX (with X fixed). We have $|X|^\omega$ choices for X_1 as a subset of X because X_1 is countable. For each choice of X_1 we have at most 2^ω choices for $diag(A)$ because A is countable. So we have at most $2^\omega|X|^\omega$ possibilities for $diag(AX)$ because $AX = A \amalg_{X_1} X$. Now we note that by Proposition 4.3.2 $diag(AX)$ determines $tp(AX/\emptyset)$, in fact we can apply this proposition because $AX \leq \mathcal{N}$ and $\mathcal{N}$ satisfies $F1 \wedge F2$ by assumption. Thus we have at most $2^\omega|X|^\omega$ possibilities for $tp(AX/\emptyset)$.

Finally observe that as X is fixed, $tp(AX/\emptyset)$ determines $tp(A/X)$ which determines $tp(\bar{a}/X)$, thus we have $2^\omega|X|^\omega = 2^\omega\lambda^\omega = \lambda$ possibilities for $tp(\bar{a}/X)$. This proves λ-stability, in particular $Th(\mathcal{M})$ is stable.

Now we were going to prove b). The proof is similar to the proof of a), we point only to the differences. Here we take $\lambda = |X| = \omega$. By PW we can find a finite subset X_0 of X such that $d_{\mathcal{N}}(\bar{a}/X) = d_{\mathcal{N}}(\bar{a}/X_0)$ instead of X_0 countable. $A = \mathrm{cl}_{\mathcal{N}}(\bar{a}X_0)$ and $X_1 = A \cap X$ are both finite, thus we have $|X|^{<\omega}$ possibilities for the choice of X_1 as a subset of X. For each choice of X_1 we have at most ω possibilities for $diag(A)$ because there are only countably many isomorphism types in $\mathcal{C}$. Thus we get at most $\omega|X|^{<\omega} = \omega\omega^{<\omega} = \omega$ possibilities for $tp(\bar{a}/X)$. This proves that $Th(\mathcal{M})$ is ω-stable. $\square$

For proving the stability of the theory of the generic model we have made the assumption that every ω-saturated model of the theory satisfies $F1 \wedge F2$. Note that if $F1 \wedge F2$ is first order axiomatizable then the assumption made holds, but in general this needs not to be true. The next results show another way of making this assumption true. First we need some more definitions.

Definition 4.3.8. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Let $A \in \mathcal{C}$, $B \in \bar{\mathcal{C}}$ and $n \in \mathbb{N}$, then we say that $A \leq^n B$ if for all B' such that $A \subseteq B' \subseteq B$ and $|B' - A| \leq n$ we have $A \leq B'$.

Definition 4.3.9. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. If for all $n, k \in \mathbb{N}$ there is a single L-formula defining $\{A \subseteq \mathcal{M} : A \leq^n \mathcal{M} \wedge |A| = k\}$ for every $\mathcal{M} \in \bar{\mathcal{C}}$ then we say that $(\mathcal{C}, \leq)$ has the *$\leq^*$-definability property.* If for all $n, k \in \mathbb{N}$ there is a set of L-formulas defining $\{A \subseteq \mathcal{M} : A \leq^n \mathcal{M} \wedge |A| = k\}$ for every $\mathcal{M} \in \bar{\mathcal{C}}$ then we say that $(\mathcal{C}, \leq)$ has the *$\leq^*$-type definability property.*

Definition 4.3.10. We say that $\mathcal{C}$ has the *boundedness property* if for each natural number n, $\mathcal{C}$ has finitely many isomorphism types of size n.

Remark 4.3.11. If the language L is finite then $\mathcal{C}$ has the boundedness property.

For the next proposition (and from now on) recall that by definition a given property is first order axiomatizable if there exists a set of sentences Σ such that a given structure $\mathcal{M}$ satisfies the property if an only if $\mathcal{M} \vDash \Sigma$.

Proposition 4.3.12. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. If $\mathcal{C}$ has the boundedness property then $(\mathcal{C}, \leq)$ has the $\leq^*$-definability property and $F1$ is first order axiomatizable.*

Proof. First we prove that for each $k, l \in \mathbb{N}$ there exists a single L-formula $\psi_{k,l}(\bar{x}, \bar{y})$ where $\bar{x}$ is a k-tuple and $\bar{y}$ is an l-tuple such that for every $\mathcal{M}$ satisfying $F1$ we have $\bar{a} \leq \bar{a}\bar{b}$ if and only if $\mathcal{M} \vDash \psi_{k,l}(\bar{a}, \bar{b})$. For this we consider all possible diagrams for structures in $\mathcal{C}$ with $k + l$ elements, there are only finitely many of them because $\mathcal{C}$ has the boundedness property. Also because there are only finitely many of such diagrams, each one of them is distinguished, thus expressed, by a single formula. Between these diagrams we pick the ones that imply $\bar{x} \leq \bar{x}\bar{y}$. Let $D_i(\bar{x}, \bar{y})$ with $1 \leq i \leq j$ be this set of diagrams. Now if this set is empty we let $\psi_{k,l}(\bar{x}, \bar{y})$ be the formula $\bar{x}\bar{y} \neq \bar{x}\bar{y}$, otherwise we let $\psi_{k,l}(\bar{x}, \bar{y})$ be the formula $\bigvee_{1 \leq i \leq j} D_i(\bar{x}, \bar{y})$. We have then $\bar{a} \leq \bar{a}\bar{b}$ if and only if $\mathcal{M} \vDash \psi_{k,l}(\bar{a}, \bar{b})$, for every $\mathcal{M}$ satisfying $F1$.

Now for each $n, k \in \mathbb{N}$ we want to find a single L-formula defining $\{A \subseteq \mathcal{M} : A \leq^n \mathcal{M} \wedge |A| = k\}$ for each $\mathcal{M} \in \bar{\mathcal{C}}$. It is now easy to see that such formula is $\bigwedge_{1 \leq i \leq n} \forall y_1 \cdots y_i \psi_{k,i}(\bar{x}, \bar{y})$.

Now we want to prove that $F1$ is first order axiomatizable. For each $n \in \mathbb{N}$ let $D_1(\bar{x}), \cdots, D_{k_n}(\bar{x})$ be the set of possible diagrams of structures of size n in $\mathcal{C}$ (again we can consider each D_i as a single formula). Consider ψ_n the sentence $\forall \bar{x} \bigvee_{1 \leq i \leq k_n} D_i(\bar{x})$ and let $\Gamma = \{\psi_n : n \in \mathbb{N}\}$. Then for each L-structure $\mathcal{N}$ we have $\mathcal{N} \vDash \Gamma$ if and only if $\mathcal{N}$ satisfies $F1$. This proves that $F1$ is first order axiomatizable. $\square$

We consider the following variant of the axiom $F2$.

Definition 4.3.13. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Consider the following axiom for an L-structure $\mathcal{M}$:

$F2^*$ For any $A \leq B \in \mathcal{C}$, $n \in \mathbb{N}$ and $\varphi \in diag(B/A)$ with parameters in A there is $m = m(\textit{isom.type of } A, \textit{isom.type of } B, n, \varphi)$ such that if $A \leq^m \mathcal{M}$ then there exists a bijection $f : B \to B'$ such that $f_{|A} = Id_{|A}$, $B' \leq^n \mathcal{M}$ and $\varphi \in diag(f(B)/A)$.

Proposition 4.3.14. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Assume that $\mathcal{C}$ has the boundedness property (in particular $(\mathcal{C}, \leq)$ has the $\leq^*$- definability property and $F1$ is first order axiomatizable). Then, if $\mathcal{M}$ satisfies $F1 \wedge F2^*$ then every model of $Th(\mathcal{M})$ also satisfies $F1 \wedge F2^*$.*

Proof. For each $A \in \mathcal{C}$ let $D_A(\bar{x})$ be the quantifier free diagram of A. Because $\mathcal{C}$ has the boundedness property we can see $D_A(\bar{x})$ as a single formula. Now for each $n \in \mathbb{N}$, $A \leq B \in \mathcal{C}$ and $\phi(\bar{a}, \bar{y})$ belonging to the quantifier free diagram of B over A consider the sentence $\forall \bar{a}((D_A(\bar{a}) \wedge \bar{a} \leq^m \mathcal{M}) \to \exists \bar{b} \phi(\bar{a}, \bar{b}) \wedge \bar{a}\bar{b} \leq^n \mathcal{M})$ where m is such that this sentence belongs to $Th(\mathcal{M})$. Such m exists because $\mathcal{M}$ satisfies $F2^*$. Let $F1 \wedge F2^*(\mathcal{M})$ be the set of such sentences together with the axiomatization of $F1$. Then $\mathcal{M} \models F1 \wedge F2^*(\mathcal{M})$ so every model $\mathcal{N}$ of $Th(\mathcal{M})$ also satisfies $F1 \wedge F2^*(\mathcal{M})$, in particular we have that $\mathcal{N}$ satisfies $F1 \wedge F2^*$. □

Notation 4.3.15. Let $F1 \wedge F2^*(\mathcal{M})$ be the set of sentences as in last proof.

Proposition 4.3.16. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. If $(\mathcal{C}, \leq)$ has the $\leq^*$-type definability property then every ω-saturated model of $F1 \wedge F2^*$ also satisfies $F2$.*

Proof. Let $\mathcal{M}$ satisfy $F1 \wedge F2^*$ and assume that $A \leq \mathcal{M}$ and $A \leq B \in \mathcal{C}$. Let $\triangle(\bar{a}, \bar{x}) = diag(B/A)$ the quantifier free diagram of B with A fixed. For each $n \in \mathbb{N}$ let $\bigtriangledown_n(\bar{y}, \bar{x})$ be a set of L-formulas such that $\bar{y}\bar{x} \leq^n \mathcal{M}$ if and only if $\mathcal{M} \vDash \bigtriangledown_n(\bar{y}, \bar{x})$, this set of formulas exists because $(\mathcal{C}, \leq)$ has the $\leq^*$-type definability property. Let $\square(\bar{a}, \bar{x}) = \triangle(\bar{a}, \bar{x}) \cup \bigcup_{n\in\mathbb{N}} \bigtriangledown_n(\bar{a}, \bar{x})$. Note that $\square(\bar{a}, \bar{x})$ is consistent with $Th(\mathcal{M})$ because every finite subset of $\square(\bar{a}, \bar{x})$ is realized in $\mathcal{M}$. In fact, to prove this is enough to realize every set of formulas of the form $\{\phi(\bar{a}, \bar{x})\} \cup \bigtriangledown_n(\bar{a}, \bar{x})$ where $n \in \mathbb{N}$ and $\phi(\bar{a}, \bar{x}) \in \triangle(\bar{a}, \bar{x})$. But by $F2^*$ there exists $m = m(A, B, n, \phi)$ such that if $A \leq^m \mathcal{M}$ and $A \leq B \in \mathcal{C}$ then there exists $\bar{b} \in \mathcal{M}$ such that $\bar{a}\bar{b} \leq^n \mathcal{M}$ and $\mathcal{M} \vDash \phi(\bar{a}, \bar{b})$. Thus $A \leq \mathcal{M}$ and $A \leq B \in \mathcal{C}$ imply that there exists $\bar{b} \in \mathcal{M}$ such that $\mathcal{M} \vDash \bigtriangledown_n(\bar{a}, \bar{b})$ and $\mathcal{M} \vDash \phi(\bar{a}, \bar{b})$ as desired.

Now, because $\square(\bar{a}, \bar{x})$ is consistent with $Th(\mathcal{M})$ and by ω-saturation of $\mathcal{M}$ we get that $\square(\bar{a}, \bar{x})$ is realizable in $\mathcal{M}$. Let $\bar{b}$ be such a realization. Then we have $\mathcal{M} \vDash \triangle(\bar{a}, \bar{b})$ and $\mathcal{M} \vDash \bigtriangledown_n(\bar{a}, \bar{b})$ for all $n \in \mathbb{N}$. Thus $\bar{a}\bar{b}$ is isomorphic (over A) to B and $\bar{a}\bar{b} \leq \mathcal{M}$. This proves $F2$. $\square$

Because of Propositions 4.3.16 and 4.3.14, our assumption on Theorem 4.3.7 that 'every ω-saturated model of the theory of the generic satisfies $F1 \wedge F2$' will hold if we prove that the generic satisfies $F2^*$ (and if $\mathcal{C}$ has the boundedness property). For that we need $(\mathcal{C}, \leq)$ to satisfy a stronger amalgamation.

Definition 4.3.17. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. We say that $(\mathcal{C}, \leq)$ satisfies the $\leq^*$-amalgamation property if whenever $A \leq B \in \mathcal{C}$ and $n \in \mathbb{N}$ then there exists $m = m(\textit{isom.type of } A, \textit{isom.type of } B, n)$ such that if we have $A \leq^m C \in \mathcal{C}$ then there exist $D \in \mathcal{C}$ and embeddings $f : B \to D$, $g : C \to D$ such that $f_{|A} = g_{|A}$, $f(B) \leq^n D$ and $g(C) \leq D$.

Remark 4.3.18. If $(\mathcal{C}, \leq)$ satisfies the $\leq^*$-amalgamation property then it satisfies the $\leq$-amalgamation property.

We have the following result.

Proposition 4.3.19. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. If $(\mathcal{C}, \leq)$ satisfies the $\leq^*$-amalgamation property then the generic model satisfies $F2^*$.*

Proof. (Sketch as in [7], Proposition 2.7) We construct a countable model $\mathcal{N}$ which satisfies $F1$, $F2$ and $F2^*$. By Theorem 4.1.13 it must be isomorphic to the generic model. For constructing $\mathcal{N}$ we proceed as in the proof of Theorem 4.1.12, but instead of amalgamating merely whenever we have a situation $A \leq M_i$, $A \leq B$ in order to obtain $F2$, we also amalgamate situations $A \leq B$, $A \leq^m M_i$, where $m = m(\textit{isom.type of } A, \textit{isom.type of } B, n)$ from the $\leq^*$-amalgamation property, and this will yield $F2^*$. $\square$

We give the following theorem that summarizes the content of this section for practical purposes.

Theorem 4.3.20. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Assume that $(\mathcal{C}, \leq)$ has the $\leq^*$-amalgamation property and the boundedness property. Then:*

a) There is a generic model $\mathcal{M}_\mathcal{C}$ which is unique up to isomorphism.

b) The generic model $\mathcal{M}_\mathcal{C}$ is saturated.

c) $F1 \wedge F2^*$ is first order axiomatizable.

d) For every L-structure $\mathcal{M}$ we have:

$$\mathcal{M} \textit{ satisfies } F1 \wedge F2^* \textit{ if and only if } \mathcal{M} \vDash Th(\mathcal{M}_\mathcal{C}).$$

e) PS/PW implies the stability/ω-stability of $Th(\mathcal{M}_\mathcal{C})$.

Proof. First remember that the $\leq^*$-amalgamation property implies the $\leq$-amalgamation property, thus by 4.1.12 there is a generic model $\mathcal{M}_\mathcal{C}$. Also by 4.3.19 the generic model $\mathcal{M}_\mathcal{C}$ satisfies $F1 \wedge F2^*$. Note that as $\mathcal{C}$ has the boundedness property, by Proposition 4.3.12 we know that $F1$ is first order axiomatizable and that we have the $\leq^*$-definability property (thus the $\leq^*$-type definability property). Also, by the proof of Proposition 4.3.14, for every structure $\mathcal{M}$ satisfying $F1 \wedge F2^*$ we can talk about the set of sentences $F1 \wedge F2^*(\mathcal{M})$.

Let $\mathcal{M}$ satisfy $F1 \wedge F2^*$ and $\mathcal{N}$ be an ω-saturated elementary extension of $\mathcal{M}$. By the proof of 4.3.14 we should have $\mathcal{M} \vDash F1 \wedge F2^*(\mathcal{M})$, consequently we also have $\mathcal{N} \vDash F1 \wedge F2^*(\mathcal{M})$ because $\mathcal{N} \succ \mathcal{M}$ thus $\mathcal{N}$ satisfies $F1 \wedge F2^*$. Moreover, as $\mathcal{N}$ satisfies $F1 \wedge F2^*$ and $\mathcal{N}$ is ω-saturated, then by Proposition 4.3.16 we have that $\mathcal{N}$ satisfies $F1 \wedge F2$.

So we can apply Proposition 4.3.2 and we get that the diagrams of finite closed subsets of $\mathcal{N}$ will determine their types over the empty set. Now observe that there are only countably many isomorphism types in $\mathcal{C}$, in particular there only countably many possible diagrams for finite closed tuples in $\mathcal{N}$. Thus there are only countably many types over the empty set of finite closed tuples in $\mathcal{N}$. Moreover each finite tuple in $\mathcal{N}$ is contained in a finite closed tuple, thus there are only countably many types over the empty set of finite tuples in $\mathcal{N}$. Finally, we are not missing any type over the empty set (consistent with $Th(\mathcal{N})$) because $\mathcal{N}$ is ω-saturated, thus there are only countably many types over the empty set consistent with $Th(\mathcal{N}) = Th(\mathcal{M})$. That is, $Th(\mathcal{M})$ is small, in particular there exists a countable saturated model $\mathcal{N}'$ of $Th(\mathcal{M})$.

Now we proceed as we did for $\mathcal{N}$, that is $\mathcal{M}$ satisfies $F1 \wedge F2^*$ implies that $\mathcal{M} \vDash F1 \wedge F2^*(\mathcal{M})$ so also $\mathcal{N}' \vDash F1 \wedge F2^*(\mathcal{M})$ thus $\mathcal{N}'$ satisfies $F1 \wedge F2^*$ and as $\mathcal{N}'$ is ω-saturated we get that $\mathcal{N}'$ satisfies $F1 \wedge F2$. Thus, as $\mathcal{N}'$ is countable, by the uniqueness of the generic model we have $\mathcal{N}' \simeq \mathcal{M}_{\mathcal{C}}$, in particular as $\mathcal{N}'$ is saturated then the generic model $\mathcal{M}_{\mathcal{C}}$ is also saturated.

Notice that as $\mathcal{N}' \simeq \mathcal{M}_{\mathcal{C}}$ then we have $Th(\mathcal{M}) = Th(\mathcal{N}') = Th(\mathcal{M}_{\mathcal{C}})$. So we actually proved that if $\mathcal{M}$ satisfies $F1 \wedge F2^*$ then $\mathcal{M} \vDash Th(\mathcal{M}_{\mathcal{C}})$. Reciprocally, if $\mathcal{M} \vDash Th(\mathcal{M}_{\mathcal{C}})$ then $\mathcal{M} \vDash F1 \wedge F2^*(\mathcal{M}_{\mathcal{C}})$, thus $\mathcal{M}$ satisfies $F1 \wedge F2^*$. So we proved that $\mathcal{M}$ satisfies $F1 \wedge F2^*$ if and only if $\mathcal{M} \vDash Th(\mathcal{M}_{\mathcal{C}})$, in particular $F1 \wedge F2^*$ is first order axiomatizable. Also notice that if $\mathcal{M}$ satisfies

$F1 \wedge F2^*$ (that is, $\mathcal{M} \models Th(\mathcal{M}_\mathcal{C})$) then $F1 \wedge F2^*(\mathcal{M})$ gives us an explicit axiomatization for $F1 \wedge F2^*$ (in other words, for $Th(\mathcal{M}_\mathcal{C})$) and we can see after this proof that the set of axioms $F1 \wedge F2^*(\mathcal{M})$ do not really depend on the particular choice of $\mathcal{M}$ satisfying $F1 \wedge F2^*$, rather, it depends only on the class $(\mathcal{C}, \leq)$.

Finally, towards the the statement that PS/PW implies the stability/ω-stability of the theory of the generic, given the Theorem 4.3.7 we only need to prove that every ω-saturated model $\mathcal{N}$ of $Th(\mathcal{M}_\mathcal{C})$ satisfies $F1 \wedge F2$. But we already know that $\mathcal{N}$ satisfies $F1 \wedge F2^*$ if and only if $\mathcal{N} \models Th(\mathcal{M}_\mathcal{C})$, so actually $\mathcal{N}$ satisfies $F1 \wedge F2^*$. Thus as $\mathcal{N}$ is ω-saturated we can use 4.3.16 to conclude that $\mathcal{N}$ satisfies $F1 \wedge F2$. This concludes the proof of the theorem. □

We can obtain some general observations out of the proof of the last theorem.

Corollary 4.3.21. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. If $\mathcal{C}$ has the boundedness property then $F1 \wedge F2^*$ is first order axiomatizable.*

Proof. The proof is included in the proof of the last Theorem (4.3.20). We just need to observe that the $\leq^*$-amalgamation property is only used essentially to guarantee the satisfiability of $F1 \wedge F2^*$. But if $F1 \wedge F2^*$ is not satisfiable then it is also trivially first order axiomatizable. □

We can also state the following more general version of Theorem 4.3.20. Somewhat it shows which are really the essential conditions in order to obtain the previous results.

Theorem 4.3.22. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. Assume $F1 \wedge F2^*$ is first order axiomatizable and consistent. Assume the $\leq^*$-type definability property. Then:*

a) There is a generic model $\mathcal{M}_\mathcal{C}$.

b) The generic model $\mathcal{M}_{\mathcal{C}}$ is saturated.

c) For every L-structure $\mathcal{M}$ we have:

$$\mathcal{M} \text{ satisfies } F1 \wedge F2^* \text{ if and only if } \mathcal{M} \models Th(\mathcal{M}_{\mathcal{C}}).$$

d) PS/PW implies the stability/ω-stability of $Th(\mathcal{M}_{\mathcal{C}})$.

Proof. The proof is just a trivial modification of the proof of Theorem 4.3.20.

Actually, is even easier, we just do not need to use Proposition 4.3.14 because we are already assuming that $F1 \wedge F2^*$ is first order axiomatizable. Also the $\leq^*$-amalgamation property was only used essentially to guarantee the satisfiability of $F1 \wedge F2^*$ and this is assumed here. Notice that in the proof of Theorem 4.3.20 we construct a generic model satisfying also $F2^*$, assuming only the satisfiability of $F1 \wedge F2^*$. The reader can easily reconstruct the proof in this context. □

Basically, the boundedness property is our practical way of guarantee the first order axiomatizability of $F1 \wedge F2^*$ and the $\leq^*$-type definability property. Also the $\leq^*$-amalgamation property is our way to guarantee the satisfiability of $F1 \wedge F2^*$. We do not know if there are other techniques to obtain the conditions of Theorem 4.3.22 when the conditions for the Theorem 4.3.20 do not hold.

We conclude this section with another axiom for δ.

Definition 4.3.23. Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension. We say that δ satisfies the *modularity equation* if the following holds for every $A, B \in \mathcal{C}$:

$$(ME) \quad \delta(A \amalg_{A\cap B} B) + \delta(A \cap B) = \delta(A) + \delta(B).$$

Proposition 4.3.24. *Let $\delta : \mathcal{C} \to \mathbb{R}_0^+$ be a predimension satisfying ME. If $A \leq B \in \mathcal{C}$ and $A \leq^n C \in \mathcal{C}$ and $D = B \amalg_A C \in \mathcal{C}$ then $B \leq^n D$ and $C \leq D$.*

Proof. Let $D = B \amalg_A C$ and let $D' \subseteq D$. Note that $D = B \amalg_A C$ implies $D' = (D' \cap B) \amalg_{D' \cap A} (D' \cap C)$, thus by ME we have

$$\delta(D') = \delta(D' \cap B) + \delta(D' \cap C) - \delta(D' \cap A).$$

Now we want to prove that $C \leq D$. Suppose that $D' \supseteq C$, we want to prove that $\delta(D') \geq \delta(C)$. But $A \leq B \Rightarrow A \cap D' \leq B \cap D' \Leftrightarrow A \leq B \cap D' \Rightarrow \delta(B \cap D') \geq \delta(A)$. Thus we have $\delta(D') = \delta(C) + \delta(D' \cap B) - \delta(A) \geq \delta(C)$.

Now we want to prove that $B \leq^n D$. Suppose that $D' \supseteq B$ and $|D' - B| \leq n$, we want to prove that $\delta(D') \geq \delta(B)$. Note that $D' \cap C - A = D' - B$, so $|D' - B| \leq n$ implies that $|D' \cap C - A| \leq n$. Thus $A \leq^n C$ implies $A \leq D' \cap C$ and we get $\delta(D' \cap C) \geq \delta(A)$. So we have $\delta(D') = \delta(B) + \delta(D' \cap C) - \delta(A) \geq \delta(B)$. In particular we proved that $B \leq D'$. Thus $B \leq^n D$. $\square$

4.4 Hrushovski's Example

Let $L = \{R\}$ be a first order language, where R is a ternary relational symbol. Consider the function δ that assigns to each finite L-structure A the integer number $\delta(A) = |A| - |R[A]|$, where $R[A]$ is the set of triples in R with coordinates in A. In order to obtain a predimension we need to restrict to a class $\mathcal{C}_3$ in such a way that $\delta(A)$ is nonnegative for every $A \in \mathcal{C}_0$ and such that $\mathcal{C}_3$ is closed under substructures. So we put,

$$\mathcal{C}_3 = \{A : A \text{ is a finite } L\text{-structure and } \delta(A') \geq 0 \, \forall A' \subseteq A\}.$$

Note that $\mathcal{C}_3$ is closed under substructures and because L is finite then $\mathcal{C}_3$ has only countably many isomorphism types. To prove that $\delta : \mathcal{C}_3 \to \mathbb{R}_0^+$ is a predimension we need to check the axioms $P1 - P4$.

Proposition 4.4.1. *$\delta : \mathcal{C}_3 \to \mathbb{R}_0^+$ is a predimension and satisfies ME.*

Proof. $P3$ and $P4$ are immediate. $P1$ holds because δ is integer valued. So we only need to check $P2$. For all $A, B \in \mathcal{C}_0$ we have $\delta(AB) = |AB| - |R[AB]| = |A| + |B| - |A \cap B| - |R[AB]| \leq |A| + |B| - |A \cap B| - (|R[A]| + |R[B]| - |R[A \cap B]|) = \delta(A) + \delta(B) - \delta(A \cap B)$. Note also that we have equality if and only if $AB = A \amalg_{A \cap B} B$, so ME holds. $\square$

Proposition 4.4.2. *$(\mathcal{C}_3, \leq)$ has the $\leq^*$-amalgamation property.*

Proof. Let $A, B \in \mathcal{C}_0$, $A \cap B = C$, $C \leq A$ and $C \leq^n B$. By modularity of δ and by Proposition 4.3.24 we have $A \leq^n A \amalg_C B$ and $B \leq A \amalg_C B$, provided that $A \amalg_C B \in \mathcal{C}_0$, so the only thing that remains to prove is that $A \amalg_C B \in \mathcal{C}_0$. Let $X \subseteq A \amalg_C B$. We want to prove that $\delta(X) \geq 0$. But as $AB = A \amalg_C B$ then we still have by restricting to X that $X = (A \cap X) \amalg_{C \cap X} (B \cap X)$, so by modularity we have $\delta(X) = \delta(X \cap B) + \delta(X \cap A) - \delta(X \cap C)$. Now just observe that $\delta(X \cap B) \geq 0$ because $X \cap B \in \mathcal{C}_0$ and that $\delta(X \cap A) - \delta(X \cap C) \geq 0$ because $X \cap C \leq X \cap A$, thus $\delta(X) \geq 0$. $\square$

Corollary 4.4.3. *$(\mathcal{C}_3, \leq)$ has a generic model $\mathcal{M}$ which is saturated and PS implies the stability of $Th(\mathcal{M})$ and PW implies the ω-stability of $Th(\mathcal{M})$.*

Now we will check that δ satisfies PS and PW, proving in this way the ω-stability of the theory of the generic.

Proposition 4.4.4. *$\delta : \mathcal{C}_3 \to \mathbb{R}_0^+$ satisfies PS and PW.*

Proof. First we are going to prove PS. Let $\mathcal{M} \in \bar{\mathcal{C}}_3$ and $A, B \leq \mathcal{M}$. Assume that $AB \neq A \amalg_{A \cap B} B$. Then there is a tuple (a, b, c) in AB with $a \in A \setminus B$, $b \in B \setminus A$ and $c \in AB$, without loss of generality.

In order to prove PS let $\gamma = 1$ and $A_0 = A \cap \{a, b, c\}$ and $B_0 = B \cap \{a, b, c\}$. We want to prove that for all $A' \leq A$ and $B' \leq B$ with $A_0 \subseteq A' \in \mathcal{P}_F(A)$ and $B_0 \subseteq B \in \mathcal{P}_F(B)$ we have

$$\delta(A') + \delta(B') \geq \delta(A'B') + \delta(A' \cap B') + \gamma$$

First observe that $(a, b, c) \in R[A'B'] \setminus (R[A'] \cup R[B'])$. We can make the following computation:

$$\delta(A'B') = |A'| + |B'| - |A' \cap B'| - |R[A'B']|$$

$$\leq |A'| + |B'| - |A' \cap B'| - (|R[A']| + |R[B']| - |R[A' \cap B']| + 1)$$

$$= \delta(A') + \delta(B') - \delta(A' \cap B') - 1$$

Thus we have $\delta(A') + \delta(B') \geq \delta(A'B') + \delta(A' \cap B') + 1$ as desired. This proves that our predimension satisfies PS.

Finally we just note that $\{\delta(X) - \delta(Y) : X, Y \in \mathcal{C}_0\}$ is a discrete subset of $\mathbb{R}$ so PS implies PW. □

Corollary 4.4.5. *The theory of the generic model is ω-stable.*

Chapter 5

The uncollapsed case

In this chapter we want to understand how different variations of the original Hrushovski example relate to each other. We investigate what happens when we make the construction starting with an arbitrary n-ary relational symbol instead of a ternary one. More precisely, each construction for a fixed arity n induces a generic model with a pregeometry. The main question then is: are the pregeometries arising from different arities isomorphic?

In order to make this chapter as self contained as possible, some definitions and propositions may overlap with the general case of last chapter.

5.1 The construction and notation

We start with a brief description of the objects of our study.

Definition 5.1.1. Let L be a first order language consisting only of an n-ary relational symbol R, where n is an arbitrary but fixed natural number ($n \geq 2$). We define:

- The predimension function δ : Finite L-structures $\to \mathbb{Z}$ given by

  $$\delta(A) = |A| - |R^A|$$

  (Note that in order this to be a true predimension we need to restrict our class so that the image is nonnegative).

- The class,

  $$\mathcal{C} = \{A : A \text{ is a finite } L\text{-structure and } \delta(A') \geq 0 \text{ for all } A' \subseteq A\}$$

  Now $\delta : \mathcal{C} \to \mathbb{N}$ is a predimension.

- The bigger class,

  $$\bar{\mathcal{C}} = \{\mathcal{M} \text{ an } L\text{-structure : every finite substructure of } \mathcal{M} \text{ is in } \mathcal{C}\}.$$

- The binary relation $\leq$ in $\mathcal{C} \times \bar{\mathcal{C}}$. That is, $A \leq \mathcal{M}$ if and only if

  $$A \subseteq \mathcal{M} \text{ and } \delta(A) \leq \delta(A') \text{ for all finite } A' \text{ such that } A \subseteq A' \subseteq \mathcal{M}.$$

  We say A is *closed* in $\mathcal{M}$.

- We can coherently extend this binary relation to $\bar{\mathcal{C}} \times \bar{\mathcal{C}}$ by defining $\mathcal{M}_1 \leq \mathcal{M}_2$ if and only if

  $$\mathcal{M}_1 \subseteq \mathcal{M}_2 \text{ and } (A \leq \mathcal{M}_1 \to A \leq \mathcal{M}_2) \text{ for all finite } A \subseteq \mathcal{M}_1.$$

The following proposition contains the definition of the $\leq$-free amalgamation property.

Proposition 5.1.2. *The class $\mathcal{C}$ satisfies the boundedness property, that is, there are only finitely many isomorphism types for each size in $\mathcal{C}$. Also*

$(\mathcal{C}, \leq)$ satisfies the $\leq^*$-amalgamation property (see Definition 4.3.17) and the following $\leq$-free amalgamation property:

- *Let $A_0 \subseteq A_1 \in \mathcal{C}$ and $A_0 \leq A_2 \in \mathcal{C}$ and $A_1 \cap A_2 = A_0$. Let $A_1 \amalg_{A_0} A_2$ be the structure with underlying set $A_1 \cup A_2$ such that the only relations are the ones arising from A_1 and A_2. Then*

$$A_1 \amalg_{A_0} A_2 \in \mathcal{C} \text{ and } A_1 \leq A_1 \amalg_{A_0} A_2.$$

Proof. The class $\mathcal{C}$ satisfies the boundedness because the language is finite. The proof of the $\leq^*$-amalgamation property is the same as in Proposition 4.4.2. Now we prove that the $\leq$-free amalgamation property holds.

Let X be a subset of $A_1 \amalg_{A_0} A_2$. Observe that $X = (A_1 \cap X) \amalg_{(A_0 \cap X)} (A_2 \cap X)$. Now we compute $\delta(X)$ as follows:

$$\begin{aligned}
\delta(X) &= \delta((A_1 \cap X) \amalg_{(A_0 \cap X)} (A_2 \cap X)) \\
&= |A_1 \cap X| + |A_2 \cap X| - |A_0 \cap X| - (|R^{A_1 \cap X}| + |R^{A_2 \cap X}| - |R^{A_0 \cap X}|) \\
&= \delta(A_1 \cap X) + \delta(A_2 \cap X) - \delta(A_0 \cap X)
\end{aligned}$$

Now we observe that $A_1 \in \mathcal{C}$ implies that $\delta(A_1 \cap X) \geq 0$. Also we have that $A_0 \leq A_2$ implies $A_0 \cap X \leq A_2 \cap X$ so in particular we have $\delta(A_2 \cap X) \geq \delta(A_0 \cap X)$. We can conclude that $\delta(X) \geq 0$ thus $A_1 \amalg_{A_0} A_2 \in \mathcal{C}$.

Finally we want to prove that $A_1 \leq A_1 \amalg_{A_0} A_2 \in \mathcal{C}$. For this we assume that $A_1 \subseteq X \subseteq A_1 \amalg_{A_0} A_2 \in \mathcal{C}$ and from the previous calculations we get $\delta(X) - \delta(A_1) = \delta(X) - \delta(A_1 \cap X) = \delta(A_2 \cap X) - \delta(A_0 \cap X) \geq 0$. In particular $\delta(X) \geq \delta(A_1)$ and this proves that $A_1 \leq A_1 \amalg_{A_0} A_2 \in \mathcal{C}$. $\square$

Using the results of last chapter we can then conclude:

Theorem 5.1.3. *There is a generic model $\mathcal{M}_{\mathcal{C}}$ for $(\mathcal{C}, \leq)$ which is saturated and $Th(\mathcal{M}_{\mathcal{C}})$ is ω-stable.*

Proof. The proof is exactly the same as in the case $n = 3$ explained in section 4.4. $\square$

Next we will define two closure operators that arise from each $\mathcal{M} \in \bar{\mathcal{C}}$.

Definition 5.1.4. Let $\mathcal{M} \in \bar{\mathcal{C}}$ and $A \subseteq \mathcal{M}$. Then we define the *self-sufficient closure of* A in $\mathcal{M}$ (or just the *closure of* A in $\mathcal{M}$) as:

$$\mathrm{cl}_{\mathcal{M}}(A) = \text{smallest closed subset of } \mathcal{M} \text{ containing } A$$

Note that $\mathrm{cl}_{\mathcal{M}}(A)$ is in fact well defined, by Lemma 4.1.7. We may sometimes write $\mathrm{cl}(A)$ instead of $\mathrm{cl}_{\mathcal{M}}(A)$ when no confusion arises.

Remark 5.1.5. The following facts are true for the self-sufficient closure as we have seen in Chapter 4:

- It is well defined, that is, there exists such a smallest closed subset. Moreover the closure of a finite set is finite. This is true for a general predimension as in Chapter 4. However this can be easily seen in this case because δ is integer valued. In fact given a finite set A we can find among the finite sets containing A one set X such that $\delta(X)$ is minimal so X is closed. The closure of A is then the intersection of such finite closed subsets containing A.
- This closure is a good closure (that is, the closure of a set is the union of the closures of finite subsets). However it is not a pregeometry.
- This notion of closure matches the previous notion of closed set, that is:
$$A \leq \mathcal{M} \text{ if and only if } \mathrm{cl}_{\mathcal{M}}(A) = A.$$
- In the particular case of the generic model, the self-sufficient closure equals the algebraic closure but note that this is not the case in the collapsed case, to be described later, see page 173 in Wagner's [7].

We now want to define another closure operator for each $\mathcal{M} \in \bar{\mathcal{C}}$. But first we need some more definitions. This is the motivation for our main result in Chapter 2, namely Proposition 2.2.9. We start by defining a dimension function in the same way we have done it in chapter 2.

Definition 5.1.6. Let $\mathcal{M} \in \bar{\mathcal{C}}$. Then we define the dimension function on $\mathcal{M}$, that is, $d_{\mathcal{M}} : \mathcal{P}_F(\mathcal{M}) \to \mathbb{N}$ as follows:

$$d_{\mathcal{M}}(A) = \inf\{\delta(A') : A \subseteq A' \in \mathcal{P}_F(\mathcal{M})\}.$$

Remark 5.1.7. The following is true,

a) $d_{\mathcal{M}}(A) = \delta(\mathrm{cl}_{\mathcal{M}}(A))$. In fact as δ is integer valued we can find among the finite sets containing A a set A' such that $\delta(A')$ is minimal. In particular we have $d_{\mathcal{M}}(A) = \delta(A')$ and $A' \leq \mathcal{M}$. Then we have that $\mathrm{cl}_{\mathcal{M}}(A) \cap A' \leq \mathcal{M}$ so we must have that $\mathrm{cl}_{\mathcal{M}}(A) \subseteq A'$ otherwise $\mathrm{cl}_{\mathcal{M}}(A)$ would not be the smallest closed set containing A. Finally $\delta(\mathrm{cl}_{\mathcal{M}}(A)) \leq \delta(A')$ because $\mathrm{cl}_{\mathcal{M}}(A) \leq \mathcal{M}$. Thus we get $\delta(\mathrm{cl}_{\mathcal{M}}(A)) \leq d_{\mathcal{M}}(A)$ and the other inequality is trivial.

b) $A \leq \mathcal{M}$ if and only if $d_{\mathcal{M}}(A) = \delta(A)$. In fact if we have $A \leq \mathcal{M}$ then we have $\mathrm{cl}_{\mathcal{M}}(A) = A$ thus $d(A) = \delta(\mathrm{cl}_{\mathcal{M}}(A)) = \delta(A)$. Conversely if we have $d_{\mathcal{M}}(A) = \delta(A)$ then for all finite set X containing A we have $\delta(X) \geq d_{\mathcal{M}}(A) = \delta(A)$ thus $A \leq \mathcal{M}$.

We are now ready to define our second closure operator.

Definition 5.1.8. Let $\mathcal{M} \in \bar{\mathcal{C}}$. Then for each finite $A \subseteq \mathcal{M}$ we define the *d-closure* of A by:

$$\mathrm{cl}^{d_{\mathcal{M}}}(A) = \{c \in \mathcal{M} : d_{\mathcal{M}}(Ac) = d_{\mathcal{M}}(A)\}.$$

We can extend this definition to infinite subsets A of $\mathcal{M}$ (this agrees with the definition for finite sets) by saying that the d-closure of A is the union of the d-closures of finite subsets of A.

Remark 5.1.9. Observe the following facts:

- Note that the fact that the definition for the finite and infinite case agree can be justified observing that the predimension function δ satisfies submodularity and using Corollary 2.2.8.
- Note that the d-closure of a finite set does not need to be finite.
- Concerning the relation with the self-sufficient closure we have:

$$A \subseteq \mathrm{cl}_{\mathcal{M}}(A) \subseteq \mathrm{cl}^{d_{\mathcal{M}}}(A).$$

The following result is stated without proof in page 173 of Wagner's article [7] for the particular case when $\mathcal{M}$ is the generic model. However using our main result of Chapter 2, Proposition 2.2.9 we get directly the following more general version that works for any $\mathcal{M} \in \bar{\mathcal{C}}$.

Theorem 5.1.10. *Let $\mathcal{M} \in \bar{\mathcal{C}}$. Then, $(\mathcal{M}, \mathrm{cl}^{d_{\mathcal{M}}})$ is a pregeometry. Moreover, the dimension function (as cardinality of a basis) equals $d_{\mathcal{M}}$ on finite subsets of $\mathcal{M}$.*

Now we set up some notation.

Notation 5.1.11. Let n be a natural number with $n \geq 2$. In order to distinguish the constructions for different arities we write $\mathcal{C}_n$ instead of $\mathcal{C}$ and $\bar{\mathcal{C}_n}$ instead of $\bar{\mathcal{C}}$. Also we may write $\mathcal{M}_n$ instead of $\mathcal{M}_{\mathcal{C}_n}$ for the corresponding generic model. Finally we write $PG(\mathcal{M}_n)$ instead of $(\mathcal{M}_n, d_{\mathcal{M}_n})$ for the corresponding pregeometry.

It is our aim to study and compare the pregeometries arising from the constructions for different n-arities. Some natural questions arise. Are $PG(\mathcal{M}_n)$

and $PG(\mathcal{M}_m)$ isomorphic? Can we distinguish $PG(\mathcal{M}_n)$ and $PG(\mathcal{M}_m)$ by just looking at finite substructures of them? In order to answer these questions it is convenient to fix a first order language for the theory of pregeometries. There are several ways to do this.

Definition 5.1.12. Consider the language $LPC = \{C_n : n \in \mathbb{N}\setminus\{0\}\}$ where each C_n is an n-ary relational symbol. A pregeometry (P, cl^P) will be seen as a structure in this language by saying $C_n^P = \{(a_0, \cdots a_{n-1}) \in P : a_0 \in \mathrm{cl}^P(a_1 \cdots a_{n-1})\}$, or in the case of $n = 1$ we define $C_1^P = \mathrm{cl}^P(\emptyset)$. Of course, in the same manner, given an LPC structure P we can define the function cl^P which may in the general case not be a pregeometry.

Alternatively we can use the language $LPI = \{I_n : n \in \mathbb{N} \setminus \{0\}\}$ where each I_n is an n-ary relational symbol. A pregeometry (P, cl^P) will be seen as a structure in this language by saying $I_n^P = \{\bar{a} \in P : \bar{a} \textit{ is independent in } P\}$.

Note that when we describe a pregeometry in the language LPC we do not lose any information, since the closure of infinite sets in a pregeometry are determined by the closure of finite sets. Also when we describe a pregeometry in the language LPI we do not lose any information, since we can recover a pregeometry just by knowing its finite independent sets. In fact, we can recover the closure operator over finite sets by saying: $a_0 \in \mathrm{cl}^P(a_1, \cdots, a_{n-1})$ if and only if there is a maximal independent subset of $\{a_0, \cdots, a_{n-1}\}$ not containing a_0. Thus, for describing pregeometries, either the language LPC or the language LPI are good, that is, two pregeometries are isomorphic if and only if they are LPC isomorphic and if and only if they are LPI isomorphic. Also, a substructure of a pregeometry (either in LPC or LPI) is a pregeometry.

Another question is to know if LPC and LPI describe exactly the same objects and isomorphism types when these objects are not necessarily coming from pregeometries. In other words, are the categories associated to this languages isomorphic (extending the isomorphism given when we restrict to pregeometries!)? The answer should be in principle 'No', but this is not relevant because when we restrict to the subcategories in which the objects are pregeometries then these restricted categories are isomorphic. The isomorphism between them was described before, from the independent sets we can recover the closure operator and from the closure operator we can recover the independent sets and these actions are of course inverse of each other.

For practical purposes, in fact, we use the following remark to compare isomorphism types of pregeometries.

Remark 5.1.13. The isomorphism type of a pregeometry is completely determined by the dimension function over finite subsets.

Our aim is to compare the pregeometries arising from different arities, but we may ask as well if producing constructions where we mix arities and use weights on the predimension gives us new pregeometries. Next we generalize our construction in this sense, and set up the corresponding notation. The proof of the following proposition is just an easy generalization of section 4.4.

Proposition 5.1.14. *Let I be a countable set and $f : I \to \mathbb{N}\setminus\{0\}\times\mathbb{N}\setminus\{0\}$ be a function, say that $f(i) = (n_i, \alpha_i)$. Let $L_f = \{R_i : i \in I\}$ be a language where R_i is an n_i-ary relational symbol. Let $\delta_f :$ Finite L_f-structures $\to \mathbb{Z}\cup\{-\infty\}$ be the function defined by*

$$\delta_f(A) = |A| - \sum_{i\in I} \alpha_i |R_i^A|$$

and

$$\mathcal{C}_f = \{A : A \text{ is a finite } L_f\text{-structure and } \delta(A') \geq 0, \forall A' \subseteq A\}.$$

Then

- $\delta_f : \mathcal{C}_f \to \mathbb{N}$ *is a predimension.*

- *$(\mathcal{C}_f, \leq)$ is an amalgamation class.*
- *We can generalize our construction in this case, obtaining a generic model $\mathcal{M}_f$ and the corresponding pregeometry $PG(\mathcal{M}_f) = (\mathcal{M}_f, d_f)$.*
- *If $\mathcal{C}_f$ satisfies the boundedness property then $\mathcal{M}_f$ is saturated and $Th(\mathcal{M}_f)$ is ω-stable.*

Remark 5.1.15. In fact the proof of the statements of the last proposition are as in section 4.4. For the last statement we can use directly Theorem 4.3.20. There are examples of f where the boundedness property does not hold and such that $\mathcal{M}_f$ is saturated and $Th(\mathcal{M}_f)$ is ω-stable. In fact we expect this to be true for all f even without the boundedness property holding, but a confirmation of this will require a non trivial modification of the results in chapter 4.

Among these predimensions we would like to distinguish the following one:

Notation 5.1.16. Let $I = \mathbb{N} \setminus \{0\}$ and f a function defined by $f(i) = (i, 1)$ for each $i \in I$. In other words we are considering the predimension given by:

$$\delta_f(A) = |A| - \sum_{i \in \mathbb{N} \setminus \{0\}} |R_i^A|.$$

We may write L_ω,$\mathcal{C}_\omega$, δ_ω,d_ω and $\mathcal{M}_\omega$ instead of L_f,$\mathcal{C}_f$, δ_f,d_f and $\mathcal{M}_f$.

5.2 Pregeometries of different predimensions

In this section we show that sometimes the pregeometry on the generic model associated to a particular predimension is isomorphic to the pregeometry on the generic model associated to a 'simpler' predimension. This is motivated by the observation that the pregeometry associated to the predimension $\delta(A) = |A| - |R_3^A| - |R_4^A|$ is isomorphic to the simpler one given by

$\delta(A) = |A| - |R_4^A|$. Actually, the proof of the general result follows the same idea as the proof of this particular result, but it gets considerably more technical. We choose the option of proving the general result and get this particular result as a corollary.

Theorem 5.2.1. *Let I be a countable set. Let $f : I \to \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ be a function and $f(i) = (n_i, \alpha_i)$. Let $\sim$ be an equivalence relation on I defined by $i \sim j$ if and only if $f(i) = f(j)$. Define a partially ordered set $(\widetilde{I}, \leq)$ by saying $[i]_\sim \leq [j]_\sim$ if and only if $n_i \leq n_j$ and $\alpha_j | \alpha_i$. Let $J \subseteq I$ be such that $\widetilde{J}$ is cofinal in $\widetilde{I}$. Then*

$$PG(\mathcal{M}_f) \simeq PG(\mathcal{M}_{f_{|J}}).$$

Proof. Let $h : I \to J$ be a function such that $[i]_\sim \leq [h(i)]_\sim$ and such that $j \in J \Rightarrow h(j) = j$. Such a function exists because $\widetilde{J}$ is cofinal in $\widetilde{I}$.

Let $g : \bigcup_{n=1}^{\infty} \mathcal{M}_f^n \to \mathbb{N} \setminus \{0\}$ be defined by $g(a_1, \cdots, a_n) = |\{a_1, \cdots, a_n\}|$.

Observe that for each $A \subseteq \mathcal{M}_f$ and $n, m \in \mathbb{N} \setminus \{0\}$, if $A^n \cap g^{-1}(m)$ is nonempty then,

$$|A^n \cap g^{-1}(m)| \geq m! \geq m$$

because we can permute the m distinct coordinates of an n-tuple in $A^n \cap g^{-1}(m)$ obtaining $m!$ tuples.

Now we fix $j \in J$. Let $A \subseteq \mathcal{M}_f$ such that $|A| \leq n_j$. Our aim is to replace each tuple $\bar{a} \in R_i^A \cap g^{-1}(|A|)$ for all $i \in h^{-1}(j)$ by one or more tuples in $(A^{n_j} \setminus R_j^A) \cap g^{-1}(|A|)$. We need to prove that we have enough space for this: if $\delta_f(A) \geq 0$ then $\sum_{i \in I} \alpha_i |R_i^A| \leq |A|$. Hence $\sum_{i \in h^{-1}(j)} \alpha_i |R_i^A| \leq |A|$ and so $\sum_{i \in h^{-1}(j) \setminus \{j\}} \alpha_i |R_i^A| \leq |A| - \alpha_j |R_j^A|$. But then we have

$$\sum_{i \in h^{-1}(j) \setminus \{j\}} \alpha_i |R_i^A \cap g^{-1}(|A|)| \leq |A| - \alpha_j |R_j^A|$$

$$\leq |A^{n_j} \cap g^{-1}(|A|)| - \alpha_j |R_j^A|$$

$$\leq \alpha_j (|A^{n_j} \cap g^{-1}(|A|)| - |R_j^A \cap g^{-1}(|A|)|)$$

$$= \alpha_j |(A^{n_j} \setminus R_j^A) \cap g^{-1}(|A|)|.$$

Thus we have

$$\sum_{i \in h^{-1}(j) \setminus \{j\}} \frac{\alpha_i}{\alpha_j} |R_i^A \cap g^{-1}(|A|)| \leq |(A^{n_j} \setminus R_j^A) \cap g^{-1}(|A|)|.$$

So we do have space, that is, for each $i \in h^{-1}(j) \setminus \{j\}$ and $\bar{a} \in R_i^A \cap g^{-1}(|A|)$ we can replace $\bar{a}$ by $\frac{\alpha_i}{\alpha_j}$ distinct tuples in $(A^{n_j} \setminus R_j^A) \cap g^{-1}(|A|)$.

Finally, we do this procedure for each $j \in J$ obtaining a structure $\mathcal{M}_f^\pi$ in the restricted language $L_{f_{|J}}$ with the same underlying set as $\mathcal{M}_f$.

Now let $\pi : \mathcal{M}_f \to \mathcal{M}_f^\pi$ be the identity function. If $A \subseteq \mathcal{M}_f$ then $\pi(A)$ means the substructure of $\mathcal{M}_f^\pi$ with A as the underlying set. Reciprocally if $A \subseteq \mathcal{M}_f^\pi$ then $\pi^{-1}(A)$ means the substructure of $\mathcal{M}_f$ with A as the underlying set.

The construction of $\mathcal{M}_f^\pi$ was made so that for each finite $A \subseteq \mathcal{M}_f$ we have:

$$\delta_f(A) = \delta_{f_{|J}}(\pi(A)).$$

Or in other words for each finite $A \subseteq \mathcal{M}_f^\pi$ we have:

$$\delta_{f_{|J}}(A) = \delta_f(\pi^{-1}(A)).$$

In particular, for each finite $A \subseteq \mathcal{M}_f^\pi$ we have $\delta_{f_{|J}}(A) = \delta_f(\pi^{-1}(A)) \geq 0$, thus $\mathcal{M}_f^\pi \in \bar{\mathcal{C}}_{f_{|J}}$. Also, because the predimension function is the same, so is the dimension function. Thus

$$PG(\mathcal{M}_f^\pi) \simeq PG(\mathcal{M}_f).$$

We now want to prove that $\mathcal{M}_f^\pi$ is the generic model for the class $\mathcal{C}_{f_{|J}}$, that is, isomorphic to $\mathcal{M}_{f_{|J}}$. It remains to show the extension property (the axiom

$F2$). The diagram below will help us to follow the proof.

$$
\begin{tikzcd}
& & & \mathcal{M}_f \arrow[r, "\pi"] & \mathcal{M}_f^\pi \\
B \arrow[r, "l_1"] & B' \arrow[r, "l_2"] & l_2(B') \arrow[r, "\pi"] \arrow[ur, "\leq"] & \pi l_2(B') \arrow[ur, "\leq"] & \\
A \arrow[u, "\leq"] \arrow[r] & \pi^{-1}(A) \arrow[u, "\leq"] \arrow[r] \arrow[ur, "\leq"] & A \arrow[ur, "\leq"] & &
\end{tikzcd}
$$

Let $A \leq B \in \mathcal{C}_{f_{|J}} \subseteq \mathcal{C}_f$ and $A \leq \mathcal{M}_f^\pi$. Let B' be obtained from B just by replacing A by $\pi^{-1}(A)$ and nothing else. Let $l_1 : B \to B'$ be the identity function. We still have $\pi^{-1}(A) \leq B' \in \mathcal{C}_f$ and $\pi^{-1}(A) \leq \mathcal{M}_f$. In fact we have that $A \leq B$ implies that $\pi^{-1}(A) \leq B'$ because the relations that we add when going from $\pi^{-1}(A)$ to B' are the same we add when going from A to B. In particular we get $\emptyset \leq \pi^{-1}(A) \leq B'$ so we may conclude that $B' \in \mathcal{C}_f$. Also $A \leq \mathcal{M}_f^\pi$ implies $\pi^{-1}(A) \leq \mathcal{M}_f$ because when going from $\mathcal{M}_f$ to $\mathcal{M}_f^\pi$ we do not change the predimension values of subsets.

So we have $\pi^{-1}(A) \leq B' \in \mathcal{C}_f$ and $\pi^{-1}(A) \leq \mathcal{M}_f$ and we can apply the extension property of $\mathcal{M}_f$. Let $l_2 : B' \to \mathcal{M}_f$ be an embedding such that $l_2(B') \leq \mathcal{M}_f$ and $l_{2|\pi^{-1}(A)} = Id_{|\pi^{-1}(A)}$. Now we apply the π function to the chain $\pi^{-1}(A) \leq l_2(B') \leq \mathcal{M}_f$ obtaining $A \leq \pi l_2 l_1(B) \leq \mathcal{M}_f^\pi$ and we are done.

It is not completely obvious that $\pi l_2 l_1$ is an $L_{f_{|J}}$-embedding, so let us check that.

Let $j \in J$, we want to prove that $\bar{a} \in R_j^B$ if and only if $\pi l_2 l_1(\bar{a}) \in R_j^{\mathcal{M}_{f_{|J}}}$. For $\bar{a} \subseteq A$ this is clear. For $\bar{a} \nsubseteq A$ then we observe that:

$\bar{a} \in R_j^B \setminus R_j^A \Leftrightarrow l_1(\bar{a}) \in R_j^{B'} \setminus R_j^{\pi^{-1}(A)}$ (the changes are made inside A)

$\Leftrightarrow l_2 l_1(\bar{a}) \in R_j^{\mathcal{M}_f} \setminus R_j^{\pi^{-1}(A)}$ (l_2 is an embedding)

$\Rightarrow \pi l_2 l_1(\bar{a}) \in R_j^{\mathcal{M}_{f_{|J}}} \setminus R_j^A$ (because $j \in J$).

But the converse of the last implication also holds. This is because for $i \in I \setminus J$ we have by construction $R_i^B = \emptyset \Rightarrow R_i^{B'} \setminus R_i^{\pi^{-1}(A)} = \emptyset \Leftrightarrow R_i^{l_2(B')} \setminus R_i^{\pi^{-1}(A)} = \emptyset$ so when we apply π no tuples are added from $R_j^{l_2(B')}$ to $R_j^{\pi l_2(B')}$. Thus $\pi l_2 l_1$ is an embedding.

Finally, by the uniqueness of the generic model we have $\mathcal{M}_f^\pi \simeq \mathcal{M}_{f_{|J}}$. In particular $PG(\mathcal{M}_f^\pi) \simeq PG(\mathcal{M}_{f_{|J}})$. But we already know that $PG(\mathcal{M}_f^\pi) \simeq PG(\mathcal{M}_f)$ thus we have $PG(\mathcal{M}_f) \simeq PG(\mathcal{M}_{f_{|J}})$. □

Note that the proof of this result depends on the fact that we are working with ordered tuples. If there is a version of this result for unordered tuples that is another problem. Also we could ask if there are only countably many geometries when varying f. We obtain the following consequences the previous result.

Corollary 5.2.2. *Let I be a countable set and $f : I \to \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ be a function. Then $PG(\mathcal{M}_f)$ embeds in $PG(\mathcal{M}_\omega)$. In other words, the pregeometry associated to the predimension*

$$\delta_f(A) = |A| - \sum_{i \in I} \alpha_i |R_i^A|$$

embeds into the pregeometry associated to the predimension

$$\delta_\omega(A) = |A| - \sum_{n \geq 1} |R_n^A|.$$

Proof. We extend I to $\bar{I} = I \cup I_\omega$ with $I_\omega = \mathbb{N} \setminus \{0\}$ (we can assume $I \cap I_\omega = \emptyset$). Then we extend f to $\bar{f} : \bar{I} \to \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ by saying $f(i) = (i, 1)$ for $i \in I_\omega$. Clearly we have that $\mathcal{M}_f$ embeds (as a closed substructure) in $\mathcal{M}_{\bar{f}}$. To be more precise $\mathcal{M}_f$ can be seen as a structure in $\mathcal{C}_{\bar{f}}$ if we interpret the extra relational symbols as the emptyset, then we use the extension property of $\mathcal{M}_{\bar{f}}$ recursively because $\mathcal{M}_f$ is countable. Thus $PG(\mathcal{M}_f)$ embeds in $PG(\mathcal{M}_{\bar{f}})$. Now we observe that by Theorem 5.2.1 we have $PG(\mathcal{M}_{\bar{f}}) \simeq PG(\mathcal{M}_{\bar{f}_{|I_\omega}})$. But $\mathcal{M}_{\bar{f}_{|I_\omega}} \simeq \mathcal{M}_\omega$, thus $PG(\mathcal{M}_{\bar{f}}) \simeq PG(\mathcal{M}_\omega)$ and $PG(\mathcal{M}_f)$ embeds in $PG(\mathcal{M}_\omega)$. □

Corollary 5.2.3. *Consider the languages $L_{3,4} = \{R_3, R_4\}$ and $L_4 = \{R_4\}$. Associated to the language L_4 we have the standard construction using the predimension $\delta_4(A) = |A| - |R_4^A|$ obtaining the class $\mathcal{C}_4$ and the generic $\mathcal{M}_4$. Associated to the language $L_{3,4}$ we proceed in the same manner as in the standard case using the predimension $\delta_{3,4}(A) = |A| - |R_3^A| - |R_4^A|$, obtaining the class $\mathcal{C}_{3,4}$ and the generic $\mathcal{M}_{3,4}$. Then,*

$$PG(\mathcal{M}_{3,4}) \simeq PG(\mathcal{M}_4).$$

Proof. In the notation of Theorem 5.2.1 we just observe that $J = \{4\}$ is cofinal in $I = \{3, 4\}$. □

Now we want to prove further that $PG(\mathcal{M}_\omega)$ embeds in $PG(\mathcal{M}_3)$. First we need the following lemma.

Notation 5.2.4. Let $\mathcal{M}, \mathcal{N}$ be two L-structures and A be a subset of both $\mathcal{M}$ and $\mathcal{N}$. We write $A[\mathcal{M}]$ to denote the substructure of $\mathcal{M}$ with underlying set A and $A[\mathcal{N}]$ similarly.

Lemma 5.2.5. *Let $\mathcal{M}_{\omega(3)}$ be the generic model associated to the predimension*

$$\delta_{\omega(3)}(A) = |A| - \sum_{n \geq 3} |R_n^A|.$$

Then,

$$PG(\mathcal{M}_{\omega(3)}) \text{ embeds in } PG(\mathcal{M}_3).$$

Proof. Let $\mathcal{M}$ be an $L_{\omega(3)}$-structure and $\bar{b} \in T_4(\mathcal{M}) := \bigcup_{n \geq 4} R_n^{\mathcal{M}}$. We define an $L_{\omega(3)}$-structure $\mathcal{M}^{\bar{b}}$ obtained from $\mathcal{M}$ by replacing the tuple $\bar{b} = (b_1, \cdots, b_n)$ by the relations

$$\{(b_1, x_1, b_2), (b_2, x_2, b_3), \cdots, (b_{n-1}, x_{n-1}, b_n)\} \cup \{(x_1, \cdots, x_{n-1})\}$$

where $\{x_1, \cdots, x_{n-1}\}$ are new distinct points specifically added for this task. So as a set $\mathcal{M}^{\bar{b}} = \mathcal{M} \cup \{x_1, \cdots, x_{n-1}\}$. We say that $\bar{b}' = (x_1, \cdots, x_{n-1})$ is the derivative of $\bar{b}$.

Claim. *Let $\mathcal{M}$ be an $L_{\omega(3)}$-structure and $\bar{b} = (b_1, \cdots, b_n) \in T_4(\mathcal{M})$. Let A' be a finite subset of $\mathcal{M}^{\bar{b}}$ and let $\bar{b}' = (x_1, \cdots, x_{n-1})$ and $A' = A \cup X$ with $A = A' \cap \mathcal{M}$ and $X = A' \cap \{x_1, \cdots, x_{n-1}\}$. Then,*

$$\delta_{\omega(3)}(A') \geq \delta_{\omega(3)}(A[\mathcal{M}]).$$

In particular, if $\mathcal{M} \in \bar{\mathcal{C}}_{\omega(3)}$ then $\mathcal{M}^{\bar{b}} \in \bar{\mathcal{C}}_{\omega(3)}$.

Proof. Suppose $X \neq \{x_1, \cdots, x_{n-1}\}$. Then, when going from $A[\mathcal{M}]$ to A' we added $|X|$ points and added at most $|X|$ relations. This is because the relation $(x_1, \cdots, x_{n-1})$ is not added as $X \neq \{x_1, \cdots, x_{n-1}\}$. Thus, in this case $\delta_{\omega(3)}(A') \geq \delta_{\omega(3)}(A[\mathcal{M}])$.

Suppose that $X = \{x_1, \cdots, x_{n-1}\}$ and $\bar{b} \nsubseteq A$. Then, when going from $A[\mathcal{M}]$ to A' we added $n-1$ points and at most $n-1$ relations because as $\bar{b} \nsubseteq A$, one of the relations (b_k, x_k, b_{k+1}) is not added. Thus in this case we have also $\delta_{\omega(3)}(A') \geq \delta_{\omega(3)}(A[\mathcal{M}])$.

Finally, suppose that $X = \{x_1, \cdots, x_{n-1}\}$ and $\bar{b} \subseteq A$. Then, when going from $A[\mathcal{M}]$ to A' we added $n-1$ points and exactly n relations, but we removed the relation $\bar{b} = (b_1, \cdots, b_n)$. Thus in this case $\delta_{\omega(3)}(A') = \delta_{\omega(3)}(A[\mathcal{M}])$.

Thus we conclude that $\delta_{\omega(3)}(A') \geq \delta_{\omega(3)}(A[\mathcal{M}])$ for all finite $A' \subseteq \mathcal{M}^{\bar{b}}$. In particular, if $\mathcal{M} \in \bar{\mathcal{C}}_{\omega(3)}$ then $\mathcal{M}^{\bar{b}} \in \bar{\mathcal{C}}_{\omega(3)}$. □

Now we need to prove another claim.

Claim. *Let $\mathcal{M} \in \bar{\mathcal{C}}_{\omega(3)}$ and $\bar{b} \in T_4(\mathcal{M})$. Then the inclusion $\mathcal{M} \to \mathcal{M}^{\bar{b}}$ gives an embedding of pregeometries (for example in the LPI language).*

Proof. We are going to prove that for finite $A \subseteq \mathcal{M}$ we have $d_{\mathcal{M}}(A) = d_{\mathcal{M}^{\bar{b}}}(A)$.

Let $A \subseteq B' \subseteq \mathcal{M}^{\bar{b}}$ with B' finite and $B' = B \cup X$ as in the previous claim. Then $A \subseteq B$ and by the previous claim we have $\delta_{\omega(3)}(B[\mathcal{M}]) \leq \delta_{\omega(3)}(B')$. This shows that $d_{\mathcal{M}}(A) \leq d_{\mathcal{M}^{\bar{b}}}(A)$.

Conversely, let $A \subseteq B \subseteq \mathcal{M}$ with B finite. If $\bar{b} \nsubseteq B$ then $\delta_{\omega(3)}(B[\mathcal{M}]) = \delta_{\omega(3)}(B[\mathcal{M}^{\bar{b}}])$. If $\bar{b} \subseteq B$ then we put $B' = B[\mathcal{M}^{\bar{b}}] \cup \{x_1, \cdots, x_{n-1}\}$ where $\bar{b}' = (x_1, \cdots, x_{n-1})$. Then $\delta_{\omega(3)}(B[\mathcal{M}]) = \delta_{\omega(3)}(B')$ as when going from $B[\mathcal{M}]$ to B' we added $n-1$ points, n relations and remove the relation $\bar{b}$.

The fact that we can find a substructure of $\mathcal{M}^{\bar{b}}$ containing A with the $\delta_{\omega(3)}$ value equal to $\delta_{\omega(3)}(B[\mathcal{M}])$ shows that $d_{\mathcal{M}^{\bar{b}}}(A) \leq d_{\mathcal{M}}(A)$. Thus we have $d_{\mathcal{M}}(A) = d_{\mathcal{M}^{\bar{b}}}(A)$ and this proves that the inclusion $\mathcal{M} \to \mathcal{M}^{\bar{b}}$ is an embedding of pregeometries. $\square$

Now we make a construction by recursion, of a sequence of $L_{\omega(3)}$-structures. We start by ordering the set of tuples $T := T_4(\mathcal{M}_{\omega(3)}) = \bigcup_{n \geq 4} R_n^{\mathcal{M}_{\omega(3)}}$.

Let $\mathcal{M}^{(0)} = \mathcal{M}_{\omega(3)}$. Let $\mathcal{M}^{(1)} = (\mathcal{M}^{(0)})^{\bar{c}_0}$ where $\bar{c}_0$ is the first element of T.

Suppose that we have constructed $\mathcal{M}^{(i+1)} = (\mathcal{M}^{(i)})^{\bar{c}_i}$. Now if $\bar{c}_i$ has n-arity greater or equal than 4, then we set $\bar{c}_{i+1} = \bar{c}_i'$, otherwise we set $\bar{c}_{i+1}$ equal to the next unused element of T. Finally we put $\mathcal{M}^{(i+2)} = (\mathcal{M}^{(i+1)})^{\bar{c}_{i+1}}$.

We have constructed a sequence of $L_{\omega(3)}$-structures $\mathcal{M}^{(i)}$ with $i \in \omega$. Moreover, if we iterate this procedure ω steps we end up with a structure $\mathcal{M}^{(\omega)}$ only with 3-tuples. By the previous claims $\mathcal{M}^{(i)} \in \bar{\mathcal{C}}_{\omega(3)}$ and for each $i \in \omega$ the inclusion function $\mathcal{M}^{(i)} \to \mathcal{M}^{(i+1)}$ gives an embedding of pregeometries.

Finally we can define a pregeometry $P^{(\omega)} = \bigcup_{i\in\omega} PG(\mathcal{M}^{(i)})$. Let $d^{(\omega)}$ be the dimension function of $P^{(\omega)}$.

We need to prove the following claim.

Claim. *Given a finite $A \subseteq \mathcal{M}^{(\omega)}$ then for all i large enough we have $A[\mathcal{M}^{(i)}] = A[\mathcal{M}^{(\omega)}]$. In particular $\mathcal{M}^{(\omega)} \in \bar{\mathcal{C}}_3 \subseteq \bar{\mathcal{C}}_{\omega(3)}$.*

Proof. Let i_0 be large enough so that $A \subseteq \mathcal{M}^{(i_0)}$. Now $A[\mathcal{M}^{(i_0)}] \in \mathcal{C}_{\omega(3)}$ so $|A| - \sum_{n\geq 4} |R_n^{A[\mathcal{M}^{(i_0)}]}| \geq 0$, in particular $\bigcup_{n\geq 4} R_n^{A[\mathcal{M}^{(i_0)}]}$ is finite. Thus, by our construction there is $i_1 \geq i_0$ such that $A[\mathcal{M}^{(i_1)}]$ has only 3-tuples. Now it is clear that for all $i \geq i_i$ we have $A[\mathcal{M}^{(i)}] = A[\mathcal{M}^{(\omega)}]$. In particular $\mathcal{M}^{(\omega)} \in \bar{\mathcal{C}}_3 \subseteq \bar{\mathcal{C}}_{\omega(3)}$. $\square$

We need to prove one last claim.

Claim. $P^{(\omega)} = PG(\mathcal{M}^{(\omega)})$.

Proof. Let $A \subseteq \mathcal{M}^{(\omega)}$ and A finite. We want to prove that $d_\omega(A) = d_{\mathcal{M}^{(\omega)}}(A)$. By our last claim there is i such that $\mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)[\mathcal{M}^{(\omega)}] = \mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)[\mathcal{M}^{(i)}]$. Then,

$$
\begin{aligned}
d^{(\omega)}(A) &= d_{\mathcal{M}^{(i)}}(A) \\
&\leq d_{\mathcal{M}^{(i)}}(\mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)) \\
&\leq \delta_{\omega(3)}(\mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)[\mathcal{M}^{(i)}]) \\
&= \delta_{\omega(3)}(\mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)[\mathcal{M}^{(\omega)}]) \\
&= d_{\mathcal{M}^{(\omega)}}(\mathrm{cl}_{\mathcal{M}^{(\omega)}}(A)) \\
&= d_{\mathcal{M}^{(\omega)}}(A)
\end{aligned}
$$

Thus $d^{(\omega)}(A) \leq d_{\mathcal{M}^{(\omega)}}(A)$.

Now we want to prove the inequality in the other direction. Let B be a finite set with $A \subseteq B \subseteq \mathcal{M}^{(i)}$. Let $B^{(i)} = B$. Suppose that we have constructed $B^{(j)}$ for some $j \geq i$, then we obtain $B^{(j+1)}$ from $B^{(j)}$ in the same manner as we obtain $\mathcal{M}^{(j+1)}$ from $\mathcal{M}^{(j)}$. More precisely, if $\bar{c}_j \nsubseteq B^{(j)}$ then $B^{(j+1)} = B^{(j)}$, if $\bar{b} \subseteq B^{(j)}$ then we remove $\bar{c}_j = (b_1, \cdots, b_n)$ and add the relations $\{(b_1, x_1, b_2), \cdots, (b_{n-1}, x_{n-1}, b_n)\} \cup \{x_1, \cdots, x_{n-1})\}$ where $\bar{c}'_j = (x_1, \cdots, x_{n-1})$. Note that $\{x_1, \cdots, x_{n-1}\}$ are really new points in $B^{(j+1)} \setminus B^{(j)}$ as $\{x_1, \cdots, x_{n-1}\} \cap \mathcal{M}^{(j)} = \emptyset$.

We have constructed $B^{(j)}$ with $i \leq j \in \omega$. Clearly the sequence stabilizes for some k for which $B^{(k)}$ has only 3-tuples (as in the proof of last claim). Moreover, $B^{(k)}$ is a substructure of $\mathcal{M}^{(\omega)}$.

It is now easy to see that for each $j \geq i$ we have

$$\delta_{\omega(3)}(B^{(j)}[\mathcal{M}^{(j)}]) = \delta_{\omega(3)}(B^{(j+1)}[\mathcal{M}^{(j+1)}])$$

in particular we have,

$$\delta_{\omega(3)}(B[\mathcal{M}^{(i)}]) = \delta_{\omega(3)}(B^{(i)}[\mathcal{M}^{(i)}]) = \delta_{\omega(3)}(B^{(k)}[\mathcal{M}^{(k)}]) = \delta_{\omega(3)}(B^{(k)}[\mathcal{M}^{(\omega)}]).$$

The fact that we were able to find a finite substructure $B^{(k)}$ of $\mathcal{M}^{(\omega)}$ with $A \subseteq B^{(k)} \subseteq \mathcal{M}^{(\omega)}$ and $\delta_{\omega(3)}(B^{(k)}[\mathcal{M}^{(\omega)}]) = \delta_{\omega(3)}(B[\mathcal{M}^{(i)}])$ proves that $d_{\mathcal{M}^{(\omega)}}(A) \leq d_{\mathcal{M}^{(i)}}(A) = d^{(\omega)}(A)$.

We have proved that $d^{(\omega)}(A) = d_{\mathcal{M}^{(\omega)}}(A)$ for all finite $A \subseteq \mathcal{M}^{(\omega)}$. Thus, $P^{(\omega)} = PG(\mathcal{M}^{(\omega)})$. □

Now we observe that as $\mathcal{M}^{(\omega)} \in \bar{\mathcal{C}}_3$, then there exists a strong embedding $g : \mathcal{M}^{(\omega)} \xrightarrow{\leq} \mathcal{M}_3$, (that is, g is an embedding with $g(\mathcal{M}^{(\omega)}) \leq \mathcal{M}_3$). For this we use the extension property (axiom $F2$) of $\mathcal{M}_3$ recursively ($\mathcal{M}^{(\omega)}$ is countable). But because g is a strong embedding then g is also an embedding of pregeometries, so we have that $PG(\mathcal{M}^{(\omega)})$ embeds in $PG(\mathcal{M}_3)$.

Finally, we recall all we have proved: $PG(\mathcal{M}_{\omega(3)})$ is a subpregeometry of $P^{(\omega)}$, $P^{(\omega)} = PG(\mathcal{M}^{(\omega)})$ and $PG(\mathcal{M}^{(\omega)})$ embeds in $PG(\mathcal{M}_3)$. Thus $PG(\mathcal{M}_{\omega(3)})$ embeds in $PG(\mathcal{M}_3)$. □

Remark 5.2.6. Note that in contrast to the other results of this section, the previous lemma also holds if we work with unordered tuples.

Lemma 5.2.7. *Let $n \geq 3$ be a natural number. Then,*

$$PG(\mathcal{M}_3) \text{ embeds in } PG(\mathcal{M}_n).$$

Proof. Let $\mathcal{M}_3^h$ be the L_n-structure obtained from $\mathcal{M}_3$ by replacing each relation $(a_1, a_2, a_3) \in R_3^{\mathcal{M}_3}$ by the relation $(a_1, a_2, a_3, a_4, \cdots, a_n)$ where $a_i = a_3$ for all $i \geq 4$. Then, clearly, $\mathcal{M}_3^h \in \bar{\mathcal{C}}_n$ and $PG(\mathcal{M}_3^h) = PG(\mathcal{M}_3)$, because the predimension function is the same in both structures. But as $\mathcal{M}_3^h \in \bar{\mathcal{C}}_n$ we can use recursively the extension property to build a strong embedding $\mathcal{M}_3^h \xrightarrow{\leq} \mathcal{M}_n$, that is, with $g(\mathcal{M}_3^h) \leq \mathcal{M}_n$. But then g is also an embedding of pregeometries, thus $PG(\mathcal{M}_3) = PG(\mathcal{M}_3^h)$ embeds in $PG(\mathcal{M}_n)$. □

We get the following theorem.

Theorem 5.2.8. *Let I be a countable set and $f : I \to \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$ be a function and $n \geq 3$ a natural number. Then there exists a chain of embeddings of pregeometries as in the following diagram:*

$$PG(\mathcal{M}_f) \to PG(\mathcal{M}_\omega) \to PG(\mathcal{M}_3) \to PG(\mathcal{M}_n) \to PG(\mathcal{M}_\omega).$$

Proof. By Corollary 5.2.2 we get $PG(\mathcal{M}_f) \to PG(\mathcal{M}_\omega)$. By Theorem 5.2.1 we have $PG(\mathcal{M}_\omega) \simeq PG(\mathcal{M}_{\omega(3)})$. By Lemma 5.2.5 we have that $PG(\mathcal{M}_{\omega(3)})$ embeds in $PG(\mathcal{M}_3)$. By Lemma 5.2.7 we get $PG(\mathcal{M}_3) \to PG(\mathcal{M}_n)$. Finally, $\mathcal{M}_n \in \bar{\mathcal{C}}_n \subseteq \bar{\mathcal{C}}_\omega$ and $\mathcal{M}_n$ is countable, so we can use the extension property recursively to build a strong embedding $g : \mathcal{M}_n \to \mathcal{M}_\omega$. Then $g : PG(\mathcal{M}_n) \to PG(\mathcal{M}_\omega)$ is an embedding of pregeometries. This conclude the proof. □

We get the following corollary.

Corollary 5.2.9. *Let $m, n \geq 3$ be natural numbers. Then $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$ embed in each other. In particular, in the LPI language of pregeometries we have $age(PG(\mathcal{M}_m)) = age(PG(\mathcal{M}_n))$.*

Proof. We use Theorem 5.2.8 to build a chain of embeddings

$$PG(\mathcal{M}_m) \to PG(\mathcal{M}_\omega) \to PG(\mathcal{M}_3) \to PG(\mathcal{M}_n).$$

In the other direction we proceed similarly. □

5.3 Pregeometries and different arities

In last section we have seen that for natural numbers $n, m \geq 3$ we have that the pregeometries $PG(\mathcal{M}_n)$ and $PG(\mathcal{M}_m)$ embed in each other. This made us consider the possibility that these pregeometries might be in fact be isomorphic. However, this turns out to be false. In order to prove this we need two technical lemmas. These lemmas are important on their own, as most of the results on this work will depend on them. The first of these lemmas is in fact almost trivial.

Lemma 5.3.1 (First Changing Lemma)**.** *Keep the notation of previous the section. In particular let I be a countable set and $f : I \to \mathbb{N} \setminus \{0\} \times \mathbb{N} \setminus \{0\}$. Suppose $\mathcal{M} \in \bar{\mathcal{C}}_f$, $A \leq \mathcal{M}$ and $A' \in \bar{\mathcal{C}}_f$ where A' has the same underlying set as A. Let $\mathcal{M}'$ be the structure obtained from $\mathcal{M}$ by replacing A by A', where* *by this we mean for each $i \in I$ we have $R_i^{\mathcal{M}'} = (R_i^{\mathcal{M}} \setminus R_i^{A}) \cup R_i^{A'}$. Then*

$$A' \leq \mathcal{M}' \in \bar{\mathcal{C}}_f.$$

Proof. Let X be a finite subset of $\mathcal{M}'$. We want to prove that $\delta(X[\mathcal{M}']) \geq 0$.

We have that $A \leq \mathcal{M} \Rightarrow A \cap X[\mathcal{M}] \leq X[\mathcal{M}]$. Thus $\delta(X[\mathcal{M}]) \geq \delta(A \cap X[\mathcal{M}])$. Notice that the points and relations added when going from $A \cap X[\mathcal{M}]$ to $X[\mathcal{M}]$ are the same points and relations added when going from $A' \cap X[\mathcal{M}']$ to $X[\mathcal{M}']$. Thus we also have $\delta(X[\mathcal{M}']) \geq \delta(A' \cap X[\mathcal{M}'])$. But as $A' \cap X[\mathcal{M}']$ is a substructure of $A' \in \bar{\mathcal{C}}_f$, we have $\delta(A' \cap X[\mathcal{M}']) \geq 0$. Thus $\delta(X[\mathcal{M}']) \geq 0$. This proves that $\mathcal{M}' \in \bar{\mathcal{C}}_f$. To see that $A' \leq \mathcal{M}'$ we use a similar argument. □

Now the second lemma.

Lemma 5.3.2 (Second Changing Lemma)**.** *Let f be as in First Changing Lemma. Suppose $\mathcal{M} \in \bar{\mathcal{C}}_f$, $A \leq \mathcal{M}$ and $A' \in \bar{\mathcal{C}}_f$ where A' has the same underlying set as A. Let $\mathcal{M}'$ be the structure obtained from $\mathcal{M}$ by replacing A by A'. Then $A' \leq \mathcal{M}' \in \bar{\mathcal{C}}_f$ and*

$$\textit{if } PG(A) = PG(A') \textit{ then } PG(\mathcal{M}) = PG(\mathcal{M}').$$

Proof. Assume that $PG(A) = PG(A')$, that is, $d_A = d_{A'}$. We want to prove that $PG(\mathcal{M}) = PG(\mathcal{M}')$, that is, $d_{\mathcal{M}} = d_{\mathcal{M}'}$. The idea of this proof is to reconstruct in little steps $\mathcal{M}$ starting with A and $\mathcal{M}'$ starting with A', and at each one of this steps the dimension function will be the same in both of these parallel constructions (reconstructions). More precisely, in the first step we add the remaining points, and in each subsequent step we add one of the remaining relations.

Let $R := \bigcup_{i \in I} R_i^{\mathcal{M}} \setminus R_i^{A} = \bigcup_{i \in I} R_i^{\mathcal{M}'} \setminus R_i^{A'}$.

Let $\mathcal{M}_0$ be the structure obtained from $\mathcal{M}$ by removing all the relations in R, that is, the relations contained in $\mathcal{M}$ not entirely contained in A. Similarly, let $\mathcal{M}'_0$ be obtained from $\mathcal{M}'$ by removing all the relations in R, that is, the relations contained in $\mathcal{M}'$ not entirely contained in A'.

Clearly we have $d_{\mathcal{M}_0} = d_{\mathcal{M}'_0}$. In fact let B be a finite subset of both $\mathcal{M}$ and $\mathcal{M}'$ (which share the same underlying set) and let $X = B \setminus (A \cap B)$, then we have

$$d_{\mathcal{M}_0}(B) = d_{\mathcal{M}_0}(A \cap B) + |X|$$

$$= d_A(A \cap B) + |X|$$

$$= d_{A'}(A \cap B) + |X|$$

$$= d_{\mathcal{M}'_0}(A \cap B) + |X| = d_{\mathcal{M}'_0}(B)$$

thus $d_{\mathcal{M}_0} = d_{\mathcal{M}'_0}$.

Now we add the relations that we have removed one by one in both $\mathcal{M}_0$ and $\mathcal{M}'_0$. Let $R = (r_i)_{i \in \kappa}$ be an enumeration of R, with κ some cardinal.

For each $j \in \kappa$ let $\mathcal{M}_j$ be obtained from $\mathcal{M}_0$ by adding the relations $\{r_i : i \in j\}$ and $\mathcal{M}'_j$ be obtained from $\mathcal{M}'_0$ by adding the same set of relations $\{r_i : i \in j\}$. Clearly $\mathcal{M}_\kappa = \mathcal{M}$ and $\mathcal{M}'_\kappa = \mathcal{M}'$. We want to prove that $d_{\mathcal{M}_\kappa} = d_{\mathcal{M}'_\kappa}$. The proof is by transfinite induction.

Let $j = 0$. In this case we already proved that $d_{\mathcal{M}_0} = d_{\mathcal{M}'_0}$.

Now we want to prove that $d_{\mathcal{M}_j} = d_{\mathcal{M}'_j} \Rightarrow d_{\mathcal{M}_{j+1}} = d_{\mathcal{M}'_{j+1}}$. This follows directly from the following claim:

Claim. *Let $A_1, A_2 \in \bar{\mathcal{C}}_f$ with the same underlying set and that $PG(A_1) = PG(A_2)$. Let B_1 and B_2 be the structures obtained by adding the same relation r to A_1 and A_2. Assume further that $B_1, B_2 \in \bar{\mathcal{C}}_f$. Then $PG(B_1) = PG(B_2)$.*

Proof. Let X be a finite subset of $B_1 = B_2$ (as sets). We know that $d_{A_1}(X) = d_{A_2}(X)$ and we want to prove that $d_{B_1}(X) = d_{B_2}(X)$.

Fix $i \in \{1, 2\}$. We have either

$$d_{B_i}(X) = d_{A_i}(X) \textit{ or } d_{B_i}(X) = d_{A_i}(X) - 1$$

because we just add one relation.

Suppose we have $d_{B_i}(X) = d_{A_i}(X) - 1$. Then there is a finite Y with $X \subseteq Y \subseteq B_i$ such that $\delta(Y[A_i]) = d_{A_i}(X)$ and $Y \supseteq r$. In particular $d_{A_i}(X) \leq d_{A_i}(Y) \leq \delta(Y[A_i]) = d_{A_i}(X)$. Thus $d_{A_i}(Y) = d_{A_i}(X)$.

Conversely, if there exists a finite Y with $X \subseteq Y \subseteq B_i$ such that $d_{A_i}(Y) = d_{A_i}(X)$ and $Y \supseteq r$, then there exists a finite $Z \supseteq Y \supseteq X$ such that $\delta(Z[A_i]) = d_{A_i}(X)$. But as $Z \supseteq r$ we have $\delta(Z[B_i]) = d_{A_i}(X) - 1$, thus $d_{B_i}(X) \leq d_{B_i}(Y) \leq \delta(Z[B_i]) = d_{A_i}(X) - 1$ so we get $d_{B_i}(X) = d_{A_i}(X) - 1$.

But now we have

$$\begin{aligned}
&d_{B_1}(X) = d_{A_1}(X) - 1 \Leftrightarrow \\
&\Leftrightarrow \exists Y \supseteq X \cup r \textit{ such that } d_{A_1}(X) = d_{A_1}(Y) \\
&\Leftrightarrow \exists Y \supseteq X \cup r \textit{ such that } d_{A_2}(X) = d_{A_2}(Y) \text{ (because } d_{A_1} = d_{A_2}) \\
&\Leftrightarrow d_{B_2}(X) = d_{A_2}(X) - 1.
\end{aligned}$$

But consequently, we also have $d_{B_1}(X) = d_{A_1}(X)$ if and only if $d_{B_2}(X) = d_{A_2}(X)$. Now as we have $d_{A_1}(X) = d_{A_2}(X)$, the previous argument proves that either way we have $d_{B_1}(X) = d_{B_2}(X)$. Thus $d_{B_1} = d_{B_2}$, that is, $PG(B_1) = PG(B_2)$. □

Now we return to our recursion argument considering the case when j is a limit ordinal. Let j be a limit ordinal. Assume that $d_{\mathcal{M}_i} = d_{\mathcal{M}'_i}$ for all $i \in j$.

We want to prove that $d_{\mathcal{M}_j} = d_{\mathcal{M}'_j}$.

Let X be a finite subset of $\mathcal{M}_j = \mathcal{M}'_j$ (as sets).

Let $i_0 \in j$ be such that all the relations in $\mathrm{cl}_{\mathcal{M}_j}(X)$ are in $\mathcal{M}_{i_0}$, this is possible because $\mathrm{cl}_{\mathcal{M}_j}(X)$ contains only finitely many relations. Then,

$$d_{\mathcal{M}_{i_0}}(X) \leq d_{\mathcal{M}_{i_0}}(\mathrm{cl}_{\mathcal{M}_j}(X)) \leq \delta(\mathrm{cl}_{\mathcal{M}_j}(X)[\mathcal{M}_{i_0}]) = \delta(\mathrm{cl}_{\mathcal{M}_j}(X)[\mathcal{M}_j]) = d_{\mathcal{M}_j}(X)$$

thus $d_{\mathcal{M}_{i_0}}(X) = d_{\mathcal{M}_j}(X)$.

We have proved that for all $i \in j$ big enough (with X fixed) we have $d_{\mathcal{M}_i}(X) = d_{\mathcal{M}_j}(X)$. Similarly, for all $i \in j$ big enough we have $d_{\mathcal{M}'_i}(X) = d_{\mathcal{M}'_j}(X)$. Let $i \in j$ be big enough for both cases, then using the inductive hypothesis we get

$$d_{\mathcal{M}_j}(X) = d_{\mathcal{M}_i}(X) = d_{\mathcal{M}'_i}(X) = d_{\mathcal{M}'_j}(X).$$

This means that we proved that $d_{\mathcal{M}_j} = d_{\mathcal{M}'_j}$ for all $j \in \kappa + 1$. Finally we have $PG(\mathcal{M}) = PG(\mathcal{M}_\kappa) = PG(\mathcal{M}'_\kappa) = PG(\mathcal{M}')$, as desired. □

We have now the tools necessary to prove that the isomorphism types of pregeometries arising from different arities are different.

Theorem 5.3.3. *Let $n, m \in \mathbb{N} \setminus \{0\}$ with $n < m$. Then,*

$$PG(\mathcal{M}_n) \ncong PG(\mathcal{M}_m).$$

Proof. Suppose that there is an isomorphism of pregeometries from $PG(\mathcal{M}_n)$ to $PG(\mathcal{M}_m)$. We first observe that without loss of generality, we can assume that $\mathcal{M}_n$ and $\mathcal{M}_m$ have the same underlying set and that the isomorphism of pregeometries is the identity map. We can write d instead of $d_{\mathcal{M}_n}$ and $d_{\mathcal{M}_m}$. Let cl^d be the closure operator of the pregeometry $PG(\mathcal{M}_n) = PG(\mathcal{M}_m)$.

Let $X = \{a_1, \cdots, a_m\}$ be a subset of $\mathcal{M}_m$ such that $\widetilde{X} := X[\mathcal{M}_m] \leq \mathcal{M}_m$ and $R_m^{\widetilde{X}} = \{(a_1, \cdots, a_m)\}$ with $a_1, \cdots, a_m$ distinct. This is possible by the extension property of $\mathcal{M}_m$. Let $\widehat{X} := X[\mathcal{M}_n]$.

Now consider $Y = \mathrm{cl}_{\mathcal{M}_n}(X)$ and $\widehat{Y} := Y[\mathcal{M}_n]$ and $\widetilde{Y} := Y[\mathcal{M}_m]$. We have

$$d(Y) = d_{\mathcal{M}_n}(\mathrm{cl}_{\mathcal{M}_n}(X)) = d_{\mathcal{M}_n}(X) = d(X) = d_{\mathcal{M}_m}(X) = \delta_m(X[\mathcal{M}_m]) = m-1$$

thus

$$d(Y) = d(X) = m - 1.$$

Let $(a_{11}, \cdots, a_{1n}), \cdots, (a_{k1}, \cdots, a_{kn})$ be a list of the relations in $R_n^{\widehat{Y}}$. We have $\widehat{Y} = \mathrm{cl}_{\mathcal{M}_n}(X) \leq \mathcal{M}_n$, thus

$$d(Y) = \delta_n(\widehat{Y}) = |Y| - k.$$

Now, for each $1 \leq i \leq k$ let $Z_i := \mathrm{cl}^d(\{a_{i1}, \cdots, a_{in}\})$ and let $\widehat{Z}_i := Z_i[\mathcal{M}_n]$ and $\widetilde{Z}_i := Z_i[\mathcal{M}_m]$. Note that Z_i can be infinite and that $\mathrm{cl}^d(Z_i) = Z_i$. Also we have $\widetilde{Z}_i \subseteq \mathrm{cl}_{\mathcal{M}_m}(\widetilde{Z}_i) \subseteq \mathrm{cl}^d(Z_i)$ and $\widehat{Z}_i \subseteq \mathrm{cl}_{\mathcal{M}_n}(\widehat{Z}_i) \subseteq \mathrm{cl}^d(Z_i)$. Thus we have $\widetilde{Z}_i = \mathrm{cl}_{\mathcal{M}_m}(\widetilde{Z}_i)$ and $\widehat{Z}_i = \mathrm{cl}_{\mathcal{M}_n}(\widehat{Z}_i)$, that is, $\widetilde{Z}_i \leq \mathcal{M}_m$ and $\widehat{Z}_i \leq \mathcal{M}_n$. In particular, the pregeometries $PG(\widetilde{Z}_i)$ and $PG(\widehat{Z}_i)$ are those induced from $PG(\mathcal{M}_m) = PG(\mathcal{M}_n)$ so we get $PG(\widetilde{Z}_i) = PG(\widehat{Z}_i)$.

Now observe that $Z_i \not\supseteq X$. In fact, if $X \subseteq Z_i$ then we would have $X \subseteq \mathrm{cl}^d(\{a_{i1}, \cdots, a_{in}\})$, in particular $d(X) \leq d(\{a_{i1}, \cdots, a_{in}\})$. But we have

$$
\begin{aligned}
& d(\{a_{i1}, \cdots, a_{in}\}) \leq \delta_n(\{a_{i1}, \cdots, a_{in}\})[\mathcal{M}_n]) \\
& = |\{a_{i1}, \cdots, a_{in}\}| - |R_n^{\{a_{i1}, \cdots, a_{in}\}[\mathcal{M}_n]}| \\
& \leq n - 1 < m - 1 = d(X).
\end{aligned}
$$

Thus $Z_i \not\supseteq X$.

Let $Z_i^* \in \bar{\mathcal{C}}_m$ be the structure obtained from $\widehat{Z}_i$ by replacing each relation $(x_1, \cdots, x_n) \in R_n^{\widehat{Z}_i}$ by the relation $(x_1, \cdots, x_n, \cdots, x_n)$ with m coordinates. Then we have

$$PG(Z_i^*) = PG(\widehat{Z}_i) = PG(\widetilde{Z}_i).$$

Now we want to construct a sequence of L_m-structures

$$\mathcal{M}_m = \mathcal{M}_m^{(0)}, \cdots, \mathcal{M}_m^{(k)}$$

all with the same underlying set and with the following properties:

1. $\mathcal{M}_m^{(i)} \in \bar{\mathcal{C}}_m$
2. $PG(\mathcal{M}_m^{(i)}) = PG(\mathcal{M}_m)$
3. $R_m^{\mathcal{M}_m^{(i)}} \supseteq \{(a_1, \cdots, a_m)\} \cup \{(a_{j1}, \cdots, a_{jn}, \cdots, a_{jn}) : 1 \leq j \leq i\}$.

Let $\mathcal{M}_m^{(0)} = \mathcal{M}_m$, then for $i = 0$ the properties are satisfied, note that $(a_1, \cdots, a_m) \in R_m^{\mathcal{M}_m}$. Now suppose we have constructed $\mathcal{M}_m^{(i)}$ satisfying the properties and $i < k$. Then we obtain $\mathcal{M}_m^{(i+1)}$ from $\mathcal{M}_m^{(i)}$ by replacing $Z_{i+1}[\mathcal{M}_m^{(i)}]$ by Z_{i+1}^*. We need to prove that $\mathcal{M}_m^{(i+1)}$ satisfies the properties.

Observe that $Z_{i+1}[\mathcal{M}_m^{(i)}] \leq \mathcal{M}_m^{(i)}$. In fact we have $PG(\mathcal{M}_m^{(i)}) = PG(\mathcal{M}_m)$ so we have $Z_{i+1} \subseteq \mathrm{cl}_{\mathcal{M}_m^{(i)}}[Z_{i+1}] \subseteq \mathrm{cl}^d(Z_{i+1})$ and as $Z_{i+1} = \mathrm{cl}^d(Z_{i+1})$ we get that $Z_{i+1} = \mathrm{cl}_{\mathcal{M}_m^{(i)}}(Z_{i+1})$, that is, $Z_{i+1}[\mathcal{M}_m^{(i)}] \leq \mathcal{M}_m^{(i)}$.

Now we apply the First Changing Lemma: $Z_{i+1}^* \in \bar{\mathcal{C}}_m$ and $Z_{i+1}[\mathcal{M}_m^{(i)}] \leq \mathcal{M}_m^{(i)}$ so $Z_{i+1}^* \leq \mathcal{M}_m^{(i+1)} \in \bar{\mathcal{C}}_m$, proving property 1).

For property 2) we use the Second Changing Lemma: all we need to prove is that $PG(Z_{i+1}^*) = PG(Z_{i+1}[\mathcal{M}_m^{(i)}])$. But in fact,

$$d_{Z_{i+1}[\mathcal{M}_m^{(i)}]} = (d_{\mathcal{M}_m^{(i)}})_{|Z_{i+1}} = (d_{\mathcal{M}_m})_{|Z_{i+1}} = d_{Z_{i+1}[\mathcal{M}_m]} = d_{\widetilde{Z}_{i+1}}$$

thus
$$PG(Z_{i+1}[\mathcal{M}_m^{(i)}]) = PG(\widetilde{Z}_{i+1}) = PG(\widehat{Z}_{i+1}) = PG(Z_{i+1}^*).$$

Now we apply the Second Changing Lemma and we get $PG(\mathcal{M}_m^{(i+1)}) = PG(\mathcal{M}_m^{(i)}) = PG(\mathcal{M}_m)$, proving property 2).

Finally we need to prove property 3). First we observe that when going from $\mathcal{M}_m^{(i)}$ to $\mathcal{M}_m^{(i+1)}$ we add the relation $(a_{j1}, \cdots, a_{jn}, \cdots, a_{jn})$ for $j = i+1$, as this relation is in $R_m^{Z_{i+1}^*}$.

Observe also that $(a_1, \cdots, a_m)$ is not removed because $\{a_1, \cdots, a_m\} = X \nsubseteq Z_{i+1}$.

Now we need to see that for $j \leq i$ the relation $(a_{j1}, \cdots, a_{jn}, \cdots, a_{jn})$ is not removed when going from $\mathcal{M}_m^{(i)}$ to $\mathcal{M}_m^{(i+1)}$. If $\{a_{j1}, \cdots, a_{jn}\} \nsubseteq Z_{i+1}$ this is clear. If $\{a_{j1}, \cdots, a_{jn}\} \subseteq Z_{i+1}$ then

$$(a_{j1}, \cdots, a_{jn}) \in R_n^{\mathcal{M}_n} \Leftrightarrow (a_{j1}, \cdots, a_{jn}) \in R_n^{Z_{i+1}[\mathcal{M}_n]} = R_n^{\widehat{Z}_{i+1}}$$
$$\Leftrightarrow (a_{j1}, \cdots, a_{jn}, \cdots, a_{jn}) \in R_m^{Z_{i+1}^*}$$

thus $(a_{j1}, \cdots, a_{jn}, \cdots, a_{jn})$ is not removed. This proves that $\mathcal{M}_m^{(i+1)}$ satisfies property 3).

Finally we get the desired contradiction. $\mathcal{M}_m^{(k)}$ satisfies properties 2) and 3), that is, $PG(\mathcal{M}_m^{(k)}) = PG(\mathcal{M}_m)$ and

$$R_m^{Y[\mathcal{M}_m^{(k)}]} \supseteq \{(a_1, \cdots, a_m)\} \cup \{(a_{j1}, \cdots, a_{jn}, \cdots, a_{jn}) : 1 \leq j \leq k\}.$$

Thus we can make the following calculation (remember that $d(Y) = |Y| - k$):

$$d(Y) \leq \delta_m(Y[\mathcal{M}_m^{(k)}]) = |Y| - |R_m^{Y[\mathcal{M}_m^{(k)}]}| \leq |Y| - (k+1) = (|Y| - k) - 1 = d(Y) - 1.$$

Thus we get the desired contradiction $d(Y) \leq d(Y) - 1$. We may then conclude that $PG(\mathcal{M}_n) \ncong PG(\mathcal{M}_m)$. $\square$

5.4 The local isomorphism type

We have seen in the previous section that $PG(\mathcal{M}_n)$ is not isomorphic to $PG(\mathcal{M}_m)$. Here we go further and observe at the end of the section that $PG(\mathcal{M}_n)$ and $PG(\mathcal{M}_m)$ are not locally isomorphic. First we need some more changing lemmas.

Lemma 5.4.1. *(Third Changing Lemma) Let f be as in the First Changing Lemma. Let $\mathcal{M}_f$ be the generic model for $(\mathcal{C}_f, \leq)$ as in Proposition 5.1.14. Let $Z \leq \mathcal{M}_f$ with Z finite. Let $Z' \in \mathcal{C}_f$ be a structure with the same underlying set as Z and $\mathcal{M}'_f$ be obtained from $\mathcal{M}_f$ by replacing Z by Z'. Then we have $Z' \leq \mathcal{M}'_f \in \bar{\mathcal{C}}_f$ and*

$$\mathcal{M}'_f \simeq \mathcal{M}_f.$$

Proof. The only thing we have to show is that $\mathcal{M}'_f$ satisfies the extension property. For any set $X \subseteq \mathcal{M}_f$ we write X instead of $X[\mathcal{M}_f]$ and X' instead of $X[\mathcal{M}'_f]$. Let $A \subseteq \mathcal{M}_f$ with $A' \leq \mathcal{M}'_f$ and $A' \leq B' \in \mathcal{C}_f$ (B' arbitrary). We want to prove that there exists an embedding $g : B' \to \mathcal{M}'_f$ fixing the elements of A' and with $g(B') \leq \mathcal{M}'_f$. The next diagram illustrates the structure of the proof.

$$\begin{tikzcd}
 & & & & \mathcal{M}_f \arrow[r] & \mathcal{M}'_f \\
B' \arrow[r, "g_1"] & B'' \arrow[r, "Id"] & B \arrow[r] & B \amalg_A \mathrm{cl}_{\mathcal{M}_f}(AZ) \arrow[r, "g_2"] & g_2(B) \amalg_A \mathrm{cl}_{\mathcal{M}_f}(AZ) \arrow[r] \arrow[u, "\leq"] & g_2(B)' \amalg_{A'} \mathrm{cl}_{\mathcal{M}_f}(AZ)' \arrow[u, "\leq"'] \\
A' \arrow[r, "Id"] \arrow[u, "\leq"] & A' \arrow[r] \arrow[u, "\leq"] & A \arrow[r] \arrow[u, "\leq"] & \mathrm{cl}_{\mathcal{M}_f}(AZ) \arrow[r] \arrow[u, "\leq"] \arrow[ur, "\leq"] & \mathrm{cl}_{\mathcal{M}_f}(AZ)' \arrow[ur, "\leq"] &
\end{tikzcd}$$

Let $g_1 : B' \to B''$ be an isomorphism fixing elements of A' such that, as sets, we have $B'' \cap \mathrm{cl}_{\mathcal{M}_f}(AZ) = A$. We have $A' \leq B'' \in \mathcal{C}_f$.

Let B be the structure obtained from B'' by replacing $A' = A[\mathcal{M}'_f]$ by $A = A[\mathcal{M}_f]$. By the First Changing Lemma we have $A \leq B \in \mathcal{C}_f$.

We have $B \cap \mathrm{cl}_{\mathcal{M}_f}(AZ) = A$ so we can use the free amalgamation property of $(\mathcal{M}_f, \leq)$ to get $\mathrm{cl}_{\mathcal{M}_f}(AZ) \leq C := B \amalg_A \mathrm{cl}_{\mathcal{M}_f}(AZ) \in \mathcal{C}_f$.

Now we can use the extension property of $\mathcal{M}_f$ to construct an embedding $g_2 : C \to \mathcal{M}_f$ fixing the elements of $\mathrm{cl}_{\mathcal{M}_f}(AZ)$ and with $g_2(C) \leq \mathcal{M}_f$. Note that $g_2(C) = g_2(B) \amalg_A \mathrm{cl}_{\mathcal{M}_f}(AZ)$.

Finally we make the same changes that we did from $\mathcal{M}_f$ to $\mathcal{M}'_f$, obtaining $g_2(C)'$. Observe that we have $g_2(C)' = g_2(B)' \amalg_A \mathrm{cl}_{\mathcal{M}_f}(AZ)'$ because every relation that we possibly added is inside of $Z' \subseteq \mathrm{cl}_{\mathcal{M}_f}(AZ)'$. For this same reason, that is, the fact that we only add relations inside of $Z \subseteq \mathrm{cl}_{\mathcal{M}_f}(AZ)$ we get that $\mathrm{cl}_{\mathcal{M}_f}(AZ)' \leq g_2(C)' \leq \mathcal{M}'_f$.

However, we need to prove that $g_2(B)' \leq \mathcal{M}'_f$. For this we use our assumption that $A' \leq \mathcal{M}'_f$. In particular we get $A' \leq \mathrm{cl}_{\mathcal{M}_f}(AZ)'$. Then we use the free amalgamation property again, obtaining:

$$\begin{tikzcd}
 & g_2(C)' = g_2(B)' \amalg_{A'} \mathrm{cl}_{\mathcal{M}_f}(AZ)' & \\
g_2(B)' \arrow[ur, "\leq"] & & \mathrm{cl}_{\mathcal{M}_f}(AZ)' \arrow[ul] \\
 & A' \arrow[ul] \arrow[ur, "\leq"] &
\end{tikzcd}$$

Thus $g_2(B)' \leq g_2(C)' \leq \mathcal{M}'_f$. In particular $g_2(B)' \leq \mathcal{M}'_f$.

Finally, the function $B' \to g_2(B)' \leq \mathcal{M}'_f$ that we get from our first diagram is the desired embedding. We should check that this function is in fact an embedding. But in fact, the only changes we made when going from B'' to B are made inside of A' and we undo these changes when going from $g_2(B)$ to $g_2(B)'$. Note also that we do nothing else when going from $g_2(B)$ to $g_2(B)'$ (apart from undoing the mentionated changes) because $Z \cap g_2(B) \subseteq A$.

We may conclude that $\mathcal{M}'_f$ satisfies the extension property, thus by the uniqueness of the generic model we get $\mathcal{M}_f \simeq \mathcal{M}'_f$. □

Now we need a Fourth Changing Lemma concerning pregeometries and localization. For our purpose it would be enough to have a finite version of this lemma, however we decide to present a proof that works for the infinite case. We still show the finite case separately inside of the proof, the reason for this is that for the infinite case the proof requires the predimension to be of a particular form (but covering all the ones that we introduced), while the finite case could be more easily generalized to other contexts.

First some notation.

Notation 5.4.2. Let $\mathcal{M}$ be a structure and $PG(\mathcal{M})$ a pregeometry associated to $\mathcal{M}$. Let $Z \subseteq \mathcal{M}$. We write $PG_Z(\mathcal{M})$ to denote the localization of the pregeometry $PG(\mathcal{M})$ to the set Z.

Lemma 5.4.3. *(Fourth Changing Lemma) Let f be as in First Changing Lemma. Let $(\mathcal{C}_f, \leq)$ be as in Proposition 5.1.14 and let $\mathcal{M} \in \bar{\mathcal{C}}_f$. Let $Z \leq \mathcal{M}$ and $Z' \in \bar{\mathcal{C}}_f$ be a structure with the same underlying set as Z. Let $\mathcal{M}'$ be obtained from $\mathcal{M}$ by replacing Z by Z'. Then $Z' \leq \mathcal{M}' \in \bar{\mathcal{C}}_f$ and*

$$PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}').$$

Proof. Let us set up some notation. The value of the predimension depends on the structure. However, instead of distinguishing the structure from the underlying set (as in most of the chapter), here we use different notation for the predimension. Depending on whether we are working inside of $\mathcal{M}$ or $\mathcal{M}'$, given a finite subset $A \subseteq \mathcal{M}$ we write

$$\delta(A) = |A| - \textstyle\sum_{i\in I} \alpha_i |R_i^{A[\mathcal{M}]}|$$

$$\delta'(A) = |A| - \textstyle\sum_{i\in I} \alpha_i |R_i^{A[\mathcal{M}']}|$$

with $f(i) = (n_i, \alpha_i)$, where n_i is the arity of R_i. Also we write d instead of $d_{\mathcal{M}}$ and d' instead of $d_{\mathcal{M}'}$.

First we give a separate proof for the case when Z is finite.

Claim. *Assume Z is finite. Then $PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}')$.*

Proof. First observe that for every finite $A \subseteq \mathcal{M}$ we have

$$\delta(A) - \delta(A \cap Z) = \delta'(A) - \delta'(A \cap Z)$$

because all the changes are done inside Z.

All we have to prove is that $d(A/Z) = d'(A/Z)$. We do the following calculation:

$$\begin{aligned}
&d(A/Z) = d(AZ) - d(Z)\\
&= \delta(\mathrm{cl}^{\mathcal{M}}(AZ)) - \delta(Z) \text{ (because } Z \leq \mathcal{M})\\
&= \delta'(\mathrm{cl}^{\mathcal{M}}(AZ)) - \delta'(Z) \text{ (because } \mathrm{cl}^{\mathcal{M}}(AZ) \cap Z = Z)\\
&\geq d'(\mathrm{cl}^{\mathcal{M}}(AZ)) - d'(Z) \text{ (because } Z' \leq \mathcal{M}')\\
&\geq d'(AZ) - d'(Z)\\
&= d'(A/Z)
\end{aligned}$$

We get $d(A/Z) \geq d'(A/Z)$. Proceeding analogously we get $d'(A/Z) \geq d(A/Z)$. We then obtain $d(A/Z) = d'(A/Z)$, concluding the proof of the claim. □

Before proving the infinite case in full generality we need to prove the following weaker result, using an extra assumption.

Claim. *Let Z be possibly infinite, but Z' obtained from Z by just adding relations. Then $PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}')$.*

Proof. Let A be a finite subset of $\mathcal{M}$. Again we have

$$\delta(A) - \delta(A \cap Z) = \delta'(A) - \delta'(A \cap Z).$$

Also, all we need to prove is that $d(A/Z) = d'(A/Z)$.

Let Z_0 be a finite subset of Z such that we have $d(A/Z) = d(A/Z_0)$ and $d'(A/Z) = d'(A/Z_0)$, this can be done by choosing Z_0 large enough. Let $Z_1 = \mathrm{cl}_{\mathcal{M}'}(AZ_0) \cap Z$. Then $Z_1 \leq \mathcal{M}'$, also, because from $\mathcal{M}$ to $\mathcal{M}'$ we are just adding relations, we have that $Z_1 \leq \mathcal{M}$. In other words, if we could decrease the predimension value by taking an superset in $\mathcal{M}$, then the same superset would work as well in $\mathcal{M}'$ because we are just adding relations. In particular, for the same reason, for every set $B \subseteq \mathcal{M}$ we have

$$\mathrm{cl}_{\mathcal{M}}(B) \subseteq \mathrm{cl}_{\mathcal{M}'}(B).$$

Now we observe that $\mathrm{cl}_{\mathcal{M}'}(AZ_1) \cap Z = \mathrm{cl}_{\mathcal{M}}(AZ_1) \cap Z = Z_1$.

For this we just make the following computation:

$$\mathrm{cl}_{\mathcal{M}'}(AZ_1) \cap Z = \mathrm{cl}_{\mathcal{M}'}(A(\mathrm{cl}_{\mathcal{M}'}(AZ_0) \cap Z)) \cap Z = \mathrm{cl}_{\mathcal{M}'}(AZ_0) \cap Z = Z_1.$$

And finally we observe that

$$Z_1 \subseteq \mathrm{cl}_{\mathcal{M}}(AZ_1) \cap Z \subseteq \mathrm{cl}_{\mathcal{M}'}(AZ_1) \cap Z = Z_1.$$

Notice that we still have $d(A/Z) = d(A/Z_1)$ and $d'(A/Z) = d'(A/Z_1)$ as $Z_0 \subseteq Z_1 \subseteq Z$.

Now we are going to prove that we have $d(A/Z) = d'(A/Z)$ by proving the two inequalities separately.

$$
\begin{aligned}
& d(A/Z) = d(A/Z_1) \\
& = d(AZ_1) - d(Z_1) \\
& = \delta(\mathrm{cl}_{\mathcal{M}}(AZ_1)) - \delta(Z_1) \text{ (because } Z_1 \leq \mathcal{M}) \\
& = \delta'(\mathrm{cl}_{\mathcal{M}}(AZ_1)) - \delta'(Z_1) \text{ (because } \mathrm{cl}_{\mathcal{M}}(AZ_1) \cap Z = Z_1) \\
& \geq d'(AZ_1) - d'(Z_1) \text{ (because } Z_1 \leq \mathcal{M}') \\
& = d'(A/Z_1) \\
& = d'(A/Z)
\end{aligned}
$$

For the other direction we have

$$
\begin{aligned}
& d(A/Z) = d(A/Z_1) \\
& = d(AZ_1) - d(Z_1) \\
& \leq \delta(\mathrm{cl}_{\mathcal{M}'}(AZ_1)) - \delta(Z_1) \text{ (because } Z_1 \leq \mathcal{M}) \\
& = \delta'(\mathrm{cl}_{\mathcal{M}'}(AZ_1)) - \delta'(Z_1) \text{ (because } \mathrm{cl}_{\mathcal{M}'}(AZ_1) \cap Z = Z_1) \\
& = d'(AZ_1) - d'(Z_1) \text{ (because } Z_1 \leq \mathcal{M}') \\
& = d'(A/Z_1) \\
& = d'(A/Z).
\end{aligned}
$$

Thus $d(A/Z) = d'(A/Z)$ for every finite $A \subseteq \mathcal{M}$. This proves that $PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}')$, proving the claim. $\square$

Now we need to remove the extra assumption made in the previous claim in order to prove the full lemma.

For this we need to consider an intermediate structure Z'' with the same underlying set as Z and Z', with relations given by $R_i^{Z''} = R_i^Z \cap R_i^{Z'}$. Notice that $Z'' \in \bar{\mathcal{C}}_f$ because we are just removing relations. Let $\mathcal{M}''$ be obtained from $\mathcal{M}$ by replacing Z by Z''.

From Z'' to Z we are just adding relations, thus by the previous claim we have $PG_{Z''}(\mathcal{M}'') = PG_Z(\mathcal{M})$. But from Z'' to Z' are also just adding relations, so again by the previous claim we have $PG_{Z''}(\mathcal{M}'') = PG_{Z'}(\mathcal{M}')$. Finally we can conclude that $PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}')$, proving the full lemma. □

Corollary 5.4.4. *Let f be as in First Changing Lemma. Let $(\mathcal{C}_f, \leq)$ be as in Proposition 5.1.14 and let $\mathcal{M} \in \bar{\mathcal{C}}_f$. Let $Z \leq \mathcal{M}$ and $Z' \in \bar{\mathcal{C}}_f$ be a structure with the same underlying set as Z. Let $\mathcal{M}'$ be obtained from $\mathcal{M}$ by replacing Z by Z'. Then $Z' \leq \mathcal{M}' \in \bar{\mathcal{C}}_f$ and*

$$\text{If } \mathrm{cl}^{d_{Z'}}(\emptyset) = Z' \text{ then } PG_Z(\mathcal{M}) = PG(\mathcal{M}').$$

Proof. Let A be a finite subset of $\mathcal{M}'$. We have that $\mathrm{cl}^{d_{Z'}}(\emptyset) = Z'$ and $Z' \leq \mathcal{M}'$. Thus $Z' = \mathrm{cl}^{d_{Z'}}(\emptyset) = \mathrm{cl}^{d_{\mathcal{M}'}}(\emptyset) \cap Z' \subseteq \mathrm{cl}^{d_{\mathcal{M}'}}(A)$. So we have $\mathrm{cl}^{d_{\mathcal{M}'}}(A) = \mathrm{cl}^{d_{\mathcal{M}'}}(AZ')$, this means that $PG_{Z'}(\mathcal{M}') = PG(\mathcal{M}')$. But by the Fourth Changing Lemma we have $PG_Z(\mathcal{M}) = PG_{Z'}(\mathcal{M}')$, thus $PG_Z(\mathcal{M}) = PG(\mathcal{M}')$. □

Now we can say something about the localization in $PG(\mathcal{M}_n)$. More precisely we show that localizing over a finite subset does not change the isomorphism type of the pregeometry.

Theorem 5.4.5. *Let Z be a finite subset of $\mathcal{M}_n$. Then*

$$PG_Z(\mathcal{M}_n) \simeq PG(\mathcal{M}_n).$$

Proof. Let us first assume that $Z \leq \mathcal{M}_n$. Let Z' be a structure in $\mathcal{C}_n$ with the same underlying set as Z and $\delta(Z') = 0$. This can be done: take for example $R_n^{Z'} = \{(c, \cdots, c) : c \in Z'\}$. Let $\mathcal{M}'_n$ be obtained from $\mathcal{M}_n$ by replacing Z by Z'. By the Third Changing Lemma we have $\mathcal{M}_n \simeq \mathcal{M}'_n$, in particular $PG(\mathcal{M}_n) \simeq PG(\mathcal{M}'_n)$. But as $d_{Z'}(Z') = \delta(Z') = 0$, by the Corollary 5.4.4 of the Fourth Changing Lemma we have $PG_Z(\mathcal{M}_n) = PG(\mathcal{M}'_n)$, thus $PG_Z(\mathcal{M}_n) \simeq PG(\mathcal{M}_n)$.

Now if $Z \not\leq \mathcal{M}_n$, we consider $Y = \mathrm{cl}_{\mathcal{M}_n}(Z)$ and we get $PG_Y(\mathcal{M}_n) \simeq PG(\mathcal{M}_n)$. But as $\mathrm{cl}^{d_{\mathcal{M}_n}}(Z) = \mathrm{cl}^{d_{\mathcal{M}_n}}(Y)$ we get $PG_Y(\mathcal{M}_n) = PG_Z(\mathcal{M}_n)$. Thus we still have $PG_Z(\mathcal{M}_n) \simeq PG(\mathcal{M}_n)$. □

Remark 5.4.6. We see that in these structures, localizing over a finite subset does not change the isomorphism type. The only obstruction in order to generalize this result to structures $\mathcal{M}_f$ instead of $\mathcal{M}_n$ is the possibility that in $\mathcal{C}_f$ there are no sets with predimension value zero on all sizes, of course this is only a problem if the predimension involves weights (and for similar reasons, if we work with unordered tuples).

In respect to localizing over infinite subsets there are things to be said, but we need to impose restrictions on the infinite subset if we want to obtain the same result.

We finalize this section with a consequence of the last theorem.

Theorem 5.4.7. *Let $m \neq n$ be natural numbers $\neq 0$. Then $PG(\mathcal{M}_m)$ and $PG(\mathcal{M}_n)$ are not locally isomorphic.*

Proof. By last theorem, if they are locally isomorphic they must be isomorphic, which we know that is not true by Theorem 5.3.3. Thus they are not locally isomorphic. □

5.5 A generic model of pregeometries

The motivation for this section was an attempt to prove that $PG(\mathcal{M}_n)$ and $PG(\mathcal{M}_m)$ are isomorphic for $m, n \geq 3$. The idea was to try to recognize both pregeometries as a generic model of the same class of pregeometries and use the uniqueness of the generic model up to isomorphism: this idea was motivated by the fact that the isomorphism types of finite subpregeometries are the same in both pregeometries. Of course this idea does not work as we have seen in Theorem 5.3.3 that $PG(\mathcal{M}_n)$ and $PG(\mathcal{M}_m)$ are not isomorphic. However we can in fact see $PG(\mathcal{M}_n)$ as a generic model of a class of pregeometries, but this class changes with n.

Definition 5.5.1. Let n be a natural number greater than zero and let $(\mathcal{C}_n, \leq_n)$ be the usual amalgamation class corresponding to arity n. Here we use the notation $\leq_n$ instead of $\leq$ to distinguish self-sufficiency corresponding to different arities. We now consider $(\mathcal{C}_n, \leq_n)$ as a category where the objects are the elements of $\mathcal{C}_n$ and where the morphisms between objects $A, B \in \mathcal{C}_n$ are the strong embeddings between the A and B, more precisely the embeddings $f : A \to B$ such that $f(A) \leq_n B$.

Now we apply the forgetful functor and we obtain a class of pregeometries $(P_n, \preceq_n)$. We forget the structure and remember only the associated pregeometries. More precisely, given two finite pregeometries $A \subseteq B$ we say that $A \preceq_n B$ if and only if there are structures $\widetilde{A}, \widetilde{B} \in \mathcal{C}_n$ such that $PG(\widetilde{A}) = A$ and $PG(\widetilde{B}) = B$ and $\widetilde{A} \leq_n \widetilde{B}$. The objects are given by $P_n = \{A : A \text{ is a finite pregeometry and } \emptyset \preceq_n A\}$. Also, we can see the category $(P_n, \preceq_n)$ as a class of relational structures in the language LPI of pregeometries, the morphisms are embeddings in this language.

Remark 5.5.2. P_n is closed under isomorphism, however not closed under substructures. Also $\preceq_n$ is invariant under isomorphism, that is, if $f : B \to B'$ is an isomorphism of pregeometries and $A \subseteq B$ then $A \preceq_n B$ if and only if $f(A) \preceq_n f(B)$. This means that this class almost fits in the general framework of Chapter 4, apart from the fact that P_n is not closed under substructures.

Proposition 5.5.3. $P_3 \subsetneq P_4$.

Proof. If $A \in P_3$ then there is a structure $\widetilde{A} \in \mathcal{C}_3$ with $PG(\widetilde{A}) = A$. Now we change the structure $\widetilde{A}$ to $\widehat{A}$ by replacing each relation (a, b, c) by the relation (a, b, c, c). Notice that $\widehat{A} \in \mathcal{C}_4$ and $PG(\widehat{A}) = PG(\widetilde{A}) = A$, thus $A \in P_4$. We have proved that $P_3 \subseteq P_4$.

However this is a proper inclusion. Consider a structure $B \in \mathcal{C}_4$ consisting of 4 distinct points a, b, c, d and only one relation (a, b, c, d). The pregeometry associated to B can be described by saying that for $X \subseteq B$ and $|X| \leq 3$ we have $d_B(X) = |X|$ and $d_B(B) = 3$. We have $PG(B) \in P_4$. However, this pregeometry is not in P_3 because there is no structure in $\mathcal{C}_3$ matching this pregeometry. In fact, suppose that there is a structure $B' \in \mathcal{C}_3$ with $PG(B') = PG(B)$. We have $d_{B'}(B') = \delta(B') = 3$, thus there is exactly one relation in B', say it is (x, y, z) with $x, y, z \in B'$. We would have $d_{B'}(\{x, y, z\}) \leq \delta(\{x, y, z\}) = |\{x, y, z\}| - 1 < |\{x, y, z\}|$ which does not happen in $PG(B)$. We have proved that $P_3 \subsetneq P_4$. □

Proposition 5.5.4. *In the class $(P_n, \preceq_n)$ the relation $\preceq_n$ is transitive.*

Proof. Suppose that we have $A \preceq_n B$ and $B \preceq_n C$. Suppose that $\widehat{A}, \widehat{B}, \widetilde{B}, \widetilde{C} \in \mathcal{C}_n$ such that $PG(\widehat{A}) = A$, $PG(\widehat{B}) = PG(\widetilde{B}) = B$, $PG(\widetilde{C}) = C$ and such that $\widehat{A} \preceq_n \widehat{B}$ and $\widetilde{B} \preceq_n \widetilde{C}$.

Now we construct a structure $\widehat{C}$ obtained from $\widetilde{C}$ by replacing $\widetilde{B}$ by $\widehat{B}$. By the First Changing Lemma we have $\widehat{B} \leq_n \widehat{C} \in \mathcal{C}_n$ and as $PG(\widetilde{B}) = PG(\widehat{B})$ then by the Second Changing Lemma we have $PG(\widehat{C}) = PG(\widetilde{C}) = C$. Now we have $\widehat{A} \leq_n \widehat{B}$ and $\widehat{B} \leq_n \widehat{C}$. Thus by transitivity of $\leq_n$ we get $\widehat{A} \leq_n \widehat{C}$ with $PG(\widehat{A}) = A$ and $PG(\widehat{C}) = C$, thus $A \preceq_n C$. □

Proposition 5.5.5. *The class $(P_n, \preceq_n)$ is an amalgamation class.*

Proof. Suppose that we have $A_0 \preceq_n A_1 \in P_n$ and $A_0 \preceq_n A_2 \in P_n$. Let $\widehat{A}_0 \leq_n \widehat{A}_2 \in \mathcal{C}_n$ and $\widetilde{A}_0 \leq_n \widetilde{A}_2 \in \mathcal{C}_n$ be the corresponding lifts to $(\mathcal{C}_n, \leq_n)$. Let $\widehat{A}_2$ be the structure obtained from $\widetilde{A}_2$ by replacing $\widetilde{A}_0$ by $\widehat{A}_0$ then by the First Changing Lemma we have $\widehat{A}_0 \leq_n \widehat{A}_2 \in \mathcal{C}_n$, moreover, as $PG(\widetilde{A}_0) = PG(\widehat{A}_0)$ then by the Second Changing Lemma we have $PG(\widehat{A}_2) = PG(\widetilde{A}_2) = A_2$. Now we use the fact that $(\mathcal{C}_n, \leq_n)$ is an amalgamation class and the fact that $\widehat{A}_0 \leq_n \widehat{A}_1 \in \mathcal{C}_n$ and $\widehat{A}_0 \leq_n \widehat{A}_2 \in \mathcal{C}_n$ to construct a structure $C \in \mathcal{C}_n$ and embeddings $f_1 : \widehat{A}_1 \to C$ and $f_2 : \widehat{A}_2 \to C$ fixing elements of $\widehat{A}_0$ such that $f_1(\widehat{A}_1) \leq_n C$ and $f_2(\widehat{A}_2) \leq_n C$. We have $PG(f_i(\widehat{A}_i)) \preceq_n PG(C)$, that is, $f_i(A_i) \preceq_n PG(C) \in P_n$. In other words, $f_i : A_i \to PG(C)$ are strong embeddings of pregeometries fixing elements of A_0 and $PG(C) \in P_n$. This proves that $(P_n, \preceq_n)$ is an amalgamation class. $\square$

We would like to define a notion of generic model for the class $(P_n, \preceq_n)$, however P_n is not closed under substructures so we cannot apply the general results given in Chapter 4. This is not a real barrier, we just need to adapt the definition of generic model to this context.

Definition 5.5.6. Let $(\mathcal{C}, \leq)$ be a class of finite relational structures in a countable language L with countably many isomorphism types. Let $\leq$ a binary relation such that $A \leq B$ implies that A is a substructure of B. Assume that $\mathcal{C}$ is closed under isomorphism (but not necessarily under substructures) and that $\leq$ is invariant under isomorphism. We say that an L-structure $\mathcal{M}$ is a generic model for $(\mathcal{C}, \leq)$ if $\mathcal{M}$ is countable and satisfies $FC1$ and $FC2$ where:

$FC1$ There is a chain $M_0 \leq M_1 \leq M_2 \leq \cdots$ with $M_i \in \mathcal{C}$ and $\bigcup_{i\in\mathbb{N}} M_i = \mathcal{M}$.

$FC2$ (Extension property) If $A \leq M_i$ and $A \leq B \in \mathcal{C}$ then there are $j \in \mathbb{N}$ and an embedding $f : B \to M_j$ such that $f_{|A} = Id_{|A}$ and $f(B) \leq M_j$.

Definition 5.5.7. Let $\mathcal{M}$ be a generic model for a class $(\mathcal{C}, \leq)$ as in the above definition, with respect to a chain $M_0 \leq M_1 \leq M_2 \leq \cdots$ Let A be a finite substructure of $\mathcal{M}$ such that $A \in \mathcal{C}$. We say that $A \leq \mathcal{M}$ if and only if $A \leq M_i$ for some $i \in \mathbb{N}$. Notice that, at least apparently, the set of closed sets of $\mathcal{M}$ depends not only on the structure of $\mathcal{M}$ but also on the choice of the chain.

Now we prove the existence and uniqueness up to isomorphism of the generic model.

Proposition 5.5.8. *Let $(\mathcal{C}, \leq)$ be as in definition 5.5.6. Assume that $\emptyset \leq A$ for all $A \in \mathcal{C}$, that $\leq$ is transitive and that $(\mathcal{C}, \leq)$ is an amalgamation class. Then there is a generic model for $(\mathcal{C}, \leq)$ in the sense of definition 5.5.6 and it is unique up to isomorphism.*

Proof. The proof of the existence of a generic model is exactly the same as in Theorem 4.1.12. The proof of uniqueness is also very similar, but we give here a detailed proof.

Let $\mathcal{M}$ and $\mathcal{N}$ be two generic models of $(\mathcal{C}, \leq)$ with respect to chains $M_0 \leq M_1 \leq M_2 \leq \cdots$ and $N_0 \leq N_1 \leq N_2 \leq \cdots$ respectively. We may assume that $M_0 = \emptyset = N_0$. Let $A_0 = M_0 = \emptyset = N_0 = B_0$, then trivially we have an isomorphism $f_0 : A_0 \to B_0$ such that $M_0 \subseteq A_0 \leq \mathcal{M}$ and $N_0 \subseteq B_0 \leq \mathcal{N}$. Suppose we have constructed a chain of isomorphisms $f_i : A_i \to B_i$ with $i \leq k \in \mathbb{N}$ such that f_j extends f_i if $i \leq j \leq k$ and with $M_i \subseteq A_i \leq \mathcal{M}$ and $N_i \subseteq B_i \leq \mathcal{N}$. We want to construct $f_{k+1} : A_{k+1} \to B_{k+1}$ in the same way.

We have $A_k \leq \mathcal{M}$ so there is $j \in \mathbb{N}$ such that $A_k \leq M_j$. We can assume that $j \geq k+1$ by transitivity of $\leq$, thus $M_j \supseteq M_{k+1}$. There is a structure Q_k and an isomorphism $g_1 : M_j \to Q_k$ extending f_k. We have $A_k \leq M_j$ thus $B_k \leq Q_k \in \mathcal{C}$ because $\leq$ is invariant under isomorphism and $\mathcal{C}$ is closed under isomorphism.

Now we have $B_k \leq \mathcal{N}$ and $B_k \leq Q_k \in \mathcal{C}$ so we can apply the extension property of $\mathcal{N}$ to construct an embedding $g_2 : Q_k \to \mathcal{N}$ fixing elements of B_k and with $g_2(Q_k) \leq \mathcal{N}$. Define $h_k : M_j \to g_2(Q_k)$ as $h_k := g_2 \circ g_1$. Then h_k is an isomorphism extending f_k and with $M_{k+1} \leq M_j \leq \mathcal{M}$ and $g_2(Q_k) \leq \mathcal{N}$.

Now we proceed in the same manner in the other direction to further extend this isomorphism. We have $h_k(M_j) = g_2(Q_k) \leq \mathcal{N}$ so there is $i \in \mathbb{N}$ such that $h_k(M_j) \leq N_i$, again we may assume that $i \geq k+1$. Let $B_{k+1} := N_i \supseteq N_{k+1}$. There is a structure $Q'_k \in \mathcal{C}$ and an isomorphism $g'_1 : B_{k+1} \to Q'_k$ extending h_k^{-1}. Now we have $h_k(M_j) \leq B_{k+1}$ thus $M_j \leq g'_1(B_{k+1}) = Q'_k \in \mathcal{C}$. We have $M_j \leq \mathcal{M}$ and $M_j \leq Q'_k \in \mathcal{C}$ so we can apply the extension property of $\mathcal{M}$ to build an isomorphism $g'_2 : Q'_k \to \mathcal{M}$ fixing elements of M_j and such $g'_2(Q'_k) \leq \mathcal{M}$. Let $l_k : B_{k+1} \to g'_2(Q'_k)$ be defined by $l_k := g'_2 \circ g'_1$ and define $A_{k+1} := g'_2(Q'_k)$, finally put $f_{k+1} := l_k^{-1}$. Then $f_{k+1} : A_{k+1} \to B_{k+1}$ extends $f_k : A_k \to B_k$ and $M_{k+1} \subseteq A_{k+1} \leq \mathcal{M}$ and $N_{k+1} \subseteq B_{k+1} \leq \mathcal{N}$. Finally $f : \mathcal{M} \to \mathcal{N}$ defined by $f := \bigcup_{i\in\mathbb{N}} f_i$ is an isomorphism. □

Now we can prove the main result of this section.

Theorem 5.5.9. *There is a generic model $\mathcal{P}_n$ (unique up to isomorphism) for the class $(P_n, \preceq_n)$ and*

$$\mathcal{P}_n \simeq PG(\mathcal{M}_n)$$

where $\mathcal{M}_n$ is the generic model for the class $(\mathcal{C}_n, \leq_n)$.

Proof. There is a unique generic model $\mathcal{P}_n$ of the class $(P_n, \preceq_n)$ because we have seen that this class satisfies the conditions of last proposition. To prove that $\mathcal{P}_n \simeq PG(\mathcal{M}_n)$ we just need to prove that $PG(\mathcal{M}_n)$ is also a generic model for the class $(P_n, \preceq_n)$.

Let $A_0 \leq_n A_1 \leq_n A_2 \leq_n \cdots$ with $A_i \in \mathcal{C}_n$ and $\mathcal{M}_n = \bigcup_{i\in\mathbb{N}} A_i$. In particular we have $PG(A_0) \preceq_n PG(A_1) \preceq_n PG(A_2) \preceq_n \cdots$ with $PG(A_i) \in P_n$ and $PG(\mathcal{M}_n) = \bigcup_{i\in\mathbb{N}} PG(A_i)$. We want to prove that $PG(\mathcal{M}_n)$ is a generic model for $(P_n, \preceq_n)$ with respect to this chain. It remains to prove the extension property.

Suppose that $A \preceq_n PG(A_i)$ and $A \preceq_n B \in P_n$. We want to prove that there exist $j \in \mathbb{N}$ and an embedding of pregeometries $f : B \to PG(\mathcal{M}_n)$ fixing the elements of A and with $f(B) \preceq_n PG(A_j)$. We have $A \preceq_n PG(A_i)$, so there are structures $A', A'_i \in \mathcal{C}_n$ such that $PG(A') = A$ and $PG(A'_i) = PG(A_i)$ and such that $A' \leq_n A'_i$. Let $\mathcal{M}'_n$ be the structure obtained from $\mathcal{M}_n$ by replacing A_i by A'_i. Then by the First Changing Lemma we have $A'_i \leq_n \mathcal{M}'_n \in \bar{\mathcal{C}}_n$, by the Second Changing Lemma we have $PG(\mathcal{M}'_n) = PG(\mathcal{M}_n)$ and by the Third Changing Lemma we have $\mathcal{M}'_n \simeq \mathcal{M}_n$.

We have $A \preceq_n B \in P_n$, so there are structures $\widetilde{A}, \widetilde{B} \in \mathcal{C}_n$ such that $PG(\widetilde{A}) = A$ and $PG(\widetilde{B}) = B$ and $\widetilde{A} \leq_n \widetilde{B} \in \mathcal{C}_n$. Now we construct B' obtained from $\widetilde{B}$ by replacing $\widetilde{A}$ by A', by the the First Changing Lemma we get $A' \leq_n B' \in \mathcal{C}_n$, moreover, as $PG(A') = A = PG(\widetilde{A})$ we can apply the Second Changing Lemma and we get $PG(B') = PG(\widetilde{B}) = B$.

Now we have $A' \leq_n A'_i \leq_n \mathcal{M}'_n$ and $A' \leq_n B' \in \mathcal{C}_n$. But we have $\mathcal{M}'_n \simeq \mathcal{M}_n$ so $\mathcal{M}'_n$ satisfies the extension property. We can apply the extension property to construct an embedding $f : B' \to \mathcal{M}'_n$ fixing the elements of A' and such that $f(B') \leq_n \mathcal{M}'_n$. Now we apply the forgetful functor to this embedding and obtain an embedding $f : B \to PG(\mathcal{M}'_n) = PG(\mathcal{M}_n)$ of pregeometries. It remains to prove that $f(B) \preceq_n PG(\mathcal{M}_n)$.

For $j \geq i$ let A'_j be the substructure of $\mathcal{M}'_n$ with the same underlying set as A_j. Notice that $A_i \leq_n A_{i+1}$ and $PG(A'_i) = PG(A_i)$ imply by the first and second Changing Lemmas that $A'_i \leq_n A'_{i+1}$ and $PG(A'_{i+1}) = PG(A_{i+1})$. Then in the next step $A_{i+1} \leq_n A_{i+2}$ and $PG(A'_{i+1}) = PG(A_{i+1})$ imply that $A'_{i+1} \leq_n A'_{i+2}$ and $PG(A'_{i+2}) = PG(A_{i+2})$... We repeat this procedure a countable number of times and we obtain a chain $A'_i \leq_n A'_{i+1} \leq_n A'_{i+2} \leq_n \cdots \leq_n A'_j \leq_n \cdots$ with $PG(A'_j) = PG(A_j)$ and $A'_j \leq_n \mathcal{M}'_n$ because we have $\mathcal{M}'_n = \bigcup_{j \geq i} A'_j$.

Finally we have that $f(B') \leq_n \mathcal{M}'_n \Rightarrow f(B') \leq_n A'_j$ (for some $j \geq i$) $\Rightarrow$ $f(B) = PG(f(B')) \preceq_n PG(A'_j) = PG(A_j)$. Thus $f(B) \preceq_n PG(\mathcal{M}_n)$ as desired. We proved the extension property for $PG(\mathcal{M}_n)$ so we can conclude that $PG(\mathcal{M}_n) \simeq \mathcal{P}_n$. $\square$

Chapter 6

After the collapse

In this chapter, following Hrushovski, we focus on the class $(\mathcal{C}_3, \leq_3)$, obtaining a family of subclasses that are still amalgamation classes. It turns out that the generic model associated to each one of this subclasses is a strongly minimal structure. Then we proceed to compare the pregeometries arising from these strongly minimal structures, partially answering a question posed by Hrushovski.

6.1 The new strongly minimal structures

In his paper [4], Hrushovski constructs a family of new strongly minimal structures refuting a classification conjecture of Zilber. In this section we describe his method.

Definition 6.1.1. In a strongly minimal structure the algebraic closure is a pregeometry. Thus to each strongly minimal structure we can associate a geometry. We say that two strongly minimal structures are geometrically equivalent to each other if their associated geometries are locally isomorphic in the sense of definition 2.1.6.

There was a conjecture by Zilber saying that up to geometric equivalence there are only three types of strongly minimal structures: pure sets, vector spaces and algebraically closed fields. Hrushovski constructed a family of new strongly minimal structures that are not geometrically equivalent to any of these three types, refuting the conjecture. However we prove here that the new strongly minimal structures constructed are geometrically equivalent to each other. In fact we prove that actually the pregeometries associated to the new strongly minimal structures are isomorphic, proving that the entire family of new strongly minimal structures correspond to a single counterexample. Now we describe the construction. But first we need some definitions.

Definition 6.1.2. Let $A \leq B \in \mathcal{C}_3$. We say that the extension $A \leq B$ is simply algebraic if $\delta(A) = \delta(B)$ and $A \subsetneq B' \subsetneq B$ implies $\delta(B') > \delta(A)$.

We say that $A \leq B$ is minimally simply algebraic if it is simply algebraic and for every $A' \subsetneq A$ we have that $A' \leq (B \setminus A) \cup A'$ is not simply algebraic.

Note that the fact that $A' \leq (B \setminus A) \cup A'$ follows from the fact that whenever we have $A \leq B$ and $X \subseteq B$ then $A \cap X \leq X$. Here we just take $X = (B \setminus A) \cup A'$.

Now we define a family of subclasses of $\mathcal{C}_3$.

Definition 6.1.3. Let μ be a function that associates an integer to each minimally simply algebraic $A \leq B$ (with $A \neq B$), such that $\mu(A \leq B)$ only depends on the isomorphism type of $A \leq B$ and such that $\mu(A \leq B) \geq \delta(A)$.

We define the subclass $\mathcal{C}_\mu$ by saying that $M \in \mathcal{C}_\mu$ if $M \in \mathcal{C}_3$ and if whenever we have disjoint $E, F_i \subseteq M$ with $i \in \{1, \cdots, n\}$, $F_i \neq \emptyset$, $E \leq EF_i$ minimally simply algebraic and isomorphisms $f_i : EF_i \to EF_1$ fixing the elements of E then $n \leq \mu(E \leq EF_i)$. Note that as usual EF_i means $E \cup F_i$.

Now we summarize in the next theorem some results by Hrushovski.

Theorem 6.1.4. *The class $(\mathcal{C}_\mu, \leq)$ has the amalgamation property. There is a generic model $\mathcal{M}_\mu$ associated to this class in the usual sense. $\mathcal{M}_\mu$ is saturated and strongly minimal. Moreover it is a counterexample to Zilber's conjecture. The algebraic closure in $\mathcal{M}_\mu$ equals the d-closure, that is $\mathrm{cl}^{d_{\mathcal{M}_\mu}}$.*

6.2 The pregeometries after the collapse

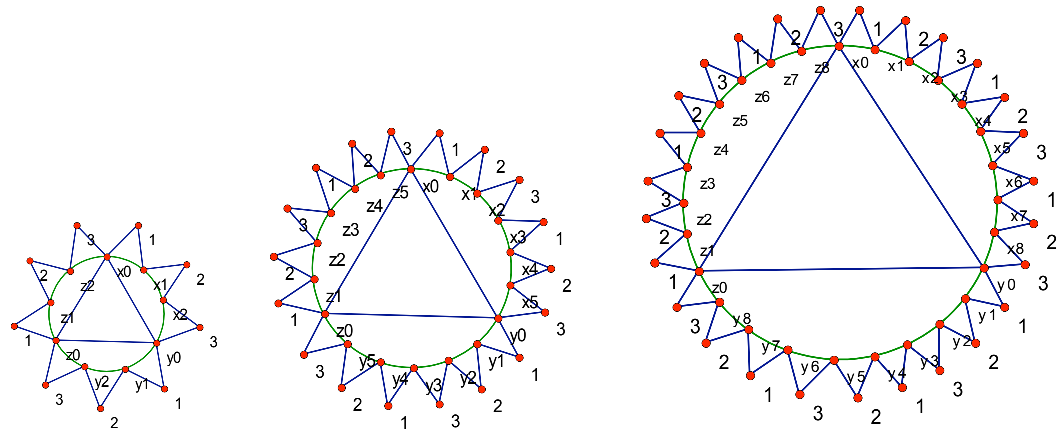
This section is entirely dedicated to proving the following theorem, answering the question of Hrushovski.

Theorem 6.2.1. *Assume that $\mu \geq 1$ for all minimally simply algebraic extensions. Then $PG(\mathcal{M}_\mu) \simeq PG(\mathcal{M}_3)$.*

In order to prove the theorem we will need to prove some technical lemmas and we will also use several of the main results from the last chapter.

Definition 6.2.2. Let $B \in \mathcal{C}_3$ and $(1,2,3) \in R_3^B$. (We do not require $1,2$ and 3 to be distinct.)

Consider the structures $D^i_{(1,2,3)}(B)$ for $i \in \mathbb{N} \setminus \{0\}$, obtained by replacing the relation $(1,2,3)$ by the corresponding picture D^i as below, where all the points in the picture apart from $1,2$ and 3 are new distinct points that are not in B. We show the pictures for $i \in \{1,2,3\}$, for bigger i we generalize in the natural way. Notice that the labelling of the vertices in the pictures might not be very clear, to avoid mistakes notice that the blue central triangle has vertices x_0, y_0 and z_0. The triangles in the picture mean that in the new structure a relation between the vertices hold. A formal definition of the structures $D^i_{(1,2,3)}(B)$ is given after the pictures.



To be more precise $D^i_{(1,2,3)}(B)$ is the structure with points

$$B \cup \{x_0, y_0, z_0, \cdots, x_{3i-1}, y_{3i-1}, z_{3i-1}\}$$

and with relations:

$$R_3^B \setminus \{(1,2,3)\} \cup \{(x_0, y_0, z_0)\}$$

together with

$$\{(x_j, x_{j+1}, \alpha_{j+1}) : j \in \{0, 1, \cdots, 3i-2\}\} \cup \{(x_{3i-1}, y_0, 3)\}$$

and

$$\{(y_j, y_{j+1}, \alpha_{j+1}) : j \in \{0, 1, \cdots, 3i-2\}\} \cup \{(y_{3i-1}, z_0, 3)\}$$

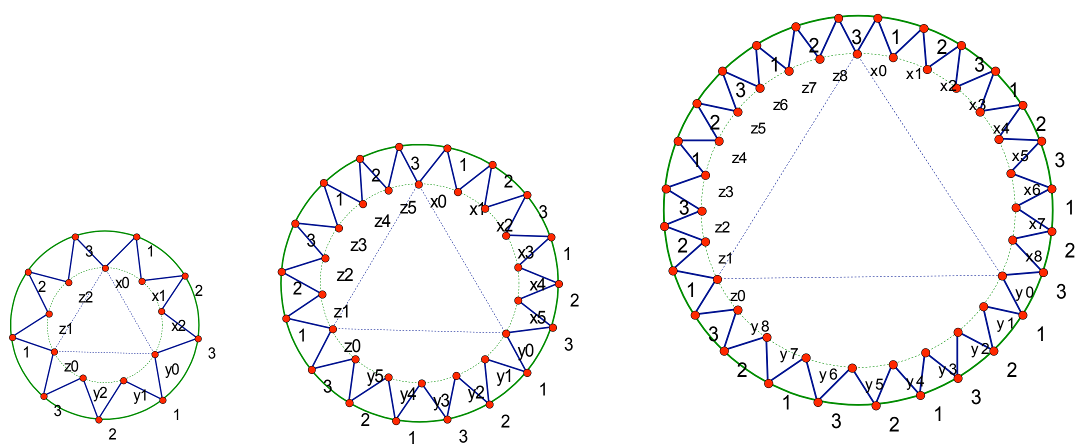
and

$$\{(z_j, z_{j+1}, \alpha_{j+1}) : j \in \{0, 1, \cdots, 3i-2\}\} \cup \{(z_{3i-1}, x_0, 3)\}$$

where $\alpha_j \in \{1, 2, 3\}$ is such that $\alpha_j \equiv j (\mathrm{MOD}\, 3)$.

We write $D(B)$ for any structure obtained by replacing each relation from a chosen set of relations by one of the pictures, eventually using different pictures (that is, different values of i) for different relations. The points added for the replacement of each relation are distinct. This notation is ambiguous because $D(B)$ depends on choice of the set of relations to replace and on the pictures used for the replacement of each relation. To be more precise we write $D(B)$ for any structure obtained recursively in the following way: we define $D_0 := B$ then we can construct $D_{n+1} := D^{i_n}_{(a_n,b_n,c_n)}(D_n)$ where i_n is some arbitrary positive natural number and (a_n, b_n, c_n) is an arbitrary relation in $R_3^{D_n} \cap R_3^B$. Instead of making the notation unnecessarily more heavy we choose to just write $D(B)$ instead of keeping track of all the choices involved in the construction.

Consider also the structures $\widehat{D}^i_{(1,2,3)}(B)$ for $i \in \mathbb{N} \setminus \{0\}$, obtained by adding (not replacing!) to the relation $(1, 2, 3)$ the corresponding picture $\widehat{D}^i$ as below. Again we only show $\widehat{D}^i$ for $i \in \{1, 2, 3\}$, for bigger i we need to generalize in the natural way. In order for us to compare $\widehat{D}^i_{(1,2,3)}(B)$ with $D^i_{(1,2,3)}(B)$ we assume that the new points introduced are the same in both cases. Notice that the dotted lines and dotted circle are not part of the picture, they are only there to facilitate the comparison between $\widehat{D}^i$ with D^i. A formal definition of the structures $\widehat{D}^i_{(1,2,3)}(B)$ is given after the pictures.



To be more precise the structure $\widehat{D}_{(1,2,3)}(B)$ is the structure with points

$$B \cup \{x_0, y_0, z_0, \cdots, x_{3i-1}, y_{3i-1}, z_{3i-1}\}$$

and with relations: R_3^B together with

$$\{(\alpha_j, \alpha_{j+1}, x_j) : j \in \{0, 1, \cdots, 3i-1\}\}$$

and

$$\{(\alpha_j, \alpha_{j+1}, y_j) : j \in \{0, 1, \cdots, 3i-1\}\}$$

and

$$\{(\alpha_j, \alpha_{j+1}, z_j) : j \in \{0, 1, \cdots, 3i-1\}\}$$

where $\alpha_j \in \{1, 2, 3\}$ is such that $\alpha_j \equiv j (\mathrm{MOD}\, 3)$.

Again we ambiguously write $\widehat{D}(B)$ if we add one of these pictures to each relation from a chosen set of relations. To be more precise we write $\widehat{D}(B)$ for any structure obtained recursively in the following way: we define $\widehat{D}_0 := B$ then we can construct $\widehat{D}_{n+1} := \widehat{D}^{i_n}_{(a_n,b_n,c_n)}(\widehat{D}_n)$ where i_n is some arbitrary positive natural number and (a_n, b_n, c_n) is an arbitrary relation in $R_3^{\widehat{D}_n} \cap R_3^B$ and such that $(a_n, b_n, c_n) \neq (a_k, b_k, c_k)$ for any $k < n$. Instead of making the notation unnecessarily more heavy we choose to just write $\widehat{D}(B)$ instead of keeping track of all the choices involved in the construction.

Lemma 6.2.3. *Let $B \in \mathcal{C}_3$. Then $\widehat{D}(B) \in \mathcal{C}_3$ and $B \leq \widehat{D}(B)$.*

Proof. We just need to observe that for each new relation we add, we also add one new specific point so $B \leq \widehat{D}(B)$ and $\emptyset \leq B \leq \widehat{D}(B)$ gives $\widehat{D}(B) \in \mathcal{C}_3$. □

Lemma 6.2.4. *Let $A \leq B \in \mathcal{C}_3$ and $D(B)$ be such that no relations inside A were replaced. Then $D(B) \in \mathcal{C}_3$ and $A \leq D(B)$. Note however that B is not a substructure of $D(B)$.*

Proof. Without loss of generality we can assume that $D(B) = D^i_{(1,2,3)}(B)$ for some relation $(1, 2, 3) \in R_3^B \setminus R_3^A$ and some i. If that is not the case we just need to apply this partial result step by step, once for each time we replace a relation and the full result will follow.

Let $B' := D(B) = D^i_{(1,2,3)}(B)$. Let $X \subseteq B$ and $X' \subseteq B' \setminus B$. If $\delta_{B'}(XX') < \delta_{B'}(X)$ then necessarily $X' = B' \setminus B$ because if X' is a proper subset of $B' \setminus B$ then when going from X to XX' inside B', we add at least as many points as relations: the reader can get convinced of this fact by analyzing the pictures. Also in this case we must have $\{1, 2, 3\} \subseteq X$. Then, in this case we have $\delta_{B'}(XX') = \delta_{B'}(X) - 1 = \delta_B(X)$.

If $\delta_{B'}(XX') \geq \delta_{B'}(X)$ then we still have $\delta_{B'}(XX') \geq \delta_{B'}(X) \geq \delta_B(X)$. Either way we always have $\delta_{B'}(XX') \geq \delta_B(X) \geq 0$ because $B \in \mathcal{C}_3$. In particular we have $\delta_{B'}(Y) \geq 0$ for all $Y \subseteq B'$. Thus $B' \in \mathcal{C}_3$.

Now we want to prove that $A \leq B'$. If not, then we would have $A \subseteq X \subseteq B$ and $X' \subseteq B' \setminus B$ with $\delta_{B'}(XX') < \delta_{B'}(A)$. But we have seen before that we have $\delta_{B'}(XX') \geq \delta_B(X)$, also we have $\delta_B(X) \geq \delta_B(A)$ because $A \leq B$ and finally we also have $\delta_B(A) = \delta_{B'}(A)$ because there are no changes inside A. Thus $\delta_{B'}(XX') \geq \delta_{B'}(A)$, which is a contradiction. $\square$

Lemma 6.2.5. *Let $B \in \mathcal{C}_3$. Then $B \preceq D(B)$. (Where by $B \preceq D(B)$ we just mean that $PG(B) \preceq PG(D(B))$ and where by $PG(B)$ we mean the pregeometry of the original structure on B and not the pregeometry of the induced substructure on B in $D(B)$.)*

Proof. By Lemma 6.2.3 we have $B \leq \widehat{D}(B)$. Thus in particular we have $PG(B) \preceq PG(\widehat{D}(B))$. So it is enough to prove that $PG(D(B)) = PG(\widehat{D}(B))$. Also without loss of generality we can assume that $D(B) = D^i_{(1,2,3)}(B)$ and $\widehat{D}(B) = \widehat{D}^i_{(1,2,3)}(B)$, because $\preceq$ is transitive.

Let $B' := D(B)$ and $\widehat{B} := \widehat{D}(B)$. Let X be a subset of B', thus of $\widehat{B}$ as B' and $\widehat{B}$ have the same underlying set. We want to prove that $d_{B'}(X) = d_{\widehat{B}}(X)$.

Let us adopt the notation $D^i(1,2,3) := D^i_{(1,2,3)}(\{1,2,3\})$ and $\widehat{D}^i(1,2,3) := \widehat{D}^i_{(1,2,3)}(\{1,2,3\})$ where $\{1,2,3\}$ is considered as a substructure of B.

Claim. *1) Let Y be a d-closed subset of B', that is $\mathrm{cl}^{d_{B'}}(Y) = Y$. If we have $|Y \cap D^i(1,2,3)| \geq 2$ then $D^i(1,2,3) \subseteq Y$.*

2) Let Y be a d-closed subset of $\widehat{B}$, that is $\mathrm{cl}^{d_{\widehat{B}}}(Y) = Y$. If we have $|Y \cap \widehat{D}^i(1,2,3)| \geq 2$ then $\widehat{D}^i(1,2,3) \subseteq Y$.

Proof. The proof is the same for both statements 1) and 2), first we prove 1). Fix two distinct points $a, b \in Y \cap D^i(1,2,3)$. Then it is easy to see that $\mathrm{cl}^{d_{B'}}(ab) \supseteq D^i(1,2,3)$, in fact, if we add the remaining points of $D^i(1,2,3)$ then we are adding $9i + |\{1,2,3\}| - 2 \leq 9i + 1$ points and at least $9i + 1$ relations. Thus $D^i(1,2,3) \subseteq \mathrm{cl}^{d_{B'}}(ab) \subseteq \mathrm{cl}^{d_{B'}}(Y) = Y$.

For statement 2) the proof is the same, we just need to justify that we have $\mathrm{cl}^{d_{\widehat{B}}}(ab) \supseteq \widehat{D}^i(1,2,3)$. In fact, if we add the remaining points of $\widehat{D}^i(1,2,3)$ then we are adding $9i + |\{1,2,3\}| - 2$ points but in order to justify that we are not adding more relations than this number we need to divide the argument in subcases: if $|\{1,2,3\}| = 3$ then we are adding $9i + 1$ relations because the relation $(1,2,3)$ is also added; if $|\{1,2,3\}| = 2$ and $|\{1,2,3\}| = \{a,b\}$ then the relation $(1,2,3)$ it is already in $\{a,b\}$ but all the remaining relations in $\widehat{D}^i(1,2,3)$ are new, thus we are adding exactly $9i$ relations; if $|\{1,2,3\}| = 2$ and $|\{1,2,3\}| \neq \{a,b\}$ then it might happen that one of the relations in $\widehat{D}^i(1,2,3)$ is already in $\{a,b\}$ but all the remaining ones plus the relation $(1,2,3)$ are new, thus we are adding at least $9i$ relations; finally if $|\{1,2,3\}| = 1$ then $\{a,b\}$ can only contain at most two of the relations in $\widehat{D}^i(1,2,3)$, possibly $(1,2,3)$ and another one, thus we are adding at least $9i - 1$ relations. In all cases the number of points added are less or equal than the number of relations added, thus $\mathrm{cl}^{d_{\widehat{B}}}(ab) \supseteq \widehat{D}^i(1,2,3)$. $\square$

Claim. *3) Let Y be a d-closed subset of B'. Then $\delta_{B'}(Y) = \delta_{\widehat{B}}(Y)$.*

4) Let Y be a d-closed subset of $\widehat{B}$. Then $\delta_{\widehat{B}}(Y) = \delta_{B'}(Y)$.

Proof. The proof is the same for both statements, so we just prove 3).

If $\{1,2,3\} \subseteq Y$ then clearly $D^i(1,2,3) \subseteq \mathrm{cl}^{d_{B'}}(\{1,2,3\}) \subseteq Y$ because Y is d-closed in B'. In particular it is easy to see that $\delta_{B'}(Y) = \delta_{\widehat{B}}(Y)$, this is just because $\delta_{B'}(D^i(1,2,3)) = \delta_{\widehat{B}}(\widehat{D}^i(1,2,3))$ and the relations that are not inside of this set are the same whether we look inside of Y as a subset of B' or Y as a subset of $\widehat{B}$.

If $\{1,2,3\} \nsubseteq Y$ then by 1) of the previous claim we have $|Y \cap D^i(1,2,3)| \leq 1$, otherwise we would have $D^i(1,2,3) \subseteq Y$. Clearly if $|Y \cap D^i(1,2,3)| = 0$ then we have $\delta_{B'}(Y) = \delta_{\widehat{B}}(Y)$. If $Y \cap D^i(1,2,3) = \{x\}$ then Y is of the form $Y = Y_0\{x\}$ with $Y_0 \subseteq B$ and $x \in D^i(1,2,3)$. If $x \in \{1,2,3\}$ then substructure induced in Y by B' and $\widehat{B}$ is the same because $\{1,2,3\} \nsubseteq Y$, thus in this case $\delta_{B'}(Y) = \delta_{\widehat{B}}(Y)$. If $x \in D^i(1,2,3) \setminus B$ then we have $\delta_{B'}(Y) = \delta_B(Y_0) + 1 = \delta_{\widehat{B}}(Y)$. In all cases we have $\delta_{B'}(Y) = \delta_{\widehat{B}}(Y)$. □

Let X be a subset of B'. Remember that B' and $\widehat{B}$ have the same underlying set and that all we need to prove is that $d_{B'}(X) = d_{\widehat{B}}(X)$. We prove the two inequalities separately:

$$d_{B'}(X) = \inf\{\delta_{B'}(Y) : X \subseteq Y \subseteq B'\}$$

$$= \inf\{\delta_{B'}(Y) : X \subseteq Y \subseteq B' \textit{ and } Y \textit{ is d-closed in } B'\}$$

$$= \inf\{\delta_{\widehat{B}}(Y) : X \subseteq Y \subseteq \widehat{B} \textit{ and } Y \textit{ is d-closed in } B'\} \text{ (by claim 3))}$$

$$\geq \inf\{\delta_{\widehat{B}}(Y) : X \subseteq Y \subseteq \widehat{B}\}$$

$$= d_{\widehat{B}}(X)$$

the other inequality is analogous,

$$
\begin{aligned}
d_{\widehat{B}}(X) &= \inf\{\delta_{\widehat{B}}(Y) : X \subseteq Y \subseteq \widehat{B}\} \\
&= \inf\{\delta_{\widehat{B}}(Y) : X \subseteq Y \subseteq \widehat{B} \textit{ and } Y \textit{ is d-closed in } \widehat{B}\} \\
&= \inf\{\delta_{B'}(Y) : X \subseteq Y \subseteq B' \textit{ and } Y \textit{ is d-closed in } \widehat{B}\} \text{ (by claim 4))} \\
&\geq \inf\{\delta_{B'}(Y) : X \subseteq Y \subseteq B'\} \\
&= d_{B'}(X)
\end{aligned}
$$

We have that $d_{B'}(X) = d_{\widehat{B}}(X)$. This proves that $PG(B') = PG(\widehat{B})$, concluding the proof of our lemma. □

Next we prove the main technical lemma of this chapter.

Lemma 6.2.6. *(Hard changing lemma) Let μ be such that $\mu \geq 1$ for all minimally simply algebraic extensions. Let $A \leq B \in \mathcal{C}_3$ and $A \in \mathcal{C}_\mu$. Then there exists $B' \in \mathcal{C}_\mu$ such that $A \leq B'$ and $B \preceq B'$ (that is, $PG(B) \preceq PG(B')$). (Note that B is a subset of B' but not a substructure of B', however A is a substructure of B'.)*

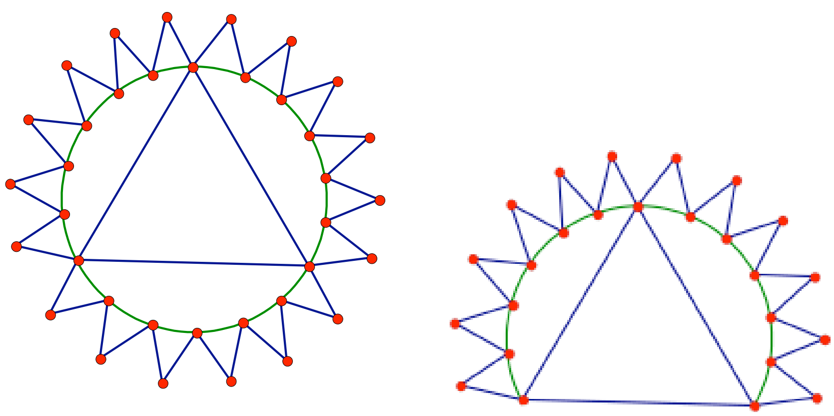
Proof. We would like to assume for our proof that for all $(\alpha, \beta, \gamma) \in R^B \setminus R^A$ we have $\alpha \neq \beta, \alpha \neq \gamma$ and $\beta \neq \gamma$, so we need to justify that we can assume this without loss of generality. Let us assume that we have proved our lemma with this extra assumption and then use this restricted lemma to prove the full one.

Let $D(B)$ be obtained from B by replacing each relation $(\alpha_1, \alpha_2, \alpha_3) \in R^B \setminus R^A$ by $D^1(\alpha_1, \alpha_2, \alpha_3)$, note however that for each relation we introduce new points specific for that relation.

By Lemma 6.2.4 we have $A \leq D(B) \in \mathcal{C}_3$ and by Lemma 6.2.5 we have that $B \preceq D(B)$. Also our extra assumption holds for $D(B)$, that is, if $(\beta_1, \beta_2, \beta_3) \in R^{D(B)} \setminus R^A$ then $\beta_1 \neq \beta_2, \beta_1 \neq \beta_3$ and $\beta_2 \neq \beta_3$. Now we use our restricted lemma to build $B' \in \mathcal{C}_\mu$ such that $A \leq B'$ and $D(B) \preceq B'$. Finally, by transitivity of $\preceq$ we have that $B \preceq D(B)$ and $D(B) \preceq B'$ imply that $B \preceq B'$, proving the full lemma.

We have just seen that in order to prove the full lemma, we may assume without loss of generality that our extra assumption holds, that is, $(\alpha, \beta, \gamma) \in R^B \setminus R^A$ implies $\alpha \neq \beta, \alpha \neq \gamma$ and $\beta \neq \gamma$. Now let us prove the lemma.

Consider a structure $\{\alpha, \beta, \gamma\}$ with $\alpha \neq \beta, \alpha \neq \gamma$, and $\beta \neq \gamma$ with only one relation (α, β, γ). Let D^i denote the isomorphism type of $D^i_{(\alpha,\beta,\gamma)}(\{\alpha, \beta, \gamma\})$. Let RD^i be the isomorphism type obtained when we remove from D^i one of the sides of the triangle: see pictures below for D^2 and RD^2 for example. Notice that there is here an abuse of notation because in fact there are three isomorphism types corresponding to RD^i depending on which side of the triangle we remove: this is because we are working with ordered tuples.



Now we construct B' by recursion.

Let $\{(\alpha_0, \beta_0, \gamma_0), \cdots, (\alpha_n, \beta_n, \gamma_n)\} = R^B \setminus R^A$. Let $B_0 := B$ and let $B_{i+1} := D^{k_i}_{(\alpha_i,\beta_i,\gamma_i)}(B_i)$ where $k_0 = 2$ and where for $i \geq 1$ we choose k_i an integer such that $k_i > k_{i-1}$ and such that RD^{k_i} does not appear in B_i as a weak substructure, that is: such that RD^{k_i} does not appear in B_i even if we remove relations of B_i. Finally, put $B' = B_{n+1}$.

The reason why we are using different k_i for each replacement of relations in $R^B \setminus R^A$ is because we want to control the number of times that each RD^{k_i} appears as weak substructure of B'.

By Lemmas 6.2.4 and 6.2.5 we already have $A \leq B'$ and $B \preceq B'$, so the only thing it remains to show is that $B' \in \mathcal{C}_\mu$.

We call the points in B the old points and the points in $B' \setminus B$ the new points. The idea of the proof is to show that for minimally simply algebraic extensions containing new points then only one copy over the same base set will appear in B', that is the reason why we assume $\mu \geq 1$ because we cannot avoid the introduction of certain minimally algebraic extensions in B', however we can force it to happen only once. For minimally simply algebraic extensions containing only old points we can prove that they are entirely inside A, so we can use the fact that $A \in \mathcal{C}_\mu$.

Let $E, F_1, \cdots, F_k$ be disjoint subsets of B' such that $E \leq EF_i$ is minimally simply algebraic and such that for each $i, j \in \{1, \cdots, k\}$ there is an isomorphism $EF_i \to EF_j$ over E. We want to prove that $k \leq \mu(E \leq EF_i)$ in order to show that $B' \in \mathcal{C}_\mu$. We divide the proof in several cases.

Claim. *(First case) Every new point in the base E is contained in one relation contained in EF_1 but which is not contained in E.*

Proof. If a new point $x \in E$ is not contained in a relation in $R^{EF_1} \setminus R^E$ then x could be removed, that is: $E \setminus \{x\} \leq E \setminus \{x\} F_1$ would be simply algebraic, contradicting the fact that $E \leq EF_1$ is minimally simply algebraic. □

So for each new point $x \in E$ there is a relation in $R^{EF_1} \setminus R^E$ containing x. There are two possibilities for this relation, either this relation has two distinct coordinates in E and one in F_1 or it has one in E and two distinct coordinates in F_1.

Claim. *(Second case) If there is a new point x in E and a relation containing x with one coordinate in F_1 and two in E then $k = 1$.*

Proof. Suppose x is a new point in $D^{k_i}(1,2,3)$ for some $(1,2,3) \in R^B \setminus R^A$. There is a relation contained in EF_1 with one point $z_1 \in F_1$ and two points in E, x and another one, say y. If $k \geq 2$ then using the isomorphism $EF_1 \to EF_2$ over E we would get a relation with coordinates x, y and $z_2 \in F_2$, but then the two relations would share two distinct points x and y and we can see from the definition that this does not happen, because x is a new point. However we need to be careful, this is only true because $1 \neq 2, 1 \neq 3$ and $2 \neq 3$ by our extra assumption. □

Claim. *(Third case) If there is a new point $x \in E$ and a relation containing x and with two distinct coordinates in F_1 then $k = 1$.*

Proof. Here we use the fact that we are working with ordered tuples, however I believe this result would be also true if we work with unordered tuples, but the proof should get more complicated.

In order to simplify the proof we assume a particular form for the relation in question that is representative of what happens in most cases. Then we consider the particular forms of the relation that do not fit with this argument. Moreover we assume that $x \in D^2(1,2,3)$, but this is also representative of what happens if $x \in D^i(1,2,3)$.

As for the representative case we choose the relation in the claim to be $(x_0, x_1, 1)$ with $x = x_1$. Thus $x_1 \in E$ and $x_0, 1 \in F_1$. Suppose that $k \geq 2$. Then there is an isomorphism $EF_1 \to EF_2$ over E. Thus there must exist a relation containing x_1 and with two distinct coordinates in F_2. The only possibility for this relation is $(x_1, x_2, 2)$ as it is the only other relation that contains x_1. But then the isomorphism $EF_1 \to EF_2$ should send the relation $(x_0, x_1, 1)$ to $(x_1, x_2, 2)$, which contradicts the fact that the image of x_1 is

x_1. This representative argument covers all the cases for values of x inside of $D^2(1,2,3)$ apart from $x = x_0, x = y_0$ and $x = z_0$.

If $x = x_0$ then the relations containing x are $(x_0, y_0, z_0), (z_5, x_0, 3)$ and $(x_0, x_1, 1)$. If our relation is $(z_5, x_0, 3)$ then clearly $k = 1$ because the image of x_0 must be x_0 and the other two relations are not compatible with this. Clearly we already have that $k \leq 2$. But if $k = 2$ then we can assume without loss of generality that our relation is (x_0, y_0, z_0) with the isomorphism $EF_1 \to EF_2$ sending (x_0, y_0, z_0) to $(x_0, x_1, 1)$, where $y_0, z_0 \in F_1$ and $x_1, 1 \in F_2$. Now we observe that the relation $(x_1, x_2, 2)$ must be in EF_2, otherwise x_1 could be removed contradicting the fact that $E \leq EF_2$ is simply algebraic. Notice that our isomorphism $EF_1 \to EF_2$ sends y_0 to x_1, so if we consider the inverse isomorphism $EF_2 \to EF_1$ the image of x_1 by this isomorphism is y_0. Now we observe that the isomorphism $EF_2 \to EF_1$ must send the relation $(x_1, x_2, 2)$ to some relation. There are three candidates that contain y_0: (x_0, y_0, z_0), which is already taken by $(x_0, x_1, 1)$; $(x_5, y_0, 3)$ is not good because $EF_2 \to EF_1$ would send $x_1 \to x_5 \neq y_0$ and the remaining option $(y_0, y_1, 1)$ also does not work as $1 \notin EF_1$ because we know that $1 \in F_2$. We may conclude that if $x = x_0$ then $k = 1$.

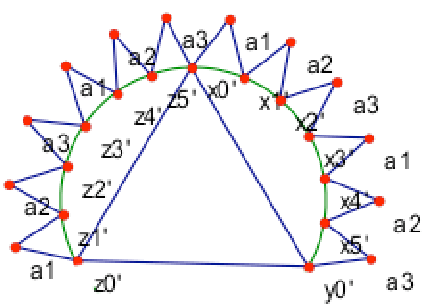
If $x = y_0$ that argument is similar. We have three relations containing y_0: $(x_0, y_0, z_0), (x_5, y_0, 3)$ and $(y_0, y_1, 1)$. If our relation is $(y_0, y_1, 1)$ then clearly $k = 1$. If $k = 2$ then without loss of generality we would have $EF_1 \to EF_2$ sending (x_0, y_0, z_0) in $(x_5, y_0, 3)$ with $y_0 \in E$, $x_0, z_0 \in F_1$ and $x_5, 3 \in F_2$. Again because $E \leq EF_2$ is simply algebraic then the relation $(x_4, x_5, 2)$ must be in EF_2 otherwise x_5 could be removed. Now, $EF_2 \to EF_1$ sends x_5 to x_0 and the relation $(x_4, x_5, 2)$ to some relation compatible with this. The three possible candidates containing x_0 are $(x_0, y_0, z_0), (z_5, x_0, 3)$ and $(x_0, x_1, 1)$. But (x_0, y_0, z_0) is already taken, with $(x_0, x_1, 1)$ we would have $EF_2 \to EF_1$ sending $x_5 \to x_1 \neq x_0$ and $(z_5, x_0, 3)$ is also not good because $3 \notin EF_1$ as we know that $3 \in F_2$. Thus if $x = y_0$ then we may conclude that $k = 1$.

Finally, if $x = z_0$ then the relations containing z_0 are (x_0, y_0, z_0), $(z_0, z_1, 1)$ and $(y_5, z_0, 3)$. There is no isomorphism between these relations fixing z_0, so $k = 1$. This concludes the proof. $\square$

So if $k \geq 2$ then there are no new points in E. However we must consider the possibility of new points in the F_i. Before considering this case we need to prove the following claim.

Claim. *For each k_i we can see RD^{k_i} exactly three times as a weak substructure of B', one time for each one of the three isomorphism types of RD^{k_i}.*

Proof. By construction each RD^{k_i} appears at least three times as a weak substructure of B', one for each one of the three isomorphism types of RD^{k_i}. The difficulty is to show that these structures do not appear again by chance. Suppose that we see one of the isomorphism types of RD^{k_i} as a weak substructure of B', as in the following picture for example:



First we observe that this picture must contain new points. If not, we would have RD^{k_i} as a weak substructure of B', then if we denote by $B[B']$ the substructure of B' with underlying set B we would have RD^{k_i} as a weak substructure of $B[B']$. Finally $B[B']$ itself is a weak substructure of B, because from B to $B[B']$ we just removed relations, thus RD^{k_i} would be a weak substructure of B, which is not true by choice of k_i in the construction of B'.

Now we observe that a_1, a_2, a_3 are old points, because each a_i appears in at least four relations (because $k_i \geq 2$) and new points appear in at most three relations. We may conclude that the in the set $RD^{k_i} \setminus \{a_1, a_2, a_3\}$ there is a new point x. Now we observe that as x is a new point and a_1, a_2, a_3 are old points then all the points inside of RD^{k_i} are determined by this because each new point is contained in exactly two relations that contain an old point. Clearly, this copy of RD^{k_i} must be the one introduced in the construction of B' in the step where x was introduced. This concludes the proof of the claim. $\square$

Now we consider the possibility that a new point appears in one of the F_i.

Claim. *(Fourth case) If there is a new point in one of the F_i and no new points in E then $k = 1$.*

Proof. Assume without loss of generality that x is a new point in F_1 and that there are no new points in E.

Observe that if there were zero or just one relation in EF_1 containing x then x could be removed, contradicting the fact that $E \leq EF_1$ is simply algebraic. Thus x is contained in at least two relations inside of EF_1. This proves that there is another new point y inside of F_1 because E have no new points. But now we could apply the same argument to this new point y to prove the existence of a third new point in F_1 and so on.. If we repeat the argument enough times we get a copy of RD^{k_i} as a weak substructure of EF_1 with all the new points inside of F_1 (we cannot prove that D^i appears in EF_1, this is the reason why we work with the RD^i). Finally, if $k \geq 2$ then the isomorphism $EF_1 \to EF_2$ would produce another copy of RD^{k_i} inside of EF_2. But this contradicts the previous claim. We may then conclude that in this case we also have $k = 1$. $\square$

We proved that if there is a new point in E or in some of the F_i then $k = 1$, so we can assume from now on that $E, F_i \subseteq B$ (as sets).

Claim. *(Fifth case) If there are no new points in E or in the F_i then we have $k \leq \mu(E \leq EF_1)$.*

Proof. We have that $EF_i \subseteq B[B']$, but in this case we can prove that actually we have $EF_i \subseteq A$. In fact if $x \in EF_i$ is not in A then by construction of B' this point is in no relation that have all the coordinates in old points, because this kind of relation was replaced. Thus this point x could be removed from EF_i, contradicting the fact that $E \leq EF_i$ is minimally simply algebraic.

So we have EF_i as a substructure of A for all $i \in \{1, \cdots, k\}$, thus we can use the fact that $A \in \mathcal{C}_\mu$ to conclude that $k \leq \mu(E \leq EF_1)$. $\square$

We went through all the cases, proving that $k \leq \mu(E \leq EF_1)$ thus $B' \in \mathcal{C}_\mu$. This concludes the proof of the lemma. $\square$

We have now the tools necessary to prove our main theorem.

Theorem 6.2.7. *Let $\mu \geq 1$ for all minimally simply algebraic extensions. Then $PG(\mathcal{M}_\mu) \simeq PG(\mathcal{M}_3)$.*

Proof. By the proof of Theorem 5.5.9 there exists a chain $\widetilde{M}_0 \leq \widetilde{M}_1 \leq \widetilde{M}_2 \leq \cdots$ with $\widetilde{M}_i \in \mathcal{C}_3$, $\bigcup_{i \in \mathbb{N}} \widetilde{M}_i = \mathcal{M}_3$ and such that $PG(\mathcal{M}_3)$ is a generic model of the class of pregeometries $(P_3, \preceq)$ with respect to the chain $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$, where $M_i := PG(\widetilde{M}_i)$. Let us fix this chain of pregeometries. So when we talk about $PG(\mathcal{M}_3)$ as a generic model of pregeometries and $\preceq$-closed subpregeometries of $PG(\mathcal{M}_3)$ it is always with respect to this chain.

The proof of the existence of an isomorphism between $PG(\mathcal{M}_\mu)$ and $PG(\mathcal{M}_3)$ will be by back and forth. However the argument will be written by combining the back step and the forth step into a single step.

Let $f_0 : A \to B$ be an isomorphism of pregeometries with $A \leq \mathcal{M}_\mu$ and $B \preceq PG(\mathcal{M}_3)$: we mean by this that $f_0 : PG(A) \to B$ is an isomorphism of pregeometries. Notice that B has a structure induced from $\mathcal{M}_3$, but we do not have necessarily $B \leq \mathcal{M}_3$ so the pregeometry of this structure is not necessarily B. Anyway this induced structure is irrelevant for the argument. We have the following diagram:

$$\begin{tikzcd} \mathcal{M}_\mu & PG(\mathcal{M}_3) \\ A \arrow[u, "\leq"] \arrow[r, "I.P", "f_0"'] & B \arrow[u, "\preceq"'] \end{tikzcd}$$

Notation 6.2.8. We use the abbreviations $I.S$ and $I.P$ for isomorphism of structures and isomorphism of pregeometries.

We have $B \preceq PG(\mathcal{M}_3)$ so $B \preceq M_i$ for some $i \in \mathbb{N}$. In particular there are some structures $\widehat{B} \in \mathcal{C}_3$ and $\widehat{M_i} \in \mathcal{C}_3$ such that $PG(\widehat{B}) = B$, $PG(\widehat{M_i}) = M_i$ and $\widehat{B} \leq \widehat{M_i}$. From now on we fix this i.

Let B' be the structure induced from A on the set B via the isomorphism of pregeometries $f_0 : A \to B$. In other words, B' is the structure with the same underlying set as B and with relations given by $R_3^{B'} := \{f_0(\bar{a}) : \bar{a} \in R_3^A\}$. Then we have that $PG(B') = B$ because $PG(B') \xrightarrow[f_0^{-1}]{} A \xrightarrow[f_0]{} B$ are isomorphisms of pregeometries. So we have that $f_0 : A \to B'$ is an isomorphism of structures with $PG(B') = B$.

Now we obtain M_i' from $\widehat{M_i}$ by replacing $\widehat{B}$ by B'. Thus by the first and second Changing Lemmas, (5.3.1 and 5.3.2) we get $B' \leq M_i' \in \mathcal{C}_3$ and $PG(M_i') = PG(\widehat{M_i}) = M_i$. Finally we construct $\mathcal{M}'$ from $\mathcal{M}_3$ by replacing $\widetilde{M_i}$ by M_i'. By the first and second Changing Lemmas we get $M_i' \leq \mathcal{M}' \in \bar{\mathcal{C}}_3$ and $PG(\mathcal{M}') = PG(\mathcal{M}_3)$. Moreover, by the Third Changing Lemma (5.4.1) we get that $\mathcal{M}' \simeq \mathcal{M}_3$.

Claim. *For $k \geq i$ let M_k' be the structure induced from $\mathcal{M}'$ in M_k. Then $M_i' \leq M_{i+1}' \leq \cdots \leq M_k' \leq \cdots$ In particular we have $PG(M_k') = M_k$ for all*

$k \geq i$.

Proof. We have $M'_i \leq \mathcal{M}'$ and $PG(M'_i) = M_i$. Suppose that we have $M'_i \leq M'_{i+1} \leq \cdots \leq M'_k$ and that $PG(M'_j) = M_j$ for $i \leq j \leq k$. Clearly M'_{j+1} coincides with the structure obtained from $\widetilde{M}_{j+1}$ by replacing $\widetilde{M}_j$ by M'_j. Thus as $\widetilde{M}_k \leq \widetilde{M}_{k+1}$ and $PG(M'_k) = M_k = PG(\widetilde{M}_k)$, then by the first and second Changing Lemmas we get $M'_k \leq M'_{k+1}$ and $PG(M'_{k+1}) = PG(\widetilde{M}_{k+1}) = M_{k+1}$. This argument proves that we have $M'_k \leq M'_{k+1}$ and $PG(M'_k) = M_k$ for all $k \geq i$. □

Now we return to our back and forth argument.

So we have $A \leq \mathcal{M}_\mu$, $B' \leq \mathcal{M}'$ and an isomorphism of structures $f_0 : A \to B'$. Let $c \in \mathcal{M}_\mu \setminus A$ and $A \leq A_1 \leq \mathcal{M}_\mu$ with $c \in A_1$. Then as $\mathcal{M}' \simeq \mathcal{M}_3$ we can use the extension property of $\mathcal{M}'$ to extend $f_0 : A \to B'$ to an isomorphism $f_1 : A_1 \to B_1$ where $B' \leq B_1 \leq \mathcal{M}'$, as in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_\mu & \mathcal{M}' \\
A_1 \arrow[u, "\leq"] \arrow[r, "I.S", "f_1"'] & B_1 \arrow[u, "\leq"] \\
A \arrow[u, "\leq"] \arrow[r, "I.S", "f_0"'] & B' \arrow[u, "\leq"]
\end{tikzcd}$$

Now let $d \in \mathcal{M}' \setminus B_1$ and $B_1 \leq B_2 \leq \mathcal{M}'$ with $d \in B_2$. We have $B_1 \simeq A_1 \in \mathcal{C}_\mu$, thus $B_1 \in \mathcal{C}_\mu$, also we have $B_1 \leq B_2$ thus by the Hard Changing Lemma we can construct $B'_2 \in \mathcal{C}_\mu$ such that $B_1 \leq B'_2$ and $B_2 \preceq B'_2$. In particular if we define $f_2 : A_1 \to B'_2$ by $f_2(a) := f_1(a)$ for all $a \in A_1$, then f_2 is a strong embedding, that is $f_2(A_1) = f_1(A_1) = B_1 \leq B'_2$. We have now the following commutative diagram:

$$\begin{tikzcd}
 & B_2' & \\
 & & B_2 \arrow[ul, "\preceq"'] \\
A_1 \arrow[uur, "\leq", "f_2"'] \arrow[rr, "I.S", "f_1"'] & & B_1 \arrow[u, "\leq"]
\end{tikzcd}$$

Now we are going to use the respective extension properties of $\mathcal{M}_\mu$ and $PG(\mathcal{M}_3)$.

First we notice that $A_1 \leq \mathcal{M}_\mu$ and that $f_2 : A_1 \to B_2' \in \mathcal{C}_\mu$ is a strong embedding. So we can use the extension property of $\mathcal{M}_\mu$ to build an isomorphism $h : B_2' \to A_3$ such that $h \circ f_2 = Id_{|A_1}$ and $A_1 \leq A_3 \leq \mathcal{M}_\mu$, as in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}_\mu & & \\
A_3 \arrow[u, "\leq"] & B_2' \arrow[l, "I.S"', "h"] & \\
 & & B_2 \arrow[ul, "\preceq"'] \\
A_1 \arrow[uu, "\leq"] \arrow[uur, "\leq", "f_2"'] \arrow[rr, "I.S", "f_1"'] & & B_1 \arrow[u, "\leq"]
\end{tikzcd}$$

Now we notice that $B_2 \leq \mathcal{M}'$. Thus by our claim $B_2 \leq M_k'$ for some $k \geq i$, but then $PG(B_2) \preceq M_k$ so $PG(B_2) \preceq PG(\mathcal{M}_3)$.

Finally we have $B_2 \preceq B_2'$ and $B_2 \preceq PG(\mathcal{M}_3)$. Thus by the extension property of $PG(\mathcal{M}_3)$ we can construct an isomorphism of pregeometries $g : B_2' \to B_3$ over B_2 such that $B_3 \preceq PG(\mathcal{M}_3)$. We obtain the following diagram:

$$\begin{tikzcd}
\mathcal{M}_\mu & & PG(\mathcal{M}_3) \\
A_3 \arrow[u, "\leq"] & B_2' \arrow[l, "I.S"', "h"] \arrow[r, "I.P", "g"'] & B_3 \arrow[u, "\preceq"'] \\
& & B_2 \arrow[u, "\preceq"'] \arrow[ul, "\preceq"] \\
A_1 \arrow[uu, "\leq"] \arrow[uur, "\leq", "f_2"'] \arrow[rr, "I.S", "f_1"'] & & B_1 \arrow[u, "\leq"']
\end{tikzcd}$$

In particular if we define $f_3 := g \circ h^{-1}$ then $f_3 : A_3 \to B_3$ is an isomorphism of pregeometries extending $f_1 : A_1 \to B_1$, where here we see f_1 as an isomorphism of pregeometries. Moreover we have $A_3 \leq \mathcal{M}_\mu$ and $B_3 \preceq PG(\mathcal{M}_3)$. We get the following commutative diagram, proving the back and forth:

$$\begin{tikzcd}
\mathcal{M}_\mu & PG(\mathcal{M}_3) \\
A_3 \arrow[u, "\leq"] \arrow[r, "I.P", "f_3"'] & B_3 \arrow[u, "\preceq"] \\
A_1 \arrow[u, "\leq"] \arrow[r, "I.P", "f_1"'] & B_1 \arrow[u, "\preceq"] \\
A \arrow[u, "\leq"] \arrow[r, "I.P", "f_0"'] & B \arrow[u, "\preceq"]
\end{tikzcd}$$

This back and forth argument proves the existence of an isomorphism of pregeometries between $PG(\mathcal{M}_\mu)$ and $PG(\mathcal{M}_3)$, concluding the proof. $\square$

Chapter 7

Two open problems

7.1 The generic model of pregeometries

In this section we consider another way of defining the generic model of $(P_n, \preceq)$ and then we ask if such a generic model exists.

Definition 7.1.1. Let

$$\bar{P}_n := \{A : A \text{ is pregeometry and there exists } A' \in \bar{\mathcal{C}}_n \text{ with } PG(A') = A\}$$

and let P_n be as usual, that is, the class of finite pregeometries of $\bar{P}_n$.

Now we define an embedding relation $\sqsubseteq$ on $\bar{P}_n$ by saying: $A \sqsubseteq B$ if and only if there are $A', B' \in \bar{\mathcal{C}}_n$ such that $PG(A') = A, PG(B') = B$ and $A' \leq B'$.

Notice that for finite pregeometries $\sqsubseteq$ coincides with $\preceq$. In other words we have $(P_n, \sqsubseteq) = (P_n, \preceq)$.

Proposition 7.1.2. *Let $\mathcal{M}$ be a countable pregeometry. Then $\mathcal{M} \in \bar{P}_n$ if and only if there exists a chain $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ such that $M_i \in P_n$ and $\mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i$.*

Proof. Assume that $\mathcal{M} \in \bar{P}_n$. Then there is $\mathcal{M}' \in \bar{\mathcal{C}}_n$ with $PG(\mathcal{M}') = \mathcal{M}$. Let $M'_0 \leq M'_1 \leq M'_2 \leq \cdots$ with $M'_i \in \mathcal{C}_n$ and $\mathcal{M}' = \bigcup_{i \in \mathbb{N}} M'_i$. In particular we have $PG(M'_0) \preceq PG(M'_1) \preceq PG(M'_2) \preceq \cdots$ with $PG(M'_i) \in P_n$ and $\mathcal{M} = PG(\mathcal{M}') = \bigcup_{i \in \mathbb{N}} PG(M'_i)$. This proves the direct forward of our proposition.

For the other implication let us assume that we have $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ with $M_i \in P_n$ and $\mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i$. We have $M_0 \preceq M_1$ so let $M'_0, M'_1 \in \mathcal{C}_n$ be such that $PG(M'_0) = M_0, PG(M'_1) = M_1$ and $M'_0 \leq M'_1$. Now suppose we have constructed $M'_0 \leq M'_1 \leq \cdots M'_i \leq \cdots \leq M'_k$ with $PG(M'_i) = M_i$ for $i \leq k$. We have $M_k \preceq M_{k+1}$ so there are $\widetilde{M_k}, \widetilde{M_{k+1}} \in \mathcal{C}_n$ with $PG(\widetilde{M_k}) = M_k$, $PG(\widetilde{M_{k+1}}) = M_{k+1}$ and $\widetilde{M_k} \leq \widetilde{M_{k+1}}$. Let M'_{k+1} be obtained from $\widetilde{M_{k+1}}$ by replacing $\widetilde{M_k}$ by M'_k. By the First Changing Lemma we get $M'_k \leq M'_{k+1} \in \mathcal{C}_n$ and by the Second Changing Lemma we get $PG(M'_{k+1}) = PG(\widetilde{M_{k+1}}) = M_{k+1}$. This recursion argument proves the existence of a chain $M'_0 \leq M'_1 \leq M'_2 \leq \cdots$ with $M'_i \in \mathcal{C}_n$ and $PG(M'_i) = M_i$. Finally we put $\mathcal{M}' := \bigcup_{i \in \mathbb{N}} M'_i$. We have $\mathcal{M}' \in \bar{\mathcal{C}}_n$ and $PG(\mathcal{M}') = \bigcup_{i \in \mathbb{N}} PG(M'_i) = \bigcup_{i \in \mathbb{N}} M_i = \mathcal{M}$. Thus $\mathcal{M} \in \bar{P}_n$. $\square$

Remember the following definition from before.

Definition 7.1.3. Let $\mathcal{M}$ be a pregeometry and $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ with $\mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i$. We define a notion of closed finite subpregeometries of $\mathcal{M}$ relatively to this chain by:

$$A \preceq \mathcal{M} \text{ if and only if } A \preceq M_i \text{ for some } i.$$

However we have another notion of closed subpregeometries arising from $(\bar{P}_n, \sqsubseteq)$ that does not depend on the choice of any chain. The next proposition shows the relation between these notions, that is, between the notions of $\preceq$-closed subpregeometries and $\sqsubseteq$-closed subpregeometries.

Proposition 7.1.4. *Let $\mathcal{M}$ be a countable pregeometry in $\bar{P}_n$ and A a finite subpregeometry of $\mathcal{M}$, then $A \sqsubseteq \mathcal{M}$ if and only if*

$$A \preceq \mathcal{M} \text{ for some chain } M_0 \preceq M_1 \preceq \cdots \text{ with } M_i \in P_n \text{ and } \mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i.$$

Proof. Suppose that $A \sqsubseteq \mathcal{M}$. Then we have $A' \in \mathcal{C}_n$, $\mathcal{M}' \in \bar{\mathcal{C}}_n$ with $PG(A') = A, PG(\mathcal{M}') = \mathcal{M}$ and $A \leq \mathcal{M}'$. Let $M'_0 \leq M'_1 \leq M'_2 \leq \cdots$ with $M'_i \in \mathcal{C}_n$ and $\mathcal{M}' = \bigcup_{i \in \mathbb{N}} M'_i$. In particular we have $PG(M'_0) \preceq PG(M'_1) \preceq PG(M'_2) \preceq \cdots$ with $PG(M'_i) \in P_n$ and $\mathcal{M} = PG(\mathcal{M}') = \bigcup_{i \in \mathbb{N}} PG(M'_i)$. This is our chain. Finally we observe that as $A' \leq \mathcal{M}'$ then $A' \leq M'_i$ for some i. In particular $PG(A') \preceq PG(M'_i)$, that is $A \preceq PG(M'_i)$ for some i. Thus $A \preceq \mathcal{M}$ relatively to our chain. This proves the direct implication of our proposition.

For the other implication let us assume that we have $M_0 \preceq M_1 \preceq M_2 \preceq \cdots$ with $M_i \in P_n$, $\mathcal{M} = \bigcup_{i \in \mathbb{N}} M_i$ and $A \preceq M_j$ for some j. By a recursion argument like in the proof of last proposition we can get a chain $M'_0 \leq M'_1 \leq M'_2 \leq \cdots$ with $M'_i \in \mathcal{C}_n$ and $PG(M'_i) = M_i$. Then we define $\mathcal{M}' := \bigcup_{i \in \mathbb{N}} M'_i$ and we get $PG(\mathcal{M}') = \bigcup_{i \in \mathbb{N}} PG(M'_i) = \bigcup_{i \in \mathbb{N}} M_i = \mathcal{M}$.

We have $A \preceq M_j$ so there are $\widetilde{A}, \widetilde{M_j} \in \mathcal{C}_n$ such that $PG(\widetilde{A}) = A, PG(\widetilde{M_j}) = M_j$ and $\widetilde{A} \leq \widetilde{M_j}$. Let us obtain $\widetilde{\mathcal{M}}$ from $\mathcal{M}'$ by replacing M'_j by $\widetilde{M_j}$. Then by the First Changing Lemma we get $\widetilde{M_j} \leq \widetilde{\mathcal{M}} \in \bar{\mathcal{C}}_n$ and by the Second Changing Lemma we have $PG(\widetilde{\mathcal{M}}) = PG(\mathcal{M}') = \mathcal{M}$. Finally we get $\widetilde{A} \leq \widetilde{M_j} \leq \widetilde{\mathcal{M}}$, in particular $PG(A) \sqsubseteq PG(\widetilde{\mathcal{M}})$, that is $A \sqsubseteq \mathcal{M}$, proving our proposition. □

This way of defining $\sqsubseteq$, somewhat more natural than the way of defining $\preceq$ relatively to a chain, allow us a natural way of defining another version of generic model of the class $(P_n, \preceq) = (P_n, \sqsubseteq)$. We write $\preceq$-generic model and $\sqsubseteq$-generic model to distinguish the two versions.

Definition 7.1.5. Let $\mathcal{M}$ be a countable pregeometry. We say that $\mathcal{M}$ is a $\sqsubseteq$-generic model of $(P_n, \preceq)$ if:

- $\mathcal{M} \in \bar{P}_n$
- ($\sqsubseteq$-extension property) $A \sqsubseteq \mathcal{M}$, $A \sqsubseteq B \in P_n$ imply that there exists an embedding of pregeometries $f : B \to \mathcal{M}$ over A such that $f(B) \sqsubseteq \mathcal{M}$.

The first axiom in this definition of $\sqsubseteq$-generic agrees with the first axiom of the $\preceq$-generic, by Proposition 7.1.2. However it is not clear if the $\sqsubseteq$-generic exists because the standard proof of the existence of a generic does not work here. In fact, in the standard construction we build a chain of finite pregeometries forcing that the extension property holds in relation to subpregeometries $\preceq$-closed relatively to this chain. However after the construction, we get at least apparently, more $\sqsubseteq$-closed subpregeometries (closed relatively to other chains) that were not considered during the construction. Thus the $\sqsubseteq$-extension property does not follow.

On other hand, if the $\sqsubseteq$-generic exists, then it is unique up to isomorphism by the standard proof of uniqueness of the generic. Actually, the same back and forth argument allow us to prove the following proposition:

Proposition 7.1.6. *If the $\sqsubseteq$-generic of $(P_n, \preceq)$ exists, then it is isomorphic to the $\preceq$-generic, that is, isomorphic to $PG(\mathcal{M}_n)$.*

The following problem arises naturally:

Problem. *(Open problem) Does the $\sqsubseteq$-generic model exists? In other words, does $PG(\mathcal{M}_n)$ satisfies the $\sqsubseteq$-extension property?*

7.2 A variation of Hrushovski construction

In Theorem 6.2.7 we have seen that if $\mu \geq 1$ then $PG(\mathcal{M}_\mu) \simeq PG(\mathcal{M}_3)$. In other words we prove that collapsing does not change the isomorphism type of the pregeometry, thus of the geometry. However there is a variation of this kind of construction, also by Hrushovski, in which this does not happen. This variation appears in Proposition 18 of the article [4], but we describe it briefly here.

We work with a ternary relational symbol R and with unordered tuples. More precisely we consider only structures A such that if $A \vDash R(a_1, a_2, a_3)$ then $a_1 \neq a_2, a_1 \neq a_3, a_2 \neq a_3$ and $A \vDash R(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation $\sigma : \{a_1, a_2, a_3\} \to \{a_1, a_2, a_3\}$. On the class of finite structures that satisfy these requirements we define a predimension given by $\delta(A) = |A| - 1/6|R^A|$, that is $\delta(A)$ is the number of elements of A minus the number of unordered tuples for which R holds in A. We get a binary relation of self-sufficiency $\leq$ as usual. Finally we restrict our class to the following one:

$$\mathcal{C} := \{A : \text{for any } B \subseteq A \text{ with } |B| \leq 3 \text{ we have } B \leq A\}.$$

Proposition 7.2.1. $(\mathcal{C}, \leq)$ *is an amalgamation class. Thus there exists a corresponding generic model* $\mathcal{M}$*, unique up to isomorphism.*

Now we proceed, as in last chapter, to produce a family of subclasses $\mathcal{C}_\mu$, one for each choice of μ.

Proposition 7.2.2. $(\mathcal{C}_\mu, \leq)$ *is an amalgamation class and the corresponding generic model* $\mathcal{M}_\mu$ *is strongly minimal.*

We may ask if we can obtain the same result as before, that is, $PG(\mathcal{M}_\mu) \simeq PG(\mathcal{M})$. But this is not true because of the following result which is contained in the discussion of Proposition 18 in the Hrushovski's article [4]:

Proposition 7.2.3. *For a structure $A \in \bar{\mathcal{C}}$ we have that $A \vDash R(a_1, a_2, a_3)$ if and only if $\{a_1, a_2, a_3\}$ is dependent. In particular $\mathcal{M}_\mu$ can be recovered from $PG(\mathcal{M}_\mu)$.*

We may conclude that for different μ_1, μ_2 we have $PG(\mathcal{M}_{\mu_1}) \ncong PG(\mathcal{M}_{\mu_2})$ because $\mathcal{M}_{\mu_1} \ncong \mathcal{M}_{\mu_2}$. In this case collapsing does affect the isomorphism type of the pregeometries/geometries. Notice that in this context the pregeometries are geometries. It is still unsolved the following question by Hrushovski:

Problem. *(Open problem) For different μ_1, μ_2 are the geometries of $\mathcal{M}_{\mu_1}$ and $\mathcal{M}_{\mu_2}$ locally isomorphic?*

It is our intuition that the answer is affirmative, we believe that by localizing to a well chosen finite subset we will get the geometry of the unordered version of $\mathcal{M}_3$. However this situation is still not clear.

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