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PRIMES IN SEQUENCES ASSOCIATED TO POLYNOMIALS (AFTER LEHMER)

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Abstract

In a paper of 1933, D. H. Lehmer continued Pierce's study of integral sequences associated to polynomials generalizing the Mersenne sequence. He developed divisibility criteria, and suggested that prime apparition in these sequences — or in closely related sequences — would be denser if the polynomials were close to cyclotomic, using a natural measure of closeness.

We review briefly some of the main developments since Lehmer's paper, and report on further computational work on these sequences. In particular, we use Mossinghoff's collection of polynomials with smallest known measure to assemble evidence for the distribution of primes in these sequences predicted by standard heuristic arguments.

The calculations lend weight to standard conjectures about Mersenne primes, and the use of polynomials with small measure permits much larger numbers of primes to be generated than in the Mersenne case.

1. *Introduction*

Let $f \in \mathbb{Z}[x]$ be a monic polynomial with factorization

$$f(x) = (x - \alpha_1) \dots (x - \alpha_d) \tag{1}$$

over the complex numbers. Following Pierce [19] and Lehmer [12], define a sequence of integers by

$$\Delta_n(f) = \prod_{i=1}^{d} |\alpha_i^n - 1|. \tag{2}$$

For example, if $f(x) = x - 2$, then $\Delta_n(f) = 2^n - 1$ is the classical Mersenne sequence. Pierce and Lehmer studied the possible factors of $\Delta_n(f)$, and Lehmer in particular used these results to compute large primes. For our purposes, the detailed arguments concerning possible factors are not relevant, but three key observations by Lehmer are:

1. if $|\alpha_i| \neq 1$ for $i = 1, \dots, d$ then $\Delta_n(f)/\Delta_{n-1}(f) \to M(f) = \prod_{i:|\alpha_i|>1} |\alpha_i|$;
2. if $M(f)$ is close to 1, then $\Delta_n(f)$ may be expected to be prime often;
3. prime factors of Δ_n satisfy (essentially) linear congruences.

It is clear from Kronecker's lemma that $M(f) = 1$ if and only if f is cyclotomic. Lehmer made an extensive search for non-cyclotomic polynomials with measure close to 1, and

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his example of degree 10 (referred to below as f_1) with $M(f) = 1.176\ldots$ is still the closest known. He also made the prescient remark that for non-cyclotomic polynomials, a zero on the unit circle *'contributes an oscillating factor which, although it never vanishes or becomes infinite, cannot be estimated readily'* and went on to use $M(f)$ as a natural measure of growth in this case also (cf. the convergence (3), discovered later).

Many subsequent authors have shed new light on various aspects of the sequence $(\Delta_n(f))$ and the associated growth rate $M(f)$. Mahler [**15**] pointed out that Jensen's formula gives the integral form

$$m(f) = \log M(f) = \int_0^1 \log|f(e^{2\pi i t})|\mathrm{d}t$$

for the measure, which is now called the (logarithmic) *Mahler measure* of f. A huge amount of work has gone into attempts to resolve *Lehmer's problem*: are there polynomials with arbitrarily small positive logarithmic measure? For an overview of this circle of results from a theoretical perspective, see [**2**] and [**8**]. The view of polynomials with small measure as being small perturbations of cyclotomic ones is explored in [**17**]. For recent results on computations of Mahler measures and their connections with other parts of mathematics, see [**4**], [**7**], [**14**] and [**16**].

To each polynomial of the form (1) there is an associated endomorphism T_f of the d-torus, given by the natural action of the companion matrix of f. If no zero of f is a root of unity, then T_f is an ergodic transformation with respect to Lebesgue measure, and $\Delta_n(f)$ is the number of points of period n under T_f. Expansiveness of T_f as a topological dynamical system corresponds to Lehmer's condition that $|\alpha_i| \neq 1$ for $i = 1, \ldots, d$. Finally, the topological entropy of T_f is equal to $m(f)$. This links arithmetic properties of the sequence to dynamical properties of the corresponding toral endomorphism — see [**13**]. Accordingly, we call the polynomial f *expansive* if $|\alpha_i| \neq 1$ for $i = 1, \ldots, d$, *ergodic* if no α_i is a root of unity, and *quasihyperbolic* if it is ergodic but not expansive.

Finally, the convergence observed by Lehmer in the expansive case does not extend to the quasihyperbolic case (see [**6, 8**, Theorem 2.16], but the more robust convergence

$$\frac{1}{n}\log \Delta_n(f) \longrightarrow m(f) \tag{3}$$

extends to the quasihyperbolic case by Gelfond's Diophantine results (see [**9**] and [**13**]). Some measure of the Diophantine subtlety involved in convergence (3) may be seen in the sequence corresponding to f_1 (defined below): $\Delta_n(f_1)$ behaves asymptotically like $(1.176...)^n$ but $\Delta_n(f_1) = 1$ for values of n as large as 74. These dramatically small values for relatively large values of n are reflected in the graphs below by the irregular early behaviour.

2. *Arithmetic of* Δ_n

The polynomial (1) is said to be *reciprocal* if $x^d f(x^{-1}) = f(x)$. Boyd [**1, 3**] and Mossinghoff [**16**] have carried out extensive calculations of Mahler measures; from [**16**] we use the list of the 100 irreducible polynomials with smallest known positive Mahler measure. These are all reciprocal (a beautiful result of Smyth [**21**] shows that if f is non-reciprocal and $f(0)f(1) \neq 0$, then $m(f) \geqslant m(x^3 - x - 1) = 0.281\ldots$), and are known to divide polynomials with coefficients in $\{0, \pm 1\}$. If f is a reciprocal polynomial, then $\Delta_n(f)/\Delta_1(f)$ is a perfect square for n odd, by the following argument. If α is a zero of f, let $K = \mathbb{Q}(\alpha)$ and $K' = \mathbb{Q}(\alpha + \alpha^{-1})$. Then

$$\begin{aligned}\Delta_n(f) &= |N_{K/\mathbb{Q}}(\alpha^n - 1)| \\ &= |N_{K/\mathbb{Q}}(\alpha - 1) N_{K/\mathbb{Q}}(1 + \alpha + \ldots + \alpha^{n-1})| \\ &= \Delta_1(f) \times |N_{K/\mathbb{Q}}(\alpha^{(n-1)/2}) N_{K/\mathbb{Q}}(\alpha^{-(n-1)/2} + \ldots + \alpha^{(n-1)/2})|.\end{aligned}$$

Now $\xi = \alpha^{-(n-1)/2} + \ldots + \alpha^{(n-1)/2}$ is an integral element of K', so

$$N_{K/\mathbb{Q}}(\xi) = \left(N_{K'/\mathbb{Q}}(\xi)\right)^2$$

is a square. Accordingly, define $\Gamma_n(f)$ by $\Gamma_n(f)^2 = \Delta_n(f)/\Delta_1(f)$ for odd $n \geqslant 1$.

Prime values of $\Gamma_n(f)$ may arise for composite values of n, and such values are called *anomalous*. In the expansive case it is clear that the anomalous primes are finite in number, and this remains so in the quasihyperbolic case, for a deeper reason.

Proposition 1. *If f is an ergodic polynomial, then there are only finitely many anomalous primes in the sequence $(\Gamma_n(f))$, or in $(\Delta_n(f)/\Delta_1(f))$ in the non-reciprocal case.*

Proof. First notice that the sequence is multiplicative. Write $M = M(f)^{1/2}$ for the square root of the Mahler measure of f, and Γ_n for $\Gamma_n(f)$. (Note that a similar argument holds for $(\Delta_n(f)/\Delta_1(f))$ in the non-reciprocal case.) By Baker's theorem (see [8] for references), there are constants $A, B, C > 0$ with

$$AM^n > \Gamma_n > BM^n/n^C.$$

It follows that only finitely many n can have $\Gamma_n = 1$.

Now an anomalous prime occurs when Γ_{mn} is prime with $m, n > 1$. If Γ_m and Γ_n are both 1, then m and n are bounded by the previous paragraph. On the other hand,

$$\Gamma_{mn}/\Gamma_m > BM^{mn}/A(mn)^C M^m = DM^{n(m-1)}/(mn)^C.$$

If the left-hand-side is 1, then there is an upper bound of the form

$$E + F(\log n + \log m)$$

for $n(m-1)$, which bounds both m and n.

This precludes $\Gamma_{mn} = \Gamma_m$ for all but finitely many m and n. □

Recall that K is the field defined by the chosen irreducible polynomial f, and let

$$\begin{aligned} h_K &= \text{class number of } K; \\ r_1 &= \text{the number of real embeddings of } K; \\ r_2 &= \text{half the number of complex embeddings of } K; \\ w_K &= \text{the number of unit roots in } K; \\ R_K &= \text{the regulator of } K; \\ d_K &= \text{the discriminant of } f; \\ \rho_K &= \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K\sqrt{|d_K|}}. \end{aligned}$$

Define as usual the Dedekind zeta-function for K by

$$\zeta_K(s) = \sum_{\mathfrak{q}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{q})^s}, \tag{4}$$

where $\mathfrak{q}$ runs through the ideals of O_K, with Laurent expansion at $s = 1$ given by

$$\zeta_K(s) = \frac{\rho_K}{s-1} + \gamma_K + \ldots \tag{5}$$

and Euler product form

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^s}\right)^{-1} \tag{6}$$

where $\mathfrak{p}$ runs through the prime ideals of O_K. Finally, there is the number-field analogue of Merten's theorem (see [10], [11] or [20]).

Proposition 2.

$$\sum_{N_{K/\mathbb{Q}}(\mathfrak{p}) \leqslant x} -\log\left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})}\right) = \log\log x + \gamma + \log \rho_K + O(1/\log x)$$

where $\mathfrak{p}$ runs through the prime ideals of O_K, and $\gamma = 0.577\ldots$ is the classical Euler constant.

3. *Heuristic arguments*

The *Mersenne numbers* $M_n = 2^n - 1$ are well-known, and 38 values of n are known for which M_n is prime. An elegant probabilistic argument due to Wagstaff [22] gives the following expected distribution of prime values of M_n. If $n_1, n_2, \ldots$ are the primes for which M_{n_j} is prime, then $j/\log_2 \log_2 M_{n_j}$ is conjectured to converge to a constant. This is a consequence of the simple linear congruences satisfied by factors of M_n (from the Euler–Fermat theorem), and Merten's theorem.

In the Lehmer case, essentially the same argument may be applied, but the arithmetic of the sequence and the analytic properties of the corresponding zeta function are more involved. The calculations described below give the following results.

1. There is compelling numerical evidence to suggest that

$$\frac{j}{\log\log \Gamma_{n_j}} \longrightarrow E_f \tag{7}$$

for some positive limit E_f as $j \to \infty$, where $n_1, n_2, \ldots$ is the sequence of prime indices for which Γ_{n_j} is prime.

2. A naive number-field analogue of Wagstaff's heuristics suggests that E_f is given by $W_f = 2e^{\gamma}/m(f)$, which is compatible with the numerical evidence.
3. The more subtle quantity $C_f = 2e^{\gamma_{K'}}/m(f)$ (or $2e^{\gamma_K}/m(f)$ in the non-reciprocal case) is sometimes closer to the observed E_f, though we do not have a heuristic rationale for this, and the calculation of $\gamma_{K'}$ (or γ_K) itself presents considerable difficulties for extensions of large degree.
4. The *discrepancy* between the observed value of E_f and either of the heuristic constants is substantial enough to suggest that more subtle arithmetic phenomena are at work.

To explain the heuristic argument, we follow essentially Caldwell's exposition of the Wagstaff heuristics (available on the WWW 'Prime Pages' site — see [5]). Assume that p is prime. If $\mathfrak{p}$ is a prime ideal in O_K with

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \mid N_{K/\mathbb{Q}}(\alpha^p - 1)$$

then $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv 1 \bmod p$. It follows that the probability of $\Gamma_p(f)$ being prime is increased by the ratio $N_{K/\mathbb{Q}}(\mathfrak{p})/(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1)$ for each prime ideal $\mathfrak{p}$ of O_K with prime norm $N_{K/\mathbb{Q}}(\mathfrak{p}) \leqslant p$. The set

$$\{\mathfrak{r} \mid \mathfrak{r} \text{ is an ideal of } O_K \text{ with } N_{K/\mathbb{Q}}(\mathfrak{r}) \leqslant x\}$$

has asymptotically $\rho_K x$ members, of which $x/\log x$ are prime ideals with prime norm. It follows that the probability that an integral ideal $\mathfrak{r}$ is a prime ideal with prime norm in O_K is $1/(\rho_K \log N_{K/\mathbb{Q}}(\mathfrak{r}))$.

In the Mersenne case, the resulting product is estimated using Merten's theorem; here we use Proposition 2 instead. The discussion above suggests that the probability that $\Gamma_p(f)$ is prime is approximately

$$\begin{aligned} P_f(p) &= \left(\frac{2\rho_K^{-1}}{p\,m(f)}\right) \prod_{N_{K/\mathbb{Q}}(\mathfrak{p})\leqslant p} \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{p})}{(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1)}\right) \\ &= \left(\frac{2\rho_K^{-1}}{p\,m(f)}\right) \left(e^{\gamma} \rho_K \log p + O(1/p)\right). \end{aligned}$$

So the expected number of (non-anomalous) prime values of $\Gamma_p(f)$ with $p \leqslant x$ is given by (p running through the rational primes)

$$\begin{aligned} \sum_{p\leqslant x} P_f(p) &= \frac{2\rho_K^{-1}}{m(f)} \sum_{p\leqslant x} \frac{1}{p} \cdot \prod_{N_{K/\mathbb{Q}}(\mathfrak{p})\leqslant p} \left(\frac{N_{K/\mathbb{Q}}(\mathfrak{p})}{N_{K/\mathbb{Q}}(\mathfrak{p}) - 1}\right) \\ &\sim \frac{2e^{\gamma}}{m(f)} \left(\sum_{p\leqslant x} \frac{\log p}{p}\right) \\ &\sim \left(\frac{2e^{\gamma}}{m(f)}\right) \log x. \end{aligned}$$

Notice that in the Mersenne case, the sum is taken over all n, weighted according to the probability that n is prime; summing instead over primes p without weighting, as we have done here, gives the same estimate.

If we write $n_1, n_2, \ldots$ for the sequence of indices for which Γ_{n_j} is prime, this suggests that the number of prime values of Γ_{n_j} with $n_j \leqslant x$ is approximately $(2e^{\gamma}/m(f)) \log x$. It follows that

$$\frac{\log\log \Gamma_{n_j}}{j} \to \frac{m(f)}{2e^{\gamma}}. \tag{8}$$

Notice that the effect of any further congruence conditions on possible factors of $\Gamma_n(f)$ will be to asymptotically *increase* the number of primes appearing in the sequence, so the relationship

$$E_f \geqslant \frac{2e^{\gamma}}{m(f)} \tag{9}$$

between convergences (7) and (8) is expected. However, the results shown in Table 2 do not give a consistent inequality; if anything, they suggest the reverse (see Section 6).

In the case of non-reciprocal polynomials, the factor 2 (which came from the fact that Γ_n is logarithmically half of Δ_n) needs to be removed, so for non-reciprocal f the letters E_f, W_f, C_f will be used for the analogous quantities also.

Three questions were therefore examined numerically. Firstly, is the sequence associated to a polynomial with small Mahler measure very rich in primes? Secondly, do calculations suggest the distribution (7) for prime apparition in these sequences with some limiting constant? Thirdly, does the 'limiting constant' observed lend support to the heuristic argument?

The results are — unsurprisingly — mixed. The first question can be answered with an emphatic 'yes': in a short search on modest equipment, sequences have been found containing over one hundred primes. The second is answered with an equivocal 'yes': the analogous plots for the polynomials of small measure do look linear (details of the statistical method used are given below). The third question probably requires a deeper understanding of the arithmetic of Γ_n, but the numbers agree fairly well. In particular, the number of primes found does decrease as the Mahler measure increases.

In light of this, it would be of interest to find a reformulation of the Mersenne heuristics in which γ appears, not via Merten's Theorem, but as the second coefficient of the Laurent expansion of the Riemann zeta function at $s = 1$.

A feature of this work is that the use of polynomials with very small measure gives significant data on Mersenne-like problems without the difficulty of testing excessively large numbers for primality. The idea of using polynomials with small measure in this way comes directly from Lehmer's paper.

4. *Description of the calculations*

Given a candidate polynomial f with small Mahler measure, the prime values of n for which $\Gamma_n(f)$ is prime up to some limit were computed. Composite values of n for which $\Gamma_n(f)$ is prime give rise to the *anomalous* primes. Primality testing was for pseudo-primality to ten randomly chosen bases: in particular, the lack of an analogue of the Lucas–Lehmer test means that the primality test used is the Miller–Rabin test. Thus, in this paper, prime values of Γ_n or Δ_n/Δ_1 are *probable* primes. All the calculations were done using PARI-GP; see [**18**] for more details.

For the first two polynomials in the Mossinghoff list,

$$f_1(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

and

$$f_2(x) = x^{18} + x^{17} + x^{16} + x^{15} - x^{12} - x^{11} - x^{10} - x^9 - x^8 - x^7 - x^6 + x^3 + x^2 + x + 1,$$

the calculations were performed for n up to 200 000. For each of the remaining polynomials $f_3, \ldots, f_{40}$, ranging in degree from 10 to 52, the calculations were performed for n up to 50 000. The full list of polynomials is in the paper [**16**]. In order to gain more insight into how much of the prime behaviour is governed simply by the field arithmetic, the same calculation was also carried out for the 'negative' polynomials, $f_j^-(x) = f_j(-x)$.

The constant $\gamma_{K'}$ has also been computed in some cases (as has γ_K in some non-reciprocal cases), though this requires extensive calculation itself. The method adopted is to use the Laurent expansion (5) and estimate $\gamma_{K'} = \lim_{s \to 1^+}((s-1)\zeta_{K'}(s) - \rho_{K'})$, with K replacing K' in the non-reciprocal case, using GP's ability to compute values of the Dedekind zeta functions for number fields of small degree.

The empirical constant E_f is found using a least-squares linear regression.

5. *Results*

We present several graphs of $\log\log \Gamma_{n_j}(f)$ against j, which indicate the asymptotic linearity (Figures 1–9). On each graph, the number on the abscissa is the total number of non-anomalous primes found for that polynomial. In each case the values of C_f, W_f and E_f are given. The graphs have been chosen from the (small) sample of polynomials for which γ_K (or $\gamma_{K'}$) can be computed. As mentioned above, the non-reciprocal polynomials do not have the factor 2 in the expressions for W and C. The numerical constants have been rounded to three decimal places.

Table 1 gives some data for the Mersenne case, some simple non-reciprocal polynomials, and for those $f_{\pm j}$ for which C could be computed (the polynomial f_2, of degree 18, is included here despite the fact that we have been unable to compute C_{f_2}). For the non-reciprocals, the growth rate is much higher by Smyth's result, and so the calculations are limited. In addition to the Mersenne case and some polynomials from [**16**] for which C could be found, some non-reciprocal polynomials of small height have been chosen. These non-reciprocal polynomials are those with smallest Mahler measure in the list of irreducible non-cyclotomic factors of trinomials with smallest known Mahler measures — we thank David Boyd for providing this list of trinomials. Table 1 is thus a mixed bag of polynomials selected on the basis of having small measure for polynomials of a certain shape, or for being of relatively small degree. Table 1 records

1. the polynomial f;
2. the Mahler measure $M(f)$;
3. the range searched, $1 \leqslant n \leqslant R$;
4. the number N of non-anomalous primes found;
5. the empirical constant E_f found using least-squares;
6. the heuristic constant W_f;
7. the heuristic constant C_f.

The polynomials in Table 1 are arranged in order of increasing Mahler measure.

Table 2 summarises the bulk of our results. It lists the following quantities:

1. the number j of the polynomial in the list from [**16**];
2. the Mahler measure $M(f_j^{\pm})$;
3. $N(f_j^{\pm})$, the number of non-anomalous prime values of $\Gamma_n(f_j^{\pm})$;
4. $E_{f_j^{\pm}}$, the least-squares estimate;
5. $W_{f_j^{\pm}}$, the value computed using the heuristic argument above.

The polynomials are again arranged in order of increasing Mahler measure.

6. *Open problems*

Several problems are suggested by this work, of which the most pressing seem to be the following. What is behind the examples in which E_f is *smaller* than W_f? Can a heuristic argument be found that predicts E_f with the same level of accuracy as that seen in the Mersenne case? In particular, significant differences between E_{f_j} and $E_{f_j^-}$ in Table 2 suggest that more accurate heuristics must involve the polynomial itself, and cannot depend only on the arithmetic of the field defined by the polynomial.

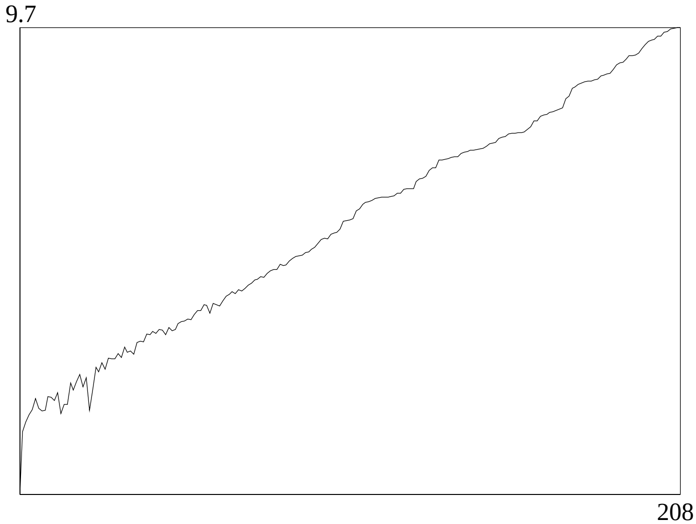

Figure 1: *Graph of* $\log \log \Gamma_{n_j}(f_1)$ *against* j *for* $n \leqslant 200,000$*;* $E_{f_1} = 25.719$, $W_{f_1} = 21.949$,$C_{f_1} = 24.767$.

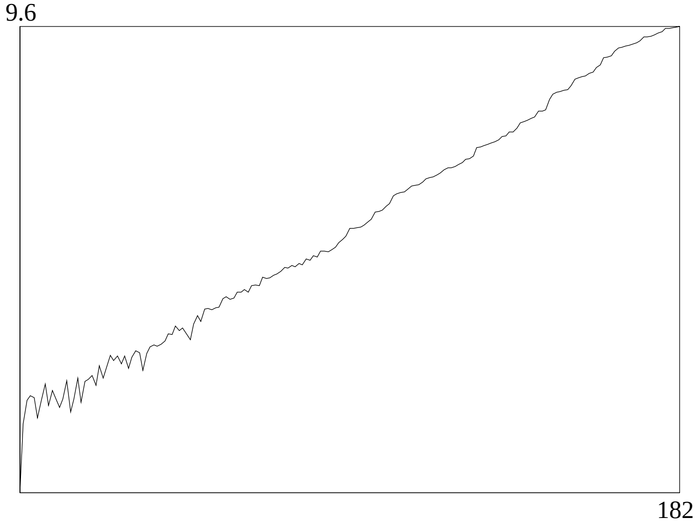

Figure 2: *Graph of* $\log \log \Gamma_{n_j}(f_2)$ *against* j *for* $n \leqslant 200,000$*;* $E_{f_1} = 21.852$, $W_{f_1} = 20.640$.

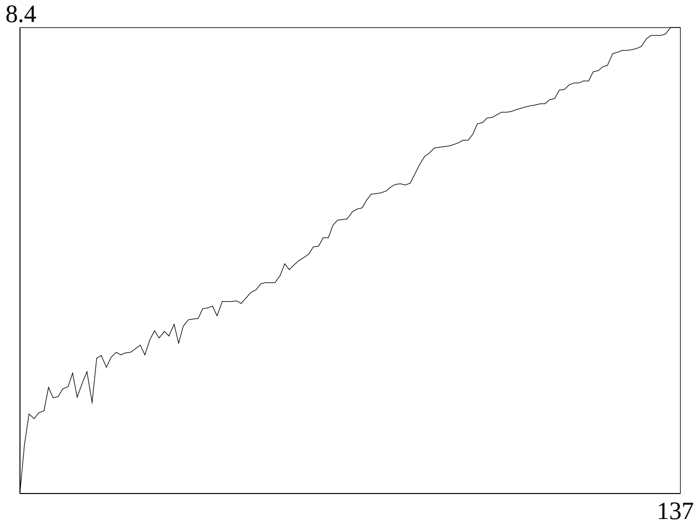

Figure 3: *Graph of* $\log\log\Gamma_{n_j}(f_{10})$ *against* j *for* $n \leqslant 50,000$*;* $E_{f_{10}} = 18.507$, $W_{f_{10}} = 18.191$,$C_{f_{10}} = 18.844$.

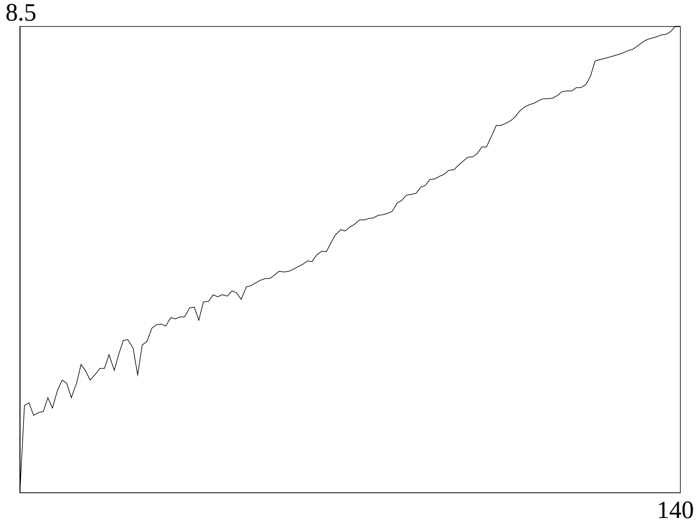

Figure 4: *Graph of* $\log\log\Gamma_{n_j}(f_{26})$ *against* j *for* $n \leqslant 50,000$*;* $E_{f_{26}} = 19.384$, $W_{f_{26}} = 17.364$,$C_{f_{26}} = 18.782$.

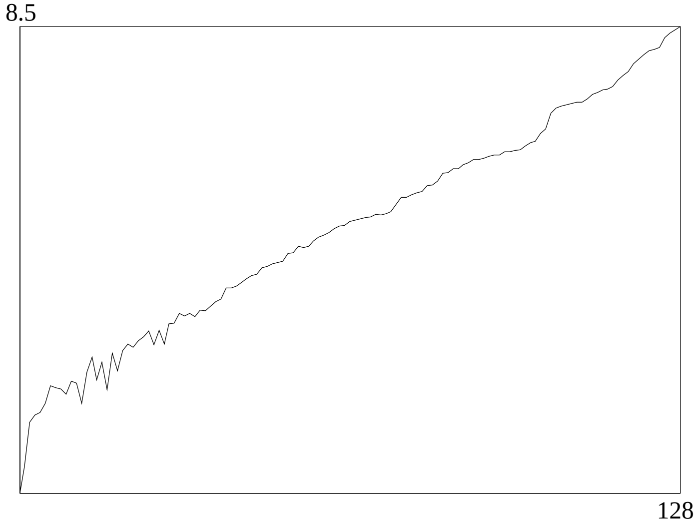

Figure 5: *Graph of* $\log\log\Gamma_{n_j}(f_{33})$ *against* j *for* $n \leqslant 50,000$*;* $E_{f_{33}} = 18.984$, $W_{f_{33}} = 17.187$,$C_{f_{33}} = 17.869$.

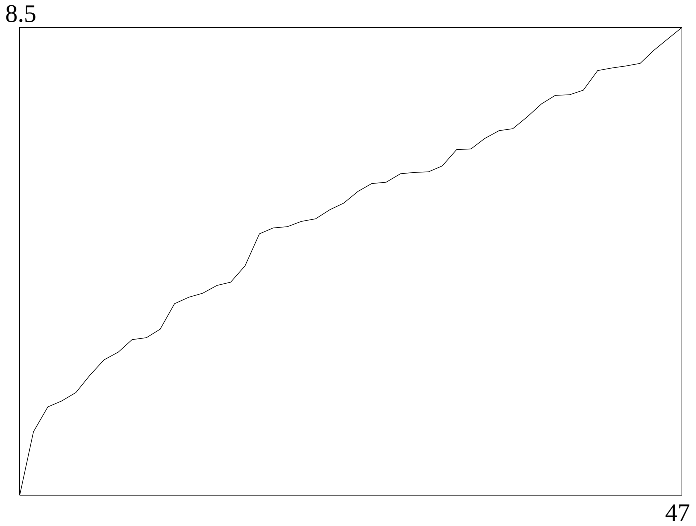

Figure 6: *Graph of* $\log\log\Gamma_{n_j}(x^3-x-1)$ *against* j *for* $n \leqslant 20,000$*;* $E_f = 6.807$, $W_f = 6.334$,$C_f = 6.398$.

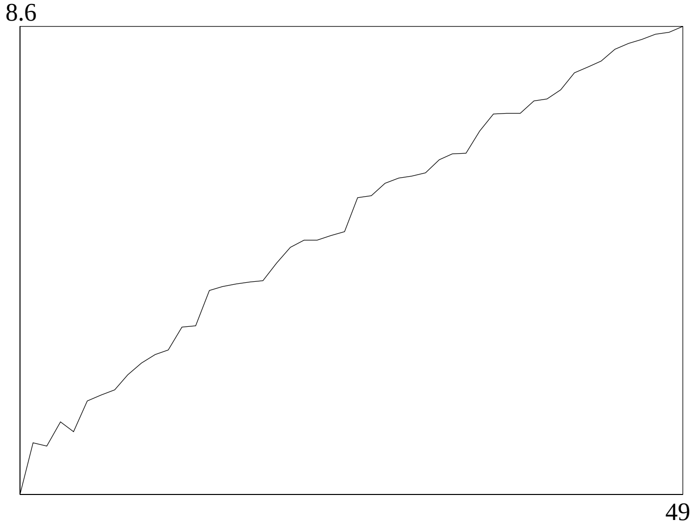

Figure 7: *Graph of* $\log\log \Gamma_{n_j}(x^5 - x^4 + x^2 - x + 1)$ *against* j *for* $n \leqslant 20,000$*;* $E_f = 6.128$, $W_f = 5.939$, $C_f = 5.930$.

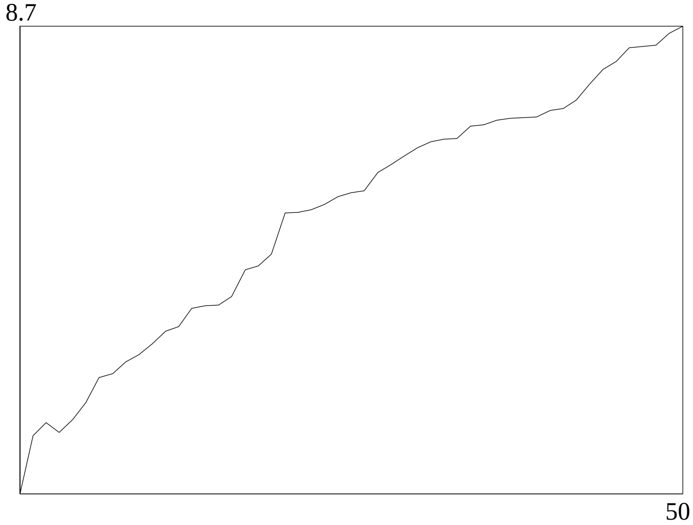

Figure 8: *Graph of* $\log\log \Gamma_{n_j}(x^6 - x^5 + x^3 - x^2 + 1)$ *against* j *for* $n \leqslant 20,000$*;* $E_f = 6.519$, $W_f = 5.793$, $C_f = 5.942$.

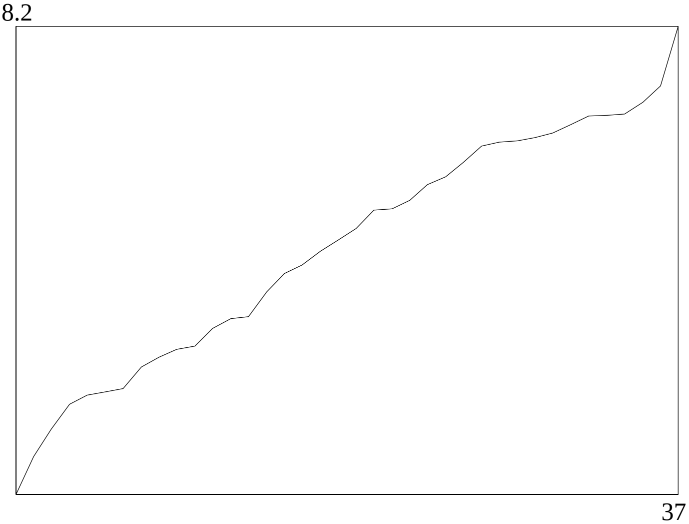

Figure 9: *Graph of* $\log\log\Gamma_{n_j}(x^5+x^2-1)$ *against* j *for* $n \leqslant 20,000$*;* $E_f = 5.411$*,* $W_f = 5.735$*,*$C_f = 5.968$.

Table 1: Mahler measure M, numbers N of prime values of Γ_n or Δ_n found for $n \leqslant R$, empirical constant E, and two heuristic constants W and C for selected polynomials.

f	$M(f)$	R	N	E_f	W_f	C_f
f_1	1.176	200,000	208	25.719	21.940	24.767
f_2	1.188	200,000	182	21.852	20.640	
f_{10}	1.216	50,000	137	18.507	18.184	18.884
f_{-10}	1.216	50,000	133	18.219	18.184	18.884
f_{26}	1.227	50,000	140	19.384	17.358	18.782
f_{-26}	1.227	50,000	145	21.297	17.358	18.782
f_{33}	1.230	50,000	128	18.984	17.180	17.869
f_{-33}	1.230	50,000	132	18.083	17.180	17.869
x^3-x-1	1.325	20,000	47	6.807	6.334	6.398
x^3-x^2+1	1.325	20,000	46	5.963	6.334	6.398
$x^5-x^4+x^2-x+1$	1.350	20,000	49	6.128	5.939	5.930
$x^5+x^4-x^2-x-1$	1.350	20,000	51	6.479	5.939	5.930
$x^6-x^5+x^3-x^2+1$	1.360	20,000	50	6.519	5.793	5.942
$x^6+x^5-x^3-x^2+1$	1.360	20,000	51	7.474	5.793	5.942
x^5+x^2-1	1.364	20,000	37	5.411	5.735	5.968
$x-2$	2	3,021,377	37	2.549	2.569	2.569

Table 2: Number N of prime values of (Γ_n), Mahler measure M, empirical constant E, and heuristic constant W for polynomials $f_1, \ldots, f_{40}$ and $f_1^-, \ldots, f_{40}^-$, $n \leqslant 50,000$.

j	$M(f_j) = M(f_j^-)$	$N(f_j)$	E_{f_j}	$W_{f_j} = W_{f_j^-}$	$E_{f_j^-}$	$N(f_j^-)$
1	1.1762	173	25.899	21.940	23.493	166
2	1.1883	151	22.482	20.640	23.420	156
3	1.2000	137	18.912	19.535	19.724	133
4	1.2013	171	24.618	19.413	20.803	146
5	1.2026	126	19.644	19.307	22.004	155
6	1.2050	148	21.374	19.100	18.356	132
7	1.2079	128	18.211	18.854	21.109	144
8	1.2128	136	19.127	18.461	18.905	136
9	1.2149	145	22.572	18.291	18.542	128
10	1.2163	137	18.507	18.184	18.219	133
11	1.2183	134	19.974	18.032	19.211	135
12	1.2188	135	18.619	17.998	19.594	140
13	1.2190	122	16.996	17.983	19.885	135
14	1.2194	114	16.258	17.954	21.704	151
15	1.2197	137	18.941	17.934	17.399	130
16	1.2202	115	16.667	17.892	16.919	124
17	1.2234	145	20.884	17.663	19.529	136
18	1.2237	136	18.806	17.639	15.666	113
19	1.2242	133	19.967	17.603	20.437	141
20	1.2255	145	19.655	17.517	19.093	132
21	1.2256	143	19.681	17.509	17.947	124
22	1.2258	125	17.293	17.495	17.837	128
23	1.2260	142	20.807	17.475	19.863	146
24	1.2264	138	20.496	17.447	15.450	111
25	1.2269	125	16.902	17.413	17.207	118
26	1.2277	140	19.384	17.358	21.297	145
27	1.2281	108	14.658	17.333	19.296	129
28	1.2294	136	19.935	17.242	14.921	105
29	1.2295	124	17.069	17.236	19.872	135
30	1.2300	128	17.973	17.207	18.011	123
31	1.2302	128	18.594	17.189	17.003	116
32	1.2302	119	16.009	17.187	17.521	129
33	1.2303	128	18.984	17.180	18.083	132
34	1.2307	125	17.453	17.157	17.693	125
35	1.2313	127	17.617	17.117	17.708	129
36	1.2322	121	17.901	17.059	17.297	122
37	1.2326	143	19.657	17.032	16.448	123
38	1.2326	128	17.987	17.031	18.130	125
39	1.2336	122	17.154	16.963	17.194	127
40	1.2343	116	15.852	16.918	16.316	112

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