1

On ◁*-maximality

Mirna Džamonja[a,*], Saharon Shelah[b]

[a] *School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK*
[b] *Mathematics Department, Hebrew University of Jerusalem, 91904 Givat Ram, Israel*

Abstract

This paper investigates a connection between the semantic notion provided by the ordering ◁* among theories in model theory and the syntactic (N)SOP$_n$ hierarchy of Shelah. It introduces two properties which are natural extensions of this hierarchy, called SOP$_2$ and SOP$_1$. It is shown here that SOP$_3$ implies SOP$_2$ implies SOP$_1$. In Shelah (Ann. Pure Appl. Logic 80 (1996) 229) it was shown that SOP$_3$ implies ◁*-maximality and we prove here that ◁*-maximality in a model of GCH implies a property called SOP$_2''$. It has been subsequently shown by Shelah and Usvyatsov that SOP$_2''$ and SOP$_2$ are equivalent, so obtaining an implication between ◁*-maximality and SOP$_2$. It is not known if SOP$_2$ and SOP$_3$ are equivalent.

Together with the known results about the connection between the (N)SOP$_n$ hierarchy and the existence of universal models in the absence of GCH, the paper provides a step toward the classification of unstable theories without the strict order property.
ⓒ 2003 Published by Elsevier B.V.

0. Introduction

This paper investigates a connection between the ordering ◁* among theories in model theory and the (N)SOP$_n$ hierarchy of Shelah and as such provides a step toward the classification of unstable theories without the strict order property. The thesis we pursue is that the syntactic property SOP$_2$ is closely related to the semantic property of being maximal in the ◁*-order. We shall now give the relevant definitions and explain

* Corresponding author.
E-mail addresses: m.dzamonja@uea.ac.uk (M. Džamonja), shelah@sunset.huji.ac.il (S. Shelah).
URLs: http://www.mth.uea.ac.uk/people/md.html, http://shelah.logic.at/

1 the motivation behind the paper as well as noting our main results. For the purpose of
   this introductory discussion we shall limit ourselves to countable (complete first order)
3 theories.
   The following order among theories was introduced and investigated by Keisler in
5 [7].

Definition 0.1. (1) For any cardinal λ, the Keisler order $\lessdot_\lambda$ among theories is defined
7 as follows: $T_0 \lessdot_\lambda T_1$ if whenever $M_l(l<2)$ is a model of T_0, T_1 respectively and $\mathscr{D}$ is a
   regular ultrafilter over λ, *then* the λ^+-compactness of $M_1^\lambda/\mathscr{D}$ implies the λ^+-compactness
9 of $M_0^\lambda/\mathscr{D}$.
   (2) We say $T_0 \lessdot T_1$ if for all λ we have $T_0 \lessdot_\lambda T_1$.

11 The relevance of this order to the project of classifying unstable theories without
   strict order property lies in the two following theorems of Shelah (note that the second
13 one implies the first).

Theorem 0.2 (Shelah [11, V14.3]). *Any (countable) theory with the strict order prop-*
15 *erty is $\lessdot$-maximal.*

As stated in [11], p. xiv, Chapter VI of [11] gives a rather complete picture of
17 Keisler's order and to complete it we should know more about unstable theories with-
   out the strict order property. Paper [13] started a classification of such theories by
19 introducing the hierarchy SOP$_n$ for $n \geqslant 3$ and in particular it is stated there that being
   maximal in the Keisler order is not a characterisation of theories with the strict order
21 property.

Theorem 0.3 (Shelah [13], see also Shelah and Usvyatsov [14]). *Any theory with*
23 SOP$_3$ *is $\lessdot$-maximal.*

Details of the proof are given in [14]. Precise definitions of properties SOP$_n$ for
25 $n \geqslant 3$ will be repeated below in Section 2 but for the moment we note that it was
   proved in [13] that for $n \geqslant 3$

27   strict order property $\Rightarrow SOP_{n+1} \Rightarrow SOP_n \Rightarrow$ not simple

and that all the implications are irreversible. One may now wonder if having SOP$_3$ is a
29 characterisation of theories that are maximal in the Keisler order, giving us a semantic
   equivalent to the syntactic notion of SOP$_3$. This would be consistent with what is
31 known about this order, see the Introduction to Chapter VI of [11]. This question
   remains open but instead one may attempt to give a characterisation of SOP$_3$ or SOP$_n$
33 in terms of some other similarly defined order. This is suggested by [13] which in fact
   gives a theorem stronger than 0.3, namely.

35 **Theorem 0.4** (Shelah [13], see also Shelah and Usvyatsov [14]). *Any theory with*
   SOP$_3$ *is $\vartriangleleft^*$-maximal.*

1   The definition of this order will be recalled in Section 1 where we shall also prove
that being $\lhd^*$-maximal implies being maximal in the Keisler order. Given this fact one
3   may now ask if being $\lhd^*$-maximal characterises theories with SOP_3. To test this claim
it is natural to investigate a prototypical example of an $NSOP_3$ theory that is still not
5   simple, which is T^*_{feq}. In Section 1 we shall recall the definition of this theory and
show that in fact it is not $\lhd^*$-maximal, as it is consistently strictly below the theory of
7   a dense linear order with no first or last element (all we need for the consistency is a
partial GCH assumption).
9   This naturally leads to the question of the possibility of refining the distinction
between simplicity and SOP_3. Definition of the SOP_n hierarchy from [13] does not
11   immediately give way to such a refinement as SOP_n is roughly speaking, defined in
terms of omitting loops of size n. However in Section 2 we introduce two properties
13   SOP_2 and SOP_1 that in fact satisfy

$$SOP_3 \Rightarrow SOP_2 \Rightarrow SOP_1 \Rightarrow \text{not simple}.$$

15   We then ask if these properties in any way characterise the maximality in $\lhd^*$. To this
end in Section 3 we prove that any theory that is $\lhd^*$-maximal in a model of a sufficient
17   amount of GCH must satisfy a syntactic property SOP_2''. Together with a subsequent
result of Shelah and Usvyatsov in [14] that proved that SOP_2'' is equivalent to SOP_2
19   we hence obtain that $\lhd^*$-maximality in any model of a sufficiently rich fragment of
GCH implies SOP_2. (See Section 3 for the definition of SOP_2'' and the exact reference
21   from [14]). To summarise, our main result, appearing as Corollary 3.9(1) below is

Theorem 0.5. *Suppose that T is a theory that is $\lhd^*$-maximal in some universe of set*
23   *theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ. Then T has* SOP_2.

Several questions remain open. The main one of course is if SOP_2 is actually equiv-
25   alent to $\lhd^*$-maximality. Recall from the discussion above that we know that SOP_3
implies $\lhd^*$-maximality. It is not known if SOP_3 and SOP_2 are actually equivalent.
27   We also note that Shelah and Usvyatsov have proved in [14] a local version of the
implication $SOP_2 \Rightarrow \lhd^*$-maximality, see Section 3 for a more detailed discussion.
29   A burning question also is that we in fact do not know almost anything about the
reverse of other implications in the (consistent) diagram

31        $$SOP_3 \Rightarrow \quad \lhd^*\text{-maximality} \Rightarrow SOP_2 \Rightarrow SOP_1 \Rightarrow \text{not simple},$$

apart that not all of them may be equivalences, as T^*_{feq} is not simple but is $NSOP_3$. In
33   fact [14] proves that T^*_{feq} is not even SOP_1.
Before laying down the organisation of the paper let us also mention the connection
35   of the SOP_n hierarchy with another semantic property, which is the possibility of having
a universal model at λ in some universe of set theory where a sufficient amount of
37   GCH fails (under GCH every countable first order theory has a universal model in
every uncountable cardinal). The connection between this property and unstable theories
39   without the strict order property has been investigated in a series of papers, notably in
[9] where it is proved that if GCH fails sufficiently then there are no universal dense

1   linear orders. It was proved in [13] that SOP_4 is already sufficient for such a negative universality result. The question of universality is interesting also for classes that are

3   not elementary classes of models of a first order theory, for example for classes without amalgamation the most interesting case is the strong limit singular μ of cofinality $\aleph_0$.

5   In [5] it is proved that for such μ and $\lambda < \mu$ a strongly compact cardinal the class of models of any $L_{\lambda,\mu}$-theory of cardinality $< \mu$ admits a universal model of cardinality

7   μ. A rather detailed description of what is known about the connection of unstable theories without the strict order property and the universality problem may be found

9   in the introduction to [4].

     The paper is organised as follows. In the first section we investigate the theory T^*_{feq}.

11   This is simply the model completion of the theory of infinitely many parametrised equivalence relations. We show that under a partial GCH assumption, this theory is

13   not maximal with respect to $\lhd^*_\lambda$, as it is strictly below the theory of a dense linear order. In the second section of the paper we extend Shelah's $NSOP_n$ hierarchy by

15   introducing two further properties SOP_1 and SOP_2, and we show that their names are justified by their position in the hierarchy. Namely $SOP_3 \Rightarrow SOP_2 \Rightarrow SOP_1$. Furthermore,

17   SOP_1 theories are not simple. The last section of the paper contains the main result showing that $\lhd^*$-maximality in a model of a sufficiently rich fragment of GCH implies

19   SOP''_2, and hence SOP_2 by Shelah–Usvyatsov.

     The following conventions will be used in the paper.

21   **Convention 0.6.** Unless specified otherwise, a "theory" stands for a first order complete theory. An unattributed T stands for a theory. We use $\tau(T)$ to denote the vocabulary

23   of a theory T, and $\mathscr{L}(T)$ to denote the set of formulae of T.

     By $\mathfrak{C} = \mathfrak{C}_T$ we denote a $\bar{\kappa}$-saturated model of T, for a large enough regular cardinal

25    $\bar{\kappa}$ and we assume that any models of T that we mention are elementary submodels of $\mathfrak{C}$.

     λ, μ, κ stand for infinite cardinals.

27   ## 1. On the order $\lhd^*_\lambda$

     Definition 1.1. (1) For (first order complete) theories T_0 and T we say that $\bar{\varphi} = \langle \varphi_R$

29    $(\bar{x}_R)$: R a predicate of $\tau(T_0)$ or a function symbol of $\tau(T_0)$ or $= \rangle$, (where we have $\bar{x}_R = (x_0, \ldots, x_{n(R)-1})$), *interprets* T_0 in T, or that $\bar{\varphi}$ *is an interpretation of* T_0 *in* T, or

31   that

       $T \vdash$ "$\bar{\varphi}$ is a model of T_0",

33   *if* each $\varphi_R(\bar{x}_R) \in \mathscr{L}(T)$, and for any $M \models T$, the model $M^{[\bar{\varphi}]}$ described below is a model of T_0. Here, $N = M^{[\bar{\varphi}]}$ is a $\tau(T_0)$ model, whose set of elements is $\{a : M \models \varphi_=(a, a)\}$

35   (so $M^{[\bar{\varphi}]} \subseteq M$) and $R^N = \{\bar{a} : M \models \varphi_R[\bar{a}]\}$ for a predicate R of T_0.

     For any function symbol f of $\tau(T_0)$ we have that $N \models$ "$f(\bar{a}) = b$" iff $M \models \varphi_f(\bar{a}, b)$,

37   while

       $M \models$ "$\varphi_f(\bar{a}, b) = \varphi_f(\bar{a}, c) \Rightarrow b = c$"

39   for all $\bar{a}, b, c$.

(2) We say that the interpretation $\bar{\varphi}$ is *trivial* if $\varphi_R(\bar{x}_R) = R(\bar{x}_R)$ for all $R \in \tau(T_0)$, so $M^{[\bar{\varphi}]} = M \upharpoonright \tau(T_0)$, for any model M of T.

(The last clause in Definition 1.1(1) shows that we can in fact restrict ourselves to vocabularies without function or constant symbols.)

We use the notion of interpretations to define a certain relation among theories. This relation was introduced by Shelah in [13, Section 2] and one can see [14] for a more detailed exposition. The reason we are interested in this ordering is Shelah's Theorem 0.3 quoted in the Introduction and we shall now start developing methods for the proof of our main result 3.9.

Definition 1.2. For (complete first order) theories T_0, T_1 we define:

(1) A triple $(T, \bar{\varphi}_0, \bar{\varphi}_1)$ is called a (T_0, T_1)-*superior* iff T is a theory and $\bar{\varphi}_l$ is an interpretation of T_l in T, for $l < 2$.

(2) For a cardinal κ, a (T_0, T_1)-superior $(T, \bar{\varphi}_0, \bar{\varphi}_1)$ is called κ-*relevant* iff $|T| < \kappa$.

(3) For regular cardinals λ, μ we say $T_0 \lhd^*_{\lambda, \mu} T_1$ if there is a $\min(\mu, \lambda)$-relevant (T_0, T_1)-superior triple $(T, \bar{\varphi}_0, \bar{\varphi}_1)$ such that in every model M of T in which $M^{[\bar{\varphi}_1]}$ is μ-saturated, the model $M^{[\bar{\varphi}_0]}$ is λ-saturated. If this happens, we call the triple a witness for $T_0 \lhd^*_{\lambda, \mu} T_1$.

(4) We say that $T_0 \lhd^*_{\lambda, \mu} T_1$ *over* θ if $\theta \leqslant \lambda, \theta \leqslant \mu$ and $T_0 \lhd^*_{\lambda, \mu} T_1$ as witnessed by a $(T, \bar{\varphi}_0, \bar{\varphi}_1)$ with $|T| < \theta$.

(5) If $\lambda = \mu$, we write $\lhd^*_\lambda$ in place of $\lhd^*_{\lambda, \mu}$.

(6) We say that $T_1 \lhd^*_\lambda T_2$ iff $T_1 \lhd^*_\lambda T_2$ holds for all large enough regular λ.

(6) T^* is $\lhd^*_\lambda$-maximal iff $T \lhd^*_\lambda T^*$ holds for all T. The notion of $\lhd^*$-maximality is defined analogously.

(7) We say $T_0 \lhd^*_{\lambda, \neq} T_1$ iff $T_0 \lhd^*_\lambda T_1$ but $\neg(T_1 \lhd^*_\lambda T_0)$.

Although in this paper we do not consider this in its own right, it is natural to define the local versions of the $\lhd^*$-relation. This is used by Shelah and Usvyatsov in [14] to obtain their local converse to the implication $\lhd^*$-maximality $\Rightarrow \mathrm{SOP}_2$, see Section 3 for more discussion on this.

Definition 1.3. Relations $\lhd^{*,1}_{\lambda, \mu}$ and $\lhd^{*,1}_\lambda$ are the local versions of $\lhd^*_{\lambda, \mu}$ and $\lhd^*_\lambda$ respectively, where by a local version we mean that in the definition of the relations, only types of the form

$$p \subseteq \{\pm \vartheta(x, \bar{a}): \ \bar{a} \in {}^{\lg(\bar{y})}M\}$$

for some fixed $\vartheta(x, \bar{y})$ are considered.

Observation 1.4. (0) If $T_0 \lhd^*_{\lambda, \mu} T_1$ and $l < 2$. *then* there is a witness $(T, \bar{\varphi}^0, \bar{\varphi}^1)$ such that $\bar{\varphi}^l$ is trivial, hence $T_l \subseteq T$.

(1) $\lhd^*_\lambda$ is a partial order among theories (note that $T \lhd^*_\lambda T$ for every complete T of size $< \lambda$, and that the strict inequality is written as $T_1 \lhd^*_{\lambda, \neq} T_2$).

(2) If $T_0 \lhd^*_{\lambda, \mu} T_1$ over θ and $T_1 \lhd^*_{\mu, \kappa} T_2$ over θ, *then* $T_0 \lhd^*_{\lambda, \kappa} T_2$ over θ.

1   [Why? (0) Trivial.

(1) Suppose that $T_l \lhd^*_\lambda T_{l+1}$ for $l < 2$ over θ, as exemplified by $(T^*, \bar{\varphi}_0, \bar{\varphi}_1)$ and
3   $(T^{**}, \bar{\psi}_1, \bar{\psi}_2)$ respectively. Without loss of generality, $\bar{\varphi}_1$ is trivial (apply part (0)), so
as T^* is complete we have $T_1 \subseteq T^*$. Similarly, without loss of generality, $\bar{\psi}_1$ is trivial
5   and so, as T^{**} is complete, we have $T_1 \subseteq T^{**}$. As T_1 is complete, without loss of
generality, T^* and T^{**} agree on the common part of their vocabularies, and hence
7   by Robinson Consistency Criterion, $T \overset{\text{def}}{=} T^* \cup T^{**}$ is consistent. Also $|T^*| + |T^{**}| < \theta$,
hence $|T| < \theta$. Clearly T interprets T_0, T_1, T_2 by $\bar{\varphi}_0, \bar{\varphi}_1 = \bar{\psi}_1$ and $\bar{\psi}_2$ respectively and
9   T is complete. We now show that the triple $(T, \bar{\varphi}_0, \bar{\psi}_2)$ is a (T_0, T_2)-superior which
witnesses $T_0 \lhd^*_\lambda T_2$ over θ. So suppose that M is a model of T in which $M^{[\bar{\psi}_2]}$ is λ-
11 saturated. As $(T^{**}, \bar{\psi}_1, \bar{\psi}_2)$ witnesses $T_1 \lhd^*_\lambda T_2$, we can conclude that $M^{[\bar{\varphi}_1]} = M^{[\bar{\psi}_1]}$ is
λ-saturated. We can argue similarly that $M^{[\bar{\varphi}_0]}$ is λ-saturated.
13   (2) is proved similarly to (1).]

In this section we consider an example of a theory which is a prototypical example
15 of an NSOP$_3$ theory that is not simple (see [12]). It is the model completion of the
theory of infinitely many (independent) parametrised equivalence relations, formally
17 defined below. We shall prove that for λ such that $\lambda = \lambda^{<\lambda}$ and $2^\lambda = \lambda^+$, this theory is
strictly $\lhd^*_{\lambda^+}$-below the theory of a dense linear order with no first or last element.

Definition 1.5. (1) T_{feq} is the following theory in $\{P, Q, E, R, F\}$
19
(a) Predicates P and Q are unary and disjoint, and $(\forall x)[P(x) \lor Q(x)]$,
21 (b) E is an equivalence relation on Q,
(c) R is a binary relation on $Q \times P$ such that

23          $[x R z \ \& \ y R z \ \& \ x E y] \Rightarrow x = y.$

(so R picks for each $z \in P$ (at most one) representative of any E-equivalence class).
25 (d) F is a (partial) binary function from $Q \times P$ to Q, which satisfies

$$F(x,z) \in Q \ \& \ F(x,z)R z \ \& \ x E F(x,z).$$

27   (so for $x \in Q$ and $z \in P$, the function F picks the representative of the E-equivalence
class of x which is in the relation R with z).
29 (2) T^+_{feq} is T_{feq} with the requirement that F is total.
(3) For $n < \omega$, we let T^n_{feq} be T^+_{feq} enriched by the sentence saying that over any n
31 elements, any (not necessarily complete) quantifier free type consisting of basic (atomic
and negations of the atomic) formulae with no direct contradictions, is realised.

33 **Note 1.6.** One may easily check that every model of T_{feq} can be extended to a model
of T^+_{feq} and that T^+_{feq} has the amalgamation property and the joint embedding property.
35 This theory also has a model completion, which can be constructed directly, and which
we denote by T^*_{feq}. It follows that T^*_{feq} is a complete theory with infinite models, in
37 which F is a full function.

Remark 1.7. Notice that T_{feq} has been defined somewhat differently than in [12, Section 1], but the difference is non-essential, as the following Claim 1.8 shows that the two theories have the same model completion. This claim also shows the origin of the name "infinitely many independent equivalence relations" for T_{feq}^*.

Claim 1.8. *Let T be the theory defined (in [12,1]) by*

(a) *T has unary predicates P and Q and a three place relation E written as $y E_x z$,*

(b) *the universe of any model of T is a disjoint union of P and Q,*

(c) *$y E_x z \Rightarrow P(x) \ \& \ Q(y), Q(z)$,*

(d) *for any fixed $x \in P$ the relation E_x is an equivalence relation on Q.*
    Then T_{feq}^* is the model completion of T.

Proof. Let M be a model of T_{feq}, we shall extend M to a model of T as follows. Each E-equivalence class $e = a/E$ gives rise to an equivalence relation E_e on P given by

$$z_1 E_e z_2 \ \text{iff} \ z_1, z_2 \in P \quad \text{and} \quad F(a, z_1) = F(a, z_2).$$

This definition does not depend on a, just on a/E. Let P^N and Q^N be Q^M and P^M respectively. Define $y E_x^N z$ iff $y E_e z$ where $e = x/E^M$. Clearly N is a model of T.

Now suppose that we have a model M of T and we shall extend it to a model N of T_{feq}. Let P^N and Q^N be Q^M and P^M respectively. Define $x E^N x'$ iff for every y, z we have $y E_x z$ iff $y E_{x'} z$. Choose a representative of each E-equivalence class and for any $z \in Q^N$ and such a representative x let $F(x, z) = x$. Then for $x' \in Q^N$ which has not been chosen as a representative of any equivalence class, find x which has been chosen as its representative and define $F(x', z) = F(x, z)$ for all $z \in P^N$.

This shows that T_{feq} and T are cotheories [1, 3.5.6(2)]. Being the model completion of T_{feq}, T_{feq}^* is its cotheory, and hence a cotheory of T. Hence T_{feq}^* is a model companion of T. In order to prove that it is the model completion of T it suffices to show that T has the amalgamation property [1, 3.5.18] which is easily seen directly. □

Observation 1.9. T_{feq}^* has elimination of quantifiers and for any n, any model of T_{feq}^* is a model of T_{feq}^n.

Notation 1.10. T_{ord} stands for the theory of a dense linear order with no first or last element.

The following convention will make the notation used in this section simpler.

Convention 1.11. Whenever considering $(T_{\text{ord}}, T_{\text{feq}}^*)$-superiors $(T, \bar{\varphi}, \bar{\psi})$ we shall abuse the notation and assume $\bar{\varphi} = (I, <_0)$ and $\bar{\psi} = (P, Q, E, R, F)$. In such a case we may also write P^M in place of $P^{M[\bar{\psi}]}$, etc., and we may simply say that T is a $(T_{\text{ora}}, T_{\text{feq}}^*)$-superior.

We intend to prove that for λ satisfying $\lambda^{<\lambda}$ and $2^\lambda = \lambda^+$ the theory T_{feq}^* is strictly $\lhd_{\lambda^+}^*$-below T_{ord} (Theorem 1.17 below). This will be done by a diagonalisation argument

1   where for a given λ-relevant $(T_{\mathrm{ord}}, T^*_{\mathrm{feq}})$-superior T we inductively construct a model of T that is saturated for T^*_{feq} but not for T_{ord}. Main Claim 1.13 provides one step in
3   the required induction. In Stage A of its proof we use the elimination of quantifiers in T^*_{feq} to reduce the situation to T_{feq}-types of four prescribed kinds, and then we show
5   that we may in fact work only with three of them. Stage B contains the main point of the proof, which is the construction of a certain tree of models and embeddings.
7   Once this is done in Stage C we use the analysis from Stage A to show that the T^*_{feq}-type defined by the union of the embeddings is consistent. In Stage D we take $N \prec \mathfrak{C}$
9   of size λ that realises this type and show that N must omit most of the Dedekind cuts induced by the tree of embeddings, and that most of these cuts are not definable
11   over N. After an application of an appropriate automorphism of $\mathfrak{C}$ this finishes the proof of the Main Claim. The proof of the theorem then follows by induction. The
13   cardinal arithmetic assumptions are used in Stage D and in the inductive proof of the theorem.

15   **Definition 1.12.** For a λ-relevant $(T_{\mathrm{ord}}, T^*_{\mathrm{feq}})$-superior T, *the statement*

$$*[M, \bar{a}, \bar{b}] = *[M, \bar{a}, \bar{b}, T, \lambda]$$

17   means:

    (i) M is a model of T of size λ,
19   (ii) $\bar{a} = \langle a_i \colon i < \lambda \rangle$, $\bar{b} = \langle b_i \colon i < \lambda \rangle$, are sequences of elements of $I^{M[\bar{\varphi}]}$ such that

$$i < j < \lambda \ \Rightarrow \ a_i <_0 a_j <_0 b_j <_0 b_i,$$

21   (iii) there is no $x \in M^{[\bar{\varphi}]}$ such that for all i we have $a_i <_0 x <_0 b_i$,
    (iv) the Dedekind cut $\{x \colon \bigvee_{i < \lambda} x <_0 a_i\}$ is not definable by any formula of $\mathscr{L}(M)$
23   with parameters in M.

   Main Claim 1.13. *Assume $\lambda^{<\lambda} = \lambda$ and $(T, \bar{\varphi}, \bar{\psi})$ is a λ-relevant $(T_{\mathrm{ord}}, T^*_{\mathrm{feq}})$-superior.*
25   *Further assume that $*[M, \bar{a}, \bar{b}]$ holds, and $p = p(z)$ is a (consistent) T^*_{feq}-type over $M^{[\bar{\psi}]}$. Then there is $N \models T$ with $M \prec N$, such that $p(z)$ is realised in $N^{[\bar{\psi}]}$ and $*[N, \bar{a}, \bar{b}]$*
27   *holds.*

   Proof.
29   **Stage A.** Without loss of generality, p is complete in the T^*_{feq}-language over $M^{[\bar{\psi}]}$.
   (By Convention 1.11, we can consider p to be a type over M (rather than $M^{[\bar{\psi}]}$). We
31   shall use this Convention throughout the proof). If p is realised in M, our conclusion follows by taking $N = M$, so let us assume that this is not the case. Using the elimina-
33   tion of quantifiers for T^*_{feq}, we can without loss of generality assume that $p(z)$ consists of quantifer free formulae with parameters in M. This means that one of the following
35   four cases must happen:
    Case 1: (This will be the main case) $p(z)$ implies that $z \in P$ and it determines which
37   elements of Q^M are R-connected to z. Hence for some function $f \colon Q^M \to Q^M$ which

1   respects E, i.e.

$$a E b \Rightarrow f(a) = f(b),$$

3   and

$$f(a) \in a/E^M;$$

5   we have

$$p(z) = \{P(z)\} \cup \{b R z : b \in \mathrm{Rang}(f)\}$$

7   and no $a \in P^M$ satisfies p.

   Case 1A: Like Case 1, but f is a partial function and

$$p(z) = \{P(z)\} \cup \{f(b) R z : b \in \mathrm{Dom}(f)\}$$
$$\cup \{\neg(b R z) : (b/E^M \cap \mathrm{Rang}(f)) = \emptyset\}.$$

9   (This case will be reduced to Cases 1–3 in Subclaim 1.15).

   Case 2: $p(z)$ determines that $z \in Q$ and that it is E-equivalent to some $a^* \in Q^M$, but
11 not equal to any "old" element. Note that in this case if b^* realises $p(z)$, we cannot
   have $b^* R c$ for any $c \in P^M$, as this would imply $F(a^*, c) = b^* \in M^{[\bar{\psi}]}$ (and we have
13 assumed that p is not realised in $M^{[\bar{\psi}]}$). Hence, the complete M-information is given
   by

15 $$p(z) = \{Q(z)\} \cup \{a^* E z\} \cup \{a \neq z : a \in a^*/E^M\}.$$

   Case 3: $p(z)$ determines that $z \in Q$, but it has a different E-equivalence class than
17 any of the elements of Q^M. As p is complete, it must determine for which $c \in P^M$
   we have $z R c$, and for which $c, d \in P^M$ we have $F(z, c) = F(z, d)$. Hence, for some
19 $f : P^M \to \{\mathrm{yes}, \mathrm{no}\}$ and some $\mathscr{E}$, an equivalence relation on P^M such that $c \mathscr{E} d \Rightarrow f(c)$
   $= f(d)$, we have

$$p = \{Q(z)\} \cup \{\neg(a E z) : a \in Q^M\} \cup \{(z R b)^{f(b)} : b \in P^M\}$$
$$\cup \{(F(z, c) = F(z, d))^{\mathrm{if} c \mathscr{E} d} : c, d \in P^M\}.$$

21   In the above description, we have used

   Notation 1.14. For a formula ϑ we let $\vartheta^{\mathrm{yes}} \equiv \vartheta$ and $\vartheta^{\mathrm{no}} \equiv \neg\vartheta$.

23 **Subclaim 1.15.** *It suffices to deal with Cases* 1–3, *ignoring the Case* 1A.

   Proof. Suppose that we are in the Case 1A. Let

25 $$\{d_i/E^M : i < i^* \leqslant \lambda\}$$

   list the d/E^M for $d \in Q^M$ such that $d' \in d/E^M \Rightarrow \neg(d' R z) \in p(z)$. We choose by in-
   duction on $i \leqslant i^*$ a pair (M_i, c_i) such that
27
   (a) $M_0 = M$, $\|M_i\| = \lambda$,

(b) $\langle M_i \colon i \leqslant i^* \rangle$ is an increasing continuous elementary chain,

(c) $*[M_i, \bar{a}, \bar{b}]$

(d) $c_i \in (d_i/E^{M_{i+1}}) \backslash M_i$, for $i < i^*$.

For i limit or $i = 0$, the choice is trivial. For the situation when i is a successor, we use Case 2.

Let $\langle c_i/E^{M_{i^*}} \colon i \in [i^*, i^{**}) \rangle$ list without repetitions the $c/E^{M_{i^*}}$ which are disjoint to M. Note that $|i^{**}| \leqslant \lambda$. Let

$$p^+(z) \stackrel{\text{def}}{=} p(z) \cup \{c_i R z \colon i < i^{**}\}.$$

Then $p^+(z)$ is a complete type (for $M_{i^*}^{[\bar{\psi}]}$), and $*[M_{i^*}, \bar{a}, \bar{b}]$ holds by (c). If $p^+(z)$ is realised in M_{i^*}, we can let $N = M_{i^*}$ and we are done. Otherwise, $p^+(z)$ is not realised in M_{i^*} and is a type of the form required in Case 1, so we can proceed to deal with it using the assumptions on Case 1. $\quad\square$

Stage B. Let us assume that p is a type as in one of the Cases 1, 2 or 3, which we can do by Subclaim 1.15. We shall define $\langle M_\alpha \colon \alpha < \lambda \rangle$, an $\prec$-increasing continuous sequence of elementary submodels of M, each of size $< \lambda$, and with union M, such that:

(a) In Case 1, each M_α is closed under f,

(b) In Case 2, $a^* \in M_0$,

(c) For every $\alpha < \lambda$,

$$(M_\alpha, \{(a_j, b_j) \colon j < \lambda\} \cap M_\alpha) \prec (M, \{(a_j, b_j) \colon j < \lambda\}).$$

Hence, for some club C of λ consisting of limit ordinals δ, we have that for all $\delta \in C$,

$$a_j \in M_\delta \;\Leftrightarrow\; b_j \in M_\delta \;\Leftrightarrow\; j < \delta,$$
$$(\forall c \in I^{M_\delta})(\exists j < \delta)[c <_0 a_j \vee b_j <_0 c].$$

Let $C = \{\delta_i \colon i < \lambda\}$ be an increasing continuous enumeration.

Now we come to the *main point* of the proof.

By induction on $i = lg(\eta) < \lambda$ we shall choose $\bar{h} = \langle h_\eta \colon \eta \in {}^{\lambda >}2 \rangle$, a sequence such that

(α) h_η is an elementary embedding of $M_{\delta_{lg(\eta)}}$ into $\mathfrak{C}_T$, whose range will be denoted by N_η.

(β) $v \triangleleft \eta \Rightarrow h_v \subseteq h_\eta$.

(γ) If $\eta_l \in {}^{\lambda >}2$ for $l = 0, 1$ and $\eta_0 \cap \eta_1 = \eta$, *then*:

(i) $N_{\eta_0} \cap N_{\eta_1} = N_\eta$.

(ii) In addition, if $a_l \in Q^{N_{\eta_l}}$ for $l = 0, 1$ and $a_0 E^{\mathfrak{C}_T} a_1$, *then* for some $a \in Q^{N_\eta}$ we have $a_l E^{\mathfrak{C}_T} a$ for $l = 0, 1$. (Equivalently, if $a_l \in Q^{N_{\eta_l}}$ and $\neg(\exists a \in N_\eta)(\bigwedge_{l < 2} a_l E a)$, then $\neg(a_0 E a_1)$).

(δ) $\models$ "$h_{\eta \frown \langle 0 \rangle}(b_{\delta_{lg(\eta)}}) <_0 h_{\eta \frown \langle 1 \rangle}(a_{\delta_{lg(\eta)}})$" (see Convention 1.11 on $<_0$).

1 Note that the requirement of h_η being elementary and onto N_η in particular implies that

3 (δ') If for some $l < 2$ and $\eta \in {}^{\lambda >}2$ we have $a \in N_{\eta \frown \langle l \rangle} \setminus N_\eta$ and $b \in N_\eta$, *then* $a E^{\mathfrak{C}_T} b$ iff $a = h_{\eta \frown \langle l \rangle}(a')$ for some a' such that $a' E^{\mathfrak{C}_T} h_\eta^{-1}(b)$.

5 We now describe the inductive choice of h_η for $\eta \in {}^{\lambda >}2$, the induction being on $i = lg(\eta)$. Let $h_{\langle \rangle} = \mathrm{id}_{M_0}$. If i is a limit ordinal, we just let $h_\eta \overset{\text{def}}{=} \bigcup_{j < lg(\eta)} h_{\eta \restriction j}$. Hence,

7 the point is to handle the successor case.

Fixing $i < \lambda$, let $\langle \eta_{i,\alpha} : \alpha < \alpha^* \leqslant \lambda \rangle$ list ${}^{i+1}2$, in such a manner that $\eta_{i,2\alpha} \restriction i = \eta_{i,2\alpha+1} \restriction i$

9 and $\eta_{i,2\alpha+l}(i) = l$ for $l < 2$ (we are using the assumption $\lambda^{<\lambda} = \lambda$). Now we choose $h_{\eta_{i,2\alpha+l}}$ by induction on α. Hence, coming to α, let us denote by η_l the sequence $\eta_{i,2\alpha+l}$,

11 and let $\eta_0 \cap \eta_1 = \eta$ (so $\eta_0 \restriction i = \eta_1 \restriction i = \eta$). Let $M_{\delta_{i+1}} \setminus M_{\delta_i} = \{d_j^i : j < j_i^*\}$, so that $d_0^i = a_{\delta_i}$ and $d_1^i = b_{\delta_i}$. We consider the type Γ, which is the union of

13 (a)

$$\Gamma_0 \overset{\text{def}}{=} \left\{ \begin{array}{l} \varphi(x_{j_0}^0, \ldots, x_{j_{n-1}}^0; h_\eta(\bar{c})) : n < \omega \ \& \ \bar{c} \subseteq M_{\delta_i} \ \& \ j_0, \ldots, j_{n-1} < j_i^* \ \& \\ M_{\delta_{i+1}} \models \varphi(d_{j_0}^i, \ldots, d_{j_{n-1}}^i; \bar{c}) \end{array} \right\},$$

15 (taking care of one "side" (for η_0 or η_1) of the part (α) above).

(b) Γ_1, defined analogously to Γ_0, but with $x_{j_0}^0, \ldots, x_{j_{n-1}}^0$ replaced everywhere by

17 $x_{j_0}^1, \ldots, x_{j_{n-1}}^1$

(taking care of the remaining "side" of (α) above, interchanging η_0 and η_1).

19 (c) $\Gamma_2 = \{(x_0^0, x_1^0)_I \cap (x_0^1, x_1^1)_I = \emptyset\}$

(this says that the above intervals in $<_0$ are disjoint, which after the right choice of

21 $h_{\eta_0}(d_j^i) = $ a realisation of x_j^0 or $h_{\eta_0}(d_j^i) = $ a realisation of x_j^1 ($j < 2$), and similarly for h_{η_1}, will take care of the part (δ) above.)

23 (d) $\Gamma_3 = \Gamma_3^0 \cup \Gamma_3^1$, where for $l < 2$

$$\Gamma_3^l = \{x_j^l \neq c : j < j_i^*, c \in \cup\{\mathrm{Rang}(h_\rho) : h_\rho \text{ already defined}\}\}.$$

25 (e) $\Gamma_4 = \{x_{j_1}^0 \neq x_{j_2}^1 : j_0, j_1 < j_i^*\}$

((d)+(e) are taking care of (γ) above, part (i)).

27 (f)

$$\Gamma_5 = \left\{ \begin{array}{l} \neg(x_{j_0}^0 E x_{j_1}^1) : \quad \text{if } j_0, j_1 < j_i^* \\ \text{but there is no } a \in M_{\delta_i} \text{ with } [d_{j_0}^i E a \vee d_{j_1}^i E a] \end{array} \right\}.$$

29 (together with Γ_6 below, taking care of part (γ)(ii), see below. Note that the type is defined using $\vee$ rather than $\wedge$, but this will turn out to be sufficient.)

31 (g) $\Gamma_6 = \Gamma_6^0 \cup \Gamma_6^1$, where

$$\Gamma_6^l = \left\{ \begin{array}{l} \neg(x_j^l E b) : j < j_i^* \text{ and } b \text{ is an element of the set} \\ \cup\{\mathrm{Rang}(h_\rho) : h_\rho \text{ already defined and } \neg(\exists c \in N_\eta)[b \, E \, c]\} \end{array} \right\}.$$

33 First note that requiring $\Gamma_5 \cup \Gamma_6$ throughout the construction indeed guarantees that

35 (γ)(ii) can be satisfied. Namely, suppose that the realisations of $x_{j_0}^0$ and $x_{j_1}^1$ are

1 E-equivalent. Then by Γ_5 we must have that for some $l < 2$ and $a \in M_{\delta_i}$ we have that
$d^i j_l E a$. By transitivity then the realisation of $x^{1-l}_{j_{1-l}}$ would have to be E-equivalent to

3 $h_\eta(a)$, which might only be precluded by $d^i_{j_{1-l}}$ being E equivalent to some c such that
a and c are not E-equivalent. This cannot happen by Γ_6.

5   We conclude that, if Γ is consistent, as $\mathfrak{C}$ is $\bar{\kappa}$-saturated, the functions h_{η_l} can be
defined. Namely, for a realisation $\{c^l_j : j < j^*_i, \ l < 2\}$ of Γ, we can define $g_l(d^i_j) = c^l_j$,

7 and then we let $h_{\eta_0} = g_0$ if $c^0_1 <_0 c^1_0$, and g_1 otherwise. We let $h_{\eta_1} = g_{1-l}$ if $h_{\eta_0} = g_l$.
This guarantees that (δ) above is satisfied.

9   Let us then show that Γ is consistent. Suppose for contradiction that this is not so, so
we can find a finite $\Gamma' \subseteq \Gamma$ which is inconsistent. Let $\{j_0, \ldots, j_{n-1}\}$ be an increasing

11 enumeration of a set including all $j < j^*_i$ such that x^l_j is mentioned in Γ' for some
$l < 2$ and let $\bar{d} = \langle d^i_{j_0}, \ldots, d^i_{j_{n-1}} \rangle$. Without loss of generality, 0 and 1 appear in the

13 list $\{j_0, \ldots, j_{n-1}\}$ and hence $j_0 = 0$ while $j_1 = 1$. By closing under conjunctions and
increasing Γ' (retaining that $\Gamma' \subseteq \Gamma$ is finite) if necessary, we may assume that for

15 some formula $\sigma(x_0, \ldots, x_{n-1}; \bar{c}) \in \text{tp}(\bar{d}/M_{\delta_i})$, we have

$$\Gamma' \cap \Gamma_l = \{\vartheta_l(x^l_{j_0}, \ldots, x^l_{j_{n-1}}; h_\eta(\bar{c}))\}$$

17 for $l < 2$, where $\vartheta_l(x^l_{j_0}, \ldots, x^l_{j_{n-1}}; h_\eta(\bar{c}))$ is the formula obtained from σ by replacing x_k
by $x^l_{j_k}$ and $\bar{c}$ by $h_\eta(\bar{c})$.

19   Let ϑ_2 be the formula comprising Γ_2 and $\vartheta^l_3(\bar{x}^l; \bar{c}^l_3) = \bigwedge(\Gamma^l_3 \cap \Gamma')$, while $\vartheta_4 = \bigwedge(\Gamma_4$
$\cap \Gamma')$ and $\vartheta_5 = \bigwedge(\Gamma_5 \cap \Gamma')$. Let $\vartheta_3 = \vartheta^0_3 \wedge \vartheta^1_3$ and $\vartheta = \bigwedge_{k < 6, \ k \neq 2,3} \vartheta_k$. Without loss of

21 generality, ϑ includes statements $x^l_0 \neq \cdots \neq x^l_{n-1}$ and $x^l_0 <_0 x^l_1$ for $l < 2$. We may also
assume that $(x^0_0, x^0_1)_I \cap (x^1_0, x^1_1)_I = \emptyset$ is included in Γ'. The choice of n may be assumed

23 to have been such that for some $c^l_0, \ldots, c^l_{n-1}$ (for $l < 2$) from $\bigcup\{\text{Rang}(h_\rho) : h_\rho \text{ already}$
defined$\}$, we have

25     $$\Gamma' \cap \Gamma_3 = \{x^l_{j_m} \neq c^l_k : l < 2, \ k < n, \ m < n\},$$

and finally that

27     $$\Gamma' \cap \Gamma_5 = \{\neg(x^0_{j_k} E x^1_{j_m}) : k, m < n \ \&\ \neg(\exists a \in M_{\delta_i})[d^i_{j_k} E a \vee d^i_{j_m} E a]\}.$$

By extending h_η to an automorphism $\hat{h}_\eta$ of $\mathfrak{C}$, and applying $(\hat{h}_\eta)^{-1}$, we may assume

29 that $h_\eta = \text{id}_{M_{\delta_i}}$. We can also assume that no c^l_k is an element of M_{δ_i}, as otherwise the
relevant inequalities can be absorbed by σ.

31   We shall use the following general

Fact 1.16. *Suppose that $N \prec \mathfrak{C}$ and $\bar{e} \in {}^m\mathfrak{C}$ is disjoint from N, while $N \subseteq A$. Then*

$$r(\bar{x}) \stackrel{\text{def}}{=} \text{tp}(\bar{e}, N) \cup \{x_k \neq d : d \in A \backslash N, \ k < m\}$$
$$\cup \{\neg(x_k E d) : d \in A \ \& \ (d/E) \cap N = \emptyset, \ k < m\},$$

33 *is consistent (and in fact, every finite subset of it is realised in N).*

Proof. Otherwise, there is a finite $r'(\bar{x}) \subseteq r(\bar{x})$ which is inconsistent. Without loss of generality, $r'(\bar{x})$ is the union of sets of the following form (we have a representative type of the sets for each clause)

- $\{\varrho(\bar{x}, \bar{c})\}$ for some $\bar{c} \subseteq N$ and ϱ such that $\models \varrho[\bar{e}, \bar{c}]$.
- $\{x_k \neq \hat{c}_k^s \colon k < m\}$ for some $\hat{c}_0^s, \ldots, \hat{c}_{m-1}^s \in A \setminus N$ and $s < s^* < \omega$,
- $\{\neg(x_k E \hat{d}_k^t) \colon k < m\}$ for some $\hat{d}_0^t, \ldots, \hat{d}_{m-1}^t \in A \setminus N$ and $t < t^* < \omega$ such that $(\hat{d}_k^t / E) \cap N = \emptyset$.

By the elementarity of N, there is $\bar{e}' \in N$ with $N \models \varrho[\bar{e}', \bar{c}]$. By the choice of the rest of the formulae in $\vec{r}'(\bar{x})$, we see that $\bar{e}'$ satisfies them as well, as which is a contradiction. □

Let $\bar{x}^l = (x_0^l, \ldots, x_{n-1}^l)$ for $l < 2$. Let

$$\Phi_0 \overset{\text{def}}{=} \{\varphi(\bar{x}^0) \colon \varphi(x_{j_0}^0, \ldots, x_{j_{n-1}}^0) \in \Gamma' \cap (\Gamma_0 \cup \Gamma_3^0 \cup \Gamma_6^0)\}.$$

Applying the last phrase in the above Fact to $\Phi_0(\bar{x}^0)$, the model M_{δ_i} and $\bar{d}$, we obtain a sequence $\bar{e}^0 = (e_0^0, \ldots, e_{n-1}^0) \in M_{\delta_i}$ which realises $\Phi_0(\bar{x}^0)$. For k, m such that $\neg(x_{j_k}^0 E x_{j_m}^0) \in \Gamma_5$ we have $\neg(\exists a \in M_{\delta_i})(a E d_{j_k}^i \vee a E d_{j_m}^i)$. So

$$\neg(x_k E e_m^0) \wedge \neg(e_m^0 E x_m) \in \mathrm{tp}(\bar{d}/M_{\delta_i}).$$

Let now

$$\Phi_1(\bar{x}^1) = \{\neg(x_k^1 E e_m^0) \wedge \neg(e_k^0 E x_m^1) \colon \neg(x_{j_k}^0 E x_{j_m}^1) \in \Gamma_5\}$$

$$\cup \{x_k^1 \neq e_m^0 \colon k, m < n\} \cup \{\varphi(\bar{x}^1) \colon \varphi(x_{j_0}^1, \ldots, x_{j_{n-1}}^1)$$

$$\in \Gamma' \cap (\Gamma_1 \cup \Gamma_3^1 \cup \Gamma_6^1)\}.$$

$\Phi_1(\bar{x}^1)$ is a finite set to which we can apply the last phrase of Fact 1.16. In this way we find $\bar{e}^1 = (e_0^1, \ldots, e_{n-1}^1) \in M_{\delta_i}$ realising $\Phi_1(\bar{x}^1)$. Now we show that $\bar{e}^0 \frown \bar{e}^1$ realises $\Gamma' \setminus \Gamma_2$. So suppose $\neg(x_{j_k}^0 E x_{j_m}^1) \in \Gamma' \cap \Gamma_5$, then $\neg(x_k^1 E e_m^0) \in \Phi_1$, hence $\neg(e_k^l E e_m^0)$. Also $\bigwedge_{k, m < n}(e_k^1 \neq e_m^0)$ holds, by the choice of Φ_1, so $\bar{e}^0 \frown \bar{e}^1$ realises $\Gamma' \cap \Gamma_4$. Now we need to deal with Γ_2. Let

$$\mathscr{D} \overset{\text{def}}{=} \{(\bar{u}^0, \bar{u}^1) \colon (\bar{u}^0, \bar{u}^1) \text{ satisfies } \vartheta\}.$$

So $\mathscr{D}$ is first order definable with parameters in M_{δ_i} and we have just shown that $\mathscr{D} \cap M_{\delta_i} \neq \emptyset$. Also if $\bar{e}^0 \frown \bar{e}^1 \subseteq M_{\delta_i}$ satisfies ϑ, it necessarily realises $\Gamma' \setminus \Gamma_2$ (as no $c_k^l \in M_{\delta_i}$, see the definition). As Γ' is presumed to be inconsistent, no $(\bar{u}^0, \bar{u}^1) \in \mathscr{D} \cap M_{\delta_i}$ can realise Γ', i.e. satisfy ϑ_2, and hence for no $(\bar{u}^0, \bar{u}^1) \in \mathscr{D} \cap M_{\delta_i}$ are the intervals $(u_0^0, u_1^0)_I$ and $(u_0^1, u_1^1)_I$ disjoint. Now we claim that if $(\bar{u}^0, \bar{u}^1) \in M_{\delta_i} \cap \mathscr{D}$, then $(u_0^0, u_1^0)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$.

Indeed, suppose otherwise, say $u_1^0 <_0 d_0^i = a_{\delta_i}$, so $u_1^0 <_0 x_0 \in \mathrm{tp}(\bar{d}/M_{\delta_i})$. Arguing as above, with $\bar{u}^0$ in place of $\bar{e}^0$ and $\Phi_1(\bar{x}) \cup \{u_1^0 <_0 x_0^1\}$ in place of $\Phi_1(\bar{x}^1)$, we can find $\bar{u} \in M_{\delta_i}$ satisfying $(u_1^0 <_0 u_0)$ and such that $(\bar{u}^0, \bar{u})$ satisfies ϑ. So $(\bar{u}^0, \bar{u}) \in \mathscr{D} \cap M_{\delta_i}$

1   and the intervals $(u_0^0, u_1^0)_I$ and $(u_0, u_1)_I$ are disjoint, a contradiction. A similar contra-
diction can be derived from the assumption $b_{\delta_i} = d_1^{\delta_i} <_0 u_0^0$. Now note that $(\vec{u}^0, \vec{u}^1) \in$
3    $\mathcal{D} \Rightarrow (\vec{u}^1, \vec{u}^0) \in \mathcal{D}$, so if $(\vec{u}^0, \vec{u}^1) \in \mathcal{D} \cap M_{\delta_i}$ we also have $(u_1^1, u_1^1)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$.

By the choice of C, there is no $x \in M_{\delta_i}$ with $d_0^{\delta_i} \leqslant_0 x \leqslant_0 d_1^{\delta_i}$, hence

5   if $(\vec{u}^0, \vec{u}^1) \in M_{\delta_i} \cap \mathcal{D}$ and $l < 2$, we have $u_0^1 <_0 d_0^{\delta_i} <_0 d_1^{\delta_i} <_0 u_1^l$. $(*)$

Let $\sigma^*(\bar{Y})$ be $\exists \bar{x}((\bar{x}, \bar{y}) \in \mathcal{D})$. Hence if

7        $\varrho_0(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge z \leqslant_0 y_0]$

and

9        $\varrho_1(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge y_1 \leqslant_0 z]$,

then

11        $M_{\delta_i} \models (\forall z_0, z_1)[\varrho_0(z_0) \wedge \varrho_1(z_1) \Rightarrow z_0 <_0 z_1]$.

Of course, this holds in $\mathfrak{C}$ as well, as

13   (a) $\varrho_0(z)$ defines an initial segment of I,
      (b) $\varrho_1(z)$ defines an end segment of I,
15   (c) the segments defined by $\varrho_0(z)$ and $\varrho_1(z)$ are disjoint,
      (d) $\varrho_0(M_{\delta_i}) \cup \varrho_1(M_{\delta_i}) = I \cap M_{\delta_i}$.
17   [Why? Note that $(\bar{e}^0, \bar{d}) \in \mathcal{D}$. Hence $\sigma^*(\bar{d})$ holds. As for every $a \in I \cap M_{\delta_i}$ we have
      $a <_I a_{\delta_i}$ or $a >_I b_{\delta_i}$, the conclusion follows.]
19   (e) $\varrho_0(a_{\delta_i})$ and $\varrho_1(b_{\delta_i})$ hold.
        [Why? Again because $\sigma^*(\bar{d})$ holds.]

21   The above arguments show that $\{x : (\exists \bar{y})[(\sigma^*(\bar{y}) \wedge x <_0 y_0)]\}$ defines the Dedekind
cut $\{x : x <_0 a_{\delta_i}\}$ over M_{δ_i}, which contradicts the choice of C and the fact that the
23   Dedekind cut induced by $(\bar{a}, \bar{b})$ is not definable (which is a part of the definition of
$*[M, \bar{a}, \bar{b}]$).
25   **Stage C.** Now we have shown that the trees $\langle N_\eta : \eta \in {}^{\lambda >}2 \rangle$, $\langle h_\eta : \eta \in {}^{\lambda >}2 \rangle$ of models
and embeddings can be defined as required, and we consider

27        $p^* \overset{\text{def}}{=} \bigcup_{\eta \in {}^{\lambda >}2} h_\eta(p \restriction M_{\delta_{\lg(\eta)}})$.

We shall show that p^* is finitely satisfiable, hence satisfiable. Let $\Gamma' \subseteq p^*$ be finite.
29   Recalling the analysis of p from Stage A, we consider each of the cases by which p
could have been defined (ignoring Case 1A, as justified by Subclaim 1.15.)
31   *Case* 1: In this case there is a function $f : Q^M \to Q^M$ respecting E, with $aEf(a)$
for all $a \in Q^M$, and without loss of generality there are some $n_0, \ldots, \eta_{m-1} \in {}^{\lambda >}2$ and
33    $\{b_i^j : j < m, i < n_j\} \subseteq \text{Rang}(f)$ such that

$$\Gamma' = \{P(z)\} \cup \bigcup_{j < m} \{h_{\eta j}(b_i^j) R z : i < n_j\},$$

1   and for each j we have $\{b_i^j : i < n_j\} \subseteq M_{\delta_{\lg(\eta_j)}}$. Let $n \overset{\text{def}}{=} \Sigma_{j<m} n_j$, hence Γ' is a quantifier free (partial) type over n variables in $\mathfrak{C}^{[\bar\psi]}$. By Observation 1.9, we only need to check

3   that in Γ' there are no direct contradictions with the axioms of T_{feq}^+.

The only possibility for such a contradiction is that for some j_0, j_1 and $b_i^{j_0}, b_k^{j_1}$ we

5   have

$$h_{\eta_{j_0}}(b_i^{j_0}) \neq h_{\eta_{j_1}}(b_k^{j_1}) \wedge h_{\eta_{j_0}}(b_i^{j_0}) E h_{\eta_{j_1}}(b_k^{j_1})$$

7   and $h_{j_0}(b_i^{j_0}) R z, h_{j_1}(b_k^{j_1}) R z \in \Gamma'$. In such a case, any c which would realise Γ' would contradict part (c) of the definition of T_{feq}^+. Suppose that $b_i^{j_0}, b_k^{j_1}$ and η_0, η_1 are as above.

9   Let $\eta_l \overset{\text{def}}{=} \eta_{j_l}$ for $l < 2$ and let $\eta = \eta_0 \cap \eta_1$. By part (γ)(ii) in the definition of $\bar h$, there is $\hat b \in N_\eta$ such that $h_{\eta_0}(b_i^{j_0}) E \hat b$ and $h_{\eta_1}(b_k^{j_1}) E \hat b$. For some $b \in M_{\delta_{\lg(\eta)}}$ we have

11     $h_{\eta_0}(b) = h_{\eta_1}(b) = h_\eta(b) = \hat b,$

so applying the elementarity of the maps, we obtain

13     $b_i^{j_0} \, E \, b \, E \, b_k^{j_1}.$

On the other hand, by the definition of p^* we have $b_i^{j_0} R z \in p(z)$ and $b_k^{j_1} R z \in p(z)$.

15   By the demands on p this implies that $b_i^{j_0} = b_k^{j_1} \notin M_{\delta_{\lg(\eta)}}$ and $f(b) = b_i^{j_0}$, contradicting the fact that $M_{\delta_{\lg(\eta)}}$ is closed under f.

17   *Case* 2: For a fixed $a^* \in M_0$ we have

$$p(z) = \{Q(z)\} \cup \{a^* E z\} \cup \{z \neq c : c \in a^*/E^M\},$$

19   so without loss of generality

$$\Gamma' = \{Q(z)\} \cup \{a^* E z\} \cup \{z \neq h_{\eta_j}(c_j) : \ j < m\}$$

21   for some $c_0, \ldots, c_{m-1} \in a^*/E^M$ and $\eta_0, \ldots, \eta_{m-1} \in {}^{\lambda >}2$, as $h_{\langle\rangle} = \text{id}_{M_0}$. As a^*/E is infinite in any model of T_{feq}^*, the set Γ' is consistent.

23   *Case* 3: We may assume that for some equivalence relation $\mathscr{E}$ on P^M, a function f from P^M into $\{\text{yes}, \text{no}\}$, sequences $\eta_0, \ldots, \eta_{n-1} \in {}^{\lambda >}2$, and $\{a_i^k : i < m, k < n\} \subseteq Q^M$ and

25   $\{b_i^k, c_i^k, d_i^k : i < m, k' < n\} \subseteq P^M$ we have $e_1 \mathscr{E} e_2 \Rightarrow f(e_1) = f(e_2)$ and

$$\Gamma'(z) = \{Q(z)\} \cup \bigcup_{k<n} \{\neg(h_{\eta_k}(a_i^k) E z) : i < m\} \cup \bigcup_{k<n} \{(zRh_{\eta_k}(b_i^k))^{f(b_i^k)} : i < m\}$$

$$\cup \bigcup_{k<n} \{[F(z, h_{\eta_k}(c_i^k)) = F(z, h_{\eta_k}(d_i^k))]^{\text{if } c_i^k \mathscr{E} d_i^k} : i < m\}.$$

We could have a contradiction if for some k_1, k_2, i_1, i_2 we had $f(b_{i_1}^{k_1}) = \text{yes}$, $f(b_{i_2}^{k_2}) = \text{no}$,

27   but $h_{\eta_{k_1}}(b_{i_1}^{k_1}) = h_{\eta_{k_2}}(b_{i_2}^{k_2})$, which cannot happen by γ(i) and the fact that each h_η is 1–1. Another possibility is that for some $b_{i_1}^{k_1}, b_{i_2}^{k_2}$ we have $f(b_{i_1}^{k_1}) = f(b_{i_2}^{k_2}) = \text{yes}$, but

29   $h_{\eta_{k_1}}(b_{i_1}^{k_1}) \neq h_{\eta_{k_2}}(b_{i_2}^{k_2})$ while $h_{\eta_{k_1}}(b_{i_1}^{k_1}) E h_{\eta_{k_2}}(b_{i_2}^{k_2})$. To see that this cannot happen, we distinguish various possibilities for $b_{i_1}^{k_1}, b_{i_2}^{k_2}$ and use part (γ)(ii) in the choice of $\bar h$.

1   Yet another possible source of contradiction could come from a similar consideration involving the last clause in the definition of $\Gamma'(z)$, which cannot happen for similar
3   reasons.

Stage D. Now we can conclude, using $\lambda = \lambda^{<\lambda}$ and $|T| < \lambda$, that there is a model
5    $N^* \prec \mathfrak{C}$ of size λ with $\bigcup_{\eta \in \,^{i>}2} N_\eta \subseteq N^*$, such that p^* is realised in N^*. For $v \in \,^\lambda 2$, let
$h_v \overset{\text{def}}{=} \bigcup_{i < \lambda} h_{v \restriction i}$, and let $N_v \overset{\text{def}}{=} \mathrm{Rang}(h_v)$, while $p_v \overset{\text{def}}{=} h_v(p)$.
7   For such v, let

$$q_v(x) \overset{\text{def}}{=} \{I(x)\} \cup \{h_v(a_i) <_0 x <_0 h_v(b_i) : i < \lambda\}.$$

9   Hence we have that for $v \neq \rho$ from $^\lambda 2$, the types q_v and q_ρ are contradictory, by (δ) above. As $\|N^*\| + |L(T)| \leqslant \lambda$, there are only $\leqslant \lambda$ definable Dedekind cuts of $<_0$ over
11   N^*, and only $\leqslant \lambda$ types q_v are realised in N^*. Hence there is $v \in \,^\lambda 2$ (actually 2^λ many) such that the Dedekind cut $\{x : \vee_{i < \lambda} x <_0 h_v(a_i)\}$ is not definable over N^* and q_v is
13   not realised in N^*. So N^* omits q_v and realises p_v. We let $N = h(N^*)$, where h is an automorphism of $\mathfrak{C}$ extending h_v^{-1}.   □

15   **Theorem 1.17.** *Assume that* $\lambda^{<\lambda} = \lambda$ *and* $2^\lambda = \lambda^+$.

     (1) *For any* λ-*relevant* $(T_{\mathrm{ord}}, T_{\mathrm{feq}}^*)$-*superior* $(T, \bar{\varphi}, \bar{\psi})$, *the theory* T *has a model* M^*
17   *of cardinality* λ^+ *such that*
         (i) $\bar{\varphi}^{M^*}$ *is not* λ^+-*saturated,*
19   (ii) $\bar{\psi}^{M^*}$ *is* λ^+-*saturated.*
     (2) *We can strengthen the claims in* (i) *and* (ii) *to include any interpretations of a*
21   *dense linear order and* T_{feq}^*-*respectively in* M^*, *even with parameters.*

     Proof. We prove (1), and (2) is proved similarly. Using the Main Claim 1.13, we
23   can construct M^* of size λ^+, by an $\prec$-increasing continuous sequence $\langle M_i^* : i \leqslant \lambda^+ \rangle$, with $\|M_i^*\| = \lambda$ satisfying $*[M_i, \bar{a}, \bar{b}]$ for each $i \leqslant \lambda^+$, and letting $M^* = M_{\lambda^+}$. The Main
25   Claim 1.13 is used in the successor steps. To assure that M^* is λ^+-saturated for T_{feq}^*, we use the assumption $2^\lambda = \lambda^+$, to do the bookkeeping of all T_{feq}^*-types involved.   □

27   **Conclusion 1.18.** Under the assumptions of Theorem 1.17, the theory T_{feq}^* is $\lhd_{\lambda^+}^*$ strictly below the theory of a dense linear order with no first or last elements.
29   [Why? It is below by Shelah's Theorem 0.4 above.]

     We recall that our motivation for studying $\lhd^*$ is to try to characterise SOP_3 (or SOP_2)
31   theories by the $\lhd^*$-maximality. As we explained in the Introduction this has origins in the connection between the maximality in the Keisler order and having the strict order
33   property, so we should show here what is the connection between the maximality in Keisler's order and the maximality in the order $\lhd^*$. The following Claim 1.19 does
35   that for countable theories.

     Claim 1.19. *Suppose that* T *is a countable theory that is* $\lhd_{\lambda^*}^*$-*maximal. Then it is*
37   *maximal in the Keisler order* $\leqslant_\lambda$.

Proof. Suppose otherwise and let T_1 be a theory that is $\lhd^*$-maximal but not maximal in the Keisler order $\ll_\lambda$. In particular we have $T_{\mathrm{ord}} \lhd^*_\lambda T_1$, so there is a λ-relevant (T_{ord}, T_1)-superior triple $(T, \bar{\varphi}_0, \bar{\varphi}_1)$-exemplifying this. By Observation 1.4(0) we may assume that the interpretation $\bar{\varphi}_1$ is trivial, so $T_1 \subseteq T$-for simplicity.

Since T is not maximal in the Keisler order $\ll_\lambda$, by [11, 4.2(1)] there is a regular ultrafilter $\mathscr{D}$ which is not good and a model M of T such that $M^\lambda/\mathscr{D}$ is nevertheless λ^+-compact. We can extend M to a model N of T and consider $N^* = N^\lambda/\mathscr{D}$. This is a model of T and by the Extension Theorem for ultrafilters we have that $[N^*]^{\bar{\varphi}_1} = M^\lambda/\mathscr{D}$, so it is λ^+-compact and hence it is λ^+-saturated. Again by the Extension Theorem we have that $[N^*]^{\bar{\varphi}_1} = (N^{\bar{\varphi}_1})^\lambda/\mathscr{D}$. Now on the one hand we have by the $\lhd^*_{\lambda^*}$-maximality of T_1 that $(N^{\bar{\varphi}_1})^\lambda/\mathscr{D}$ must be λ^+-saturated, hence λ^+-compact. But on the other hand $(N^{\bar{\varphi}_1})^\lambda/\mathscr{D}$ cannot be λ^+-compact because $\mathscr{D}$ is not a good ultrafilter and T_{ord} is maximal in the Keisler order, contradicting [11, 4.2(1)]. $\square$

2. On the properties SOP$_2$ and SOP$_1$

In his paper [13], S. Shelah investigated a hierarchy of properties unstable theories without strong order property may have. This hierarchy is named NSOP$_n$ for $3 \leqslant n < \omega$, where the acronym NSOP stands for "not strong order property". The negation of NSOP$_n$ is denoted by SOP$_n$. It was shown in [13] that SOP$_{n+1} \Rightarrow$ SOP$_n$, that the implication is strict and that SOP$_3$ theories are not simple. In this section we investigate two further notions, which with the intention of furthering the above hierarchy, we name SOP$_2$ and SOP$_1$. The original definitions of SOP$_n$ for $n \geqslant 3$ do not immediately lend themselves to extending the hierarchy for $n = 1, 2$, but the properties we define nevertheless fulfill that role. In Section 3, a connection between this hierarchy and $\lhd^*_\lambda$-maximality will be established.

Recall from [13] one of the equivalent definitions of SOP$_3$. (The equivalence is established in Claim 2.19 of [13]).

Definition 2.1. A (complete) theory T has SOP$_3$ iff there is an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ and formulae $\varphi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{y})$ such that

(a) $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ is contradictory,
(b) for some sequence $\langle \bar{b}_j : j < \omega \rangle$ we have

$$i \leqslant j \Rightarrow \models \varphi[\bar{b}_j, \bar{a}_i] \quad \text{and} \quad i > j \Rightarrow \models \psi[\bar{b}_j, \bar{a}_i],$$

(c) for $i < j$, the set $\{\varphi(\bar{x}, \bar{a}_j), \psi(\bar{x}, \bar{a}_i)\}$ is contradictory.

Definition 2.2. (1) T has SOP$_2$ if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C} = \mathfrak{C}_T$, which means:

There are $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in {}^{\omega>}2$ such that

(a) for every $\rho \in {}^\omega 2$, the set $\{\varphi(\bar{x}, \bar{a}_{\rho \restriction n}) : n < \omega\}$ is consistent,
(b) if $\eta, v \in {}^{\omega>}2$ are incomparable, $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_v)\}$ is inconsistent.

1   (2) T has SOP$_1$ if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this in $\mathfrak{C}$, which means:

3   There are $\bar{a}_n \in \mathfrak{C}$, for $\eta \in {}^{\omega >}2$ such that:

(a) for $\rho \in {}^{\omega}2$ the set $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}): n < \omega\}$ is consistent.

5   (b) if $v \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega >}2$, then $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{v \frown \langle 1 \rangle})\}$ is inconsistent.

(3) NSOP$_2$ and NSOP$_1$ are the negations of SOP$_2$ and SOP$_1$ respectively.

7   The following claim establishes the relative position of the properties introduced in Definition 2.2 within the (N)SOP hierarchy.

9   **Claim 2.3.** *For any complete first order theory T, we have*

$$\text{SOP}_3 \Rightarrow \text{SOP}_2 \Rightarrow \text{SOP}_1.$$

11   **Proof.** Suppose that T is SOP$_3$, as exemplified by $\langle \bar{a}_i: i < \omega \rangle$, $\langle \bar{b}_j: j < \omega \rangle$ and formulae $\varphi(\bar{x}, \bar{y})$ and $\psi(\bar{x}, \bar{y})$ (see Definition 2.1), and we shall show that T satisfies SOP$_2$. We

13   define

$$\vartheta(\bar{x}, \bar{y}^0 \frown \bar{y}^1) \equiv \varphi(\bar{x}, \bar{y}^0) \wedge \psi(\bar{x}, \bar{y}^1), \quad \text{where } lg(\bar{y}^0) = lg(\bar{y}^1).$$

15   Let us first prove the consistency of

$$\Gamma \overset{\text{def}}{=} \begin{array}{l} T \cup \{\neg(\exists \bar{x})[\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_v)]: \eta \perp v \text{ in } {}^{\omega >}2\} \\ \cup \bigcup_{n < \omega} \{(\exists \bar{x})\left[\bigwedge_{k \leqslant n} \vartheta(\bar{x}, \bar{y}_{\eta \upharpoonright k})\right]: \eta \in {}^n 2\}. \end{array}$$

17   Suppose for contradiction that Γ is not consistent, then for some $n < \omega$, the following set is inconsistent:

$$\Gamma' \overset{\text{def}}{=} \begin{array}{l} T \cup \{\neg(\exists \bar{x})[\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_v)]: \eta, v \text{ incomparable in } {}^{n \geqslant}2\} \\ \cup \{(\exists \bar{x})\left[\bigwedge_{k \leqslant n} \vartheta(\bar{x}, \bar{y}_{\eta \upharpoonright k})\right]: \eta \in {}^n 2\}. \end{array}$$

19

Fix such n. We pick ordinals $\alpha_\eta, \beta_\eta < \omega$ for $\eta \in {}^{n \geqslant}2$ so that

21   (i) $v \triangleleft \eta \Rightarrow \alpha_v < \alpha_\eta < \alpha_\eta + 1 < \beta_\eta < \beta_v$,

(ii) $\beta_{\eta \frown \langle 0 \rangle} < \alpha_{\eta \frown \langle 1 \rangle}$.

23   For $\eta \in {}^{n \geqslant}2$ let $\bar{a}_\eta^* \overset{\text{def}}{=} \bar{a}_{\alpha_\eta} \frown \bar{a}_{\beta_\eta}$. We show that $\mathfrak{C}$ and $\{\bar{a}_\eta^*: \eta \in {}^{n \geqslant}2\}$ exemplify that Γ' is consistent. So, if $\eta \in {}^{n \geqslant}2$ then we have $\bigwedge_{k \leqslant n} \vartheta[\bar{b}_{\alpha_\eta} + 1, \bar{a}_{\eta \upharpoonright k}^*]$ as for every $k \leqslant n$

25   we have $\alpha_{\eta \upharpoonright k} < \alpha_n + 1$, so $\varphi[\bar{b}_{\alpha_\eta + 1}, \bar{a}_{\alpha_{\eta \upharpoonright k}}]$ holds, but also for all $k \leqslant n$, as $\eta \upharpoonright k \trianglelefteq \eta$, we have $\beta_{\eta \upharpoonright k} > \alpha_\eta + 1$, so $\psi[\bar{b}_{\alpha_\eta + 1}, \bar{a}_{\beta_{\eta \upharpoonright k}}]$ holds. Hence $(\exists \bar{x})[\bigwedge_{k \leqslant n} \vartheta(\bar{x}, \bar{a}_{\eta \upharpoonright k}^*)]$. On the other

27   hand, if $v \frown \langle l \rangle \trianglelefteq \eta_l$ for $l < 2$, then $\{\vartheta(\bar{x}, \bar{a}_{\eta_0}^*), \vartheta(\bar{x}, \bar{a}_{\eta_1}^*)\}$ is contradictory as the conjunction implies $\psi(\bar{x}, \bar{a}_{\beta_{\eta_0}}) \wedge \varphi(\bar{x}, \bar{a}_{\alpha_{\eta_1}})$, which is contradictory by $\beta_{\eta_0} < \alpha_{\eta_1}$ and (c) of

29   Definition 2.1. This shows that Γ' is consistent, hence we have also shown that Γ is consistent.

1   Having shown that Γ is consistent, we can find witnesses $\{\bar{a}_\eta^*: \eta \in {}^{\omega>}2\}$ in $\mathfrak{C}$ realising
Γ. Now we just need to show that $\{\vartheta(\bar{x}, \bar{a}_{\eta\restriction n}^*): n < \omega\}$ is consistent for every $\eta \in {}^\omega 2$.

3   This follows by the compactness theorem and the definition of Γ. Hence we have
shown that $\mathrm{SOP}_3 \Rightarrow \mathrm{SOP}_2$.

5   The second part of the claim is obvious (and the witnesses for SOP_2 can be used
for SOP_1 as well).   □

7   **Question 2.4.** Are the implications from Claim 2.3 reversible?

Claim 2.5. *If T satisfies* SOP_1, *then T is not simple. In fact, if $\varphi(\bar{x}, \bar{y})$ exemplifies*

9   SOP_1 *of T, then the same formula exemplifies that T has the tree property.*

Proof. Let $\varphi(\bar{x}, \bar{y})$ and $\{\bar{a}_\eta: \eta \in {}^{\omega>}2\}$ exemplify SOP_1. Then

11    $$\Gamma_\eta \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_{\eta \frown \langle 0 \rangle n \frown \langle 1 \rangle}): n < \omega\}$$

for $\eta \in {}^{\omega>}2$ consists of pairwise contradictory formulae. (Here $\langle 0 \rangle_n$ denotes a sequence

13   consisting of n zeroes.) For $n < \omega$ and $v \in {}^n\omega$ let

$$\rho_v \stackrel{\text{def}}{=} \langle 0 \rangle_{v(0)+1} \frown \langle 1 \rangle \frown \langle 0 \rangle_{v(1)+1} \ldots \frown \langle 0 \rangle_{v(n-1)+1} \frown \langle 1 \rangle,$$

15   so $\rho_v \in {}^{\omega>}2$ and $v \trianglelefteq \eta \Rightarrow \rho_v \trianglelefteq \rho_\eta$. For $v \in {}^n\omega$ let $\bar{b}_v = \bar{a}_{\rho_v}$. We observe first that $\{\varphi(\bar{x},$
$\bar{b}_{v \frown \langle k \rangle}): k < \omega\}$ is a set of pairwise contradictory formulae, for $v \in {}^n\omega$; namely, if $k_0 \neq k_1$,

17   then $\varphi(\bar{x}, \bar{b}_{v \frown \langle k_l \rangle})$ for $l < 2$ are two different elements of Γ_{ρ_v}. On the other hand,
$\{\varphi(\bar{x}, \bar{b}_{v \restriction n}): n < \omega\}$ is consistent for every $v \in {}^\omega\omega$. Hence $\varphi(\bar{x}, \bar{y})$ and $\{\bar{b}_v: v \in {}^{\omega>}\omega\}$

19   exemplify that T has the tree property, and so T is not simple.   □

This ends the discussion of the properties of SOP_1 and SOP_2 that are directly relevant

21   to the main thesis of the paper—the reader only interested in the connection with the
order $\vartriangleleft^*$ can now turn directly to Section 3. The rest of this section however contains

23   some further syntactic developments of these properties which are of interest if one
wishes to understand the type theory induced by them. The indescernibility results we

25   have here were recently used by Shelah and Usvyatsov [14] to define a rank function
on NSOP_1 theories (see Theorem 2.18).

27   The definition of when a theory has SOP_1 can be made in another equivalent fashion.

29   **Definition 2.6.** Let $\varphi(\bar{x}, \bar{y})$ be a formula of $\mathscr{L}(T)$. We say $\varphi(\bar{x}, \bar{y})$ has SOP_1' iff there
is $\langle \bar{a}_\eta: \eta \in {}^{\omega>}2 \rangle$ in $\mathfrak{C}_T$ such that

31   (a) $\{\varphi(\bar{x}, \bar{a}_{\eta \restriction n})^{\eta(n)}: n < \omega\}$ is consistent for every $\eta \in {}^\omega 2$ where we use the notation

$$\varphi^l = \begin{cases} \varphi & \text{if } l = 1, \\ \neg\varphi & \text{if } l = 0 \end{cases}$$

33   for $l < 2$.

1 (b) If $v \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega >}2$, *then* $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_v)\}$ *is inconsistent.*

We say that T has property SOP'_1 iff some formula of $\mathscr{L}(T)$ has it.

3 **Claim 2.7.** (1) *If* $\varphi(\bar{x}, \bar{y})$ *exemplifies* SOP_1 *of* T *then* $\varphi(\bar{x}, \bar{y})$ *(hence* T*) has property* SOP'_1.

5   (2) *If* T *has property* SOP'_1 *then* T *has* SOP_1.

Proof. (1) Let $\{\bar{a}_\eta : \eta \in {}^{\omega >}2\}$ and $\varphi(\bar{x}, \bar{y})$ exemplify that T has SOP_1. For $\eta \in {}^{\omega >}2$ we

7 define $\bar{b}_\eta \overset{\mathrm{def}}{=} \bar{a}_{\eta \frown \langle 1 \rangle}$. We shall show that $\varphi(\bar{x}, \bar{y})$ and $\{\bar{b}_\eta : \eta \in {}^{\omega >}2\}$ exemplify SOP'_1.
   Given $\hat{\eta} \in {}^{\omega}2$. Let $\bar{c}$ exemplify that item (a) from Definition 2.2(2) holds for $\hat{\eta}$.

9 Given $n < \omega$, we consider $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \restriction n}]^{\hat{\eta}(n)}$. If $\hat{\eta}(n) = 1$, then, as $\bar{b}_{\hat{\eta} \restriction n} = \bar{a}_{\hat{\eta} \restriction n \frown \langle 1 \rangle} = \bar{a}_{\hat{\eta} \restriction (n+1)}$,
   we have that $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \restriction n}]^{\hat{\eta}(n)} = \varphi[\bar{c}, \bar{a}_{\hat{\eta} \restriction (n+1)}]$ holds. If $\hat{\eta}(n) = 0$, then

11    $$(\hat{\eta} \restriction n) \frown \langle 0 \rangle = \hat{\eta} \restriction (n+1).$$

As $\varphi[\bar{c}, \bar{a}_{\hat{\eta} \restriction (n+1)}]$ holds, by (b) of Definition 2.2(2), we have that $\varphi[\bar{c}, \bar{a}_{\hat{\eta} \restriction n \frown \langle 1 \rangle}]$ can-

13 not hold, showing again that, $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \restriction n}]^{\hat{\eta}(n)} = \neg \varphi[\bar{c}, \bar{a}_{\hat{\eta} \restriction n \frown \langle 1 \rangle}]$ holds. This shows that
   $\{\varphi(\bar{x}, \bar{b}_{\hat{\eta} \restriction n})^{\hat{\eta}(n)} : n < \omega\}$ is consistent, as exemplified by $\bar{c}$.

15   Suppose $v \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega >}2$ and that $\varphi[\bar{d}, \bar{b}_\eta] \wedge \varphi[\bar{d}, \bar{b}_v]$ holds. So both $\varphi[\bar{d}, \bar{a}_{\eta \frown \langle 1 \rangle}]$
   and $\varphi[\bar{d}, \bar{a}_{v \frown \langle 1 \rangle}]$ hold. On the other hand, as $v \frown \langle 0 \rangle \trianglelefteq \eta$, clearly $v \frown \langle 0 \rangle \trianglerighteq \eta\langle 1 \rangle$, and

17 so (b) of Definition 2.2(2) implies that $\{\varphi(\bar{x}, \bar{a}_{\eta \frown \langle 1 \rangle}), \varphi(\bar{x}, \bar{a}_{v \frown \langle 1 \rangle})\}$ is contradictory,
   a contradiction. Hence the set $\{\varphi(\bar{x}, \bar{b}_\eta), \varphi(\bar{x}, \bar{b}_v)\}$ is contradictory

19   (2) Define first for $\eta \in {}^{\omega \geq}2$ an element $\rho_\eta \in {}^{\omega \geq}2$ by letting

$$\rho_\eta(3k) = \eta(k),$$

$$\rho_\eta(3k+1) = 0,$$

$$\rho_\eta(3k+2) = 1,$$

and if $lg(\eta) = k < \omega$, then $lg(\rho_\eta) = 3k$. Note that for $\eta \in {}^{\omega}2$ and $k < \omega$ we have

21 $\rho_{\eta \restriction k} = \rho_\eta \restriction (3k)$.
   Let $\varphi(\bar{x}, \bar{y})$ and $\{\bar{a}_\eta : \eta \in {}^{\omega >}2\}$ exemplify property SOP'_1. We pick $c_0 \neq c_1$ and define

23 for $\eta \in {}^{\omega >}2$

$$\bar{b}_{\eta \frown \langle 1 \rangle} \overset{\mathrm{def}}{=} \bar{a}_{\rho_\eta} \frown \bar{a}_{\rho_\eta \frown \langle 1 \rangle} \frown \langle c_0, c_1 \rangle,$$

$$\bar{b}_{\eta \frown \langle 0 \rangle} \overset{\mathrm{def}}{=} \bar{a}_{\rho_\eta \frown \langle 0, 0 \rangle} \frown \bar{a}_{\rho_\eta} \frown \langle c_0, c_1 \rangle,$$

$$\bar{b}_{\langle \rangle} \overset{\mathrm{def}}{=} \langle c_0 \rangle_{2n+2},$$

where $\langle c \rangle_k$ stands for the sequence of k entries, each of which is c, and $n = lg(\bar{y})$ in

25 $\varphi(\bar{x}, \bar{y})$. We define

$$\psi(\bar{x}, \bar{z}) \equiv \psi(\bar{x}, \bar{z}^0 \frown \bar{z}^1 \frown w^0 \frown w^1)$$
$$\equiv [w^0 = w^1] \vee [\varphi(\bar{x}, \bar{z}^0) \wedge \neg \varphi(\bar{x}, \bar{z}^1)],$$

where $\bar{z} = \bar{z}^0 \frown \bar{z}^1 \frown \langle w^0, w^1 \rangle$ and $lg(\bar{z}^0) = lg(\bar{z}^1) = lg(\bar{y})$. We now claim that $\psi(\bar{x}, \bar{z})$ and $\{\bar{b}_\eta : \eta \in {}^{\omega >}2\}$ exemplify that SOP_1 holds for T. Before we start checking this, note that for $\eta \in {}^{\omega >}2$ we have

- $\bullet_1$ $\psi(\bar{x}, \bar{b}_{\langle \rangle})$ holds for any $\bar{x}$,
- $\bullet_2$ $\psi(\bar{x}, \bar{b}_{\eta \frown \langle 0 \rangle})$ holds iff $\varphi(\bar{x}, \bar{a}_{\rho_{\eta \frown \langle 0,0 \rangle}}) \wedge \neg \varphi(\bar{x}, \bar{a}_{\rho_\eta})$ holds,
- $\bullet_3$ $\psi(\bar{x}, \bar{b}_{\eta \frown \langle 1 \rangle})$ holds iff $\neg \varphi(\bar{x}, \bar{a}_{\rho_{\eta \frown \langle 1 \rangle}}) \wedge \varphi(\bar{x}, \bar{a}_{\rho_\eta})$ holds.

Let us verify 2.2(2)(a), so let $\eta \in {}^\omega 2$. Pick $\bar{c}$ such that $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright n}}]^{\rho_\eta(n)}$ holds for all $n < \omega$. We claim that

$$\psi[\bar{c}, \bar{b}_{\eta \upharpoonright n}] \quad \text{holds for all } n. \tag{$*$}$$

The proof is by a case analysis of n.

If $n = 0$, this is trivially true. If $n = k + 1$ and $\eta(k) = 0$, then we need to verify that $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k} \frown \langle 0,0 \rangle}]$ holds and $\neg \varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k}}]$ holds. We have

$$\rho_{\eta \upharpoonright k} \frown \langle 0, 0 \rangle = \rho_\eta \upharpoonright (3k + 2),$$

and $\rho_\eta(3k + 2) = 1$. Hence $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k} \frown \langle 0,0 \rangle}]$ holds by the choice of $\bar{c}$. On the other hand, we have $\rho_{\eta \upharpoonright k} = \rho_\eta \upharpoonright (3k)$, and $\rho_\eta(3k) = \eta(k) = 0$, hence $\neg \varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k}}]$ holds.

If $n = k + 1$ and $\eta(k) = 1$, then we need to verify that $\varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k}}]$ holds while $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k) \frown \langle 1 \rangle}]$ does not. As $\rho_{\eta \upharpoonright k} = \rho_\eta \upharpoonright (3k)$, and $\rho_\eta(3k) = \eta(k) = 1$, we have that $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k}]$ holds. Note that $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+2)}]$ holds as $\rho_\eta(3k + 2) = 1$. We also have $(\rho_\eta \upharpoonright (3k + 1)) \frown \langle 0 \rangle \trianglelefteq \rho_\eta \upharpoonright (3k + 2)$. Hence $\neg \varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+1)}]$ by part (b) in Definition 2.6. But

$$\neg \varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+1)}] \equiv \neg \varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k) \frown \langle 1 \rangle}] \equiv \neg \varphi[\bar{c}, \bar{a}_{\rho_{\eta \upharpoonright k} \frown \langle 1 \rangle}]$$

holds, so we are done proving $(*)$.

Let us now verify 2.2(2)(b). So suppose $v \frown \langle 0 \rangle \trianglelefteq \eta$ and consider $\{\psi(\bar{x}, \bar{b}_{v \frown \langle 1 \rangle}), \psi(\bar{x}, \bar{b}_\eta)\}$. Let $\eta = \sigma \frown \langle l \rangle$.

Case 1: $v = \sigma$. Hence $l = 0$. So $\psi(\bar{x}, \bar{b}_\eta) \Rightarrow \neg \varphi(\bar{x}, \bar{a}_{\rho_v})$ and $\psi(\bar{x}, \bar{b}_{v \frown \langle 1 \rangle}) \Rightarrow \varphi(\bar{x}, \bar{a}_{\rho_v})$, by $\bullet_2$ and $\bullet_3$ respectively, showing that $\{\psi(\bar{x}, \bar{b}_\eta), \psi(\bar{x}, \bar{b}_{v \frown \langle 1 \rangle})\}$ is inconsistent.

Case 2: $v \triangleleft \sigma$ and $l = 0$. Hence $v \frown \langle 0 \rangle \trianglelefteq \sigma$. Clearly $\rho_v \frown \langle 0 \rangle \trianglelefteq \rho_\sigma \frown \langle 0, 0 \rangle$, as

$$\rho_\sigma(lg(\rho_v)) = \sigma(lg(v)) = 0.$$

We have $\psi(\bar{x}, \bar{b}_{v \frown \langle 1 \rangle}) \Rightarrow \varphi(\bar{x}, \bar{a}_{\rho_v})$ by $\bullet_3$ and $\psi(\bar{x}, \bar{b}_\eta) \Rightarrow \varphi(\bar{x}, \bar{a}_{\rho_\sigma \frown \langle 0,0 \rangle})$ by $\bullet_2$, and the two formulae being implied are contradictory, by (b) in the definition of SOP_1'.

Case 3: $v \triangleleft \sigma$ and $l = 1$. Observe that $\psi(\bar{x}, \bar{b}_\eta) \Rightarrow \varphi(\bar{x}, \bar{a}_{\rho_\sigma})$ by $\bullet_3$ and $\psi(\bar{x}, \bar{b}_{v \frown \langle 1 \rangle}) \Rightarrow \varphi(\bar{x}, \bar{a}_{\rho_v})$. As above, using $v \frown \langle 0 \rangle \trianglelefteq \sigma$, we show that the set $\{\varphi(\bar{x}, \bar{a}_{\rho_v}), \varphi(\bar{x}, \bar{a}_{\rho_\sigma})\}$ is inconsistent. $\square$

Conclusion 2.8. T has SOP_1 iff T has property SOP_1' from Claim 2.7.

Question 2.9. Is the conclusion of 2.8 true when the theory T is replaced by a formula φ?

1   It turns out that witnesses to being SOP_1 can be chosen to be highly indiscernible.

3   **Definition 2.10.** (1) Given an ordinal α and sequences $\bar{\eta}_l = \langle \eta_0^l, \eta_1^l, \ldots, \eta_{n_l}^l \rangle$ for $l = 0, 1$
       of members of $^{\alpha >}2$, we say that $\bar{\eta}_0 \approx_1 \bar{\eta}_1$ iff

5   (a) $n_0 = n_1$,
       (b) the truth values of

7      $$\eta_k^l = \langle \, \rangle, \eta_{k_3}^l \trianglelefteq \eta_{k_1}^l \cap \eta_{k_2}^l, \quad \eta_{k_1}^l \cap \eta_{k_2}^l \triangleleft \eta_{k_3}^l, \quad (\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 0 \rangle \trianglelefteq \eta_{k_3}^l,$$

       (hence also of $(\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 1 \rangle \trianglelefteq \eta_{k_3}^l$, as the tree is binary) for $k_1, k_2, k_3 \leqslant n_0$, do
9   not depend on l.
       (2) We say that the sequence $\langle \bar{a}_\eta : \eta \in {}^{\alpha >}2 \rangle$ of elements of $\mathfrak{C}$ (for an ordinal α) is 1-
11   *fully binary tree indiscernible* (1-*fbti*) *iff whenever* $\bar{\eta}_0 \approx_1 \bar{\eta}_1$ *are sequences of elements
       of* $^{\alpha >}2$, *then*

13     $$\bar{a}_{\bar{\eta}_0} \stackrel{\text{def}}{=} \bar{a}_{\eta_0^0} \frown \cdots \frown \bar{a}_{\eta_{n_0}^0}$$

       and the similarly defined $\bar{a}_{\bar{\eta}_1}$, realise the same type in $\mathfrak{C}$.
15   (3) We replace 1 by 2 in the above definitions iff $(\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 0 \rangle \trianglelefteq \eta_{k_3}^l$ (and hence
       also $(\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 1 \rangle \trianglelefteq \eta_{k_3}^l$) is omitted from clause (b) above.

17   **Claim 2.11.** *If* $t \in \{1, 2\}$ *and* $\langle \bar{b}_\eta : \eta \in {}^{\omega >}2 \rangle$ *are given, and* $\delta \geqslant \omega$, *then we can find*
       $\langle \bar{a}_\eta : \eta \in {}^{\delta >}2 \rangle$ *such that*

19   (a) $\langle \bar{a}_\eta : \eta \in {}^{\delta >}2 \rangle$ *is* t-fbti,
       (b) *If* $\bar{\eta} = \langle \eta_m : m < n \rangle$, *where each* $\eta_m \in {}^{\delta >}2$ *is given, and* Δ *is a finite set of formulae*
21   *of* T, *then we can find* $v_m \in {}^{\omega >}2 (m < n)$ *such that with* $\bar{v} \stackrel{\text{def}}{=} \langle v_m : m < n \rangle$, *we have*
       $\bar{v} \approx_t \bar{\eta}$ *and sequences* $\bar{a}_{\bar{\eta}}$ *and* $\bar{b}_{\bar{v}}$ *realise the same* Δ-*types.*

23   **Proof.** By the compactness theorem it suffices to assume $\delta = \omega$. The proof goes through
       a series of steps through which we obtain increasing degrees of indiscernibility. We
25   shall need some auxiliary definitions.

       Definition 2.12. (1) Given $\bar{\eta} = \langle \eta_0, \ldots, \eta_{k-1} \rangle$, a sequence of elements of $^{\alpha >}2$, and an
27   ordinal γ. We define $\bar{\eta}' = \mathrm{cl}_\gamma(\bar{\eta})$ as follows:

       $$\bar{\eta}' = \langle \langle \, \rangle, \eta_0, \eta_0 \restriction \gamma, \eta_1, \eta_1 \restriction \gamma, \eta_0 \cap \eta_1, \eta_2, \eta_2 \restriction \gamma, \eta_0 \cap \eta_2, \eta_1 \cap \eta_2 \ldots \rangle.$$

29   (2) We say that $\bar{\eta} \approx_{\gamma, n} \bar{v}$ iff $\bar{\eta}' \stackrel{\text{def}}{=} \mathrm{cl}_\gamma(\bar{\eta})$ and $\bar{v}' \stackrel{\text{def}}{=} \mathrm{cl}_\gamma(\bar{v})$ satisfy

       (i) $\bar{\eta}' = \langle \eta_l' : l < m \rangle$ and $\bar{v}' = \langle v_l' : l < m \rangle$ are both in $^m(^{\alpha >}2)$ for some m,
31   (ii) for $l < m$ we have $\eta_l' \in {}^{\gamma \geqslant}2 \Leftrightarrow v_l' \in {}^{\gamma \geqslant}2$, and for such l we have $\eta_l' = v_l'$,
       (iii) $n \geqslant |u_1| = |u_2|$, where we let $u_1 = \{lg(\eta_l') : l < m\} \setminus (\gamma + 1)$ and $u_2 = \{lg(v_l') : l < m\}$
33     $\setminus (\gamma + 1)$,
       (iv) $lg(\eta_l'), lg(\eta_k') \in u_1 \Rightarrow lg(\eta_1') < lg(\eta_k') \Leftrightarrow lg(v_l') < lg(v_k')$,

1   (v) $\eta'_{l_1} \trianglelefteq \eta'_{l_2} \Leftrightarrow v'_{l_1} \trianglelefteq v'_{l_2}$, and the same holds for the equality,
     (vi) $\eta'_{l_1} {}^\wedge \langle 0 \rangle \trianglelefteq \eta'_{l_2} \Leftrightarrow v'_{l_1} {}^\wedge \langle 0 \rangle \trianglelefteq v'_{l_2}$ (and hence the same holds with 1 in place of 0).

3   (3) $\langle \bar{a}_\eta \colon \eta \in {}^{\alpha >}2 \rangle$ is (γ, n)-*indiscernible* iff for every k, for every $\bar{\eta}, \bar{v} \in {}^k({}^{\alpha >}2)$ with
     $\bar{\eta} \approx_{\gamma, n} \bar{v}$, the tuples $\bar{a}_{\bar{\eta}}$ and $\bar{a}_{\bar{v}}$ realise the same type.

5   (4) $(\leqslant \gamma, n)$-*indiscernibility* is the conjunction of (β, n)-indiscernibility for all $\beta \leqslant \gamma$.
     (5) We say that $\langle \bar{a}_\eta \colon \eta \in {}^{\alpha >}2 \rangle$ is 0-*fbti* iff it is (γ, n)-indiscernible for all γ and n.

7   **Subclaim 2.13.** *If $\bar{a}_\eta \in {}^k\mathfrak{C}$ for $\eta \in {}^{\omega >}2$, then for any $\alpha \geqslant \omega$ we can find $\bar{a}' = \langle \bar{a}'_\eta \colon \eta \in {}^{\alpha >}2 \rangle$ such that*

9    $(x) \bar{a}'$ *is 0-fbti,*
     (xx) *for every m and a finite set Δ of formulae, we can find $h \colon {}^{m \geqslant}2 \to {}^{\omega >}2$ such*
11   *that*
         (α) *$\langle \bar{a}'_\eta \colon \eta \in {}^{m \geqslant}2 \rangle$ and $\langle \bar{a}_{h(\eta)} \colon \eta \in {}^{m \geqslant}2 \rangle$ realise the same Δ-type,*
13   (β) *h satisfies $h(\eta) {}^\wedge \langle l \rangle \trianglelefteq h(\eta {}^\wedge \langle l \rangle)$ for $\eta \in {}^{m >}2$ and $l < 2$, and*

$$lg(\eta_1) = lg(\eta_2) \Rightarrow lg(h(\eta_1)) = lg(h(\eta_2)).$$

15   **Proof.** By the compactness theorem it suffices to work with $\alpha = \omega$.
     Let $(*)_{\gamma, n}$ be the conjunction of the statement $(x)_{\gamma, n}$ given by

17       $\bar{a}'$ is $(\leqslant \gamma, n)$-indiscernible,

     and (xx) above. We prove by induction on n and then γ that for any $\gamma \leqslant \omega$ we can
19   find $\bar{a}'$ for which $(*)_{\gamma, n}$ holds.
     $n = 0$. We use $\bar{a}'_\eta = \bar{a}_\eta$.
21   $n+1$. By induction on $\gamma \leqslant \omega$, we prove that there is $\bar{a}'$ for which $(*)_{\gamma, n+1} + (*)_{\omega, n} + (xx)$
     holds.
23   $\gamma = 0$ (or just $\gamma < \omega$).
     Without loss of generality, the sequence $\langle \bar{a}_\eta \colon \in {}^{\omega >}2 \rangle$ is $(\leqslant \omega, n)$-indiscernible, as
25   (xx) as a relation between $\langle \bar{a}_\eta \colon \eta \in {}^{\omega >}2 \rangle$ and $\langle \bar{a}'_\eta \colon \eta \in {}^{\omega >}2 \rangle$ is transitive. Suppose we
     are given $\bar{\eta}^*, \bar{v}^*$ satisfying $\bar{\eta}^* \approx_{\gamma, n+1} \bar{v}^*$, in particular the appropriately defined u_1, u_2
27   have size $\leqslant n + 1$. For simplicity in notation, we assume $\bar{\eta}^*, \bar{v}^*$ to be the same as their
     cl_γ closures and the same convention will hold for any $\bar{\eta}, \bar{v}$ that we mention in this
29   context.
     If $|u_1| \leqslant n$, the conclusion follows by the assumptions. We shall assume $|u_1| > n$.
31   Moreover, if $\min(u_1) = \min(u_2)$ and $lg(\eta^*_l) = \min(u_1) \Rightarrow \eta^*_1 = v^*_l$, using $(x)_{\min(u_1), n}$, we
     get that $\bar{a}_{\bar{\eta}^*}$ and $\bar{a}_{\bar{v}^*}$ realise the same type. By the same argument, fixing a finite set
33   Δ of formulae, for every $\bar{\eta}$, defining u_1 appropriately, we get that the $tp_\Delta(\bar{a}_{\bar{\eta}})$ depends
     just on the

35       $\bar{\eta}/\approx_{\gamma, n+1} \overset{\text{def}}{=} \Upsilon$ and $\{\eta_l \colon l < lg(\bar{\eta})\} \cap {}^{\min(u_1)}2 = \{\eta_l \colon l \in v^\Upsilon\}$

     for some $v^\Upsilon \subseteq lg(\bar{\eta})$. Let us define $F^0_{\Upsilon, \Delta}$ by $F^0_{\Upsilon, \Delta}(\langle \eta_l \colon l \in v^\Upsilon \rangle) = tp_\Delta(\bar{a}_{\bar{\eta}})$. By the closure
37   properties of $\bar{\eta}$ and the definition of $\approx_{\gamma, n+1}$, we get that for $l_1 \neq l_2 \in v^\Upsilon$ the truth
     values of $\eta_{l_1} \restriction (\gamma + 1) = \eta_{l_2} \restriction (\gamma + 1)$ depends only on Υ. We can hence replace v^Υ by

1   a set $v_*^{\Upsilon} \subseteq v^{\Upsilon}$ such that $\langle \eta_l : l \in v_*^{\Upsilon} \rangle$ are the representatives under the equality of the restrictions to $\gamma + 1$.

3   As we have fixed Δ, there is a finite set A of Υs that can be used as representatives for the values of $F_{\Upsilon,\Delta}^0$. Let $k^* = 2^{\gamma+1}$ (so finite) and let $\{v_k^* : k < k^*\}$. We define a

5   partial function $F_{\Upsilon,\Delta}$ by

$$F_{\Upsilon,\Delta}(v_0, \ldots, v_k, \ldots)_{k<k^*} \overset{\text{def}}{=} F_{\Upsilon,\Delta}^0(\langle \eta_l : l \in v_*^{\Upsilon} \rangle),$$
$$\text{where } \eta_l \restriction (\gamma + 1) = v_k^* \Rightarrow \eta_l = v_k^* \frown v_k.$$

    Define a function F with arity k^* so that $F((\ldots, x_k, \ldots)_{k<k^*})$ is defined iff for some

7     $m < \omega$ we have $\{x_k : k < k^*\} \subseteq {}^m 2$ and then

$$F((\ldots, x_k, \ldots)_{k<k^*}) = \langle F_{\Upsilon,\Delta}((\ldots, x_k, \ldots)_{k<k^*}) : \Upsilon \in A \rangle.$$

9   Now we use the Halpern–Lauchli [6] theorem. We get a function $h : {}^{\omega>}2 \to {}^{\omega>}2$ such that

11   • $lg(h(\eta))$ depends just on $lg(\eta)$ (not on η),
    • $h(\eta) \frown \langle l \rangle \triangleleft h(\eta \frown \langle l \rangle)$ for $l = 0, 1$,

13   • for some c we have that for all $m < \omega$

$$\{\eta_k : k < k^*\} \subseteq {}^m 2 \Rightarrow F((h(\eta_0), h(\eta_1), \ldots, h(\eta_k), \ldots)_{k<k^*}) = c.$$

15   Let $\bar{a}_\eta'$ for $\eta \in {}^{\omega>}2$ be defined to be: $\bar{a}_\eta$ if $\eta \in {}^{\gamma\geqslant}2$, and $\bar{a}_{h(v)}$ if $\eta \restriction \gamma = v_k^*$ and $\eta = v_k^* \frown v$.
    We have obtained the desired conclusion, but localized to Δ. The induction step ends

17   by an application of the compactness theorem.
    $\gamma = \omega$. Follows by the induction hypothesis and the compactness.

19   The conclusion of the subclaim follows by the compactness theorem. $\square$

    Now we go *back to the proof of the claim*. Let us first work with $t = 1$. Given

21     $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2 \rangle$ as in the assumptions, by the subclaim we can assume that they are 0-fbti. We choose by induction on n a function $h_n : {}^{n\geqslant}2 \to {}^{\omega>}2$ as follows. Let $h_0(\langle \rangle) = \langle \rangle$.

23   If h_n is defined, let

$$k_n \overset{\text{def}}{=} \max\{lg(h_n(\eta)) + 1 : \eta \in {}^{n\geqslant}2\}$$

25   and let

$$h_{n+1}(\langle \rangle) = \langle \rangle, \quad h_{n+1}(\langle 1 \rangle^\frown v) = \langle 1 \rangle^\frown h_n(v), \quad h_{n+1}(\langle 0 \rangle^\frown v) = \langle 0, \ldots, 0 \rangle^\frown h_n(v),$$

27   where the sequence of 0s in the last part of the definition has length k_n. The point of the definition of h_n is that if $\bar{\eta}^l = \langle \eta_0^l, \ldots, \eta_{n_l}^l \rangle$ for $l = 0, 1$ are given and $n^* = lg(cl_0(\bar{\eta}^0))$,

29   then

$$\bar{\eta}^0 \approx_1 \bar{\eta}^1 \Rightarrow \langle h_n(\eta_0^0), \ldots, h_n(\eta_{n_0}^0) \rangle \approx_{0,n^*} \langle h_n(\eta_0^1), \ldots, h_n(\eta_{n_1}^1) \rangle.$$

    To check this, we verify the six relevant items of the definition of $\approx_{0,n^*}$.

31

    (i) Follows because $n_0 = n_1$ by the definition of $\approx_1$.

1 (ii) If $h_n(\eta_i^0) \cap h_n(\eta_j^0) = \langle \rangle$ then $\eta_i^0 \cap \eta_j^0 = \langle \rangle$ so $\eta_i^1 \cap \eta_j^1 = \langle \rangle$ by the definition of $\approx_1$,
     and hence $h_n(\eta_i^1) \cap h_n(\eta_j^1) = \langle \rangle$. The opposite implication holds by symmetry.
3 (iii) Follows by the definition of n^*.
     (iv) Suppose

5     $$0 < lg(h_n(\eta_i^0) \cap h_n(\eta_j^0)) < lg(h_n(\eta_k^0) \cap h_n(\eta_s^0)).$$

     Let $m \leqslant n$ be the first such that

7     $$0 < lg(h_n(\eta_i^0 \upharpoonright m) \cap h_n(\eta_j^0 \upharpoonright m)) < lg(h_n(\eta_k^0 \upharpoonright m) \cap h_n(\eta_s^0 \upharpoonright m)).$$

     Clearly, $m > 0$. To simplify the notation, let us assume that $m = n$. Let $\eta_t^0 =$
9    $\langle l_t \rangle \frown v_t^0$ for $t \in \{i, j, k, s\}$ and for some $l_t \in \{0, 1\}$ depending on t. The situa-
     tion we describe can happen iff $l_i = l_j = 1$ and $l_k = l_s = 0$, by the definition of h_n.
11   By the definition of $\approx_1$ this can be recognised by the $\approx_1$-type of $\bar{\eta}^0$.
     (v), (vi) Follow because the corresponding properties are preserved by h_n. Fix an $n < \omega$
13   and define $\bar{a}_\eta = \bar{b}_{h_n(\eta)}$ for $\eta \in {}^{n \geqslant} 2$. By the above argument it follows that $\langle \bar{a}_\eta : \eta \in {}^{n \geqslant} 2 \rangle$
     are 1-fbti. As n was arbitrary, we can finish by compactness.
15   For $t = 2$, we use the same proof, except that we let

     $$h_{n+1}(\langle 1 \rangle^\frown v) = \langle 0, 1 \rangle^\frown h_n(v). \qquad \square$$

17 **Claim 2.14.** *If $t \in \{1, 2\}$ and $\varphi(\bar{x}, \bar{y})$ exemplifies that T has SOP_t, we can without
   loss of generality assume that the witnesses $\langle \bar{a}_\eta : \eta \in {}^{\omega >} 2 \rangle$ for this fact are t-fbti.*

19 **Proof.** Let $\bar{b} = \langle \bar{b}_\eta : \eta \in {}^{\omega >} 2 \rangle$ be any witnesses to the fact that $\varphi(\bar{x}, \bar{y})$ exemplifies that
   T has SOP_t. Let $\bar{a} = \langle \bar{a}_\eta : \eta \in {}^{\omega >} 2 \rangle$ be t-fbti and satisfy the properties guaranteed by
21 Claim 2.11. We check that $\bar{a}$ satisfies the properties (a) and (b) from the Definition
   of SOP_t.
23   For (a), we first work with $t = 1$, the case $t = 2$ is similar. Let $\rho \in {}^\omega 2$ be given, and
   suppose that $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}) : n < \omega\}$ is inconsistent. Then there is some $n^* < \omega$ such that
25 $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}) : n < n^*\}$ is inconsistent. Let $\bar{\rho} = \langle \rho \upharpoonright n : n < n^* \rangle$, and let

   $$\Delta \overset{\text{def}}{=} \{\varphi_{n^*}^*((\bar{y}_0, \ldots, \bar{y}_k, \ldots)_{k < n^*})\} \quad \text{where } \varphi_{n^*}^* \equiv (\exists \bar{x}) \bigwedge_{k < n^*} \varphi(\bar{x}, \bar{y}_k).$$

27 Let $\bar{v} \approx_t \bar{\rho}$ be such that $\bar{b}_{\bar{v}}$ and $\bar{a}_{\bar{\rho}}$ realise the same Δ-types. As $\bar{v} \approx_t \bar{\rho}$ we have that
   $\bar{v} = \langle v_0, v_1, \ldots, v_{n^*-1} \rangle$ for some $v_0, v_1, \ldots, v_{n^*-1}$ satisfying

29   $$i < j < n^* \Rightarrow v_i \lhd v_j.$$

   Let $\eta \in {}^\omega 2$ be such that $v_1 \lhd \eta$ for all i. Hence $\{\varphi(\bar{x}, \bar{b}_{\eta \upharpoonright n}) : n < \omega\}$ is consistent, so in
31 particular

   $$\models \text{``}(\exists \bar{x}) \left[\bigwedge_{n < n^*} \varphi(\bar{x}, \bar{b}_{v_n}) \right] \text{''}.$$

1   Hence

$$\models \text{``}(\exists \bar{x}) \left[\bigwedge_{n < n^*} \varphi(\bar{x}, \bar{a}_{\rho \restriction n}) \right] \text{''},$$

3   a contradiction.

For (b), suppose that $\eta_0 \frown \langle 0 \rangle \unlhd \eta$ and let $\bar{\eta} = \langle \eta_0, \eta, \eta_0 \frown \langle 1 \rangle \rangle$, while $\varDelta = \{\varphi_2^*(\bar{y}_0,$
5   $\bar{y}_1)\}$. Let $\bar{v} = \langle v_0, v_1, v_2 \rangle$ be such that $\bar{\eta} \approx_t \bar{v}$ and $\bar{b}_{\bar{v}}$ realises the same $\varDelta$-types as $\bar{a}_{\bar{\eta}}$. If
$t = 1$, as $\bar{v} \approx_t \bar{\eta}$ we have $v_0 \frown \langle 0 \rangle \unlhd v_2$, hence $\{\varphi(\bar{x}, \bar{b}_{v_0 \frown \langle 1 \rangle}), \varphi(\bar{x}, \bar{b}_v)\}$ is contradictory,
7   hence $\{\varphi(\bar{x}, \bar{a}_{\eta_0 \frown \langle 1 \rangle}), \varphi(\bar{x}, \bar{a}_\eta)\}$ is contradictory. The case $t = 2$ is similar, as the notion
of incompatibility in $^{\omega >}2$ can be encaptured by a relevant choice of $\bar{\eta}$.   □

9   As we mentioned before, it would be really interesting to know if SOP$_2$ and SOP$_1$
are equivalent. A step towards understanding this question is provided by the next claim
11 which shows that in the case of theories which are SOP$_1$ and NSOP$_2$, the witnesses
to being SOP$_1$ can be chosen to be particularly nice.

13 **Claim 2.15.** *Suppose that $\varphi(\bar{x}, \bar{y})$ satisfies* SOP$_1$, *but for no n does the formula*
$\varphi_n(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}) \equiv \bigwedge_{k < n} \varphi(\bar{x}, \bar{y}_k)$ *satisfy* SOP$_2$. *Then there are witnesses $\langle \bar{a}_\eta \colon \eta \in {}^{\omega >}$*
$2 \rangle$ *for $\varphi(\bar{x}, \bar{y})$ satisfying* SOP$_1$ *which in addition satisfy*:
15

(c) *if $X \subseteq {}^{\omega >}2$, and there are no $\eta, v \in X$ such that $\eta \frown \langle 0 \rangle \unlhd v$, then $\{\varphi(\bar{x}, \bar{a}_\eta) \colon \eta \in$*
17   *$X\}$ is consistent.*
(d) *$\langle \bar{a}_\eta \colon \eta \in {}^{\omega >}2 \rangle$ is 1-fbti.*

19 (*In particular, such a formula and witnesses can be found for any theory satisfying*
SOP$_1$ *and* NSOP$_2$.)

21 **Proof.** We shall be using the following colouring theorem, for which we could not
find a specific reference and so we include a proof of it.

23 **Lemma 2.16.** *Suppose* $\mathrm{cf}(\kappa) = \kappa$ *and we colour* $^{\kappa >}2$ *by* $\theta < \kappa$ *colours. Then there is*
an embedding $h \colon {}^{\omega >}2 \to {}^{\kappa >}2$ such that $h(\eta)^\frown \langle l \rangle \unlhd h(\eta^\frown \langle l \rangle)$ and Rang(h) *is monochro-*
25 *matic.*

Proof. Let c be a colouring as in the assumptions and let $\{a_i \colon i < 0\}$ list Rang(c). We
27 claim that there is $v^* \in {}^{\kappa >}2$ and $j < \theta$ such that for every $v \in {}^{\kappa >}2$ satisfying $v^* \unlhd v$ there
is $\rho \in {}^{\kappa >}2$ with $v \unlhd \rho$ and $c(\rho) = j$. For otherwise, we can choose by induction on $i \leqslant \theta$
29 a member $\eta_i \in {}^{\kappa >}2$ with $i < j \Rightarrow \eta_i \unlhd \eta_j$ such that for no $\rho \in {}^\kappa 2$ do we have $\eta_{i+1} \unlhd \rho$ and
$c(\rho) = i$, using $\theta < \mathrm{cf}(\kappa)$. As $\theta < \kappa$, we obtain a contradiction.
31 Having found such v^*, j we define $h(\eta)$ for $\eta \in {}^n 2$ by induction on $n < \omega$. For $n = 0$
we choose $h(\langle \rangle)$ to satisfy $v^* \unlhd h(\langle \rangle)$ and $c(h(\langle \rangle)) = j$, which is possible by the choice
33 of v^* and j. For $n + 1$, for any $\eta \in {}^{n+1}2$ we choose for $l = 0, 1$ a member $h(\eta \frown \langle l \rangle)$
of $^{\kappa >}2$ which is above $h(\eta) \frown \langle l \rangle$ and on which c is j, which again is possible by the
35 choice of v^* and j.   □

1   Let $\varphi(\bar{x}, \bar{y})$ be a SOP_1 formula which is not SOP_2, and moreover assume that for no n does the formula φ_n defined as above satisfy SOP_2. By Claim 2.14, we can find

3 witnesses $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$ which are 1-fbti. By the compactness theorem, we can assume that we have a 1-fbti sequence $\langle \bar{a}_\eta : \eta \in {}^{\omega_1>}2 \rangle$ with the properties corresponding to (a) and (b) of Definition 2.2(2), namely

5
(a) for every $\eta \in {}^{\omega_1}2$, set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction\alpha}) : \alpha < \omega_1\}$ is consistent,

7 (b) if $v \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega_1>}2$, then $\{\varphi(\bar{x}, \bar{a}_{v\frown\langle 1 \rangle}), \varphi(\bar{x}, \bar{a}_\eta)\}$ is inconsistent.

    We shall now attempt to choose v_η and w_η for $\eta \in {}^{\omega_1>}2$, by induction on $lg(\eta) = \alpha$

9 $< w_1$ so that

    (i) $v_\eta \in {}^{\omega_1>}2$,

11   (ii) $\beta < \alpha \Rightarrow v_{\eta\restriction\beta} \vartriangleleft v_\eta$,
    (iii) $\beta < \alpha \Rightarrow v_\eta(lg(v_{\eta\restriction\beta})) = \eta(\beta)$,

13   (iv) $w_\eta \subseteq {}^{\omega_1>}2$ is finite and $v \in w_\eta \Rightarrow lg(v) < lg(v_\eta)$,
    (v) if $lg(\eta)$ is a limit ordinal > 0, then $w_\eta = \emptyset$,

15   (vi) if $\eta \in {}^\beta 2$ and $l < 2$, then $w_{\eta\frown\langle l \rangle} \subseteq \{\rho \in {}^{\omega_1>}2 : v_\eta \frown \langle l \rangle \trianglelefteq \rho\}$ and $\max\{lg(\rho) : \rho \in w_{\eta\frown\langle l \rangle}\} < lg(v_{\eta\frown\langle l \rangle})$,

17 (vii) for each η there is $\rho^* = \rho_\eta^*$ such that
      (α) $v_\eta \vartriangleleft \rho^* \in {}^{\omega_1}2$,

19   (β) $|\{\alpha < \omega_1 : \rho^*(\alpha) = 1\}| \aleph_1$,
      (γ) letting

21          $$p_\eta(\bar{x}) \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_{\eta\restriction\gamma} \text{ for some } \gamma \leqslant lg(\eta)\},$$

        we have that all large enough β^*, the set

23          $$p_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\rho^*\restriction\beta}) : \beta > \beta^* \land \rho^*(\beta) = 1\}$$

        is consistent,

25 (viii) $\rho_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta\frown\langle 0 \rangle} \cup w_{\eta\frown\langle 1 \rangle}\}$ is inconsistent.

    Before proceeding, we make several remarks about this definition. Firstly, requirements

27 (vii) and (viii) taken together imply that for each $\eta \in {}^{\omega_1>}2$ we have that $w_{\eta\frown\langle 0 \rangle} \cup w_{\eta\frown\langle 1 \rangle} \neq \emptyset$. Secondly, the definition of $w_{\eta\frown\langle l \rangle}$ for $l \in \{0, 1\}$ implies that

    $$\bigwedge_{l=0,1} \rho_l \in w_{\eta\frown\langle l \rangle} \Rightarrow \rho_0 \perp \rho_1.$$
29

    Thirdly, in (vii), any ρ^* which satisfies that $v_\eta \vartriangleleft \rho^*$ and $|\{\gamma : \rho^*(\gamma) = 1\}| = \aleph_1$ can be

31 chosen as ρ_η^*, by indiscernibility.

    Now let us assume that a choice as above is possible, and we have made it. Hence

33 for each $\eta \in {}^{\omega_1>}2$ there is a finite $q_\eta \subseteq p_\eta$ such that

    $$q_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta\frown\langle 0 \rangle} \cup w_{\eta\frown\langle 1 \rangle}\} \tag{$*$}$$

35 is inconsistent. Notice that there are q and $\eta^* \in {}^{\omega_1}2$ such that

    $$(\forall \eta_1)[\eta^* \trianglelefteq \eta_1 \in {}^{\omega_1>}2 \Rightarrow (\exists \eta_2 \in {}^{\omega_1>}2)(\eta_1 \trianglelefteq \eta_2 \land q_{\eta_2} = q)].$$

1   Namely, otherwise, we would have the following: each p_η is countable, hence for every η there is $g(\eta)$ with $\eta \lhd g(\eta) \in {}^{\omega_1 >}2$ and

3
$$g(\eta) \trianglelefteq \eta_1 \Rightarrow q_{\eta_1} \nsubseteq p_\eta.$$

Let $\eta_0 \overset{\text{def}}{=} \langle \rangle$, and for $n < \omega$ let $\eta_{n+1} = g(\eta_n)$. Let $\eta \overset{\text{def}}{=} \bigcup_{n<\omega} \eta_n$, hence $p_\eta = \bigcup_{n<\omega} p_{\eta_n}$

5   (as $w_\eta = \emptyset$), and so $q_\eta \subseteq p_{\eta_n}$ for some n, a contradiction.

Having found such q, η^*, by renaming and using Lemma 2.16, we can assume that

7    $\eta^* \overset{\text{def}}{=} \langle \rangle$ and that for all $\eta \in {}^\omega 2$ we have $q_\eta = p_{\langle \rangle} = q$ (as $\eta \trianglelefteq \nu \Rightarrow p_\eta \subseteq p_\nu$). For $\eta \in {}^{\omega >}2$ let $\bar\tau_\eta$ list w_η. Without loss of generality, by thinning and renaming, we have that for

9   all η_1, η_2,

$$\langle v_{\eta_1} \rangle \frown \bar\tau_{\eta_1 \frown \langle 0 \rangle} \frown \bar\tau_{\eta_1 \frown \langle 1 \rangle} \approx_1 \langle v_{\eta_2} \rangle \frown \bar\tau_{\eta_2 \frown \langle 0 \rangle} \frown \bar\tau_{\eta_2 \frown \langle 1 \rangle}.$$

11   Similarly to the proof of Claim 2.7, we can define a formula $\psi(\bar x, \bar y)$ and $\{\bar b_\eta : \eta \in {}^{\omega >}2\}$ such that

13
$$\psi(\bar x, \bar b_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_\eta\}.$$

We claim that $\psi(\bar x, \bar y)$ and $\langle \bar b_\eta : \eta \in {}^{\omega >}2 \rangle$ exemplify SOP$_2$ of T, which is then a con-

15   tradiction (noting that ψ is a formula of the form φ_n for some n, where φ_n was defined in the statement of the claim). We check the two properties from Definition 2.2(1).

17   To see (a), let $\eta \in {}^\omega 2$ be given. We have that p_η is consistent, and $q \subseteq p_\eta$. For $n < \omega$, we have

19
$$\psi(\bar x, \bar b_{\eta \restriction n}) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_{\eta \restriction n}\}.$$

As this is a conjunction of a set of formulae each of which is from p_η, we have that

21    $\{\psi(\bar x, \bar b_{\eta \restriction n}) : n < \omega\}$ is consistent. To check (b), suppose $\eta \perp \nu \in {}^{\omega >}2$. Let n be such that $\eta \restriction n = \nu \restriction n$ but $\eta(n) \neq \nu(n)$. Hence

23
$$\psi(\bar x, \bar b_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_{\eta \restriction n \frown \eta(n)}\}$$

and

25
$$\psi(\bar x, \bar b_\nu) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar x, \bar a_\rho) : \rho \in w_{\eta \restriction n \frown \nu(n)}\},$$

so taken together, the two are contradictory by the choice of q.

27   We conclude that the choice of v_η and w_η cannot be carried throughout $\eta \in {}^{\omega_1 >}2$. So, there is $\alpha < \omega_1$ and $\eta \in {}^\alpha 2$ such that v_η, $w_{\eta \frown \langle l \rangle}$, $v_{\eta \frown \langle l \rangle}$ for $l < 2$ cannot be chosen, and

29   α is the first ordinal for which there is such η. Let $v_\eta^0 \in {}^{\omega_1 >}2 \rhd \bigcup_{\beta < \alpha} v_{\eta \restriction \beta} \frown \langle \eta(\alpha - 1) \rangle$ if the latter part is defined, otherwise let $v_\eta^0 \rhd \bigcup_{\beta < \alpha} v_{\eta \restriction \beta}$. This choice of $v_\eta = \rho$ for any

31    $\rho \unrhd v_\eta^0$ with $\rho \in {}^{\omega_1}2$ satisfies items (i)–(iii) above. We conclude that $w_{\eta \frown \langle l \rangle}$ for $l < 2$

33   using any $\rho \unrhd v_\eta^0$ with $\rho \in {}^{\omega_1 >}2$ for v_η could not have been chosen, and examine why

this is so. Note that p_η is already defined. Let

$$\Theta \overset{\text{def}}{=} \left\{ \begin{array}{l} (\rho, \gamma, w) : v_\eta^0 \vartriangleleft \rho \in {}^{\omega_1}2, \\ \quad lg(v_\eta^0) \leqslant \gamma < \omega_1, \\ \quad (\exists^{\aleph_1}\beta < \omega_1)(\rho(\beta) = 1), \\ \quad w \subseteq \{ \Upsilon \in {}^{\omega_1 >}2 : \rho \restriction \gamma \vartriangleleft \Upsilon \} \text{ is finite and} \\ \quad \text{for some } \beta_\rho < \omega_1 \text{ the set} \\ \quad p_\eta \cup \{ \varphi(\bar{x}, \bar{a}_{\rho \restriction \beta}) : \rho(\beta) = 1 \,\&\, \beta \in [\beta_\rho, \omega_1) \} \\ \quad \cup \{ \varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w \} \\ \quad \text{is consistent} \end{array} \right\}$$

We make several *observations*:

(0) If $(\rho, \gamma, w) \in \Theta$ and $w \subseteq w'$ with w' finite and $w' \setminus w$ is contained in $\{\rho \restriction \beta : \beta_\rho \leqslant \beta \wedge \rho(\beta) = 1\}$, *then* $(\rho, \gamma, w') \in \Theta$.

[This is obvious.]

(1) If $(\rho_l, \gamma, w_l) \in \Theta$ and for some $\sigma \in {}^{\omega_1 >}2$ with $v_\eta^0 \trianglelefteq \sigma$ we have $\sigma \frown \langle l \rangle \vartriangleleft \rho_l \restriction \gamma$ for $l < 2$, while ρ_0 and ρ_1 are eventually equal, *then* $(\rho_l, lg(\sigma), w_0 \cup w_1) \in \Theta$.

[Why? We have $w_l \subseteq \{ \Upsilon \in {}^{\omega_1 >}2 : \rho_l \restriction \gamma \trianglelefteq \Upsilon \}$ is finite, so clearly $w_0 \cup w_1 \subseteq \{ \Upsilon \in {}^{\omega_1 >}2 : \sigma \trianglelefteq \Upsilon \}$ is finite. By the assumption, we have that for some $\beta_l < \omega_1$ for $l < 2$

$$p_\eta \cup \{ \varphi(\bar{x}, \bar{a}_{\rho_l \restriction \beta}) : \beta > \beta_l \wedge \rho_l(\beta) = 1 \} \cup \{ \varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_l \}$$

is consistent. Suppose that (1) is not true with $l = 0$ and let $\beta^* \geqslant \max\{\beta_0, \beta_1\}$ be such that $\beta^* < \omega_1$ and for $\beta > \beta^*$ the equality $\rho_0(\beta) = \rho_1(\beta)$ holds. Hence we have that

$$p_\eta \cup \{ \varphi(\bar{x}, \bar{a}_{\rho 0 \restriction \beta}) : \beta > \beta^* \wedge \rho_0(\beta) = 1 \} \cup \{ \varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1 \}$$

is inconsistent. By increasing w_0 if necessary, (0) implies that

$$\rho_\eta \cup \{ \varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1 \}$$

is inconsistent. Let $v_\eta \overset{\text{def}}{=} \sigma$, for $l < 2$ let $w_{\eta \frown \langle l \rangle} = w_l$, and let $v_{\eta \frown \langle l \rangle} \overset{\text{def}}{=} \rho_l \restriction \beta_l^*$ for a large enough β_l^* so that $\beta^* < \beta_l^*$ and $\max(\{lg(\Upsilon) : \Upsilon \in w_{\eta \frown \langle l \rangle}\}) < \beta_l^*$.

This choice shows that we could have chosen v_η, $w_{\eta \frown \langle l \rangle}$ as required, contradicting the choice of η.]

(2) If $v_\eta^0 \vartriangleleft \rho \in {}^{\omega_1}2$ for some ρ such that there are $\aleph_1$ many $\beta < \omega_1$ with $\rho(\beta) = 1$, and $lg(v_\eta^0) \leqslant \gamma < \omega_1$, *then* $(\rho, \gamma, \emptyset) \in \Theta$.

[Why? By the choice of p_η and the remark about the freedom in the choice of ρ^* that we made earlier.]

Now we use the choice of η to define witnesses to T being SOP$_1$ which also satisfy the requirements of the claim. For $\tau \in {}^{\omega >}2$, let $\bar{b}_\tau \overset{\text{def}}{=} \bar{a}_{v_\eta^0 \frown \tau}$. Let us check the required properties. Properties (a), (b) and (d) follow from the choice of $\{\bar{a}_\sigma : \sigma \in {}^{\omega_1 >}2\}$. Let $X^* \subseteq {}^{\omega >}2$ be such that there are no $\sigma, v \in X^*$ with $\sigma \frown \langle 0 \rangle \trianglelefteq v$, we need to show that $\{\varphi(\bar{x}, \bar{b}_\tau) : \tau \in X^*\}$ is consistent. It suffices to show the same holds when X^* replaced

1   by an arbitrary finite $X \subseteq X^*$. Fix such an X. Clearly, it suffices to show that for some
ρ, γ, letting $w = \{v_\eta^0 \frown \tau : \tau \in X\}$, we have $(\rho, \gamma, w) \in \Theta$.

3   Let $\rho^* \in {}^{\omega_1}2$ be such that $v_\eta^0 \lhd \rho^*$ and $\rho^*(\beta) = 1$ for $\aleph_1$ many β. By induction on
$n \stackrel{\text{def}}{=} |X|$ we show:

5   there is $\rho \in {}^{\omega_1}2$ such that for some $\gamma \geqslant \max\{lg(\sigma) : \sigma \in w\}$, we have $(\rho, \gamma, w) \in \Theta$
and $\beta > \gamma \Rightarrow \rho(\beta) = \rho^*(\beta)$, while $\rho(\gamma) = 1$.

7    $n = 0$. Follows by observation (2) above.
$n = 1$. Let $X = \{\tau\}$ and $\gamma = lg(\tau) + lg(v_\tau^0)$. Let $\rho \in {}^{\omega_1}2$ be such that $\rho \restriction \gamma = v_\eta^0 \frown \tau$,

9    $\rho(\gamma) = 1$ and $\beta > \gamma \Rightarrow \rho(\beta) = \rho^*(\beta)$. By observation (2) above, we have that $(\rho, \gamma, \emptyset) \in$
Θ. Then, by observation (0), we have $(\rho, \gamma, w) \in \Theta$.

11    $n = k + 1 \geqslant 2$. *Case 1.* w is linearly ordered by $\lhd$.
Let $\tau \in w$ be of maximal length, so clearly $\sigma \in w \setminus \{\tau\} \Rightarrow \sigma \frown \langle 1 \rangle \unlhd \tau$. Let $\rho \in {}^{\omega_1}2$ be

13   such that $\tau \frown \langle 1 \rangle \lhd \rho$ and $\beta > lg(\tau)$, while $\rho(\beta) = \rho^*(\beta)$. Now continue as in the case
$n = 1$.

15   *Case 2.* Not Case 1.
Let $\sigma \in {}^{\omega_1 >}2$ be $\lhd$-maximal such that $(\forall \tau)(\tau \in w \Rightarrow \sigma \unlhd \tau)$. This is well defined, as

17    $w \neq \emptyset$ is finite. Let $w_l \stackrel{\text{def}}{=} \{\tau \in w : \sigma \frown \langle l \rangle \unlhd \tau\}$, so $w_0 \cap w_1 = \emptyset$ but neither of w_0, w_1 is
empty. Now we have that $\sigma \notin w$, as otherwise we could choose $\tau \in w_0$ such that $\sigma \frown$

19    $\langle 0 \rangle \unlhd \tau$, obtaining an easy contradiction with our assumptions on X. Hence $w = w_0 \cup w_1$.
We can now use observation (1) and the inductive hypothesis.   □

21   To complete this discussion of the syntactic properties (N)SOP1, 2 we shall quote
a result from [14] in which the understanding of SOP'_1 and the witnesses for SOP_1

23   developed here was used to show that NSOP_1 theories admit a rank function.

Definition 2.17. Given (partial) types $p(\bar{x}), q(\bar{y})$ and a formula $\varphi(\bar{x}, \bar{y})$. By induction

25   on $n < \omega$ we define when

$$\text{rk}^1_{\varphi(\bar{x}, \bar{y})}(p(\bar{x}), q(\bar{y})) \geqslant n.$$

27    $n = 0$. This happens iff both $p(\bar{x})$ and $q(\bar{y})$ are consistent.
$n + 1$. The rank is $\geqslant n + 1$ iff for some $\bar{c}$ realising $q(\bar{y})$ both

29    $$\text{rk}^1_{\varphi(\bar{x}, \bar{y})}(p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}, q(\bar{y})) \geqslant n$$

and

31    $$\text{rk}^1_{\varphi(\bar{x}, \bar{y})}(p(\bar{x}), q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}) \geqslant n.$$

If the rank is $\geqslant n$ for all n then we say it is infinite, otherwise we say it is finite.

33   **Theorem 2.18** (Shelah–Usvyatsov [14]). *A theory T is* NSOP_1 *iff*

$$\text{rk}^1_{\varphi(\bar{x}, \bar{y})}(\bar{x} = \bar{x}, \bar{y} = \bar{y}) < \infty$$

35   *for every formula* $\varphi(\bar{,} \bar{y})$.

1 ## 3. ◁*-maximality revisited

In this section we come back to our main thesis, which is that properties SOP_2 and
3 the maximality in the ◁*-order are closely connected.

Our main proof will use two auxiliary notions. The first is the order $\lhd_\lambda^{**}$, which is
5 a version of the $\lhd_\lambda^*$-order.

Definition 3.1. (1) For (complete first order theories) T_1, T_2 and a regular cardinal
7 $\lambda > |T_1|$, $|T_2|$, let $T_1 \lhd_\lambda^{**} T_2$ mean:

There is a λ-relevant (T_1, T_2)-superior $(T^*, \bar\varphi, \bar\psi)$ (see Definition 1.2) such that T^*
9 has Skolem functions and if $T^{**} \supseteq T^*$ is complete with $|T^{**}| < \lambda$ *then*

($\oplus$) there is a model M of T^{**} of size λ and an $M^{[\bar\psi]}$-type p omitted by M such
11 that for every elementary extension N of M of size λ which omits p and a type q (in
one variable) over $N^{[\bar\varphi]}$, there is an elementary extension of N of size λ which realises
13 q and omits p.

(2) Let $T_1 \lhd^{**} T_2$ mean that $T_1 \lhd_\lambda^{**} T_2$ holds for all large enough regular λ.
15 (3) T_1 is said to be $\lhd_\lambda^{**}$-maximal iff there is no T_2 such that $T_1 \lhd_\lambda^{**} T_2$. Similarly
for ◁**.

17 The connection between this notion and ◁* is given by the following claim:

Claim 3.2. *Suppose that T_1, T_2 are theories and $\lambda > |T_1|$, $|T_2|$ satisfies $2^\lambda = \lambda^+$. Then*

19 $$T_1 \lhd_{\lambda^+}^* T_2 \Rightarrow \neg(T_2 \lhd_\lambda^{**} T_1).$$

Proof. This statement is just a reformulation of the beginning of the proof of Theo-
21 rem 1.17. In other words, let $(T, \bar\varphi_1, \bar\varphi_2)$ show that $T_1 \lhd_{\lambda^+}^* T_2$. This means that $|T| < \lambda^+$
but since $\lambda^{<\lambda} = \lambda$ and $\lambda > |T_1|, |T_2|$ we may assume that $|T^*| < \lambda$. Namely since there
23 is a consistent theory $T \supseteq \bar\varphi_1 \cup \bar\varphi_2$ in which $\bar\varphi_l$ interprets T_l, and each T_l has size
$< \lambda$, there is a consistent theory T' of size $< \lambda$ which does the same. Without loss of
25 generality $T' \subseteq T$. In particular $|\tau(T')| < \lambda$ so by extending T' to a complete subtheory
of T and renaming we may assume T' is complete. Any model M of T has a reduct
27 N that is a model of T' and that satisfies $M^{[\bar\varphi]} = N^{[\bar\varphi]}$ and similarly for $\bar\psi$. Hence
$(T', \bar\varphi, \bar\psi)$ is a λ-relevant (T_1, T_2)-superior that exemplifies $T_1 \lhd_{\lambda^+}^* T_2$, so by renaming
29 we may assume $|T| < \lambda$.

Suppose for contradiction that $T_2 \lhd_\lambda^{**} T_1$ and let $(T^*, \bar\varphi, \bar\psi)$ exemplify this. Without
31 loss of generality, $\bar\varphi_1 = \bar\psi$ and $\bar\varphi_1 = \bar\varphi$ and the common vocabulary of T and T^* is
$\tau(\bar\varphi_1) \cup \tau(\bar\varphi_2)$. Hence $T^{**} = T \cup T^*$ is consistent by Robinson Consistency Criterium.
33 Without loss of generality T^{**} is complete. Hence let M be a model of T^{**} of size
λ and p be a $M^{[\bar\psi]}$ type omitted by M exemplifying the definition of $\lhd_\lambda^{**}$. Using the
35 assumption $2^\lambda = \lambda^+$ we can build by induction an elementary extension N of M with
$|N| = \lambda^+$, with N omitting p and being $\bar\varphi$-saturated. This is a contradiction with the
37 choice of T. $\square$

Corollary 3.3. *Suppose that for all large enough regular λ we have $2^\lambda = \lambda^+$. Then*
39 *any ◁*-maximal theory is also ◁**-maximal.*

Proof. Suppose otherwise and let T exemplify this. Hence for every κ there is regular $\lambda > \kappa$ such that T is not $\triangleleft^{**}$-maximal and $2^\lambda = \lambda^+$. Hence T is not $\triangleleft^*_{\lambda^+}$-maximal by Claim 3.2, a contradiction. $\square$

The next notion we need is a syntactic property.

Definition 3.4. Let T be a theory.

(1) For a formula $\sigma(x, \bar{y})$ we say that $\sigma(x, \bar{y})$ has SOP_2'' iff for some [by compactness equivalently all] regular $\lambda > |T|$ there is a sequence

$$\langle \bar{e}_{\bar{\eta}} : \bar{\eta} = \langle \eta_0, \ldots, \eta_{n^*-1} \rangle, \eta_0 \triangleleft \cdots \triangleleft \eta_{n^*-1} \in {}^{\lambda >}\lambda \text{ and } lg(\eta_i) \text{ a successor} \rangle$$

such that

(α) for each $\eta \in {}^\lambda\lambda$, the set

$$\left\{ \begin{array}{c} \sigma(x, \bar{e}_{\bar{\eta}} : \bar{\eta} = \langle \eta \upharpoonright (\alpha_0 + 1), \eta \upharpoonright (\alpha_1 + 1), \ldots, \eta \upharpoonright (\alpha_{n^*-1} + 1) \rangle \\ \text{and } \alpha_0 < \alpha_1 < \cdots \alpha_{n^*-1} < \lambda \end{array} \right\}$$

is consistent

(β) for every large enough m, if $g : {}^{n^* \geq} m \to {}^{\lambda >}\lambda$ satisfies

$$\rho \triangleleft v \Rightarrow g(\rho) \triangleleft g(v)$$

and

$$\rho \in {}^{n \geq} m \Rightarrow lg(g(\rho)) \text{ is a successor,}$$

while for $l < n^* - 1$

$$(g(\rho)) \frown \langle l \rangle \trianglelefteq g(\rho \frown \langle l \rangle),$$

then

$$\{ \sigma(x, \bar{e}_{\langle g(\rho \upharpoonright 1), g(\rho \upharpoonright 2), \ldots, g(\rho) \rangle}) : \rho \in {}^{n^*} m \}$$

is inconsistent. Here $n^* = lg(\bar{y})$ in $\sigma(x, \bar{y})$.
(2) T is said to have SOP_2'' iff some $\sigma(x, \bar{y})$ exemplifies it.

Our Theorem 3.6 is phrased in terms of SOP_2''. Answering a question from an earlier version of this paper Shelah and Usvyatsov proved in [14] the following Theorem 3.5, which then can be used together with Theorem 3.6 to prove Corollary 3.9 which states that $\triangleleft^*$-maximality implies SOP_2.

Theorem 3.5 (Shelah–Usvyatsov [14]). *For any theory T, T has SOP_2 iff it has SOP_2''.*

Main Theorem 3.6. *For any theory T and regular cardinal $\lambda > |T|$, if T is $\triangleleft^{**}_\lambda$-maximal then T has SOP_2''.*

Proof. Let T be a given theory and let $\lambda = \mathrm{cf}(\lambda) > |T|$. We shall assume that T is $\vartriangleleft_\lambda^{**}$-maximal and prove that T has SOP_2''. To make the reading of the proof easier we shall break it into stages.

Stage A. Let $T_{\mathrm{tree}}^n \overset{\mathrm{def}}{=} \mathrm{Th}(^{n\geqslant}2, <_{\mathrm{tr}})$ for $n < \omega$, where $< = <_{\mathrm{tr}}$ stands for the relation of "being an initial segment of", and let $T_{\mathrm{tree}} \overset{\mathrm{def}}{=} \lim\langle T_{\mathrm{tree}}^n : n < \omega\rangle$, that is to say the set of all ψ which are in T_{tree}^n for all large enough n. In order to use our assumptions at a later point, let us fix a theory T^* which is a λ-relevant (T_{tree}, T)-superior with Skolem functions (such a T^* is easily seen to exist), and let $\bar\varphi, \bar\psi$ be the interpretations of T_{tree} and T in T^*, respectively. We can without loss of generality, by renaming if necessary, assume that $\mathscr{L}(T) \subseteq \mathscr{L}(T^*)$, so the interpretation $\bar\psi$ is trivial.

As $|T|, |T^*| < \lambda$, we can find $A \subseteq \lambda$ which codes T and T^*. Working in $\mathbf{L}[A]$, we shall define a model M of T^* of size λ as follows. Let

$$
\begin{aligned}
\Gamma \overset{\mathrm{def}}{=} {}& T^* \cup \{\varphi = (x_\eta, x_\eta): \eta \in {}^{\lambda>}\lambda\} \\
& \cup\{x_\eta <_\varphi x_v: \eta \vartriangleleft v \in {}^{\lambda>}\lambda\} \\
& \{\neg(x_\eta <_\varphi x_v): \neg(\eta \vartriangleleft v) \text{ for } \eta, v \in {}^{\lambda>}\lambda)\}.
\end{aligned}
$$

By a compactness argument and the fact that $\bar\varphi$ interprets T_{tree} in T^*, we see that Γ is consistent. Let M be a model of Γ of size $\lambda = \lambda^{<\lambda}$ (as we are in $\mathbf{L}[A]$). For $\eta \in {}^{\lambda>}\lambda$ let a_η be the realisation of x_η in M. For $\eta \in {}^{\lambda>}\lambda$, let

$$
p_\eta(x) \overset{\mathrm{def}}{=} \{a_{\eta\restriction\alpha} <_\varphi x : \alpha < \lambda\}.
$$

By the choice of M and the compactness argument it follows that each p_η is a (consistent) type. Note that for $\eta_0 \neq \eta_1 \in {}^\lambda\lambda$, types p_{η_0} and p_{η_1} are contradictory. Let

$$
p_\eta'(x) = \{a <_\varphi x : \text{for some } \alpha < \lambda,\ a <_\varphi a_{\eta\restriction\alpha}\}.
$$

By the axioms of T_{tree}, we have that p_η and p_η' are equivalent. Now we observe that by the size of M there is $\eta^* \in {}^\lambda\lambda$ such that the type p_{η^*}' is omitted in M, and p_{η^*}' is not definable in M, i.e. for no formula $\vartheta(y, \bar z)$ and $\bar c \subseteq M$ do we have: for $a \in M$, the following are equivalent: $[a <_\varphi x] \in p_{\eta^*}'$ and $M \models \vartheta[a, \bar c]$. Let $p \overset{\mathrm{def}}{=} p_{\eta^*}'$ for such a fixed η^*. For $\alpha < \lambda$, let $a_\alpha \overset{\mathrm{def}}{=} a_{\eta^*\restriction\alpha}$. We now go back to V and make an observation about M.

Subclaim 3.7. T_{tree} *satisfies the following property*:
for any formula $\vartheta(x, \bar y)$ *we have that* $T_{\mathrm{tree}} \vdash \sigma = \sigma(\vartheta)$, *where*

$$
\begin{aligned}
\sigma \equiv {}& (\forall \bar y)[[(\forall x_1, x_2))\vartheta(x_1, \bar y) \wedge \vartheta(x_2, \bar y) \Rightarrow x_1 \leqslant_{\mathrm{tr}} x_2 \vee x_2 \leqslant_{\mathrm{tr}} x_1] \\
& \Rightarrow (\exists z)(\forall x)(\vartheta(x, \bar y) \Rightarrow x \leqslant_{\mathrm{tr}} z)].
\end{aligned}
$$

Proof. Let $\vartheta(x, \bar y)$ be given. By the definition of T_{tree} we only need to show that $T_{\mathrm{tree}}^n \vdash \sigma$ for all large enough n, which is obvious as for every n the tree $^{n\geqslant}2$ has the top level. $\square$

1   Hence the interpretation $\bar{\varphi}$ of T_{tree} in T^* satisfies the same statement claimed about
T_{tree}. We conclude:

3   ⊗ if $M \prec N$ and p is not realised in N, *then* there is no $\vartheta(x, \bar{c})$ with $\bar{c} \subseteq N$ such
that $\vartheta(a_{\eta^* \upharpoonright \alpha}, \bar{c})$ for all $\alpha < \lambda$ holds and every two elements of N satisfying $\vartheta(x, \bar{c})$ are

5    $<_\varphi$-comparable.

Stage B. We shall choose a filtration $\bar{M} = \langle M_i : i < \lambda \rangle$ of M, and an increasing
sequence $\langle \alpha_i : i < \lambda \rangle$, requiring:

7

(a) $M_i \prec M$ and M_i are $\prec$-increasing continuous of size $< \lambda$, with M being the $\bigcup_{i<\lambda} M_i$,

9   (b) $a_{\alpha_i} \in M_{i+1} \setminus M_i$.

We may note that the branch induced by $\{a_{\alpha_i} : i < \lambda\}$ is the same as the one induced

11   by $\{a_\alpha : \alpha < \lambda\}$. Hence p is realised in any model in which $p'(x) \overset{\text{def}}{=} \{a_{\alpha_i} <_\varphi x : i < \lambda\}$ is
realised (or even the similarly defined type using any unbounded subset of $\{\alpha_i : i < \lambda\}$).

13   Hence, by renaming, without loss of generality we have $\alpha_i = i$ for all $i < \lambda$.

Stage C. At this point we shall use the $\lhd_\lambda^{**}$-maximality of T, which implies that it

15   is not true that $T \lhd_\lambda^{**} T_{\text{tree}}$. In particular, our T^*, M and p do not exemplify this, hence
there is N with $M \prec N$ and $\|N\| = \lambda$, such that N omits p, but for some $N^{[\bar{\psi}]}$-type q

17   over N, whenever $N \prec N^+$ and N^+ realises q, also N^+ realises p. By ⊗, the branch
induced by $\{a_{\eta^* \upharpoonright \alpha} : \alpha < \lambda\}$ is not definable in N, so without loss of generality $N = M$.

19   We can also assume that q is a complete type over $M^{[\bar{\psi}]}$. Let us now use the choice
of q to define for each club E of λ a family of formulae associated with it, and to

21   show that each of these families is inconsistent. We use the abbreviation c.d. for "the
complete diagram of".

23   For any club E of λ we define

$$\Gamma_E \overset{\text{def}}{=} \text{c. d. } (M) \cup q(x) \cup \{\neg(a_i <_\varphi \tau(x, \bar{b})) : i \in E, \tau \text{ a term of } T^*, \bar{b} \subseteq M_i\}.$$

25   Clearly, for any club E, if Γ_E is consistent then there is a model N in which Γ_E
is realised. Identifying any $b \in M$ with its interpretation in N and letting a^* be the

27   interpretation of x from Γ_E, we can assume that N is an elementary extension of M
in which q is realised by a^*. As T^* has Skolem functions, we have $M \prec N$. Let N_1

29   be the submodel of N with universe

$$A^* \overset{\text{def}}{=} M \cup \bigcup_{i \in E} \{\tau(a^*, \bar{b}) : \bar{b} \subseteq M_i \text{ and } \tau \text{ a term of } T^*\}.$$

31   Note that the size of N_1 is λ. Clearly, N_1 is closed under the functions of T^*, so
$M \subseteq N_1 \subseteq N$. As T^* has Skolem functions, we get that $M \prec N_1 \prec N$. By the third

33   part of the definition of Γ_E, p is omitted in N_1. This is in contradiction with our
assumptions, as $a^* \in N_1$ realises $q(x)$.

35   Hence we can conclude

for every club E of λ, the set Γ_E is inconsistent.

37   **Stage D.** Now we start our search for a formula that exemplifies that T has SOP$_2''$.
In the following definitions, we shall use the expression "an almost branch" or the

1   abbreviation a.b. to stand for a set linearly ordered by $<_\varphi$ (but not necessarily closed under $<_\varphi$-initial segments and not necessarily unbounded). Let

$$\Theta^0_{T^*} \overset{\text{def}}{=} \left\{ \begin{array}{l} \vartheta(x,y,\bar z): \quad \text{there is } l = l_\vartheta < \omega \text{ such that} \\ \qquad \text{for every } M^* \models T^*, a \in M^*, \bar c \subseteq M^*, \text{ the set} \\ \qquad \vartheta(a,y,\bar c)^{M^*} \text{ is the union of } \leq l \text{ a.b. in } M^{*[\bar\varphi]} \end{array} \right\},$$

and let Θ_{T^*} be the set of all $\vartheta(x,\bar y,\bar z)$ of the form $\bigvee_{j<n} \vartheta_j(x,y_j,\bar z_j)$ for some ϑ_0,

5    $\ldots, \vartheta_{n-1} \in \Theta^0_{T^*}$ (where $\bar y = \langle y_j : j < n \rangle$ and $\bar z = \underset{j<n}{\frown} \bar z_j$). The formulae in Θ_{T^*} will be called candidates. For every candidate

$$\vartheta(x,\bar y,\bar z) \equiv \bigvee_{j<n} \vartheta_j(x,y_j,\bar z_j)$$

and a $\bar\psi$-formula $\sigma(x,\bar t)$, we consider the following game $\partial_{n,\sigma,\vartheta}$ (whose definition also

9   depends on our fixed p,q and $\bar M$), played by two players $\exists$ and $\forall$. The game starts by $\exists$ playing $\bar b^0$ from $^{\lg(\bar z_0)}M$, then $\forall$ playing $\alpha_0 < \lambda$. After that $\exists$ chooses $\beta_0 \in (\alpha_0, \lambda)$ and

11    $\bar b^1 \in {}^{\lg(\bar z_1)}M$ such that $\bar b^0 \in {}^{\lg(\bar z_0)}M_{\beta_0}$, after which $\forall$ chooses $\alpha_1 < \lambda$, etc., finishing by $\exists$ choosing $\bar b^{n-1} \in {}^{\lg(\bar z_{n-1})}M$ and $\forall$ choosing α_{n-1}, while $\exists$ chooses $\beta_{n-1} \in (\alpha_{n-1}, \lambda)$ such

13   that $\bar b^{n-1} \in {}^{\lg(\bar z_{n-1})}M_{\beta_{n-1}}$. Player $\exists$ wins this game iff for some $\bar e \in {}^{\lg(\bar t)}M$ we have

$$\sigma(x,\bar e) \in q \text{ and } M \models (\forall x)[\sigma(x,\bar e) \Rightarrow \vartheta(x, \langle a_{\beta_0}, \ldots, a_{\beta_{n-1}} \rangle, \underset{k<n}{\frown} \bar b^k)]. \qquad (\otimes_1)$$

15   (Note: the constants a_{β_k} are from the set $\{a_i : i < \lambda\}$ we chose above.) Observe that every sequence $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in {}^n\lambda$ is an admissible sequence of moves for $\forall$.

17   We shall show that for some $n \geq 1$ and σ, ϑ, player $\exists$ has a winning strategy in the game $\partial_{n,\sigma,\vartheta}$, where $\vartheta = \bigvee_{j<n} \vartheta_j$ as above. As these are determined games, it suffices

19   to show that for some $n \geq 1$ and σ, ϑ, player $\forall$ does not have a winning strategy. Suppose that this is not the case, arguing in $(\mathscr{H}(\chi), \in, <^*_\chi, \bar M, p, q)$, where χ is large

21   enough and $<^*_\chi$ is a fixed well ordering of $\mathscr{H}(\chi)$. Fix for a moment (n, σ, ϑ). Player $\forall$ has a winning strategy in $\partial_{n,\sigma,\vartheta}$, which, replacing the ordinals α_l by constants a_{α_l}, can

23   be represented by a sequence of functions $G^l_{n,\sigma,\vartheta}$ for $l < n$ (in $(\mathscr{H}(\chi), \in, <^*_\chi, \bar M, p, q)$), where for $l < n$, if the play up to time l has been $\bar b_0, \alpha_0, \beta_0, \ldots, \alpha_{l-1}, \beta_{l-1}, \bar b^l$, then

25    $G^l_{n,\sigma,\vartheta}$ applied to this play is a_{α_l} for the α_l in the choice of player $\forall$. We shall assume that these functions are the $<^*$-first which can act in this manner. Using this and

27   elementarity, we notice that for every n, σ, ϑ the values of $G^l_{n,\sigma,\vartheta}$ take place in M, and that

29    $$E_0 \overset{\text{def}}{=} \{\delta < \lambda : (\forall \sigma, \vartheta)(\forall n)(\forall l < n)[M \cap \text{Skolem}_{(\mathscr{H}(\chi), \in, \bar M, G^l_{n,\sigma,\vartheta})}(M_\delta) = M_\delta]\}$$

is a club of λ (as $|T^*|, \|M_i\| < \lambda$ for all i and $\bar M$ is increasing continuous). Let

31    $E \overset{\text{def}}{=} \text{acc}(E_0)$. Consider now the set Γ_E. It is contradictory, so there is a finite subset of it which is contradictory. Hence for some $n_0, n_1, n_2 < \omega$ and formulae $\varrho_l(\bar z_l)(l < n_0)$

33   from the c.d.(M), formulae $\sigma_k(x, \bar e_k)(k < n_1) \in q(x)$, ordinals $\delta_0 < \cdots < \delta_{n_2-1} \in E$, a

1   sequence $\langle \bar{b}_{j,l} \colon j < n_2, l < l_j \rangle$ with $\bar{b}_{j,l} \subseteq M_{\delta_j}$ and terms $\langle \tau_{j,l} \colon j < n_2, l < l_j \rangle$ of T^*, the
following is inconsistent:

$$
\bigwedge_{l < n_0} \varrho_l(\bar{z}_l) \wedge \bigwedge_{k < n_1} \sigma_k(x, \bar{e}_k) \wedge \bigwedge_{j < n_2, l < l_j} \neg(a_{\delta_j} <_\varphi \tau_{j,l}(x, \bar{b}_{j,l})).
$$

3

As ϱ_l come from the c.d.(M) and $q(x)$ is a complete type over $M^{[\bar{\psi}]}$, we may assume
5   that $n_0 = 1$ and $n_1 = 1$. Note that we must have $n_2 \geq 1$ and that there is no loss of
generality in assuming that $\bar{b}_{j,l} = \bar{b}_j$ for all $l < l_j$ for $j < n$. We shall omit the subscript
7   0 from $\varrho, \sigma, \bar{e}$. Let $n = n_2$ and let us define $\vartheta_j(x, y_j, \bar{z}_j)$ for $j < n$ by

$$
\vartheta_j(x, y_j, \bar{z}_j) \equiv \bigvee_{l < l_j} y_j <_\varphi \tau_{j,l}(x, \bar{z}_j),
$$

9   and let $\vartheta = \bigvee_{j < n} \vartheta_j$. Note that for each j we have that $\vartheta_j \in \Theta^0_{T^*}$, as $<_\varphi$ is a tree
order. Hence ϑ is a candidate, $\sigma(x, \bar{e}) \in q(x)$, and since $M \models \varrho[\bar{d}]$ for some $\bar{d}$ we have

$$
M \models (\forall x) \left[\sigma(x, \bar{e}) \Rightarrow \bigvee_{j < n} \vartheta_j(x, a_{\delta_j}, \bar{b}_j) \right]. \tag{$*$}
$$

11

Now we consider the following play of $\Game_{n, \sigma, \vartheta}$. Let $\exists$ choose $\bar{b}_0$. Recall that $\bar{b}_0 \subseteq M_{\delta_0}$.
13   The strategy $G^0_{n, \sigma, \vartheta}$ of $\forall$ yields an ordinal α_0. By the choice of E_0 we have $\alpha_0 < \delta_0$
and $\bar{b}_0 \in M_{\delta_0}$, so we can let $\exists$ choose $\beta_0 = \delta_0$. Let $\exists$ choose $\bar{b}_1$ and then let $\forall$ choose
15   α_1 according to the strategy, etc. At the end of the play, player $\forall$ should have won
(as he/she used the supposed winning strategy), but clearly $(*)$ implies that $\exists$ won, a
17   contradiction.

Stage E. We conclude that (for our $\lambda, \bar{M}, p, q$), for some σ, ϑ and $n \geq 1$ the player
19   $\exists$ has a winning strategy in the game $\Game_{n, \sigma, \vartheta}$, call it St. Let us fix $n = n^*, \sigma, \vartheta$, and St
and use them to get SOP$''_2$.
21   For any $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in {}^n\lambda$, we can let $\langle \bar{b}^{\bar{\alpha}\restriction k}, \beta^{\bar{\alpha}\restriction (k+1)} \colon k < n \rangle$ be the sequence
of moves that $\exists$ plays by following the winning strategy St in a play in which
23   $\forall$ plays $\bar{\alpha}$, as the dependence is as marked. Let E be a club of λ such that if $k \leq n$
and $\alpha_0 < \cdots < \alpha_{k-1} < \delta \in E$, then $\bar{b}^{\langle \alpha_0, \ldots, \alpha_{k-1} \rangle} \in {}^{lg(\bar{z}_j)}M_\delta$. (Such a club can be found by a
25   method similar to the one used in Stage D). Renaming the M_i and a_i's, we can without
loss of generality assume that $E = \lambda$. For $\bar{\alpha} \in {}^n\lambda$ let $\bar{e}^{\bar{\alpha}}$ be such that:

$$
M \models \forall x \left[\sigma(x, \bar{e}^{\bar{\alpha}}) \Rightarrow \bigvee_{j < n} \vartheta_j(x, a_{\beta^{\bar{\alpha}\restriction(j+1)}}, \bar{b}^{\bar{\alpha}\restriction(j+1)}_j) \right].
$$

27

Notice that σ is a formula in the language of T. We shall show that σ, together with
29   a conveniently chosen sequence of $\bar{e}_{\bar{\eta}}$'s, exemplifies SOP$''_2$. The proof now proceeds
similarly to the proof of Main Claim 1.13. Namely

31   **Lemma 3.8.** *There are sequences*

$$
\langle N_\eta \colon \eta \in {}^{\lambda >}\lambda \rangle, \langle h_\eta \colon \eta \in {}^{\lambda >}\lambda \rangle
$$

1   *such that*

(i) h_η is an elementary embedding of $M_{lg(\eta)}$ into $\mathfrak{C}_{T^*}$ with range N_η,

3   *(ii) $v \trianglelefteq \eta \Rightarrow h_v \subseteq h_\eta$,*
(iii) for $\alpha \neq \beta < \lambda$ and $\eta \in {}^{\lambda >}\lambda$ we have

5
$$h_{\eta \frown \langle \alpha \rangle}(a_{lg(\eta)}) \perp_\varphi h_{\eta \frown \langle \beta \rangle}(a_{lg(\eta)}),$$

(iv) $N_{\eta_0} \cap N_{\eta_1} = N_{\eta_0 \cap \eta_1}$ for all η_0, η_1.

7   **Proof.** This Lemma has the same proof as that of Main Claim 1.13 Stage B. In the
notation of that proof, ignore b_{δ_i}. When defining Γ use

9
$$\Gamma = \bigcup_{\alpha < \lambda} \Gamma_0^\alpha \cup \bigcup_{\alpha < \lambda} \Gamma_3^\alpha \cup \Gamma_4 \cup \Gamma_2^+,$$

where $\Gamma_2^+ = \{x_0^\alpha \perp_\varphi x_0^\beta : \alpha \neq \beta < \lambda\}$ and $\Gamma_0^\alpha, \Gamma_3^\alpha$ and Γ_4 are defined as in the proof of
11   Main Claim 1.13, allowing for the replacement of ${}^{\lambda >}2$ by ${}^{\lambda >}\lambda$ by using $\{\bar{x}^\alpha : \alpha < \lambda\}$ in
place of $\{\bar{x}^0, \bar{x}^1\}$. Assumptions on $\Gamma_0^\alpha, \Gamma_2^+$ and Γ_3^α are analogous to the ones we made in
13   that proof. Fact 1.16 still holds, except that we drop the last set from the definition of
$r(\bar{x})$. The rest of the proof is the same, recalling that the branch induced by $\{a_i : i < \lambda\}$
15   is undefinable in M.   □
Stage F. For $\eta \in {}^\lambda\lambda$, let $h_\eta \overset{\text{def}}{=} \bigcup_{\alpha < \lambda} h_{\eta \restriction \alpha}$. Let $q_\eta \overset{\text{def}}{=} h_\eta(q)$, hence each q_η is a consis-
17   tent type. For $\bar{\eta} = \langle \eta_0, \ldots, \eta_{n-1} \rangle$ and $\eta_0 \triangleleft \cdots \triangleleft \eta_{n-1}$ with $lg(\eta_i) = \alpha_i + 1$, let $\bar{e}_{\bar{\eta}} \overset{\text{def}}{=} h_{\eta_{n-1}}$
$(\bar{e}^{\langle \alpha_0, \ldots, \alpha_{n-1} \rangle})$.
19   Suppose now that $\eta \in {}^\lambda\lambda$ is given, and consider the set

$$\{\sigma(x, \bar{e}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \restriction (\alpha_0 + 1), \ldots, \eta \restriction (\alpha_{n-1} + 1) \rangle \text{ for some } \alpha_0 < \ldots \alpha_{n-1} < \lambda\}.$$

21   This set is a subset of q_η, and is hence consistent. This proves property (α) from the
definition of SOP_2''. For (β), let m be large enough and $g : {}^{n \geqslant}m \to {}^{\lambda >}\lambda$ be as in the
23   statement of (β). For $\rho \in {}^n m$ let $\bar{e}_{g_\rho} \overset{\text{def}}{=} \bar{e}_{\langle g(\rho \restriction 1), \ldots, g(\rho) \rangle}$ (note that this is always defined).
We shall now show that the set

25
$$\{\sigma(x, \bar{e}_{g_\rho}) : \rho \in {}^n m\}$$

is inconsistent. Suppose otherwise, so let $d \in \mathfrak{C}_{T^*}$ realise it. For each $p \in {}^n m$, let
27   $\eta_\rho \in {}^\lambda\lambda \supseteq g(\rho)$ and let $\bar{\alpha}^\rho \overset{\text{def}}{=} \langle \alpha_0^\rho, \ldots, \alpha_{n-1}^\rho \rangle$ satisfy $lg(g(\rho \restriction k)) = \alpha_k^\rho + 1$ for $k \leqslant n$,
so for each $k < n$ we have $g(\rho \restriction (k + 1)) = \eta_\rho \restriction (\alpha_k^\rho + 1)$. Now we have that for each
29   $\rho \in {}^n m$

(i) $\sigma(x, \bar{e}_{g_\rho}) \equiv \sigma(x, h_{\eta_\rho \restriction (\alpha_{n-1}^\rho + 1)}(\bar{e}^{\bar{\alpha}^\rho})) \in q_{\eta_\rho} \restriction \sigma_{\eta_\rho}(x)$

31   (ii) $N_{\eta_\rho} \models (\forall x)[\sigma(x, \bar{e}_{g_\rho}) \Rightarrow \vartheta(x, \langle h_{\eta_\rho}(a_{\beta^{\bar{\alpha}\rho} \restriction 1}), \ldots, h_{\eta_\rho}(a_{\beta^{\bar{\alpha}\rho}}) \rangle, \underset{j<n}{\frown} h_{\eta_\rho}(\bar{b}^{\bar{\alpha}_j^\rho}))]$ (hence the
same holds in $\mathfrak{C}_{T^*}$),

(iii)

$$\vartheta(x, \langle h_{\eta_\rho}(a_{\beta^{\bar{\alpha}^\rho\upharpoonright 1}}), \ldots, h_{\eta_\rho}(a_{\beta^{\bar{\alpha}^\rho}}) \rangle, \frown_{j<n} h_{\eta_\rho}(\bar{b}^{\bar{\alpha}^\rho_j}))$$

$$\Rightarrow \bigvee_{j<n} \vartheta_j(x, h_{\eta_\rho}(a_{\beta^{\bar{\alpha}^\rho\upharpoonright(j+1)}}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^{\rho\upharpoonright(j+1)}}))$$

for our $\vartheta_0, \ldots, \vartheta_{n-1}$.

For each $\rho \in {}^n m$ let $j(\rho) < n$ be the first such that

$$\vartheta_j(d, h_{\eta_\rho}(a_{\beta^{\bar{\alpha}^\rho\upharpoonright(j+1)}}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^{\rho\upharpoonright(j+1)}}))$$

holds. Let $l^* = \max\{l_0^\vartheta, \ldots, l_{n-1}^\vartheta\}$.

As m is large enough, there are $\rho_0, \ldots, \rho_{l^*} \in {}^n m$ such that $j(\rho_s) = j^*$ for all $s \in \{0, \ldots, l^*\}$, while $\rho_s \upharpoonright j^*$ is fixed and $\rho_s(j^*) \neq \rho_t(j^*)$ for $s \neq t \leqslant l^*$. (We use that there is a full $l^*+1 \geqslant n$ subtree t^* of ${}^{n\geqslant}m$ such that for all $\rho \in t^* \cap {}^n m$ we have $j(\rho) = j^*$. Choose ρ_s belonging to t^* and splitting at the level j^*). In particular, $\alpha_0^{\rho_s} = \alpha_0, \ldots, \alpha_{j^*-1}^{\rho_s} = \alpha_{j^*-1}$ is fixed, and so is $h_{\eta_{\rho_s}} \upharpoonright M_{\alpha_{j_{-1}^*}+1}$, but

$$g(\rho_s) \upharpoonright (\alpha_{j^*-1} + 2) \quad \text{for } s \leqslant l^* \text{ are incomparable in } {}^\lambda\lambda. \tag{$**$}$$

Let $\bar{\alpha} \overset{\text{def}}{=} \bar{\alpha}^{\rho_0}$.

For each $\rho \in {}^n m$ and $k < n$ we have that $\bar{b}^{\bar{\alpha}^{\rho\upharpoonright(k+1)}} \in M_{\alpha_{k+1}^\rho}$ (by the choice of E), so in particular $\bar{b}^{\bar{\alpha}^{\rho\upharpoonright j^*}} \in M_{\alpha_{j^*-1}^\rho+1}$, and hence $h_{\eta_{\rho_s}}(\bar{b}^{\bar{\alpha}^{\rho\upharpoonright j^*}})$ is a fixed $\bar{b}^*$. By the choice of d and definitions of j^*, l^* and Θ_{T^*}, there are $s \neq t < l_{\vartheta_{j^*}} \leqslant l^*$ such that $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s}\upharpoonright(j^*+1)}})$ and $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t}\upharpoonright(j^*+1)}})$ are on the same almost branch. Now note that for all ρ we have

$$a_{\beta^{\bar{\alpha}^{\rho\upharpoonright(j^*+1)}}} \in M_{\beta^{\bar{\alpha}^{\rho\upharpoonright(j^*+1)}}+1} \setminus M_{\beta^{\bar{\alpha}^{\rho\upharpoonright(j^*+1)}}}$$

and $\beta^{\bar{\alpha}^{\rho\upharpoonright(j^*+1)}} > \alpha_{j^*}^\rho$. Hence $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s}\upharpoonright(j^*+1)}})$ and $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t}\upharpoonright(j^*+1)}})$ are incomparable, by property (iii) in Lemma 3.8, a contradiction. This shows (β) from the definition of SOP_2'', so finishing the proof. $\square$

Putting this together with Corollary 3.3 and Shelah–Usvyatsov theorem 3.5 above we get the following Corollary 3.9.

Corollary 3.9. (1) *Suppose that T is a theory that is $\lhd^*$-maximal in some universe of set theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ. Then T has SOP_2.*

(2) *Suppose that T is a theory that is $\lhd^*_{\lambda^+}$-maximal in some universe of set theory in which λ is regular and $2^\lambda = \lambda^+$. Then T has SOP_2.*

Proof. (1) Let W be a universe of set theory in which $2^\lambda = \lambda^+$ holds for all large enough regular λ and in which T is $\lhd^*$-maximal. Hence by Corollary 3.3 T is $\lhd^{**}$-maximal in W and hence by Main Theorem 3.6 in W it satisfies SOP_2''. By Shelah–

1 Usvyatsov Theorem 3.5 above T satisfies SOP_2 in W. An application of the Compactness Theorem shows that satisfying SOP_2 is absolute, hence T satisfies SOP_2 in V.

3 (2) This follows similarly, but more directly, from Main Theorem 3.6 and the Shelah–Usvyatsov Theorem 3.5. □

5 This section hence provides us with the proof of one side of our thesis that SOP_2 and $\lhd^*$-maximality are closely connected. Recall that Shelah proved in [13] that SOP_3 7 implies $\lhd^*$-maximality. So an important open question (provided that SOP_3 are not actually equivalent, which we still do not know) is

9 **Question 3.10.** Does SOP_2 imply $\lhd^*$-maximality?

In a partial answer to this question posed in an earlier version of the paper Shelan and 11 Usvyatsov in Theorem 3.12 of [14] provided a local positive answer to this question, where by "local" we mean that they proved that any theory with SOP_2 is $\lhd^*$ above T_{tree} 13 when only types localised by a certain formula are considered (see Definition 1.3).

Acknowledgements

15 ### References

[1] C.C. Chang, H.J. Keisler, Model theory, Studies in Logic and Foundations of Mathematics, 3rd Edition, 17   vol. 73, North-Holland, Amsterdam, 1990.

[2] M. Džamonja, S. Shelah, Universal graphs at the successor of a singular cardinal, J. Symbolic Logic 19   68 (2) (2003) 366–387.

[3] M. Džamonja, S. Shelah, On the existence of universals and an application to Banach spaces and 21   triangle free graphs, submitted for publication.

[4] M. Džamonja, S. Shelah, On properties of theories which preclude the existence of universal models, 23   submitted for publication.

[5] R. Grossberg, S. Shelah, On universal locally finite groups, Israel J. Math. 44 (1983) 289–302.

25 [6] J.D. Halpern, H. Lauchli, A partition theorem, Trans. Amer. Math. Soc. 124 (2) (1966) 360–367.

[7] H.J. Keisler, Six classes of theories, J. Austral. Math. Soc. 21 (1976) 257–275.

27 [8] B. Kim, A. Pillay, Simple theories, Ann. Pure Appl. Logic 88 (2–3) (1997) 149–164.

[9] M. Kojman, S. Shelah, Nonexistence of universal orders in many cardinals, J. Symbolic Logic 57 (1992) 29   875–891.

1   [10] M. Kojman, S. Shelah, The universality spectrum of stable unsuperstable theories, Ann. Pure Appl.
         Logic, 1992.
3   [11] S. Shelah, Classification theory and the number of nonisomorphic models, in: Studies in Logic and the
         Foundation of Mathematics, vol. 92, North-Holland, Amsterdam, 1990.
5   [12] S. Shelah, The Universality Spectrum: Consistency for more classes in Combinatorics, Paul Erdös
         is Eighty, vol. 1, Bolyai Society Mathematical Studies, 1993, pp. 403–420; Proceedings of the
7   Meeting in honor of P. Erdös, Keszthely, Hungary 7, 1993; an improved version available at
         http://www.math.rutgers.edu/~she-laharch.
9   [13] S. Shelah, Towards classifying unstable theories, Ann. Pure Appl. Logic 80 (1996) 229–255.
    [14] S. Shelah, A. Usvyatsov, More on SOP$_1$ and SOP$_2$, preprint.