Some Remarks on a Question of D. H. Fremlin Regarding ϵ-Density

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Abstract

We show the relative consistency of $\aleph_1$ satisfying a combinatorial property considered by David Fremlin (in the question DU from his list) in certain choiceless inner models. This is demonstrated by first proving the property is true for Ramsey cardinals. In contrast, we show that in ZFC, no cardinal of uncountable cofinality can satisfy a similar, stronger property. The questions considered by D. H. Fremlin are if families of finite subsets of ω_1 satisfying a certain density condition necessarily contain all finite subsets of an infinite subset of ω_1, and specifically if this and a stronger property hold under $MA + \neg CH$. Towards this we show that if $MA + \neg CH$ holds, then for every family $\mathcal{A}$ of $\aleph_1$ many infinite subsets of ω_1, one can find a family $\mathcal{S}$ of finite subsets of ω_1 which is dense in Fremlin's sense, and does not contain all finite subsets of any set in $\mathcal{A}$.

We then pose some open problems related to the question.

1 Introduction

Definition 1.1 *Let $0 < \epsilon \leq 1$ be fixed but arbitrary, and let κ be an infinite cardinal. Call a family $\mathcal{S} \subseteq [\kappa]^{<\aleph_0}$* ϵ-dense open *iff $\mathcal{S}$ is downward closed, i.e., $p \in \mathcal{S}$ and $q \subseteq p$ implies $q \in \mathcal{S}$, and $\mathcal{S}$ in addition satisfies the property*

$$\forall p \in [\kappa]^{<\aleph_0} \exists q \subseteq p \, [q \in \mathcal{S} \wedge |q| \geq \epsilon \cdot |p|].$$

D. H. Fremlin in question DU from [5] asks if $\kappa = \aleph_1$ and we have ϵ and $\mathcal{S}$ as in the above, then does there necessarily exist $A \in [\aleph_1]^{\aleph_0}$ so that $[A]^{<\aleph_0} \subseteq \mathcal{S}$? In fact this particular question is credited to S. Argyros, while the original question of D. H. Fremlin is if under $MA + \neg CH$ one can find an uncountable such A. For the motivation behind the question, we refer the reader to Fremlin's note [6].

The reason one would ask the question in the above form is that an apparently folklore argument (communicated to us by Ilijas Farah) shows that CH implies that there is $\mathcal{S}$ on $\aleph_1$ which is 1/2-dense open, but contains no $[A]^{<\aleph_0}$ for an uncountable A, while an easy modification of an argument by P. Erdös and R. Rado in [3] answers negatively the analogue of Fremlin's question with $\aleph_0$ in place of $\aleph_1$ (see the appendix for these facts).

Ilijas Farah brought to our attention the fact that ϵ-density is a variation on Kelley's intersection number [9] of a family of sets, but that if in question DU, instead of requiring that $\mathcal{S}$ be ϵ-dense open, we ask that $\mathcal{S}$ be a family of finite sets obtained from some family with the intersection number positive, then the problem has a positive solution - there is an infinite set with all finite subsets in $\mathcal{S}$, and under $MA(\aleph_1)$ there is an uncountable such set. Note [6] by D. H. Fremlin deals with certain minimal ϵ-dense families and furthers the connection with intersection numbers.

In Section 2, we will define the notions of a cardinal κ being Fremlin and being strongly Fremlin. We show that modulo the existence of a Ramsey cardinal, it is consistent that $\aleph_1$ is strongly Fremlin in a model of ZF which satisfies DC. We then show in Section 3 that if the original condition in question DU is fortified, no cardinal of uncountable cofinality can be Fremlin in this stronger sense, in a model of ZFC. In Section 4 we show that under $MA + \neg CH$, for every family $\mathcal{A}$ of

$\aleph_1$ elements of $[\omega_1]^{\aleph_0}$, there is a 1/2-dense open family $\mathcal{S}$ of finite subsets of ω_1 such that for no $A \in \mathcal{A}$ do we have that $[A]^{<\aleph_0} \subseteq \mathcal{S}$. The paper ends by giving further open problems.

2 Choiceless universes

We begin by noting that if κ is a Ramsey cardinal, then the above property holds for κ. Specifically, we show the following.

Fact 2.1 *Suppose κ is a Ramsey cardinal and $\mathcal{S}$ is ϵ-dense open for an $\epsilon \in (0, 1]$. There is then $A \in [\kappa]^\kappa$ such that $[A]^{<\aleph_0} \subseteq \mathcal{S}$.*

Proof: Define $f : [\kappa]^{<\aleph_0} \to 2$ by $f(p) = 1$ iff $p \in \mathcal{S}$. Since κ is a Ramsey cardinal, let $A \subseteq \kappa$, $A \in [\kappa]^\kappa$ be such that A is homogeneous for f. We show that A is our desired set by considering the following two cases.

1. There are unboundedly many $n < \omega$ so that $f''[A]^n = \{1\}$, i.e., so that $[A]^n \subseteq \mathcal{S}$. Since $\mathcal{S}$ is downward closed, we automatically have that $[A]^{<\aleph_0} \subseteq \mathcal{S}$.

2. Case 1 doesn't hold, i.e., there is $n_0 < \omega$ so that for all $n \geq n_0$, we have $f''[A]^n = \{0\}$. This means that if $n \geq n_0$ and $p \in [A]^n$, then $p \notin \mathcal{S}$. In this situation, let $m > n_0$ be large enough so that $m \cdot \epsilon > n_0$, and let $p \in [A]^m$. By the properties of $\mathcal{S}$, there must be $q \subseteq p$, $q \in \mathcal{S}$ so that $|q| \geq \epsilon \cdot |p|$, i.e., $|q| \geq m \cdot \epsilon > n_0$. As $f(q) = 1$, this contradicts that $q \in [A]^\ell$ for some $\ell > n_0$, i.e., $f(q) = 0$.

The contradiction just obtained means that Case 2 above can't hold, i.e., that the homogeneous set A for f has the desired properties. This completes the proof of Fact 2.1.

$\square$

More generally,

Definition 2.2 *An infinite cardinal κ is called* Fremlin *if for every ϵ-dense open family $\mathcal{S}$ of finite subsets of κ, there is an infinite $A \subseteq \kappa$ all of whose finite subsets are contained in $\mathcal{S}$. If we can always guarantee that such an A can be found with $|A| = \kappa$, we say that κ is* strongly Fremlin.

The proof of Fact 2.1 shows that a Ramsey cardinal is strongly Fremlin and that an Erdös cardinal κ, i.e., a cardinal κ such that $\kappa \to (\omega)^{<\omega}$, is Fremlin. D. H. Fremlin in [6] shows that a real-valued measurable cardinal is Fremlin.

We remark that the above proof makes no use of the Axiom of Choice. Thus, a model in which $\aleph_1$ is a Ramsey cardinal will be a model in which $\aleph_1$ is strongly Fremlin. This gives us the following theorem.

Theorem 1 *Con(ZFC + κ is a Ramsey cardinal) $\implies$ Con(ZF + DC + $\aleph_1$ is strongly Fremlin).*

The model used for Theorem 1 is either Jech's model of [7] (see also [8], Theorem 90, pages 476-477) or Takeuti's model of [11]. In both models, a Ramsey cardinal κ (instead of a measurable cardinal) is symmetrically collapsed to $\aleph_1$, and the Ramseyness of κ is preserved.

We note now that there are instances of models in which the full Axiom of Choice is false and in which many successor cardinals are strongly Fremlin. The following two theorems provide examples of such models.

Theorem 2 *Let $V \models$ "ZFC + There is a proper class of supercompact cardinals". There is then a class partial ordering $\mathbb{P}$ and a symmetric inner model $N \subseteq V^{\mathbb{P}}$ such that $N \models$ "ZF + DC + Every successor cardinal is regular + Every uncountable limit cardinal is singular + The successor of every regular cardinal is strongly Fremlin".*

Theorem 3 *Let $V \models$ "ZFC + κ is almost huge + $B, C \subseteq \kappa$ are so that $B \cap C = \emptyset$ and $B \cup C = \{\alpha < \kappa : \alpha$ is a successor ordinal$\}$". There are then a partial ordering $\mathbb{P} \in V$ and a symmetric inner model $N_B \subseteq V^{\mathbb{P}}$ of height κ such that $N_B \models$ "ZF + $\neg AC_\omega$ + Every limit cardinal is singular + If $\alpha \in B$, then $\aleph_\alpha$ is strongly Fremlin + If $\beta \in C$, then $\aleph_\beta$ is a singular Rowbottom cardinal".*

The models used for Theorems 2 and 3 are the models of [1] and [2] respectively, in which the relevant cardinals in N and N_B are Ramsey cardinals. Observe that an immediate corollary of Theorem 3 is that if $C = \emptyset$, then $N_B \models$ "Every successor cardinal is strongly Fremlin".

3 Variants on containment

We have already remarked that the original question appears with a "weak" and "strong" version of the containment (containing all finite subsets of an infinite set, and containing all finite subsets of a set of size κ). If we further strengthen the containment requirement, to request all finite subsets of a *club* subset of κ to be contained in our ϵ-dense open family, then for any κ of uncountable cofinality a counterexample can be constructed just in ZFC.

Lemma 3.1 *Suppose that $\kappa \geq \aleph_1$ is regular. Then there is a 1/2-dense open family $\mathcal{S} = \mathcal{S}(\kappa)$ of finite subsets of κ such that for no club subset C of κ do we have*

$$[C]^{<\aleph_0} \subseteq \mathcal{S}.$$

Proof: Fix κ as in Lemma 3.1. By a theorem of Solovay (see Lemma 7.6 and Theorem 85 of [8]), let $\langle S_\alpha : \alpha < \kappa \rangle$ be such that $\kappa = \cup_{\alpha<\kappa} S_\alpha$, each S_α is stationary, and for $\alpha < \beta < \kappa$, $S_\alpha \cap S_\beta = \emptyset$. Let $m_0 \geq 1$ be fixed but arbitrary. Using $\langle S_\alpha : \alpha < \kappa \rangle$, enumerate all elements of $[\kappa]^{m_0}$ as $\langle F'_\delta : \delta < \kappa \rangle$ so that each $F \in [\kappa]^{m_0}$ appears stationarily often. For $\delta < \kappa$ such that $F'_\delta \subseteq \delta$, define $F_\delta = F'_\delta \cup \{\delta\}$ (otherwise, F_δ is left undefined). Now let $\mathcal{S}$ be the set of all finite subsets of κ which do not contain any F_δ as a subset.

Clearly, $\mathcal{S}$ is downward closed. Let us show that it is 1/2-dense. Given F a finite subset of κ, if F contains no F_δ, we are done, so assume that this is not the case. Let $\{\delta_0, \delta_1, \ldots, \delta_{n-1}\}$ list increasingly all δ such that $F_\delta \subseteq F$. We divide F into 2 disjoint sets A and B as follows. By induction on $k \leq n-1$, we decide for the elements of $F \cap [0, \delta_k)$ if they are in A or in B.

First, let $F \cap [0, \delta_0) \subseteq B$. Suppose that we have decided for the elements of $F \cap [0, \delta_k)$ if they are in A or in B. If F'_{δ_k} is not a subset of A, let $\delta_k \in A$. Otherwise, let $\delta_k \in B$. Let $F \cap (\delta_k, \delta_{k+1}) \subseteq B$. Also, let $F \cap (\delta_{n-1}, \kappa) \subseteq B$.

We now claim that neither A nor B contains any F_δ. Supposing otherwise, any such δ would have to be among $\{\delta_0, \delta_1, \ldots, \delta_{n-1}\}$, so let $k \leq n-1$ be the first such that $F_{\delta_k} \subseteq A$ or $F_{\delta_k} \subseteq B$. If $F_{\delta_k} \subseteq A$, then $F'_{\delta_k} \subseteq A$, so $\delta_k \in F_{\delta_k} \setminus A$, a contradiction. We similarly show that the remaining case cannot happen. So, both $A, B \in \mathcal{S}$, and as at least one of them has size $\geq 1/2 \cdot |F|$, this shows that $\mathcal{S}$ is 1/2-dense.

Given a club C of κ, let F^* be any element of $[C]^{m_0}$. Let S be the stationary set such that $\delta \in S \implies F'_\delta = F^*$. Now let $\delta \in (C \cap S) \setminus (\mathrm{Max}(F^*) + 1)$. Clearly $F'_\delta = F^* \subseteq \delta$, so in particular F_δ is defined. As $\delta \in C$ and $F'_\delta \subseteq C$, we have that $F_\delta \subseteq C$. Hence, $F_\delta \in [C]^{<\aleph_0} \setminus \mathcal{S}$.

$\square$

Lemma 3.2 *Suppose that $\epsilon \in (0,1]$ and*

$$\aleph_1 \leq \kappa = \mathrm{cof}(\mu).$$

Then, there is an ϵ-dense open family $\mathcal{S}$ of elements of $[\mu]^{<\aleph_0}$ for which there is no club C of μ with $[C]^{<\aleph_0} \subseteq \mathcal{S}$ iff there is an ϵ-dense open family $\mathcal{T}$ of elements of $[\kappa]^{<\aleph_0}$ for which there is no club E of κ with $[E]^{<\aleph_0} \subseteq \mathcal{T}$.

Proof: Let us first fix a continuous increasing function f from κ to μ whose image C^* is a club in μ. Then, given $\mathcal{S}$ as in the Lemma, let $\mathcal{T} = \{f^{-1}(F) : F \in \mathcal{S}\}$. It is easily checked that $\mathcal{T}$ is ϵ-dense open in $[\kappa]^{<\aleph_0}$. Suppose that E is a club of κ with $[E]^{<\aleph_0} \subseteq \mathcal{T}$. Then it can be seen that $C = f''E$ is a club of μ with $[C]^{<\aleph_0} \subseteq \mathcal{S}$.

Vice versa, given $\mathcal{T}$ as in the Lemma, let $\mathcal{S}$ consist of all sets of the form $(f''F) \cup G$ where $F \in \mathcal{T}$ and G is a finite subset of $\mu \setminus C^*$. It is easy to check that $\mathcal{S}$ is ϵ-dense open in $[\mu]^{<\aleph_0}$. Suppose that C is a club of μ with $[C]^{<\aleph_0} \subseteq \mathcal{S}$. Without loss of generality, $C \subseteq C^*$. Hence, $[C]^{<\aleph_0} \subseteq \{f''F : F \in \mathcal{T}\}$, and so letting $E = f^{-1}(C)$, we obtain a club of κ with $[E]^{<\aleph_0} \subseteq \mathcal{T}$.

$\square$

Theorem 4 *Suppose that λ is a cardinal of uncountable cofinality. Then there is a 1/2-dense open family $\mathcal{S} = \mathcal{S}(\lambda)$ of finite subsets of λ such that for no club subset C of λ do we have*

$$[C]^{<\aleph_0} \subseteq \mathcal{S}.$$

Proof: If λ is regular, the conclusion follows from Lemma 3.1. Otherwise, let κ be the cofinality of λ. As in the proof of Lemma 3.2, $\mathcal{S}(\kappa)$ gives rise to $\mathcal{S}(\lambda)$ as required.

$\square$

4 An MA result

Theorem 5 *Assume MA (or even just MA for Knaster property forcing and $\aleph_1$ dense sets), and $\neg CH$. Then, for any family $\mathcal{A}$ of $\aleph_1$ many elements of $[\omega_1]^{\aleph_0}$, there is a 1/2-dense open family $\mathcal{S}$ such that for no $A \in \mathcal{A}$ do we have $[A]^{<\aleph_0} \subseteq \mathcal{S}$.*

Proof: Let us assume the axioms from the statement of the Theorem, and suppose that $\mathcal{A}$ as there is given. Let $\langle A_\zeta : \zeta < \omega_1 \rangle$ be a 1-1 enumeration of $\mathcal{A}$. Let $m_0 \geq 2$ be a fixed but arbitrary integer, and let $[\omega_1]^{m_0} = \langle F_\alpha : \alpha < \omega_1 \rangle$ be a 1-1 enumeration. We define

$$\mathbb{P} = \left\{ \begin{array}{l} p : \text{(i)}\, p \text{ is a finite function from } \omega_1 \text{ into } \omega_1 \\ \quad \text{(ii)}\, p(\zeta) = \alpha \implies F_\alpha \subseteq A_\zeta \\ \quad \text{(iii)}\, \zeta \in \mathrm{Dom}(p) \implies [p(\zeta) \in \mathrm{Ran}(p \restriction \zeta) \vee \mathrm{Max}(F_{p(\zeta)}) \notin \cup_{\xi < \zeta, \xi \in \mathrm{Dom}(p)} F_{p(\xi)}] \\ \quad \text{(iv)}\, \mathcal{S}_p \stackrel{\mathrm{def}}{=} \{F \in [\omega_1]^{<\aleph_0} : \{\alpha : F_\alpha \subseteq F \ \&\ \alpha \in \mathrm{Ran}(p)\} = \emptyset\} \text{ is 1/2-dense} \end{array} \right\}$$

and let $\mathbb{P}$ be ordered by inclusion, i.e. q is stronger than p, written as $p \leq q$, iff $p \subseteq q$.

Claim 4.1 *For every $\zeta < \omega_1$, the set*

$$\mathcal{D}_\zeta \stackrel{\mathrm{def}}{=} \{p \in \mathbb{P} : \zeta \in \mathrm{Dom}(p)\}$$

is dense in $\mathbb{P}$.

Proof: Suppose we are given $\zeta < \omega_1$ and $p \notin \mathcal{D}_\zeta$. Let $F^* \in [A_\zeta]^{m_0}$ be disjoint from $\cup_{\alpha \in \mathrm{Ran}(p)} F_\alpha$, which exists as A_ζ is infinite. Let β be such that $F^* = F_\beta$ and let $q = p \cup \{(\zeta, \beta)\}$. Clearly $p \subseteq q$ and properties (i)–(iii) from the definition of $\mathbb{P}$ are satisfied. Let us verify (iv).

Let $F \in [\omega_1]^{<\aleph_0}$ be given. We can assume that $F_\beta \subseteq F$, as otherwise the situation is trivial by our assumptions. Let $i \neq j \in F_\beta$. Let $F' \subseteq F \setminus \{i, j\}$ be in $\mathcal{S}_p$ and such that $F' \geq [|F| - 2]/2$, which exists as $p \in \mathbb{P}$. Let $F'' \stackrel{\mathrm{def}}{=} F' \cup \{i\}$. Note that $F_\beta \nsubseteq F''$ as $j \notin F''$, and that adding $\{i\}$ to F' does not change the membership in $\mathcal{S}_p$, as i is not an element of any F_α for $\alpha \in \mathrm{Ran}(p)$. Hence $F'' \subseteq F$ is an element of $\mathcal{S}_q$, and $|F''| \geq |F|/2$.

$\square$

Main Claim 4.2 $\mathbb{P}$ *has the Knaster property.*

Proof: Let $\{p_i : i < \omega_1\} \subseteq \mathbb{P}$ be given. For $i < \omega_1$, let $D_i \stackrel{\mathrm{def}}{=} \mathrm{Dom}(p_i)$.

(1) Without loss of generality, we can assume that the D_i's form a Δ-system with root D^*, and $|D_i \setminus D^*| = n^*$ is constant.

(2) As for every $\zeta \in D^*$ and $i < \omega_1$, we have $p_i(\zeta) \in \{\alpha : F_\alpha \subseteq A_\zeta\}$, which is a countable set, we may assume that $p_i \restriction D^* = p^*$ is a constant.

Now the situation of $n^* = 0$ becomes trivial, so we can assume that $n^* > 0$.

(3) Using the fact that each $D_i \subseteq \omega_1$, we can also assume that the Δ-system formed by the D_i's is head-tail-tail, i.e., if $i < j < \omega_1$, then

$$\mathrm{Max}(D^*) < \min(D_i \setminus D^*) \leq \mathrm{Max}(D_i \setminus D^*) < \min(D_j \setminus D^*).$$

(N.B. This is where we use that fact that $|\mathcal{A}| = \aleph_1$, rather than just $|\mathcal{A}| < \mathfrak{m}$.)

For $i < \omega_1$, let $\{\zeta_0^i, \ldots, \zeta_{n^*-1}^i\}$ be the increasing enumeration of $D_i \setminus D^*$. Let $\alpha_k^i \stackrel{\mathrm{def}}{=} p_i(\zeta_k^i)$.

(4) By applying the Δ-system lemma n^* times if necessary, we may assume that for every $k < n^*$, the family $\{F_{\alpha_k^i} : i < \omega_1\}$ forms a Δ-system with root F_k^*, and this Δ-system is head-tail-tail.

(5) By thinning out further if necessary, we may assume

$$i < j < \omega_1 \ \&\ k < n^* \ \&\ \gamma \in F_{\alpha_k^j} \setminus F_k^* \implies \gamma > \mathrm{Max}(\cup\{F_\alpha : \alpha \in \mathrm{Ran}(p_i)\}).$$

(6) Thinning again if we have to, we may assume that for every $k < n^*$, the truth value of the statement "$\alpha^i_k \in \mathrm{Ran}(p_i \restriction \zeta^i_k)$" does not depend on i.

Let us now take $i < j < \omega_1$, and claim that $r = p_i \cup p_j \in \mathbb{P}$ is a common extension of p_i and p_j. Clearly, conditions (i) and (ii) of membership in $\mathbb{P}$ are satisfied. Let us now verify condition (iii).

Let $\zeta \in \mathrm{Dom}(r)$ and $\xi \in \mathrm{Dom}(r) \cap \zeta$ be given. If $\zeta \in \mathrm{Dom}(p_i)$, the conclusion follows from the assumptions on p_i and the fact that $\mathrm{Dom}(r \restriction \zeta) = \mathrm{Dom}(p_i \restriction \zeta)$. So suppose $\zeta \in D_j \setminus D^*$. Let $\zeta = \zeta^j_k$. If $p_j(\zeta) = r(\xi)$ for any $\xi < \zeta$, we are done. So, assume that this is not the case, in particular $|F^*_k| < m_0$. Let $\gamma \stackrel{\text{def}}{=} \mathrm{Max}(F_{p_j(\zeta)})$. By our assumption (5) above, $\gamma \notin F_\alpha$ for any $\alpha \in \mathrm{Ran}(p_i)$. Using the fact that p_j satisfies (iii), we see that $\gamma \notin F_\alpha$ for any $\alpha \in \mathrm{Ran}(r \restriction \zeta)$.

It remains to verify (iv). The proof of this is by induction on $n^* \geq 1$. We may assume that $p_i \restriction \zeta^i_{n^*-1} \cup p_j \restriction \zeta^i_{n^*-1}$ is a condition in $\mathbb{P}$ (so, if $n^* = 1$, this just says that $p^* \in \mathbb{P}$). Let $n \stackrel{\text{def}}{=} n^* - 1$. If $\alpha^i_n \in \mathrm{Ran}(p_i \restriction \zeta^i_n)$, by (6) we also have $\alpha^j_n \in \mathrm{Ran}(p_j \restriction \zeta^j_n)$, and so our conclusion easily follows. Let us assume that none of these events occurs. We consider two cases.

<u>Case 1</u>. $|F^*_n| = m_0$.

Let $\gamma \stackrel{\text{def}}{=} \mathrm{Max}(F^*_n)$. Note that $\gamma \notin \cup\{F_\alpha : \alpha \in \mathrm{Ran}(p_i \restriction \zeta^i_n)\}$, by the assumptions on p_i, but also $\gamma \notin \cup\{F_\alpha : \alpha \in \mathrm{Ran}(p_j \restriction \zeta^j_n)\}$. Let $\gamma' \in F^*_n \setminus \{\gamma\}$.

The argument is now the same as the one in the proof of density found in Claim 4.1. Namely, suppose we are given $F \in [\omega_1]^{<\aleph_0}$. If $F^*_n \nsubseteq F$, we can simply use the inductive assumption, so let us assume that this is not the case. By the inductive assumption, we can find $F' \subseteq F \setminus \{\gamma, \gamma'\}$ with $F' \in \mathcal{S}_{p_i \restriction \zeta^i_n \cup p_j \restriction \zeta^j_n}$ and $|F'| \geq [|F| - 2]/2$. Taking $F'' = F' \cup \{\gamma\}$, we obtain a set as desired.

<u>Case 2</u>. $|F^*_n| < m_0$.

Let $\beta \stackrel{\text{def}}{=} \mathrm{Max}(F_{\alpha^i_n})$ and $\gamma \stackrel{\text{def}}{=} \mathrm{Max}(F_{\alpha^j_n})$. By our assumption (5) and the fact that $p_j \in \mathbb{P}$, we have that $\gamma \notin \cup\{F_\alpha : \alpha \in \mathrm{Ran}(p_i) \cup \mathrm{Ran}(p_j \restriction \zeta^j_n)\}$. Using our assumption (4) we have $\beta \notin F^*_n$, and so using (5) and (iii) in the definition of $\mathbb{P}$, we have

$$\beta \notin \cup\{F_\alpha : \alpha \in \mathrm{Ran}(p_j) \cup \mathrm{Ran}(p_i \restriction \zeta^i_n)\}.$$

Now we continue with an argument similar to that of Case 1. Namely, let $F \in [\omega_1]^{<\aleph_0}$ be given. The most interesting case is when $F_{\alpha^i_n} \cup F_{\alpha^j_n} \subseteq F$, so let us assume that this happens. Let $\beta' \in F_{\alpha^i_n} \setminus \{\beta\}$

and $\gamma' \in F_{\alpha^j_n} \setminus \{\beta', \gamma\}$. By the inductive assumption, we can find $F' \subseteq F \setminus \{\beta, \beta', \gamma, \gamma'\}$ with $F' \in \mathcal{S}_{p_i \restriction \zeta^i_n \cup p_j \restriction \zeta^j_n}$ and $|F'| \geq [|F| - 4]/2$. Taking $F'' = F' \cup \{\beta, \gamma\}$, we obtain a set as desired.

$\square$

Let now G be a $\mathbb{P}$-filter with $G \cap \mathcal{D}_\zeta \neq \emptyset$ for all $\zeta < \omega_1$. Hence $\cup G$ is a function from ω_1 to ω_1. Let

$$\mathcal{S} \stackrel{\text{def}}{=} \{F \in [\omega_1]^{<\aleph_0} : \{\alpha : F_\alpha \subseteq F \ \&\ \alpha \in \mathrm{Ran}(\cup G)\} = \emptyset\}.$$

Clearly, $\mathcal{S}$ is open. We show that it is 1/2 dense. Suppose we are given $F \in [\omega_1]^{<\aleph_0}$. Let $\{\alpha_0, \ldots, \alpha_{n-1}\}$ list all elements α of $\mathrm{Ran}(\cup G)$ with the property $F_\alpha \subseteq F$. Let $p \in G$ be such that $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq \mathrm{Ran}(p)$. Now note that a subset of F is in $\mathcal{S}$ iff it is in $\mathcal{S}_p$, and use the fact that $p \in \mathbb{P}$.

Obviously, for no $\zeta < \omega_1$ do we have $[A_\zeta]^{<\aleph_0} \subseteq \mathcal{S}$.

$\square$

We note that in the prooof of Theorem 5, it is critical that m_0 be at least 2. If $m_0 = 1$, then assuming $\{\alpha\}, \{\beta\} \notin \mathcal{S}$, the set $\{\alpha, \beta\}$ provides a counterexample to the 1/2 density of $\mathcal{S}$.

5 Further questions and remarks

There are many open questions connected to D. H. Fremlin's. We state some of them here.

Clearly, the number 1/2 in the proof of Lemma 3.1 is somewhat arbitrary, but this does not appear to be the case with the fact that C is a club.

Question 5.1 *What happens when we simply require that $\mathcal{S}$ contains $[S]^{<\aleph_0}$ for a stationary set $S \subseteq \kappa$?*

D. H. Fremlin remarks that if κ is measurable, then for any normal measure μ on κ there is a set S with $\mu(S) = 1$ and $[S]^{<\aleph_0} \subseteq \mathcal{S}$. Of course, such an S, having μ-measure 1, is stationary.

As to how arbitrary ϵ is, it seems natural to suppose that the value of ϵ does not influence the fact that on a cardinal κ there is or there is not an ϵ-dense open family which does not satisfy the relevant version of containment. Answering a question from an earlier version of this paper D. H. Fremlin showed in [6] that in the most interesting case, this is indeed true:

Theorem 6 *(Fremlin, [6]) For a cardinal κ, let*

$$I(\kappa) \stackrel{\text{def}}{=} \{\epsilon \in (0,1] : \text{ Every } \epsilon\text{-dense open family contains } [A]^{<\aleph_0} \text{ for some } A \in [\kappa]^\kappa\}.$$

Then $I(\kappa)$ is always of the form $(0,1]$ or $\{1\}$.

Question 5.2 *Is it relatively consistent for a successor cardinal to be either Fremlin or strongly Fremlin in a model in which the full Axiom of Choice is true? In particular, if κ is not Fremlin or strongly Fremlin, then can it be inferred that κ^+ isn't either Fremlin or strongly Fremlin?*

Question 5.3 *Can the existence of an infinite cardinal which is either Fremlin or strongly Fremlin be proven in ZFC?*

Question 5.4 *Is the limit of Ramsey cardinals strongly Fremlin? What about the limit of measurable cardinals?*

Question 5.5 *Can singular cardinals be strongly Fremlin? Can a singular cardinal be either the first Fremlin cardinal or the first strongly Fremlin cardinal? Can $\aleph_\omega$ be either the first Fremlin cardinal or the first strongly Fremlin cardinal?*

Note that trivially, if κ is Fremlin, then so is any cardinal $\lambda \geq \kappa$. This does not tell us, however, if being either the first Fremlin or (the first) strongly Fremlin is a large cardinal property, nor does it tell us if it is possible for a singular cardinal to be either the first Fremlin or first strongly Fremlin cardinal.

Regarding the above questions, let us call a cardinal κ λ-*Fremlin* for $\omega < \lambda < \kappa$ iff for every $\epsilon > 0$ and every ϵ-dense open family $\mathcal{S}$ of subsets of κ, there is an $S \in [\kappa]^\lambda$ with $[S]^{<\aleph_0} \subseteq \mathcal{S}$. Then, using an appropriate variant of Fremlin's theorem above, one can prove that if κ is singular, $\kappa > \lambda \geq \mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$, and $\{\theta : \theta \text{ is not } \lambda\text{-Fremlin}\}$ is cofinal in κ, then κ is not λ-Fremlin. Namely, fixing λ, we prove this by induction on $\mathrm{cf}(\kappa)$. We fix a strictly increasing sequence $\langle \theta_i : i < \mathrm{cf}(\kappa) \rangle$

of cardinals which are not λ-Fremlin and increase to κ. By the inductive hypothesis we can assume that the sequence is continuous. By Fremlin's theorem we can find for each i a 1/2-dense open $\mathcal{S}_i \subseteq [\theta_i]^{<\aleph_0}$ which demonstrates that θ_i is not λ-Fremlin. Considering

$$\mathcal{S}_i \cup [\mathcal{S}_{i+1} \cap [[\theta_i, \theta_{i+1})]^{<\aleph_0}] \cup \{A \cup B : A \in \mathcal{S}_i, B \in \mathcal{S}_{i+1} \cap [[\theta_i, \theta_{i+1})]^{<\aleph_0}\}$$

we can see that, by changing $\langle \mathcal{S}_i : i < \mathrm{cf}(\kappa) \rangle$ inductively, we can assume that $\mathcal{S}_{i+1} \cap [\theta_i]^{<\aleph_0} = \mathcal{S}_i$, and it is also clear that for i limit we can assume that $\mathcal{S}_i = \cup_{j<i} \mathcal{S}_j$, by continuity. Now $\mathcal{S} = \cup_{i<\mathrm{cf}(\kappa)} \mathcal{S}_i$ demonstrates that κ is not λ-Fremlin.

Question 5.6 *Starting from a model of ZFC and a strongly Fremlin cardinal, can one always find a choiceless universe in which $\aleph_1$ is strongly Fremlin?*

The following important question about ϵ-density was communicated to us by D. H. Fremlin and asked by F. Galvin and K. Prikry in [4]:

Question 5.7 *If $\epsilon > 0$ and $\mathcal{S} \subseteq [\kappa]^{<\aleph_0}$ is ϵ-dense open, is there necessarily a finitely additive measure $\nu : \mathcal{P}(S) \longrightarrow [0, 1]$ such that $\nu(\{I : \xi \in I \in \mathcal{S}\}) > 0$ for all $\xi < \kappa$?*

6 Appendix

We comment on two facts mentioned earlier in the paper.

Fact 6.1 *$\aleph_0$ is not Fremlin.*

Proof: This is a simple modification of the Erdös-Rado argument [3] that $\aleph_0$ is not Ramsey. We just consider the family of all finite subsets of ω whose smallest element is at least as large as its cardinality.

$\square$

There is a paper by P. Pudlak and V. Rödl [10] in which the authors classify all the families $\mathcal{S}$ of finite subsets of ω which are downward closed, every infinite subset of ω has an initial segment in $\mathcal{S}$, yet no infinite subset of ω has all initial segments in $\mathcal{S}$.

Fact 6.2 *If CH holds, then $\aleph_1$ is not strongly Fremlin.*

Proof: This is the apparently folklore argument mentioned in Section 1 and communicated to us by Ilijas Farah. By CH, fix an increasing sequence $\langle A_\xi : \xi < \omega_1 \rangle$ of null sets which cover the unit interval. For every ξ, fix a compact set K_ξ of measure $1/2$ which is disjoint from A_ξ. Define $\mathcal{S}$ to be the family of those finite subsets F of ω_1 for which $\bigcap\{K_\xi : \xi \in F\} \neq \emptyset$. Clearly, $\mathcal{S}$ is closed under taking subsets. If $|F| \geq 2n$, then F clearly has a subset of size at least n which is in $\mathcal{S}$ - namely consider $\int \Sigma_{\xi \in F} \chi_{K_\xi}$, and apply the Mean Value Theorem.

On the other hand, any $X \subseteq \omega_1$ satisfies $[X]^{<\aleph_0} \subseteq \mathcal{S}$ iff $\bigcap\{K_\xi : \xi \in X\} \neq \emptyset$, by compactness. If such X were to be uncountable, we would have a contradiction with the choice of the A_ξ's.

$\square$

It would be interesting to know if one can generalize the above proof to show that if $2^\kappa = \kappa^+$, then κ^+ is not strongly Fremlin. The obvious analogue runs into difficulty, as we would need κ-additivity of the measure we use.

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