
Nonlinear wave dynamics in discrete space-time systems

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Philosophy

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Abstract

Nonlinear wave dynamics are ubiquitous across many fields of mathematics and physics, and are typically studied in the context of continuous space-time. In this thesis a different setting is considered, in which wave evolution is governed by discrete space-time.

In particular, an experimentally realisable discrete space-time optical lattice model, here referred to as the photonic mesh lattice, is studied. The lattice is comprised of two distinct dispersion bands, each equipped with a unique set of normal modes. The thermalisation processes among these bands are investigated in-depth, to understand the role of discrete-time on modal interactions and thermal equilibrium.

Through numerical simulations, it is shown that the system evolves toward a universal thermal equilibrium characterised by equal modal occupation among both bands, independent of lattice parameters and initial conditions. This behaviour arises from additional resonant wave interactions that exist through duplicate dispersion bands permitted by discrete-time, violating the conservation of linear energy. Consequently, the thermal equilibrium state is no longer constrained by linear energy conservation, leading to a deviation from the Rayleigh–Jeans distribution typically predicted by weak wave turbulence theory.

Furthermore, these additional discrete-time resonances are also shown to accelerate thermodynamic processes via a four-wave resonant interaction. A

systematic exploration of lattice parameters reveals a clear relationship between the size of the resonant manifold and thermalisation timescales, with larger manifolds corresponding to substantially faster modal equilibration.

Lastly, the thermal equilibrium state is explored for a driven/damped lattice model, by driving one band and dissipating the other. In doing so, the effects of discrete time on modal interaction and thermal equilibrium are diminished, returning the system to the typical Rayleigh–Jeans distribution, reminiscent of continuous-time multi-band wave systems.

To the best of the author’s knowledge, this thesis demonstrates the first observations of the effect of discrete-time on thermalisation processes and weak wave turbulence, showing that discrete-time is a novel setting for the study of thermalisation and weak wave turbulence theory.

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Introduction

Nonlinear wave dynamics arise throughout mathematics and physics, governing an enormous range of physical phenomena where waves are present. They manifest across virtually every length scale, and are typically modelled using nonlinear ordinary differential equations (ODE) or partial differential equations (PDE). As such, systems consist of many interacting wave modes coupled through mechanisms determined by the specific form of the nonlinearity. These interactions result in complex and often chaotic behaviour in which the underlying physical processes are obscured by a background of chaotic fluctuations. As a result, direct interpretation of the system's evolution can be difficult, and statistical approaches are frequently required to extract meaningful physical insight.

One such approach is known as weak wave turbulence (WWT) [[1](#), [2](#), [3](#)], a framework established in the 1960's [[4](#), [5](#)] that describes the chaotic wave interactions between the linear normal modes of a dispersive conservative nonlinear system. WWT approximates the equation of motion of a nonlinear wave system into an integro-differential equation that describes the long-time evolution of the ensemble averaged normal mode spectrum called the wave-kinetic equation. Through this approximation, the theory reveals that the dominant wave interactions between modes simultaneously satisfy the conservation of momentum and linear energy. These dominant interactions are referred to as resonant wave interactions and have been used to explain a

multitude of physical phenomena. Consequently, WWT has been applied to a plethora of physical contexts through theory, computation and experiment. Some areas include water waves [4, 5, 6, 7, 8, 9], plasma physics [10, 11, 12, 13, 14], nonlinear optics [15, 16, 17, 18], quantum fluids [15, 19, 20, 21, 22, 23], astrophysics [14, 24] and climate science [7, 8, 25, 26, 27, 28], among many others.

More often than not, physical systems are governed by continuous space-time evolution. That is, space and time can take any value along the real numbers, with no restrictions on the size of the step taken in space or time. As such, we are free to take derivatives in space and time, which represent spatio-temporal evolution in the respective ODE/PDE. However, there exists a distinct class of systems that do not undergo continuous space-time evolution, but instead progress according to discrete steps in space and time. While such systems are less common, their study has become increasingly important in modern physics, particularly within areas of optics and photonics [29, 30, 31, 32] and quantum many-body physics [33, 34, 35], highlighting the need for a deeper understanding of the governing wave dynamics.

Typically, WWT is applied to systems in the thermodynamic limit, where a continuous/dense spectrum of wavelengths ensures efficient wave mixing between modes. In contrast, systems with finite spatial structure have access to only a limited set of wave scales, as the corresponding momentum space is discrete. Often, this discreteness prevents wave numbers from interacting, with interactions scattering into length scales not accessible in the discrete momentum space. However important works in the 2000's [36, 37, 38, 39] showed that, while WWT is restricted in the finite setting, efficient wave mixing is still possible through quasi-resonant interactions, facilitated by nonlinear broadening effects. Since then several studies have applied WWT to a variety of discrete-space systems [40, 41, 42, 43, 44], successfully describing thermalisation processes among normal modes, showing that despite its limitations, the theory remains effective in capturing fundamental processes through the resonant wave

description.

With the extension of WWT to discrete-space systems, this thesis considers, for the first time, the effects of discrete space-time evolution on wave interactions and WWT in the context of nonlinear optics and photonics. Through theoretical analysis and computational study, we show that discrete-time evolution has highly non-trivial effects, altering thermodynamic states and permitting a broader class of resonant wave interactions, not possible in continuous-time. This thesis demonstrates a new avenue of research within the broader field of thermalisation processes extending the current understanding of WWT.

0.1 Thesis outline

Broadly, the thesis is divided into five distinct chapters. Chapters (1) and (2) introduce concepts from literature, with no new novelties. Chapters (3) and (4) comprise the results of this thesis and Chapter (5) are some concluding remarks. Here the general structure of the thesis is laid out more clearly.

Chapter (1) introduces the concept of discrete space-time and its consequences for wave dynamics. We examine spatial and temporal discretisation through illustrative examples, highlighting their impact on dispersion relations, Fourier space and the conservation of momentum and energy. Important concepts such as Bloch's theorem, and Floquet theory are also introduced to be utilised in later chapters.

Chapter (2) introduces WWT theory for systems with discrete-space. Starting from a Hamiltonian formulation of weakly interacting systems, we introduce the problem of statistical closure and how WWT addresses this. Next we discuss resonant interaction, with particular emphasis on the role played by discrete space-time. Then, we derive the wave-kinetic equation for the discretised nonlinear Schrödinger equation, following the derivation of Onorato and

Demmateis [42], as to introduce important concepts from WWT theory. Conservation laws, equilibrium solutions, and the conditions under which thermalisation occurs are analysed for the wave-kinetic equation, and non-trivial resonant interactions are derived analytically.

Chapter (3) presents the photonic mesh lattice, which is the primary system studied in this thesis. After introducing the lattices space-time evolution equations and the linear and nonlinear properties, we investigate thermalisation processes within a single dispersion band and between both bands simultaneously.

Single band results discuss the physical origin of the observed power leakage process, justified through asymptotic analysis of non-resonant four wave interactions and numerical analysis of six wave resonances. We then determine a power law scaling of the thermalisation time scale with the nonlinear coefficient numerically for the single band thermal state, an expected property from WWT. From this we infer what wave interactions are needed for thermalisation to occur, finding good agreement with the numerical evolution of the truncated nonlinear equations. We then demonstrate that the non-trivial resonances that are likely to facilitate thermal equilibrium are solely accessed through nonlinear broadening, using numerical measurement of the broadened spectrum.

The full thermalisation between bands, mediated by the power leakage is then investigated through numerical simulations. Here we observe unusual thermodynamic behaviour explained through the existence of extra resonant channels only accessible because of the discrete-time nature of the system. In turn these resonances violate energy conservation ensuring that the lattice always tends to a universal thermal state of equal modal occupation. We also show how these resonances can accelerate thermodynamic processes and how this relates to the geometry of the dispersion bands.

Lastly to close this chapter, we discuss potential avenues of experimental and theoretical investigation that could further verify the existence of these extra

discrete time effects.

Chapter (4) explores the dynamics of the photonic mesh lattice under external forcing and dissipation, in the pursuit of steady states. Here the implementation of the forcing and dissipation in simulation alongside the characterisation of the stationary states is discussed. This is then followed by an investigation into the effect of the size of the dispersion bands and the discrete-time effects on the driven/damped steady states. It is concluded from these numerical simulations that, many discrete-time effects are found to be suppressed when the system is externally driven, unlike the closed lattice model.

Chapter (5) summarises the main results and outlines possible directions for future research based on the results presented in this thesis.

The effect of discretisation in space and time

Space and time are more often than not treated to be continuous quantities. That is, if we have a wave described by a complex envelope $\psi(x, t)$ in space x and time t , for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real values, then ψ is defined for all real values of (x, t) , allowing arbitrarily small changes in either coordinate, see Fig. (1.0.1) panel a). In the limit that these changes become infinitesimal, derivatives can be defined in space and time describing how the wave varies in either coordinate. For example, the partial derivative of $\psi(x, t)$ in time is defined by,

$$\frac{\partial \psi(x, t)}{\partial t} = \lim_{\delta t \rightarrow 0} \frac{\psi(x, t + \delta t) - \psi(x, t)}{\delta t} \quad (1.0.1)$$

assuming $\psi(x, t)$ is smooth in t . The assumption that $t \in \mathbb{R}$ ensures that arbitrarily small increments δt are meaningful, allowing the derivative to be properly defined. Then, a generic description of temporal wave evolution of $\psi(x, t)$ can be written generically as a PDE,

$$\frac{\partial \psi}{\partial t} = \mathcal{F}[\psi] \quad (1.0.2)$$

where $\mathcal{F}$ may depend on ψ , its spatial derivatives and their complex conjugates. We note that, the evolution of waves in time can also be described by the n^{th} order temporal derivatives, however this can always be reduced to a system of n coupled first order PDEs in time of the form in equation (1.0.2).

On the contrary, if we have a space-time that is not continuous, for example when (x, t) are only defined for evenly spaced point,

$$x_n = n\delta x, \quad (1.0.3)$$

$$t_m = m\delta t, \quad (1.0.4)$$

where $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, then $\psi(x_n, t_n)$ can no longer be defined for arbitrarily small changes in (x, t) . As such, the definition in equation (1.0.1) is no longer valid and the description of the wave propagation is incremental, see Fig. (1.0.1) panel b). Now, we can describe the evolution not as a PDE but rather a recursive map updating in time,

$$\psi(x_n, t_{m+1}) = \mathcal{G}[\psi(x_n, t_m)] \quad (1.0.5)$$

where in principle $\mathcal{G}[\psi(x_n, t_m)]$ can depend on $\psi(x_n, t_m) \forall n$ and their complex conjugates.

Alongside continuous and discrete space-time wave systems, there also exists semi-discrete space-time systems where for example space could be discrete and time is continuous, see figure (1.0.1) panel c). In this third type of system, the generic evolution is given by what is sometimes referred to as a discrete-difference equation (Δ DE) [45],

$$\frac{d\psi(x_n, t)}{dt} = \mathcal{H}[\psi(x_n, t)]. \quad (1.0.6)$$

In this case $\mathcal{H}[\psi(x_n, t)]$ behaves much like $\mathcal{G}[\psi(x_n, t_m)]$, potentially including $\psi(x_n, t_m) \forall n$ along with complex conjugates, but is described by a continuous time evolution. Equivalently a wave system could exist that has continuous space and discrete time.

Across these different classes of wave systems, we find that the physics vary considerably depending on what coordinate is discretised. A significant discrepancy is the effect of discretisation on conservation laws. In 1918 Emmy Noether established one of the most important theorems to physics [46], stating

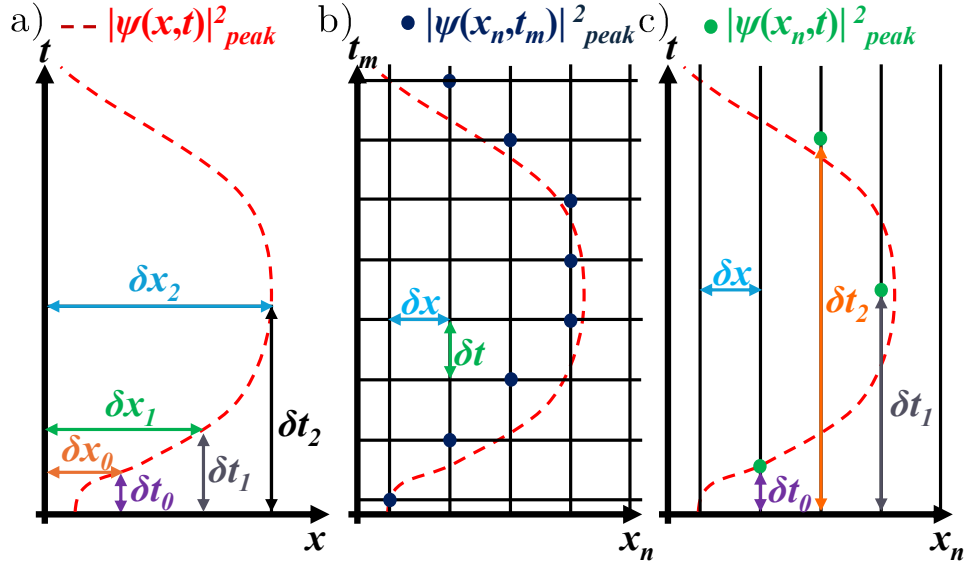


Figure 1.0.1: Illustrative trajectory of the peak intensity position of an infinitely localised wave, evolving in space-time, given by $|\psi(x, t)|^2_{peak}$. a) In continuous space-time, the peak position can be evaluated at any point $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, allowing arbitrarily small steps δx and δt to be taken along the trajectory. b) In discrete space-time the equivalent trajectory is only defined at the grid points (x_n, t_m) , with fixed spacings δx and δt . c) in discrete-space, continuous-time, the trajectory is restricted to the spatial positions x_n , but can be evaluated at any continuous time t .

that for every continuous symmetry possessed by the Lagrangian, there is a corresponding conserved quantity. As such, conservation of linear momentum corresponds to space translation invariance and the conservation of energy corresponds to continuous time translation invariance. As we shall now show, upon discretisation these properties are altered with new physical effects introduced.

1.1 Space discretisation

1.1.1 The effect on momentum conservation

To study the effects of space discretisation, we will consider one of the most common nonlinear wave equations in physics, the nonlinear Schrödinger equation (NLS). The NLS describes a slow varying wave packet subject to cubic

nonlinearity and dispersion. It has been used to describe waves in numerous contexts such as water waves [47], nonlinear optics [48] and Bose-Einstein condensates [49] to name a few. To obtain the NLS, we start from its Hamiltonian in one spatial dimension [50],

$$H_{\text{NLS}} = \int_{\Omega} \left[\left| \frac{\partial \psi}{\partial x} \right|^2 + \frac{\nu}{2} |\psi|^4 \right] dx \quad (1.1.1)$$

$\psi(x, t)$ is the complex wave field in continuous space-time $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, ν is the nonlinear parameter and Ω is some arbitrary space. We then utilise Hamilton's equation to obtain the desired wave equation,

$$i \frac{\partial \psi}{\partial t} = \frac{\delta H_{\text{NLS}}}{\delta \psi^*} \quad (1.1.2)$$

where $\delta H_{\text{NLS}}/\delta \psi^*$ represents the functional derivative of H_{NLS} with respect to conjugate momentum here given by the complex conjugate ψ^* . We then define a variation such that $\psi^* \rightarrow \psi^* + \epsilon \eta^*$ where $\eta(x)^*$ is some arbitrary complex function and ϵ is an arbitrary constant. Thus in the limit that ϵ becomes small, $\epsilon \ll \mathcal{O}(1)$ we Taylor expand:

$$H_{\text{NLS}}[\psi, \psi^* + \epsilon \eta^*] = H_{\text{NLS}}[\psi, \psi^*] + \epsilon \int_{\Omega} \eta^* \frac{\delta H_{\text{NLS}}}{\delta \psi^*} dx + \mathcal{O}(\epsilon^2). \quad (1.1.3)$$

Therefore through the variation of the Hamiltonian the equation of motion is given,

$$\delta H_{\text{NLS}} = H_{\text{NLS}}[\psi, \psi^* + \epsilon \eta^*] - H_{\text{NLS}}[\psi, \psi^*] = \epsilon \int_{\Omega} \eta^* \frac{\delta H_{\text{NLS}}}{\delta \psi^*} dx + \mathcal{O}(\epsilon^2). \quad (1.1.4)$$

For each term of $H_{\text{NLS}}[\psi, \psi^* + \epsilon \eta^*]$ we get,

$$\begin{aligned} \int_{\Omega} \frac{\partial \psi}{\partial x} \frac{\partial (\psi^* + \epsilon \eta^*)}{\partial x} dx &= \int_{\Omega} \frac{\partial \psi}{\partial x} \left[\left(\frac{\partial \psi}{\partial x} \right)^* + \epsilon \frac{d \eta^*}{dx} \right] dx, \\ \int_{\Omega} \frac{\nu}{2} \psi \psi (\psi^* + \epsilon \eta^*) (\psi^* + \epsilon \eta^*) dx &= \int_{\Omega} \frac{\nu}{2} |\psi|^4 + \epsilon \nu \psi \psi \psi^* \eta^* + \epsilon^2 (\eta^*)^2 dx. \end{aligned}$$

Therefore δH_{NLS} is,

$$\delta H_{NLS} = \epsilon \int_{\Omega} \left[\frac{\partial \psi}{\partial x} \frac{d\eta^*}{dx} + \nu |\psi|^2 \psi \eta^* \right] + \epsilon^2 (\eta^*)^2 dx. \quad (1.1.5)$$

Using integration by parts on the $\mathcal{O}(\epsilon)$ term in equation (1.1.5), we can simplify by choosing that the arbitrary function $\eta(x)^*$ vanishes at the boundaries of the Ω space domain,

$$\int_{\Omega} \frac{\partial \psi}{\partial x} \frac{d\eta^*}{dx} dx = \underbrace{\left[\frac{\partial \psi}{\partial x} \eta^* \right]_{\Omega}}_{=0} - \int_{\Omega} \frac{\partial^2 \psi}{\partial x^2} \eta^* dx. \quad (1.1.6)$$

Thus equation (1.1.5) is,

$$\epsilon \int_{\Omega} \eta^* \left[-\frac{\partial^2 \psi}{\partial x^2} + \nu |\psi|^2 \psi \right] + \epsilon^2 (\eta^*)^2 dx = \epsilon \int_{\Omega} \eta^* \frac{\delta H_{NLS}}{\delta \psi^*} + \mathcal{O}(\epsilon^2) dx \quad (1.1.7)$$

Equating coefficients in equation (1.1.7) and using equation (1.1.2) yields the integrable one dimensional NLS,

$$i \frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + \nu |\psi|^2 \psi. \quad (1.1.8)$$

Alongside the Hamiltonian, the NLS equation possess other conservation laws [50] such as norm N_{NLS} arising from invariance under a phase shift,

$$N_{NLS} = \int_{\Omega} |\psi|^2 dx, \quad (1.1.9)$$

and total linear momentum P_{NLS} arising from space translation invariance,

$$P_{NLS} = i \int_{\Omega} \left(\psi \left(\frac{\partial \psi}{\partial x} \right)^* - \psi^* \frac{\partial \psi}{\partial x} \right) dx, \quad (1.1.10)$$

alongside an infinite hierarchy of conserved quantities entering through the property of integrability [51]. To demonstrate that equations (1.1.1), (1.1.9), (1.1.10) are conserved, we evolve the NLS system according to the second order symmetrised split-step Fourier method [52, 53], in Fig. (1.1.1). Starting from an

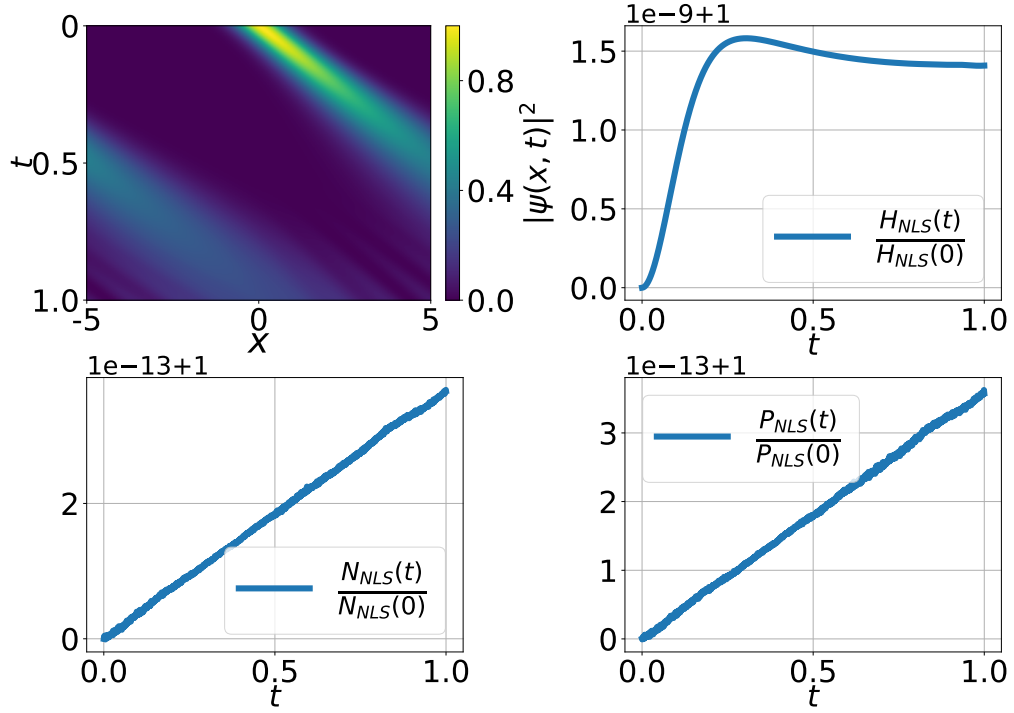


Figure 1.1.1: Evolution of the NLS via the split-step with periodic boundary conditions. $x \in [-L/2, L/2]$ where $L = 10$ and $t \in [0, 1]$ forming a $(2048, 2048)$ numerical grid and $\nu = 0.5$. Here we choose the initial condition given by $\psi(x, 0) = e^{-x^2} e^{ik_0 x}$ with $k_0 = 5$ chosen for significant off-axis propagation. To interpret the y-axis scale of these figures, which display a combined scale factor and offset (e.g. ‘1e-9 + 1’), the true data value is obtained by multiplying the y-tick label by the stated scale factor and then adding the stated offset, i.e. value = (tick label $\times$ scale factor) + offset. This scaling is used in figures throughout this thesis.

off-axis Gaussian pulse, to ensure non-zero momentum, it is clear that each quantity is well conserved up to some numerical accuracy. Alternatively, we can show quantities (1.1.1), (1.1.9), (1.1.10) are conserved through a variational argument of the action [54].

Now, let us discretise the spatial domain Ω into $N \in 2\mathbb{Z}$ equally spaced points with spacing δx ,

$$x_n = n\delta x, n = -N/2, -N/2 + 1, \dots, N/2 - 1. \quad (1.1.11)$$

Here we have intentionally centred n about $n = 0$ as this will be the common convention in chapters (3) and (4). Through discretisation, the derivative in x

can be written using first-order forward finite difference in the form of equation (1.0.1) without taking the limit $\delta x \rightarrow 0$,

$$\left. \frac{\partial \psi}{\partial x} \right|_{x_n} = \frac{\psi((n+1)\delta x, t) - \psi(n\delta x, t)}{\delta x}.$$

We now introduce the notation $\psi(n\delta x, t) = \psi_n(t)$. In discrete space the NLS Hamiltonian (1.1.1) is,

$$H_{\text{DNLS}} = \sum_{n=-N/2}^{N/2-1} c|\psi_{n+1} - \psi_n|^2 + \frac{\nu}{2}|\psi_n|^4 \quad (1.1.12)$$

where $c = |1/\delta x|^2$ and controls the strength of nearest neighbour coupling in space. From equation (1.1.12) we can obtain the Δ DE using the spatially-discretised analogue of equation (1.1.2),

$$i \frac{d\psi_n(t)}{dt} = \frac{\delta H_{\text{DNLS}}}{\delta \psi_n^*(t)} \quad (1.1.13)$$

Upon substitution of H_{DNLS} ,

$$\begin{aligned} i \frac{d\psi_n(t)}{dt} &= \frac{\delta}{\delta \psi_n^*(t)} \sum_{n=-N/2}^{N/2-1} \left[c(|\psi_{n+1}|^2 + |\psi_n|^2 - \psi_{n+1}^* \psi_n - \psi_{n+1} \psi_n^*) + \frac{\nu}{2}|\psi_n|^4 \right], \\ &= c\psi_n - c\psi_{n+1} + \nu|\psi_n|^2\psi_n + c\psi_n - c\psi_{n-1}, \\ i \frac{d\psi_n(t)}{dt} &= c(2\psi_n - \psi_{n+1} - \psi_{n-1}) + \nu|\psi_n|^2\psi_n. \end{aligned} \quad (1.1.14)$$

This equation is often referred to as the non-integrable discrete nonlinear Schrödinger equation (DNLS) and is one of several variants of the NLS. For example an integrable Δ DE version has been derived called the Ablowitz-Ladik equation [55] (AL),

$$i \frac{d\psi_n(t)}{dt} = c(2\psi_n - \psi_{n+1} - \psi_{n-1}) + \frac{\nu}{2}|\psi_n|^2(\psi_{n+1} + \psi_{n-1}). \quad (1.1.15)$$

subject to a non-locally acting cubic nonlinearity. A hybridized model of DNLS and AL has also been formulated [56] referred to here as the Salerno equation

(SAL),

$$i \frac{d\psi_n(t)}{dt} = c(2\psi_n - \psi_{n+1} - \psi_{n-1}) + (1 - \alpha)\nu |\psi_n|^2 \psi_n + \alpha \frac{\nu}{2} |\psi_n|^2 (\psi_{n+1} + \psi_{n-1}), \quad (1.1.16)$$

introducing an interpolation parameter $0 < \alpha < 1$ that can vary the dependence on the type of cubic nonlinearity.

For the DNLS, we will now assume that we can obtain a discrete norm and linear momentum via the same discretising procedure. For the norm this corresponds to,

$$N_{\text{DNLS}} = \sum_{n=-N/2}^{N/2-1} |\psi_n|^2 \quad (1.1.17)$$

and linear momentum,

$$\begin{aligned} P_{\text{DNLS}} &= \sum_{n=-N/2}^{N/2-1} ic [\psi_n (\psi_{n+1} - \psi_n)^* - \psi_n^* (\psi_{n+1} - \psi_n)] \\ &= \sum_{n=-N/2}^{N/2-1} ic [\psi_n \psi_{n+1}^* - \psi_n^* \psi_{n+1}] \end{aligned} \quad (1.1.18)$$

We can test the conservation numerically of these conserved quantities, as shown in Fig.(1.1.2); conservation can also be shown analytically (see appendix (A.1)). Here we evolve the same off-axis Gaussian pulse as the NLS, modified for discrete-space. By tracking the evolution of equations (1.1.12), (1.1.17) and (1.1.18) it is clear that space-discretisation preserve conservation of the Hamiltonian and the norm, but not the momentum. This is because conservation of momentum, according to Noether's theorem [46], requires continuous symmetry in spatial translation. By discretising space we are no longer free to arbitrarily translate in space by any amount we choose and are instead restricted to integer multiples of the spacing δx . As such conservation of linear momentum is lost.

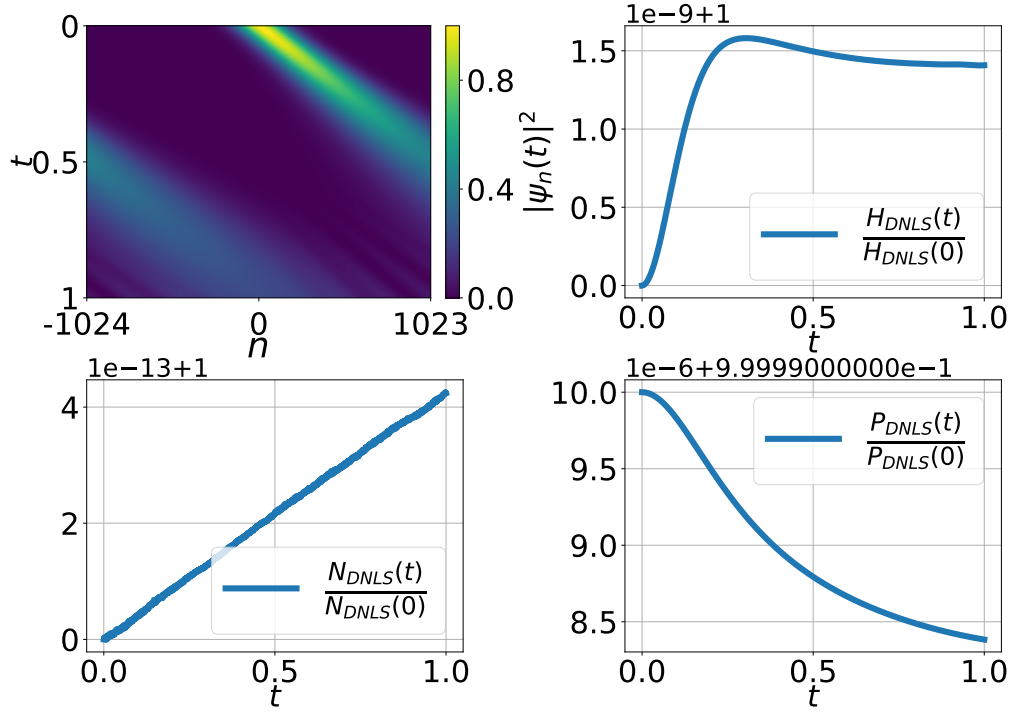


Figure 1.1.2: Evolution of the DNLS via the split-step method with periodic boundary conditions. $n = -N/2, -N/2 + 1, \dots, N/2 - 1$ for $N = 2048$ and $t \in [0, 1]$ forming a $(2048, 2048)$ grid with $\nu = 0.5$. Here we choose the same initial condition for the NLS but discretised in the space coordinate. This is given by $\psi_n(0) = e^{-(n\delta x)^2} e^{ik_0 n\delta x}$ with $k_0 = 5$, setting $\delta x = 10/2048$ to match the numerical grid of the continuous-space dynamics shown in Fig. (1.1.1).

1.1.2 The effect of space-discretisation on Fourier space and dispersion relation

Another important consequence of spatial discretisation arises in the Fourier space (or momentum space, or wave-numbers) k , which describes the accessible wavelengths λ in the system via $\lambda = 2\pi/k$. For a non-periodic continuous-space $x \in \mathbb{R}$, the Fourier transform between physical and Fourier space is,

$$\tilde{\psi}(k, t) = \int_{-\infty}^{+\infty} \psi(x, t) e^{-ikx} dx \quad (1.1.19)$$

where $\tilde{\psi}(k, t)$ is the continuous wave field in Fourier space, requiring $k \in \mathbb{R}$. As such all possible wavelengths $\lambda = 2\pi/k$ are accessible.

Alternatively, for an infinitely large discrete-space $x_n = n\delta x$ for $n \in \mathbb{Z}$ the Fourier

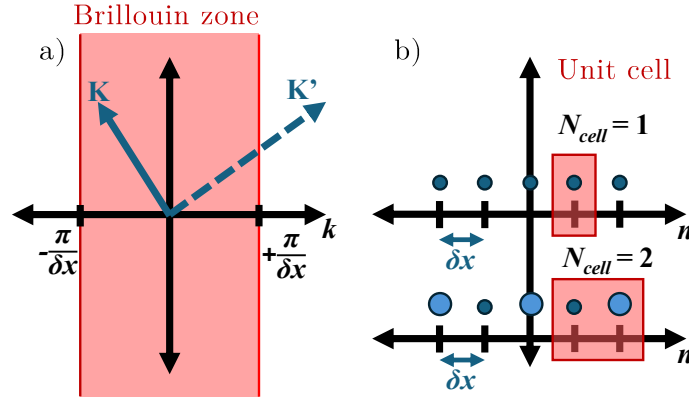


Figure 1.1.3: a) Here we show a wave number K' that exists outside of the Brillouin-zone being mapped back into the interval at K . b) We illustrate the unit cell of a chain of identical atoms $N_{cell} = 1$, and when the atoms alternate $N_{cell} = 2$.

series is,

$$\tilde{\psi}_k(t) = \sum_{n=-\infty}^{+\infty} \psi_n(t) e^{-ikn\delta x}. \quad (1.1.20)$$

In this case, the Fourier space k remains continuous, but because the underlying real-space has spacing δx , the exponential factor in equation (1.1.20) is invariant under shifts,

$$k \rightarrow k + \frac{2\pi q}{\delta x}, \quad q \in \mathbb{Z}. \quad (1.1.21)$$

Therefore, the entirety of Fourier space is defined on the finite continuous interval $k \in [-\pi/\delta x, \pi/\delta x]$ with wave-numbers outside this interval being folded back, introducing periodicity into k . No longer are all possible wavelengths accessible.

This folding is denoted by $k \bmod 2\pi/\delta x$. The finite interval for k in discrete-space is analogous to the first Brillouin-zone, and the folding procedure of wave numbers is analogous to Umklapp-processes first described in crystal lattice structures [57] (see Fig. (1.1.3) panel a)), terminology which will be used later in this thesis. We note that in some conventions the factor of δx is absorbed into k , such that $k \rightarrow \delta x k$ and k is now $k \bmod 2\pi$. This convention will be used later on in chapters (3) and (4) to simplify the notation in the discrete-space expressions.

Slightly different effects also emerge in the Fourier space if the discrete-space is

finite sized with periodic boundary conditions $\psi_{n-N/2} = \psi_{n+N/2}$. In this case we write,

$$\begin{aligned}\psi_{n-\frac{N}{2}}(t)e^{-ik(n-\frac{N}{2})} &= \psi_{n+\frac{N}{2}}(t)e^{-ik(n+\frac{N}{2})}, \\ e^{+\frac{ikN}{2}} &= e^{-\frac{ikN}{2}}, \\ e^{ikN} &= 1,\end{aligned}$$

using the new convention that $k \rightarrow k\delta x$. From the property that $e^{i2\pi q} = 1$ for $q \in \mathbb{Z}$ we find that k is no longer continuous but is also discrete,

$$k = \frac{2\pi}{N}q, \quad (1.1.22)$$

restricted to the interval $k \in [-\pi, \pi - 2\pi/N]$.

We next consider the effect on the dispersive wave property of the system, typically given by the linear dispersion relation $\omega(k)$, by comparing the dispersion relations for the NLS and DNLS in equations (1.1.8) and (1.1.14). The dispersion relation is an important quantity in any wave system relating the wave number k to the temporal frequency ω . This quantity encapsulates how waves evolve in space and time, distinguishing different wave equations from one another.

For the NLS, we obtain the dispersion relation by considering a plane wave solution in continuous-space,

$$\psi(x, t) = Ae^{i[kx - \omega(k)t]} \quad (1.1.23)$$

where A is the complex amplitude of the wave. We then substitute this into equation (1.1.8) for the linear case when $\nu = 0$, to obtain $\omega(k)$:

$$\begin{aligned}i\frac{\partial}{\partial t} [Ae^{i[kx - \omega t]}] &= -\frac{\partial^2}{\partial x^2} [Ae^{i[kx - \omega t]}], \\ \omega(k) &= k^2.\end{aligned} \quad (1.1.24)$$

Equivalently for the DNLS, we obtain the dispersion relation ω_k by considering a plane wave solution in discrete-space,

$$\psi_n(t) = Ae^{i[kx_n - \omega_k t]} = Ae^{i[kn\delta x - \omega_k t]} \quad (1.1.25)$$

substitute into the DNLS (1.1.14) for the linear case, $\nu = 0$:

$$\begin{aligned} i \frac{d}{dt} \left[Ae^{i[kx_n - \omega_k t]} \right] &= c \left[2 - e^{ik\delta x} - e^{-ik\delta x} \right] Ae^{i[kx_n - \omega_k t]}, \\ \omega_k &= 2c [1 - \cos(k\delta x_n)] = 4c \sin^2 \left(\frac{k\delta x}{2} \right), \end{aligned} \quad (1.1.26)$$

for which ω_k is periodic in the Brillouin-zone. Fig. (1.1.4) plots both dispersion relations to show the difference in dispersion relation between continuous and discrete space. As can be seen, at small k (large λ) the dispersion relations coincide well, because a long wavelength wave in discrete space varies slowly in x_n and as such is largely insensitive to the discretisation effects. This is evident when Taylor expanding equation (1.1.26) for small k in which we recover $\omega(k)$,

$$4c \sin^2 \left(\frac{k\delta x}{2} \right) \approx 4c \left(\frac{k\delta x}{2} \right)^2 + \mathcal{O}(k^4) \approx 4c \frac{k^2 \delta x^2}{4} = k^2,$$

noting that $c = |1/\delta x|^2$ and $\delta x > 0$. For large k (small λ) we see that for the continuous space, $\omega(k)$ continues to grow quadratically in k while the discrete-space remains bounded as k approaches the grid limit $\pi/\delta x$. As consequence, large k waves oscillate slower in time for discrete-space systems compared to their continuous-space analogues.

1.1.3 Bloch's theorem

To close off the section on spatial-discretisation, we consider a related concept originating from condensed matter physics: Bloch's theorem [58]. This theorem states that for a spatially periodic system (e.g. discrete-space lattices or finite sized continuous-space systems with periodic boundary conditions), the linear solutions can be expressed generically as a plane wave modulated by a periodic

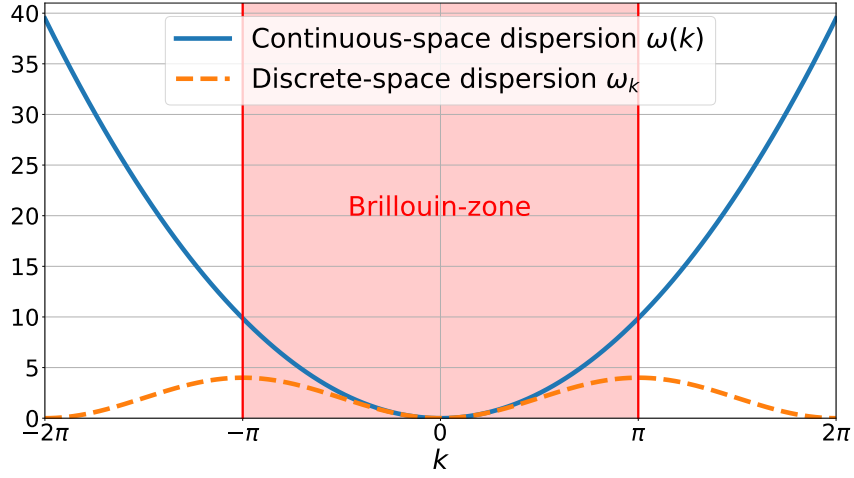


Figure 1.1.4: Comparison of the linear dispersion relation in discrete and continuous-space for the NLS and DNLS, with the Brillouin-zone sketched. Here $\delta x = 1$ and $c = 1$.

function. For example in a discrete-space lattice this would be written as,

$$\psi_n = U_n e^{ikn\delta x} \quad (1.1.27)$$

The function U_n is periodic in the unit cell,

$$U_{n+N_{cell}} = U_n \quad (1.1.28)$$

where $N_{cell} \in \mathbb{Z}$ is the number of sites in the unit cell. The unit cell is the smallest repeated structure of the lattice in space. For example consider a chain of atoms described by masses locally coupled with springs, with the masses representing the atoms and the springs the atomic bonds. If all the masses are identical then the unit cell is given by a single spatial shift and $N_{cell} = 1$ (this is the unit cell of the DNLS). If, however, the chain alternates between two different atoms, then the unit cell is given by $N_{cell} = 2$, see Fig. (1.1.3) panel b) for a comparison of the two different unit cells.

The main idea in Bloch's theorem is that by including U_n we can restrict the degrees of freedom to the solutions defined in the first Brillouin-zone, simplifying

the problem. In many cases, by solving the eigenvalue problem in U_n yields a closed-form dispersion relation that is otherwise intractable when attempted in ψ_n .

To demonstrate this, let us consider the DNLS in the linear case when $\nu = 0$:

$$i \frac{d\psi_n(t)}{dt} = c(2\psi_n - \psi_{n+1} - \psi_{n-1}). \quad (1.1.29)$$

Here let us assume harmonic time evolution such that,

$$\psi_n(t) = \phi_n e^{-i\omega t}, \quad (1.1.30)$$

but make no assumption about the spatial profile ϕ_n . Substituting into equation (1.1.29),

$$\omega \phi_n(t) = c(2\phi_n - \phi_{n+1} - \phi_{n-1}) \quad (1.1.31)$$

For a system of N lattice sites we have,

$$\Phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{N-1} \end{pmatrix} \quad (1.1.32)$$

and as such this gives an $N \times N$ matrix problem,

$$\omega \Phi = \mathbf{M} \Phi. \quad (1.1.33)$$

Assuming periodic boundary conditions $\mathbf{M}$ is,

$$\mathbf{M} = \begin{bmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & -1 \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 2 \end{bmatrix}. \quad (1.1.34)$$

Diagonalising $\mathbf{M}$ is non-trivial for large N (or $N \rightarrow \infty$). Instead invoking Bloch's theorem using equation (1.1.27),

$$\phi_n = U_n e^{ikn\delta x} \quad (1.1.35)$$

where $N_{cell} = 1$ as we have a single site unit cell (see Fig. (1.1.3) panel b). Therefore upon substitution,

$$\omega U_n e^{ikn\delta x} e^{-i\omega t} = c(2 - e^{ik\delta x} - e^{-ik\delta x}) U_n e^{ikn\delta x} e^{-i\omega t} \quad (1.1.36)$$

Now we find that we can cancel most of the terms obtaining the dispersion relation in equation (1.1.26) irrespective of the size of N . In fact we note here that Bloch's theorem in this case gives the plane wave solution $\psi_n(t) = A e^{i(kn\delta x - \omega t)}$, with $U_n = A$ because the unit cell of this lattice is given by a single spatial site ($N_{cell} = 1$) and is the simplest lattice structure we can have.

This example highlights the strength of Bloch's theorem. Independent of the size of N , the eigenvalue problem reduces to solving the behaviour inside a single unit cell, with all the wave numbers folded into the first Brillouin zone. Moreover, this theorem is general to linear lattice models, applicable to more complex unit cells that contain multiple lattice sites. This theorem will be used later in this thesis when studying the discrete space-time lattice in chapter (3).

1.2 Time discretisation

1.2.1 The effect on energy conservation

Much like spatial discretisation destroys momentum conservation, we will now show that time discretisation destroys energy conservation. To illustrate this effect of time discretisation we first consider the example of a simple harmonic oscillator in continuous-time. The Hamiltonian, which is a real quantity

representing the total energy is,

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kx^2, \quad (1.2.1)$$

where m is the mass of the oscillator and k is the spring constant, $x(t)$ is displacement and $p(t)$ is momentum, t denotes continuous-time. For simplicity we set $m = k = 1$, giving the Hamiltonian,

$$H(x, p) = \frac{p^2}{2} + \frac{x^2}{2}. \quad (1.2.2)$$

We know that from Hamilton's equations we have,

$$\frac{\partial H}{\partial q} = -\frac{dp}{dt}, \quad (1.2.3)$$

$$\frac{\partial H}{\partial p} = \frac{dq}{dt}, \quad (1.2.4)$$

where $q = x(t)$, $p = p(t)$ are the canonical coordinates. Upon substituting equation (1.2.1) in Hamilton's equations,

$$-\frac{dp}{dt} = x, \quad (1.2.5)$$

$$\frac{dx}{dt} = p. \quad (1.2.6)$$

Through equations (1.2.5) (1.2.6) we obtain standard simple harmonic motion,

$$\frac{d^2x}{dt^2} + x = 0. \quad (1.2.7)$$

With initial data $x(0) = x_0$, $p(0) = p_0$ the exact solutions are,

$$x(t) = x_0 \cos(t) + p_0 \sin(t), \quad (1.2.8)$$

$$p(t) = p_0 \cos(t) - x_0 \sin(t). \quad (1.2.9)$$

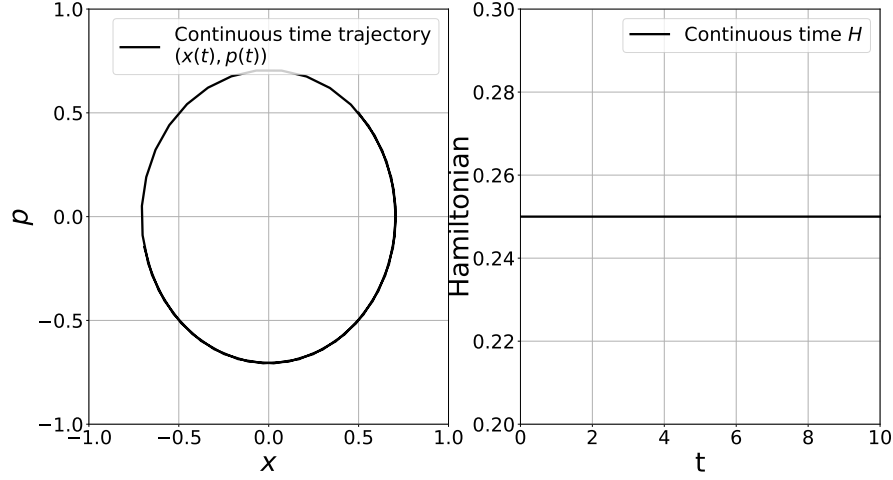


Figure 1.2.1: The left panel shows the $(x(t), p(t))$ phase space trajectory of the continuous-time harmonic oscillator and the right shows the Hamiltonian evolution. Initial conditions is $(x_0, p_0) = (0.5, 0.5)$.

Substituting these expressions of $x(t)$ and $p(t)$ into equation (1.2.2), we find that the time dependence vanishes,

$$H(x_0, p_0) = H = \frac{1}{2} [(x_0 \cos(t) + p_0 \sin(t))^2 + (p_0 \cos(t) - x_0 \sin(t))^2] = \frac{p_0^2 + x_0^2}{2} \quad (1.2.10)$$

So H , the total energy, is time invariant and therefore conserved. In the $(x(t), p(t))$ phase space, the trajectory is bound to a circle of constant radius $R = \sqrt{p_0^2 + x_0^2} = \sqrt{2H}$, defined by the value of the Hamiltonian, see Fig.(1.2.1). Each trajectory defined by equation (1.2.10) corresponds to a unique energy, determined by the initial condition. Any deviation from this closed circular orbit would correspond to a change in energy, and an open orbit corresponds to energy blow-up or decay [59].

To introduce the discrete-time evolution, equations (1.2.5) and (1.2.6) can be discretised using a forward difference to approximate the time derivatives,

$$-\left(\frac{p^{m+1} - p^m}{\delta t}\right) = x^m, \quad (1.2.11)$$

$$\frac{x^{m+1} - x^m}{\delta t} = p^m. \quad (1.2.12)$$

$\delta t > 0$ is the size of the discrete-time step and m indexes the number steps in time. Taking the limit when $\delta t \rightarrow 0$ recovers the derivatives in continuous-time, but for finite δt we obtain the linear update rule given by the recursive map,

$$\begin{pmatrix} x^{m+1} \\ p^{m+1} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & \delta t \\ -\delta t & 1 \end{bmatrix}}_{M_{FE}} \begin{pmatrix} x^m \\ p^m \end{pmatrix}. \quad (1.2.13)$$

where M_{FE} is the updating matrix for the finite element (FE) method. Applying the same reasoning then, the the Hamiltonian in discrete-time is simply,

$$H_{FE}^m = \frac{1}{2} [(p^m)^2 + (x^m)^2] \quad (1.2.14)$$

Choosing the same initial data as before $x^0 = x_0$, $p^0 = p_0$, we find that at step $m = 0$ the energies are the same in both continuous and discrete-time.

$$H_{FE}^0 = \frac{p_0^2 + x_0^2}{2}. \quad (1.2.15)$$

However, evolving the system using equation (1.2.13), shows that the energy grows with each step. After one iteration we find,

$$\begin{aligned} H_{FE}^{m+1} &= \frac{1}{2} [(-\delta t x_0 + p_0)^2 + (x_0 + \delta t p_0)^2], \\ &= \frac{\delta t^2 x_0^2 - 2\delta t x_0 p_0 + p_0^2 + x_0^2 + 2\delta t x_0 p_0 + \delta t^2 p_0^2}{2}, \\ &= \frac{(1 + \delta t^2)(x_0^2 + p_0^2)}{2}. \end{aligned} \quad (1.2.16)$$

Thus the energy increases by a multiplicative factor of $(1 + \delta t^2)$ at each step. Consequently, the discrete-time trajectory in (x^m, p^m) phase-space spirals outwards, reflecting an unbounded energy growth (see Fig. (1.2.2)).

The loss of energy conservation in Forward Euler can also be understood through the stability analysis of the update matrix M_{FE} . The eigenvalues $\lambda^\pm$ of the equation (1.2.13) are:

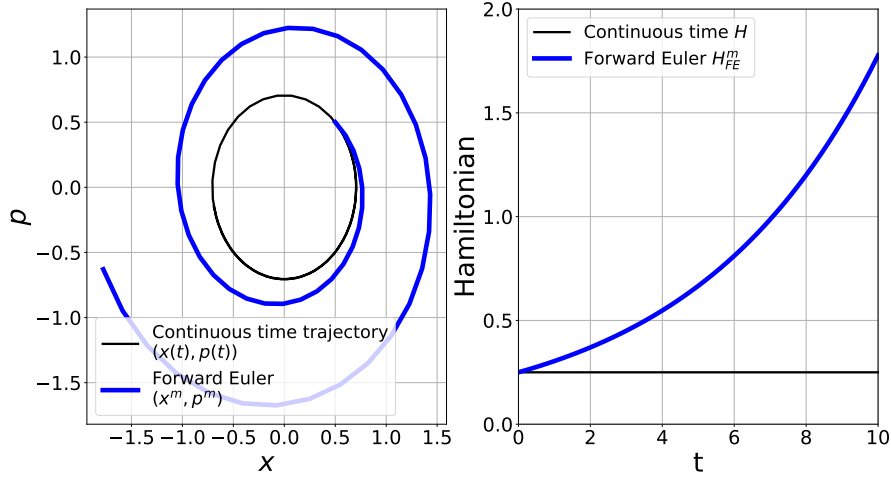


Figure 1.2.2: The left panel plots the (q^m, p^m) phase space trajectory of the Forward Euler discretised harmonic oscillator and the right panel the corresponding Hamiltonian in blue. Initial conditions is $(x_0, p_0) = (0.5, 0.5)$ and $\delta t = 0.2$. The continuous time evolution is plotted for comparison.

$$\begin{aligned}
 \det(M_{FE} - I\lambda) &= 0, \\
 (1 - \lambda)(1 - \lambda) - \delta t \cdot (-\delta t) &= 0, \\
 (1 - \lambda)^2 &= -(\delta t)^2, \\
 \lambda^\pm &= 1 \mp i\delta t.
 \end{aligned} \tag{1.2.17}$$

Notably, the magnitude $|\lambda^\pm| > 1$ for any $\delta t \neq 0$, ensuring that the dynamics are given by a spiral source, according to the standard tools of stability theory in discrete dynamical systems [60]. This corresponds precisely to the monotonically increasing energy H_{FE}^m in equation (1.2.16), reflecting that Forward Euler does not preserve the qualitative geometry of the phase space trajectory of the Hamiltonian system.

By naively discretising time using Forward Euler, we lose the exact conservation of the Hamiltonian, causing unbounded energy growth. To better preserve the qualitative features of the dynamics, it is necessary to consider discretisation methods that leave the underlying geometric structure of phase space largely unchanged. Such methods are known as symplectic integrators [61].

A symplectic integrator is one that exactly conserves the symplectic 2-form $dq \wedge dp$, or equivalently, approximately preserves the area enclosed by the trajectory in phase space by altering the trajectory to a closed contour nearby in (x, p) phase space. By preserving this area, the method ensures that the dynamics remain bounded and qualitatively similar to the continuous time system. Importantly, this property guarantees the existence of a modified version of the Hamiltonian $\tilde{H}$ which is exactly conserved under discrete-time evolution.

To address the instability and energy blow up of the Forward Euler method, we now consider an alternative scheme to discretise time for the simple harmonic oscillator, sometimes referred to as symplectic Euler or semi-implicit Euler. Time is now discretised via,

$$\frac{-p^{m+1} + p^m}{\delta t} = x^m, \quad (1.2.18)$$

$$\frac{x^{m+1} - x^m}{\delta t} = p^{m+1}. \quad (1.2.19)$$

The key change in this method is the indexing of p^{m+1} in equation (1.2.19). The asymmetry in indexing allows us to first solve the momentum map which is then used to update the space map x^{m+1} ,

$$p^{m+1} = p^m - \delta t x^m, \quad (1.2.20)$$

$$x^{m+1} = \delta t p^{m+1} + x^m = \delta t(p^m - \delta t x^m) + x^m. \quad (1.2.21)$$

Equations (1.2.20) (1.2.21) can be written as the following linear update rule,

$$\begin{pmatrix} x^{m+1} \\ p^{m+1} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 - \delta t^2 & \delta t \\ -\delta t & 1 \end{bmatrix}}_{M_{SE}} \begin{pmatrix} x^m \\ p^m \end{pmatrix} \quad (1.2.22)$$

where M_{SE} is the updating matrix for the Symplectic Euler (SE) method. Using

the symplectic Euler update rule, we write the Hamiltonian at $m + 1$ as,

$$H_{SE}^{m+1} = \frac{1}{2} [(p^m - \delta x^m)^2 + (\delta t(\delta p^m - \delta x^m) + x^m)^2]. \quad (1.2.23)$$

To establish the criteria of stability in the symplectic Euler method, we consider the eigenvalues $\lambda^\pm$ of equation (1.2.22),

$$\begin{aligned} \det(M_{SE} - I\lambda) &= 0, \\ (1 - \delta t^2)(1 - \lambda) - \delta t(-\delta t) &= 0, \\ \lambda^2 + (\delta t^2 - 2)\lambda + 1 &= 0, \\ \lambda^\pm &= \frac{2 - \delta t^2 \pm \sqrt{(\delta t^2 - 2)^2 - 4}}{2}. \end{aligned}$$

For convenience let us define $D = (\delta t^2 - 2)^2 - 4$ and write the eigenvalues in complex form,

$$\lambda^\pm = \frac{2 - \delta t^2 \pm i\sqrt{-D}}{2}.$$

In order for $\lambda^\pm$ to be complex, we require $D < 0$. This bounds δt to ensure stable dynamics,

$$\begin{aligned} (\delta t^2 - 2)^2 - 4 &< 0 \\ (\delta t^2 - 2) &< 2 \Rightarrow |\delta t| < 2 \end{aligned}$$

Therefore, provided that $|\delta t| < 2$ the eigenvalues are complex numbers, with magnitude 1:

$$\begin{aligned} |\lambda^\pm|^2 &= \frac{(2 - \delta t^2)^2 - D}{4} = \frac{(2 - \delta t^2)^2 - (\delta t^2 - 2)^2 + 4}{4} \\ &= \frac{4 - 4\delta t^2 + \delta t^4 - \delta t^4 + 4\delta t^2 - 4 + 4}{4} = 1 \end{aligned} \quad (1.2.24)$$

This criteria ensures a closed contour in (x^m, p^m) phase space [60]. In Fig. (1.2.3) it is clear that while this contour is closed, given by a tilted ellipse, it does not conserve H_{SE}^m exactly, and instead H_{SE}^m evolves according to bounded periodic

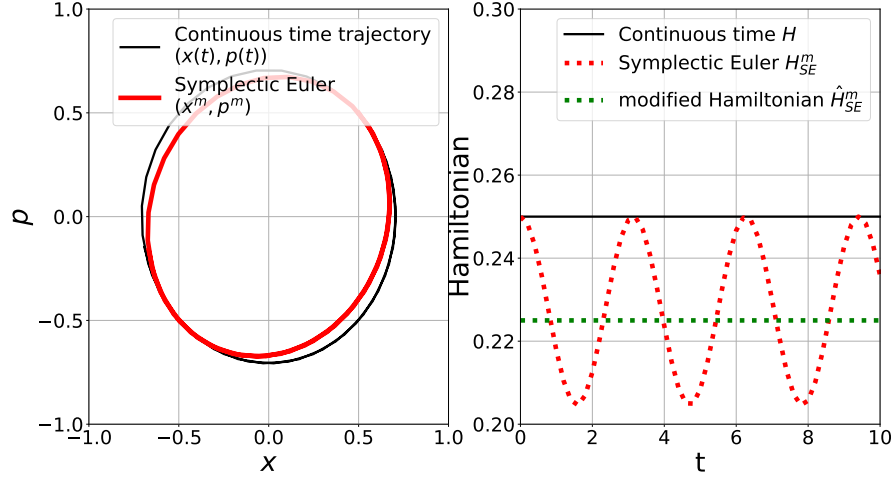


Figure 1.2.3: The left panel shows the (x^m, p^m) phase space for the symplectic Euler method, with the continuous time trajectory for comparison. The right panel shows the evolution of H_{SE}^m in red and the modified Hamiltonian $\hat{H}_{SE}^m$ in green. Initial conditions is $(x_0, p_0) = (0.5, 0.5)$ and $\delta t = 0.2$.

motion.

Instead, this discrete-time evolution exactly conserves a new quantity $\hat{H}_{SE}^m$:

$$\hat{H}_{SE}^m(x^m, p^m) = \frac{1}{2} \left((x^m)^2 - \delta t x^m p^m + (p^m)^2 \right), \quad (1.2.25)$$

demonstrated in Fig. (1.2.3), and is derived in appendix (A.2). Equation (1.2.25) describes the tilted ellipse observed in the phase-space trajectory, that possess approximately the same area as the circle defined by the continuous-time Hamiltonian in equation (1.2.10). Since equation (1.2.25) possesses a cross term $(-\delta t x^m p^m)$, the constant value of the modified Hamiltonian for initial data $(x^0, p^0) = (x_0, p_0)$ differs slightly,

$$\hat{H}(x_0, p_0) = \hat{H} = \frac{1}{2} \left[x_0^2 - \delta t x_0 p_0 + p_0^2 \right], \quad (1.2.26)$$

which reduces to the continuous-time Hamiltonian in equation (1.2.10) when $\delta t \rightarrow 0$. Therefore, smaller δt yields better approximations of the continuous-time energy.

We can show that the symplectic Euler method is a good approximation of the continuous-time dynamics by comparing the areas enclosed by the phase-space orbits. We first compute the area of the circle corresponding to continuous-time evolution,

$$A = \pi R^2 = \pi \left(\sqrt{p_0^2 + x_0^2} \right)^2 = 2\pi H \quad (1.2.27)$$

For the symplectic Euler discretisation, the area of the tilted elliptical contour is,

$$A_{SE} = \pi ab = \pi \sqrt{\frac{2\hat{H}}{(1 - \frac{\delta t}{2})}} \sqrt{\frac{2\hat{H}}{(1 + \frac{\delta t}{2})}} = 2\pi \frac{\hat{H}}{\sqrt{1 - (\frac{\delta t}{2})^2}}, \quad (1.2.28)$$

obtained in appendix (A.2). For small δt , $A_{SE} \approx A$, demonstrating that the symplectic Euler method preserves the phase- well providing a good approximation of the continuous-time dynamics.

This analysis shows that time discretisation alters the conservation of the Hamiltonian, potentially causing unstable dynamics. It also emphasises that the method of discretisation requires careful consideration as to not drastically change the system. We have shown that by considering methods of symplectic integration, we best preserve the qualitative features of the dynamics and can formulate a new modified conserved quantity consistent with the discrete-time evolution. These ideas will be of relevance in later discussion of the discrete space-time optical lattice model of section (3).

1.2.2 Floquet theory

To finish the section on time discretisation, we will consider Floquet theory [62]. This theory is typically used when linear systems are time-periodic, stating that for first order linear systems with periodic coefficients $A(t + T) = A(t)$,

$$\frac{dx(t)}{dt} = A(t)x(t) \quad (1.2.29)$$

the solutions of $x(t)$ can always be decomposed into the product of an exponential and a time-periodic function,

$$x(t) = e^{\mu t} p(t), \quad (1.2.30)$$

where $p(t+T) = p(t)$ is the periodic function. $\mu \in \mathbb{C}$ is called the Floquet exponent and characterises the growth/decay of the system in time. Analogous to Bloch's theorem, Floquet theory allows one to capture the full system dynamics across all time by focusing on a single period T , greatly simplifying the analysis of time-periodic linear systems. As a result, Floquet theory has been frequently used in the study of dynamical systems such as Hill's equation [63], and it has found important applications in quantum mechanics for time-periodic Hamiltonians [64] and the emergent field of Floquet engineering [65, 66].

Floquet theory is not exclusive to continuous-time systems and can also be applied to discrete-time evolution. For example, consider the following iterative map:

$$X^{m+1} = Q^m X^m \quad (1.2.31)$$

where X^m is a column vector of length N and Q^m is some $N \times N$ matrix that iterates X^m to the next step in time. While there is no restriction on the form of Q^m at each step, many discrete-time systems of interest involve coefficients that are periodic in the iteration index m , either by definition of the map or due to an underlying periodic drive. In the former case, iterative matrix Q^m satisfies $Q^{m+M} = Q^m$ for some integer period M , allowing Floquet theory to be applied directly. As a simple illustration consider the case where Q^m has period $M = 2$, such that:

$$Q^m = \begin{cases} A & \text{if } m \text{ is even,} \\ B & \text{if } m \text{ is odd,} \end{cases} \quad (1.2.32)$$

where A and B are some arbitrary matrices. Then, assuming we start from an

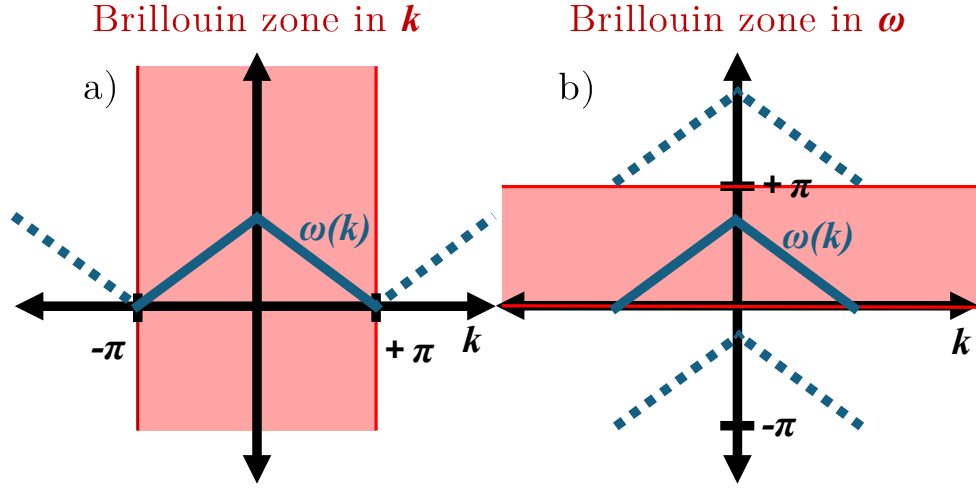


Figure 1.2.4: Here we compare the periodicity effects of a) Bloch's theorem where $k \bmod 2\pi$ and b) Floquet theory where $\omega \bmod \pi$. In either sketch, the dotted components of $\omega(k)$ represent the values that are folded back into the Brillouin-zone given by the red regions.

even step in m , the stroboscopic evolution of X^m is given by,

$$X^{m+2} = BAX^m, \quad (1.2.33)$$

where one stroboscopic step is M single time steps. Note that the order of A and B matters as they do not necessarily commute. We can then use Floquet theory to re-write X^m ,

$$X^m = P^m e^{\mu m} \quad (1.2.34)$$

choosing $P^{m+2} = P^m$ to be periodic in the stroboscopic step. Upon substitution of the Floquet ansatz into equation (1.2.33) we can form an eigenvalue problem

$$P^{m+2} e^{\mu(m+2)} = BAP^m e^{\mu m},$$

$$\underbrace{e^{2\mu}}_{\lambda} P^m = BAP^m. \quad (1.2.35)$$

Once A and B are specified, we can solve the eigenvalue problem to obtain λ and find an expression for μ :

$$\mu = \frac{1}{2} [\ln(\lambda) + i2\pi q], \quad (1.2.36)$$

for $q \in \mathbb{Z}$, where the $i2\pi q$ term reflects the periodicity of the complex exponential, $e^{2\mu+2i\pi q} = e^{2\mu}$. The Floquet exponent μ characterises the temporal behaviour of each linear normal mode under the stroboscopic evolution, with $Re[\mu]$ governing growth or decay and $Im[\mu]$ the oscillatory response. Dynamical stability requires $|\lambda| = 1$ [60], implying $Re[\mu] = 0$ and hence $\mu = i\omega$, which serves as the linear dispersion relation of the discrete-time map in the stroboscopic step:

$$\omega = -\frac{i}{2} \ln(\lambda) + q\pi. \quad (1.2.37)$$

This demonstrates a property unique to Floquet dynamics - the dispersion relation ω is periodic (in this case mod π), introducing a Brillouin-zone effect in the ω direction of the (k, ω) space. This periodicity of ω is sometimes referred to as the quasi-energy spectrum [64], emphasising its analogy with the energy bands of spatially periodic systems, see Fig.(1.2.4) for a comparison sketch.

As will be demonstrated in chapter (3) when discussing the discrete space-time photonic mesh lattice, the periodicity in the dispersion relation introduced by discrete-time plays a non-trivial role on the thermalisation processes, and are essential to understanding the full thermodynamic processes among the linear normal modes of the lattice.

2

Weak wave turbulence theory in discrete space

In this next chapter, we will now give an introduction to the theory of weak wave turbulence (WWT). All of the material in this chapter is in some sense based on the following texts [1, 2, 3], and the derivation of the wave-kinetic equation (WKE) for the discrete nonlinear schrödinger equation (DNLS) published by Onorato and Dematteis in 2020 [42]. Therefore, none of the material in this chapter is new and is mainly a summary of what is already known.

A natural starting point for WWT is the Hamiltonian description of weakly nonlinear wave systems. Through this description we shall introduce the different types of wave interactions that can be present in nonlinear waves systems and more formally explain what is meant by weak in WWT theory.

2.1 The Hamiltonian description of weakly nonlinear wave systems

To begin let us consider a generic Hamiltonian system defined on a discrete real space lattice containing $N \in \mathbb{N}$ sites. Let us then say that the N normal modes in the discrete Fourier space k , of this lattice are given by $a_k(t) \in \mathbb{C}$. Now we

write Hamilton's equations in terms of the normal modes,

$$i \frac{da_k}{dt} = \frac{\partial H}{\partial a_k^*}, \quad (2.1.1)$$

as this is the most natural setting for WWT theory often called the interaction representation. Here (a_k, a_k^*) are the canonical coordinates with the $*$ representing the complex conjugate. We now make our first assumption: that the amplitude of these normal modes, given by $|a_k|$, is small such that the system is weakly interacting. In this case, the Hamiltonian can be expanded as a hierarchy of wave interactions typically written as:

$$H = \sum_{j=2}^{\infty} \mathcal{H}_j = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \dots \quad (2.1.2)$$

where the subscript j represents the number of waves involved in each interaction. Through the assumption that the mode amplitudes are small, higher order terms that involve higher products of mode amplitudes become progressively smaller in magnitude. As a result, each term in the hierarchy is expected to govern wave dynamics on progressively longer timescales, with lower-order interactions dominating at shorter timescales. This separation of timescales underpins the asymptotic structure of WWT; this will be demonstrated in the WKE derivation in the DNLS shortly. Note that equation (2.1.2) starts from $j = 2$ as the zeroth order term corresponds to a constant in the Hamiltonian and therefore not present in the derived wave equation. The first order term is linear in a_k making it a constant in the equation of motion playing no role in the wave interactions, and is often omitted in the WWT setting.

The first term in the expansion is $\mathcal{H}_2$:

$$\mathcal{H}_2 = \sum_{k'k_1} \omega_{k'} a_{k'}^* a_{k_1} \delta_{k'}^{k_1}, \quad (2.1.3)$$

describes the linear wave evolution. Here $\omega_{k'}$ is the dispersion relation which is specific to the type of wave equation as discussed in chapter (1). This wave

interaction involves two waves, an incoming wave given by $a_{k'}^*$ and an outgoing wave given by a_{k_1} . The Kronecker delta $\delta_{k'}^{k_1} = \delta_{k',k_1}$ enforces conservation of wave number selecting k' to be k_1 , not allowing for different modes to interact. As such equation (2.1.3) simplifies:

$$\mathcal{H}_2 = \sum_1 \omega_1 a_1^* a_1, \quad (2.1.4)$$

for which we have introduced a simplified notation in the sum $\sum_{k_1} \rightarrow \sum_1$ and in the indexing of $(\omega_{k_1}, a_{k_1}^*, a_{k_1}) \rightarrow (\omega_1, a_1^*, a_1)$ for simplification. Using equation (2.1.1), $\mathcal{H}_2$'s contribution to the equation of motion is therefore,

$$i \frac{da_1}{dt} = \omega_1 a_1. \quad (2.1.5)$$

This term governing the linear wave interactions is sketched as a Feynman diagram in panel a) of Fig. (2.1.1).

The next term $\mathcal{H}_3$ is the first nonlinear term in the equation of motion involving three waves:

$$\mathcal{H}_3 = \sum_{123} \frac{1}{2} (U_1^{23} a_1^* a_2 a_3 + c.c.) \delta_1^{23} + \sum_{123} \frac{1}{3} (U_{123} a_1^* a_2^* a_3^* + c.c.) \delta_{123}. \quad (2.1.6)$$

$U_1^{23}, U_{123} \in \mathbb{C}$ are interaction coefficients associated to different three wave interactions. In this notation, the subscripts denote the incoming wave numbers and superscripts outgoing wave numbers. Further, we have adopted the reduced notation for the Kronecker delta as well e.g. $\delta_{k_1}^{k_2 k_3} = \delta_{k_1, k_2 + k_3} \rightarrow \delta_1^{23}$ for simplicity, alongside *c.c.* to represent the complex conjugate of each respective term. Equation (2.1.6) introduces new types of wave interactions that enable wave mixing: A decay and confluence given by terms with coefficients U_1^{23} and $(U_1^{23})^*$ respectively, panels b) and c) in Fig. (2.1.1), and a creation and annihilation wave processes given by U_{123} and (U_{123}^*) , panels d) and e) in Fig. (2.1.1). The complex conjugate here inverts the wave process by swapping the order of incoming and outgoing waves. In $\mathcal{H}_3$ we also include factors of 1/2 and

1/3 to account for permutation symmetry in the dummy indices of the sum. For example the interaction $U_1^{23}a_1^*a_2a_3$ is equivalent to $U_1^{32}a_1^*a_3a_2$, so to avoid double counting we multiply by 1/2. These are given by the following symmetries in the interaction coefficients under index permutations:

$$U_1^{23} = U_1^{32}$$

$$U_{123} = U_{213} = U_{231}$$

resulting in the following quadratic contributions to the equation of motion:

$$i\frac{da_1}{dt} = \frac{1}{2} \sum_{23} U_1^{23} a_2 a_3 \delta_1^{23} + \sum_{23} (U_1^{13})^* a_2 a_3^* \delta_{13}^2 + \sum_{23} U_{123} a_2^* a_3^* \delta_{123} \quad (2.1.7)$$

$\mathcal{H}_4$ is then the second nonlinear term in the equation of motion entering in the form of cubic nonlinearity. In general:

$$\begin{aligned} \mathcal{H}_4 = & \frac{1}{3} \sum_{1234} (U_1^{234} a_1^* a_2 a_3 a_4 + c.c.) \delta_1^{234} + \frac{1}{4} \sum_{1234} (U_{1234} a_1^* a_2^* a_3^* a_4^* + c.c.) \delta_{1234} \\ & + \frac{1}{4} \sum_{1234} (U_{12}^{34} a_1^* a_2^* a_3 a_4 + c.c.) \delta_{12}^{34} \end{aligned} \quad (2.1.8)$$

Terms with coefficients U_1^{234} , $(U_1^{234})^*$, U_{1234} , $(U_{1234})^*$ are the four wave equivalents of decay/confluence and annihilation/creation respectively. $\mathcal{H}_4$ now introduces a new type of wave interaction referred to as scattering given by U_{12}^{34} and $(U_{12}^{34})^*$ sketched in panel f) in Fig. (2.1.1). This type of wave interaction is the exact form of cubic nonlinearity in the DNLS equation of motion. The interaction coefficients have the following symmetries:

$$U_1^{234} = U_1^{324} = U_1^{342}$$

$$U_{1234} = U_{2134} = U_{2314} = U_{2341}$$

$$U_{12}^{34} = U_{21}^{34} = (U_{34}^{12})^*$$

and thus $\mathcal{H}_4$ yields the following contributions to the equation of motion,

$$\begin{aligned} i\frac{\partial a_1}{\partial t} = & \frac{1}{3} \sum_{234} U_1^{234} a_2 a_3 a_4 \delta_1^{234} + \sum_{234} (U_1^{234})^* a_2^* a_3^* a_4 \delta_{123}^4 \\ & + \sum_{234} U_{1234} a_2^* a_3^* a_4^* \delta_{1234} + \sum_{234} U_{12}^{34} a_2^* a_3^* a_4 \delta_{12}^{34}. \end{aligned} \quad (2.1.9)$$

We are free to continue this process of expanding higher order terms, each time involving more complicated expressions. By choosing the nonlinear wave interactions to be weak, we enforce a separation of time scales between successive interaction processes, making the hierarchical representation of the Hamiltonian natural. This hierarchy also demonstrates the enormous class of wave systems that WWT describes by separating out each order of wave interaction. The choice of a particular wave system is therefore entirely determined by the ω_k and the scattering coefficients U_a^{bc} , U_{ab}^{cd} , etc...

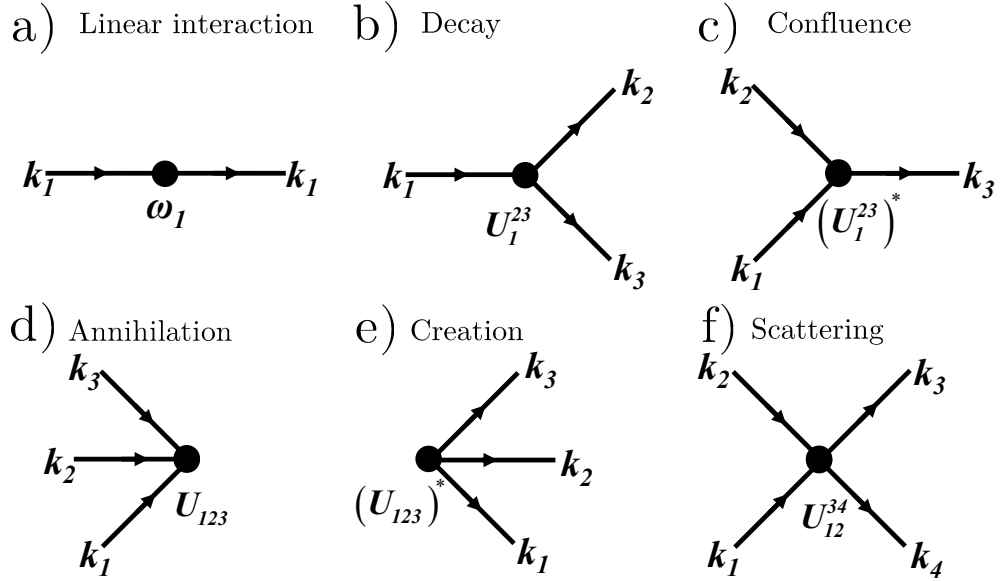


Figure 2.1.1: Feynman diagrams of the different types of wave interactions emerging in the first three terms $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$. Here 'decay' is referred to as a one-to-two wave process and a 'confluence' as a two-to-one wave process, following standard weak wave turbulence terminology. This naming is independent of the creation/annihilation operator analogy used in quantum field theory.

2.1.1 DNLS Hamiltonian and equation of motion

Of particular interest to this chapter is the DNLS equation that will be the setting in which we derive the WKE later on, an essential statistical tool in WWT. To write the DNLS in the normal mode form, we start from the Hamiltonian in real space:

$$H = \sum_{n=1}^N |\psi_{n+1} - \psi_n|^2 + \frac{\nu}{2} |\psi_n|^4. \quad (2.1.10)$$

Previously H was written in equation (1.1.12), with n starting from $-N/2$ to $N/2 - 1$ and with a linear coupling constant c . Here we shift n and omit c to match the form of the Hamiltonian in [42] - which is the derivation this chapter will recount. This particular discrete-space variant is favourable for study since the nonlinearity acts locally in space, making it representative of a variety of physical lattice systems [67]. In contrast, the Ablowitz–Ladik (AL) equation in (1.1.15) involves non-local spatial coupling in its nonlinear term, and is therefore less directly used in physical modelling.

To move equation (2.1.10) into the normal mode representation, we simply apply the discrete Fourier transform,

$$\psi_n = \sum_{k=1}^N \hat{\psi}_k e^{+i \frac{2\pi k n}{N}}, \quad \hat{\psi}_k = \sum_{n=1}^N \psi_n e^{-i \frac{2\pi k n}{N}}, \quad (2.1.11)$$

where k is the index in Fourier space. Hence we obtain:

$$\begin{aligned} H &= N \sum_1 \left(2 - e^{-i \frac{2\pi k_1}{N}} - e^{+i \frac{2\pi k_1}{N}} \right) |\hat{\psi}_1|^2 + \sum_{n=1}^N \left[\frac{\nu}{2} \sum_{1234} \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 e^{i \frac{2\pi (k_3 + k_4 - k_2 - k_1)n}{N}} \right], \\ &= N \sum_1 \omega_1 |\hat{\psi}_1|^2 + \sum_{n=1}^N \left[\frac{\nu}{2} \sum_{1234} \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 e^{i \frac{2\pi (k_3 + k_4 - k_2 - k_1)n}{N}} \right], \end{aligned}$$

where $\psi_1 = \psi_{k_1}$ and $\omega_1 = 4 \sin^2(\pi k_1/N)$ is the linear dispersion relation of the DNLS. The sum over n in the quartic term leads to N Kronecker deltas, which

we absorb into H for convenience via the rescaling $H/N \rightarrow H$:

$$\begin{aligned} H &= N \left[\sum_1 \omega_1 |\hat{\psi}_1|^2 + \frac{\nu}{2} \sum_{1234} \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \delta_{12}^{34} \right], \\ \frac{H}{N} \rightarrow H &= \sum_1 \omega_1 |\hat{\psi}_1|^2 + \frac{\nu}{2} \sum_{1234} \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \delta_{12}^{34}. \end{aligned} \quad (2.1.12)$$

It is noted that, for consistency with the Fourier transforms, if we return to physical space we must multiply through by N . In the case of the DNLS the normal modes $a_k(t)$ are simply the Fourier transformed wave fields $\hat{\psi}_k(t)$ and so equation (2.1.12) is in the typical interaction representation. Comparing to the Hamiltonian expansion above, equation (2.1.12) is comprised of $\mathcal{H}_2$ and the scattering term in $\mathcal{H}_4$ given by the interaction coefficient U_{12}^{34} .

Equivalently, we can now write the equation of motion of the DNLS in the interaction representation as:

$$i \frac{d\hat{\psi}_1}{dt} = \omega_1 \hat{\psi}_1 + \nu \sum_{234} \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \delta_{12}^{34}. \quad (2.1.13)$$

2.2 The statistical approach of weak wave turbulence and closure

We now move from a deterministic perspective of the wave dynamics to a statistical one. To begin, the assumption of weak nonlinearity is made such that the nonlinear wave dynamics must be weak in comparison to the linear wave dynamics:

$$\mathcal{H}_2 \gg \sum_{j=3}^{\infty} \mathcal{H}_j. \quad (2.2.1)$$

In this regime then the normal amplitudes $\hat{\psi}_k(t)$ are written in terms of an action angle decomposition,

$$\hat{\psi}_1(t) = \sqrt{I_1(t)} e^{-i\theta_1(t)}, \quad (2.2.2)$$

in which $I_1(t)$ is the action that evolves slowly in time due to the assumption

of weak nonlinearity and $\theta_1(t)$ is the angle that oscillates rapidly in accordance with the frequency of the linear dispersion relation ω_1 and some small nonlinear correction. As such, the fast phase dynamics do not directly determine the long time evolution of the action I_1 and therefore may be averaged out.

To perform the phase averaging, we assume the phases of the Fourier modes $\theta_k(t)$ are statistically independent and uniformly distributed on $[0, 2\pi)$. As such we define the average as,

$$\langle f(\theta_1, \theta_2, \theta_3, \dots, \theta_N) \rangle_\theta = \frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} f(\theta_1, \theta_2, \theta_3, \dots, \theta_N) d\theta_1 d\theta_2 d\theta_3, \dots d\theta_N, \quad (2.2.3)$$

where the subscript θ denotes that we are averaging over the phases. Upon taking the averages we find the following terms are zero:

$$\langle \hat{\psi}_1 \rangle_\theta = \langle \sqrt{I_1} e^{-i\theta_1} \rangle_\theta = 0, \quad (2.2.4)$$

$$\langle \hat{\psi}_1 \hat{\psi}_2 \rangle_\theta = \langle \sqrt{I_1} \sqrt{I_2} e^{-i(\theta_1 + \theta_2)} \rangle_\theta = 0, \quad (2.2.5)$$

$$\langle \hat{\psi}_1^* \hat{\psi}_2^* \rangle_\theta = \langle \sqrt{I_1} \sqrt{I_2} e^{+i(\theta_1 + \theta_2)} \rangle_\theta = 0, \quad (2.2.6)$$

noting that because equation (2.2.4) is zero then all odd correlators must also be zero. However when we have complex conjugate pairs:

$$\langle \hat{\psi}_1 \hat{\psi}_2^* \rangle_\theta = \langle \sqrt{I_1} \sqrt{I_2} e^{-i(\theta_1 - \theta_2)} \rangle_\theta = \sqrt{I_1 I_2} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} e^{-i(\theta_1 - \theta_2)} d\theta_1 d\theta_2, \quad (2.2.7)$$

there are two possible outcomes. If the modes are distinct $k_1 \neq k_2$, the phase factors are independent and the above factorises into a product of vanishing integrals. The second outcome is that the modes are not distinct $k_1 = k_2$, with the phases cancelling identically and the integrand reducing to 1. Ultimately the correlators of conjugate pairs can be written using a Kronecker delta in the mode labels:

$$\langle \hat{\psi}_1 \hat{\psi}_2^* \rangle_\theta = I_1 \delta_1^2. \quad (2.2.8)$$

In effect, equation (2.2.8) says that under the assumption of random phases, only correlations between identical Fourier modes are non-zero.

At this stage, no assumptions have been made about the action, which so far has been treated as a deterministic quantity. In the weakly nonlinear regime, it is often further assumed that the complex wave fields $\hat{\psi}_1(t)$ are random variables, possessing quasi-Gaussian statistics with the action being a statistically independent random variable. Now, when taking averages, we also average over the action denoted by the subscript I :

$$\langle \hat{\psi}_1 \hat{\psi}_2^* \rangle_{\theta, I} = n_1 \delta_1^2. \quad (2.2.9)$$

This defines a new quantity called the wave action spectrum $n_k(t)$ that describes the averaged dynamics of the normal modes.

By assuming weak nonlinearity and therefore quasi-Gaussian statistics, we can ensure the closure of the evolution equation of the wave action spectrum. For example, let us consider the time evolution of $n_1(t)$:

$$\begin{aligned} \frac{dn_1}{dt} &= \left[\left\langle \frac{d\hat{\psi}_1}{dt} \hat{\psi}_1^* \right\rangle_{\theta, I} + \left\langle \frac{d\hat{\psi}_1^*}{dt} \hat{\psi}_1 \right\rangle_{\theta, I} \right] \\ &= -i\omega_1 n_1 - i\nu \sum_{234} \langle \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \rangle_{\theta, I} \delta_{12}^{34} + i\omega_1 n_1 + i\nu \sum_{234} \langle \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3^* \hat{\psi}_4^* \rangle_{\theta, I} \delta_{34}^{12}. \end{aligned} \quad (2.2.10)$$

Problems immediately arise in equation (2.2.10), as the evolution of the second order moment n_1 depends on fourth order moments such as $\langle \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \rangle_{\theta, I}$. To evolve this equation requires solving the evolution equation for fourth order moments, which in turn require solving the evolution equation for even higher order moments and so on. However, because we are in the weakly nonlinear regime, and the statistics of $\hat{\psi}_1$ are quasi-Gaussian, we can decompose higher order correlators into second order ones. For example:

$$\langle \hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3^* \hat{\psi}_4^* \rangle_{\theta, I} = n_1 n_2 (\delta_1^3 \delta_2^4 + \delta_1^4 \delta_2^3) + C_{1234}$$

Here we get two sets of Kronecker deltas when written in terms of second order correlators that correspond to $\theta_1 = \theta_3$ and $\theta_2 = \theta_4$ or $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$, and an extra terms C_{1234} called the fourth order cumulant. C_{1234} is small compared to the second order correlators through the assumption of quasi-gaussian statistics but non-vanishing. The consequences of this term are very important to WWT as it allows one to find a non-trivial evolution equation for the wave action spectrum that can describe modal interactions. It is through the decomposition of higher order correlators into second order correlators and weak cumulants, alongside separation of time scales, that we achieve statistical closure of our evolution. This procedure provides a systematic way to obtain statistical solutions to the wave equation described in terms of second order correlators, and is a standard approach within the theory.

Thus, to obtain statistical solutions to the DNLS, we must re-formulate the evolution equation in terms of second order correlators. As will be shown shortly, this involves a perturbative method in time scales valid through the assumption of weak nonlinearity. However, this derivation is unique in that it does not follow the standard approach described above, but yields an equivalent end result.

2.3 Resonant interactions

Before proceeding to the derivation of the WKE, the importance of resonant wave interactions will be emphasized. These are the exact wave interactions that dictate the nonlinear wave dynamics in WWT, brought about through the higher order weak cumulants, and are responsible for important physical phenomena such as thermal equilibrium.

For an interaction to be considered resonant, it must simultaneously satisfy two criteria - The conservation of wave number and conservation of linear energy. For a four wave scattering interaction, as in the DNLS, conservation of wave number

is given by:

$$k_1 + k_2 = k_3 + k_4, \text{ mod } N \quad (2.3.1)$$

and conservation of linear energy is given by:

$$\omega_1 + \omega_2 = \omega_3 + \omega_4. \quad (2.3.2)$$

The criteria in equation (2.3.1) is already evident in the deterministic equation when written in Fourier space, see equation (2.4.6). The criteria in equation (2.3.2) however, will emerge naturally once the WKE is derived.

The modulo 2π in equation (2.3.1) is a feature of the discrete space, and as such can permit extra resonant interactions that are inaccessible in continuous space systems. However, it is important to note that when dealing with discrete spaces the resonant interactions can often be restricted, as many wave numbers are inaccessible in the discrete Fourier space [36, 37, 38, 39]. As such this can inhibit thermodynamic processes, with resonances not being connected. For example it has been shown that for FPUT system the resonant quartets are not connected and cannot efficiently transfer wave action between modes at a four wave [40].

In discrete time systems, as will be presented in chapter (3), a similar folding effect can arise through the application of Floquet dynamics because of discrete-time, introducing periodicity in ω_k , as discussed in section (1.2.2), leading to unusual thermodynamic effects not previously reported in literature.

2.4 Deriving the wave-kinetic equation for the DNLS

We will now move onto the full derivation of the WKE for the DNLS given by Onorato and Demmateis [42], that presents an original method of obtaining a WKE for discrete-space systems not previously reported.

To begin the physical system we are considering is a one-dimensional chain of complex oscillators that are coupled to nearest neighbours only. This chain is

length $L \in \mathbb{R}$ in physical space with $N \in \mathbb{N}$ oscillators and periodic boundary conditions. The physical spacing between oscillators is $L/N = a$, where for convenience we choose $a = 1$. We also have an equivalent discrete Fourier space with the spacing between wave numbers of $\Delta k = 2\pi/L$. The wave system we use to describe this chain of oscillators is the DNLS with the Hamiltonian written as,

$$H = \sum_1 \omega_1 |\hat{\psi}_1|^2 + \frac{\nu}{2} \sum_{1234} \hat{\psi}_1^* \hat{\psi}_2^* \hat{\psi}_3 \hat{\psi}_4 \delta_{12}^{34}. \quad (2.4.1)$$

$\omega_1 = 4 \sin^2(\pi k_1/N)$ and $\nu \ll \mathcal{O}(1)$ is the way in which we ensure weakly nonlinear wave dynamics. Equation (2.4.1) is a real quantity describing the total energy of the chain and is the starting point for the derivation. To proceed, apply the action-angle decomposition to the wave fields:

$$H = \sum_1 \omega_1 I_1 + \frac{\nu}{2} \sum_{1234} \sqrt{I_1 I_2 I_3 I_4} e^{+i\Delta\theta_{12}^{34}} \delta_{12}^{34}, \quad (2.4.2)$$

where we introduce the shorthand notation, $\Delta\theta_{12}^{34} = \theta_1 + \theta_2 - \theta_3 - \theta_4$. We note that no assumptions have yet been made about $I_k(t)$ or $\theta_k(t)$, which are still deterministic quantities. As we expect the Hamiltonian to be a real quantity, we project the exponent entirely long the real axis introducing a cosine:

$$H = \sum_1 \omega_1 I_1 + \frac{\nu}{2} \sum_{1234} \sqrt{I_1 I_2 I_3 I_4} \cos(\Delta\theta_{12}^{34}) \delta_{12}^{34}. \quad (2.4.3)$$

As (I_k, θ_k) are a canonical transformation of (ψ_k, ψ_k^*) (see appendix (A.3)), we can write Hamilton's equations as:

$$\begin{aligned} \frac{dI_k}{dt} &= -\frac{\partial H}{\partial \theta_k}, \\ \frac{d\theta_k}{dt} &= \frac{\partial H}{\partial I_k}. \end{aligned}$$

Substitution of equation (2.4.3) into the action angle variant of Hamilton's

equations we obtain,

$$\frac{dI_1}{dt} = 2\nu \sum_{234} \sqrt{I_1 I_2 I_3 I_4} \sin [\Delta\theta_{12}^{34}] \delta_{12}^{34}, \quad (2.4.4)$$

$$\frac{d\theta_1}{dt} = \omega_1 + \nu \sum_{234} \sqrt{\frac{I_2 I_3 I_4}{I_1}} \cos [\Delta\theta_{12}^{34}] \delta_{12}^{34}. \quad (2.4.5)$$

Upon differentiating, a factor of 4 emerges in either coefficient as the dummy variables in k can be relabelled upon taking the derivative. This gives four equivalent contributions (See appendix (A.4)).

To obtain equations (2.4.4) and (2.4.5) we point out that the Hamiltonian is not required and thus technically not needed to obtain the WKE. These equations can be obtained directly from the real and imaginary parts of the equation of motion in action angle coordinates:

$$i \left[\frac{d\sqrt{I_1}}{dt} - i \frac{d\theta_1}{dt} \sqrt{I_1} \right] e^{-i\theta_1} = \omega_1 \sqrt{I_1} e^{-i\theta_1} + \nu \sum_{234} \sqrt{I_2 I_3 I_4} e^{i(\theta_2 - \theta_3 - \theta_4)} \delta_{12}^{34}. \quad (2.4.6)$$

with the full details given in appendix (A.5).

To avoid secular growth for the evolution of equation (2.4.5), we extract out from the sum instances where the nonlinearity evolves independent of the phase. If left in the summation, these terms would evolve on a linear time scale dominating the dynamics. Secular terms are found when the scattering events are trivial, i.e. $k_1 = k_2 = k_3 = k_4$, $k_1 = k_3$ and $k_2 = k_4$ or equivalently $k_1 = k_4$ and $k_2 = k_3$. The extracted terms formulate the re-normalised dispersion relation:

$$\Omega_1 = \omega_1 + \nu \left[2 \sum_2 I_2 - I_1 \right], \quad (2.4.7)$$

noting that I_1 is subtracted in order not to count the same interaction twice.

2.4.1 Outline of wave kinetic equation derivation

Now that the equations of motion for the action and angle are written in full, and

the linear dispersion relation renormalised to avoid secular terms, we will derive the WKE. Before proceeding with this derivation however, it is useful to briefly outline the strategy that will be used. The system exhibits a natural separation of timescales due to weak nonlinearity, with linear dynamics dominating on a time scale $t \sim \mathcal{O}(1)$, while nonlinear interactions accumulate over much longer timescales τ dictating the wave action evolution.

In order to capture this behaviour, we will perform a perturbative expansion in the small parameter ν , which will implicitly introduce the desired nonlinear timescale τ on which the wave action evolves. The purpose of this expansion is to systematically separate contributions at different orders of ν and identify which terms survive after the random phase averaging. This will eliminate the non-resonant terms leaving only the resonant wave interactions that evolve on the nonlinear timescale τ . This procedure is equivalent to the multiple time-scale expansions typically used in derivations of the WKE in WWT. We now proceed by expanding both the action and angle in powers of ν .

2.4.2 Weakly nonlinear expansion in ν

With the assumption that $\nu \ll \mathcal{O}(1)$, expansions of both $I_k(t)$ and $\theta_k(t)$ can now be carried out:

$$I_k(t) = I_k^{(0)} + \nu I_k^{(1)} + \nu^2 I_k^{(2)} + \mathcal{O}(\nu^3) \dots \quad (2.4.8)$$

$$\theta_1(k) = \theta_k^{(0)} + \nu \theta_k^{(1)} + \nu^2 \theta_k^{(2)} + \mathcal{O}(\nu^3) \dots \quad (2.4.9)$$

Applying the expansions (2.4.8) and (2.4.9) to Hamilton's equations:

$$\frac{dI_k}{dt} = \frac{dI_k^{(0)}}{dt} + \nu \frac{dI_k^{(1)}}{dt} + \nu^2 \frac{dI_k^{(2)}}{dt} + \mathcal{O}(\nu^3) \dots \quad (2.4.10)$$

$$\frac{d\theta_k}{dt} = \frac{d\theta_k^{(0)}}{dt} + \nu \frac{d\theta_k^{(1)}}{dt} + \nu^2 \frac{d\theta_k^{(2)}}{dt} + \mathcal{O}(\nu^3) \dots \quad (2.4.11)$$

We now want to solve the evolution equations at each order of ν . Suppose that we have some initial data:

$$I_1(t=0) = \bar{I}_1, \quad \theta_1(t=0) = \bar{\theta}_1. \quad (2.4.12)$$

The zeroth order terms at $\mathcal{O}(\nu^0)$ are,

$$\frac{dI_1^{(0)}}{dt} = 0 \rightarrow I_1^{(0)} = \bar{I}_1 \quad (2.4.13)$$

$$\frac{d\theta_1^{(0)}}{dt} = \bar{\Omega}_1 \rightarrow \theta_1^{(0)} = \int \bar{\Omega}_1 dt = \bar{\theta}_1 + \bar{\Omega}_1 t \pmod{2\pi} \quad (2.4.14)$$

where $\bar{\Omega}_1 = \omega_1 + \nu [2 \sum_2 \bar{I}_2 - \bar{I}_1]$. Then through substitution of the zeroth order terms we can obtain expressions at $\mathcal{O}(\nu^1)$, integrating over the time interval 0 to t :

$$\begin{aligned} \frac{dI_1^{(1)}}{dt} &= 2\nu \sum'_{234} \sqrt{I_1^{(0)} I_2^{(0)} I_3^{(0)} I_4^{(0)}} \sin [\Delta \theta_{12}^{34(0)}] \delta_{12}^{34}, \\ I_1^{(1)} &= 2\nu \sum'_{234} \sqrt{\bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4} \frac{\cos [\Delta \bar{\theta}_{12}^{34}] - \cos [\Delta \bar{\theta}_{12}^{34} + \Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{12}^{34}} \delta_{12}^{34}, \end{aligned} \quad (2.4.15)$$

$$\begin{aligned} \frac{d\theta_1^{(1)}}{dt} &= \nu \sum'_{234} \sqrt{\frac{I_2^{(0)} I_3^{(0)} I_4^{(0)}}{I_1^{(0)}}} \cos [\Delta \theta_{12}^{34(0)}] \delta_{12}^{34}, \\ \theta_1^{(1)} &= \nu \sum'_{234} \sqrt{\frac{\bar{I}_2 \bar{I}_3 \bar{I}_4}{\bar{I}_1}} \frac{\sin [\Delta \bar{\theta}_{12}^{34} + \Delta \bar{\Omega}_{12}^{34} t] - \sin [\Delta \bar{\theta}_{12}^{34}]}{\Delta \bar{\Omega}_{12}^{34}} \delta_{12}^{34}. \end{aligned} \quad (2.4.16)$$

Now a prime is introduced into the sum to denote that the secular terms have been extracted out. More care must be taken when determining $I_k^{(2)}$ with contributions coming from both previous orders of the expansion:

$$\begin{aligned} \frac{dI_1^{(2)}}{dt} &= 2\nu \sum'_{234} \sqrt{(I_1^{(0)} + \nu I_1^{(1)})(I_2^{(0)} + \nu I_2^{(1)})(I_3^{(0)} + \nu I_3^{(1)})(I_4^{(0)} + \nu I_4^{(1)})} \times \\ &\quad \sin [\Delta \theta_{12}^{34(0)} + \nu \Delta \theta_{12}^{34(1)}] \delta_{12}^{34}. \end{aligned}$$

Each bracket in the square root can be rewritten as a binomial expansion and the sine term can be rewritten using the double angle formula:

$$\sin \left[\Delta\theta_{12}^{34(0)} + \nu\Delta\theta_{12}^{34(1)} \right] = \sin(\Delta\theta_{12}^{34(0)}) \cos(\nu\Delta\theta_{12}^{34(1)}) + \sin(\nu\Delta\theta_{12}^{34(1)}) \cos(\Delta\theta_{12}^{34(0)})$$

Since $\Delta\theta_{12}^{34(0)} \gg \nu\Delta\theta_{12}^{34(1)}$, small angle approximations can be made for the terms containing $\nu\Delta\theta_{12}^{34(1)}$. Expanding out the square root fully and collecting terms that go as ν only, yields:

$$\begin{aligned} \frac{dI_1^{(2)}}{dt} = 2\nu^2 \sum'_{234} \sqrt{\bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4} \left[\Delta\theta_{1,2}^{3,4(1)} \cos [\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t] \right. \\ \left. + \frac{1}{2} \left(\frac{I_1^{(1)}}{\bar{I}_1} + \frac{I_2^{(1)}}{\bar{I}_2} + \frac{I_3^{(1)}}{\bar{I}_3} + \frac{I_4^{(1)}}{\bar{I}_4} \right) \sin [\Delta\bar{\theta}_{12}^{34} + \Delta\bar{\Omega}_{12}^{34}t] \right] \delta_{12}^{34}. \end{aligned} \quad (2.4.17)$$

Equation (2.4.17) can be brought to a more compact form, making use of equations (2.4.15) and (2.4.16), double angle formulae, and careful relabelling of some of the dummy variables:

$$\begin{aligned} \frac{dI_1^{(2)}}{dt} = 2\nu^2 \sum'_{234} \sum'_{567} \sum_{q=1}^4 \frac{1}{\sqrt{\bar{I}_q}} \sqrt{\bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \bar{I}_5 \bar{I}_6 \bar{I}_7} \delta_{12}^{34} \delta_{q5}^{67} \times \\ \left[\frac{\sin [\sigma_q \Delta\bar{\theta}_{q5}^{67} - \Delta\bar{\theta}_{12}^{34} + (\sigma_q \Delta\bar{\Omega}_{q5}^{67} - \Delta\bar{\theta}_{12}^{34})t] + \sin [\Delta\bar{\theta}_{12}^{34} - \sigma_q \Delta\bar{\theta}_{q5}^{67} + \Delta\bar{\theta}_{12}^{34}t]}{\Delta\bar{\Omega}_{q5}^{67}} \right]. \end{aligned} \quad (2.4.18)$$

With $\sigma_q = [+ , + , - , -]$ for $q = 1, 2, 3, 4$ respectively.

2.4.3 Phase averaging and the discrete wave kinetic equation

To remove the phase dependence, we take the average over the initial phases $\bar{\theta}_k$, making the assumption that phases are statistically independent and uniformly distributed on $[0, 2\pi)$. In doing so, the terms that evolve on the linear time scale t vanish, eliminating non-resonant contributions. As we shall see this leaves terms that evolve on the nonlinear timescale $\tau = \nu^2 t$ in which thermal equilibrium among the modes is reached. We recall that the averaging over the phases is

given by:

$$\langle f(\theta_1, \theta_2, \theta_3, \dots, \theta_N) \rangle_\theta = \frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} f(\theta_1, \theta_2, \theta_3, \dots, \theta_N) d\theta_1 d\theta_2 d\theta_3, \dots d\theta_N.$$

We then average over every $\bar{\theta}_k$ for each term in the expansion of I_1 only, since we ultimately want to derive an equation for the wave action spectrum n_1 . At the zeroth order $\mathcal{O}(\nu^0)$ we obtain:

$$\frac{d\langle I_1^{(0)} \rangle_{\bar{\theta}_1}}{dt} = 0, \quad (2.4.19)$$

For the first order term $\mathcal{O}(\nu^1)$:

$$\frac{d\langle I_1^{(1)} \rangle_{\bar{\theta}_1}}{dt} = 2 \left\langle \sum'_{234} \sqrt{I_1^{(0)} I_2^{(0)} I_3^{(0)} I_4^{(0)}} \sin \left[\Delta \theta_{12}^{34(0)} \right] \delta_{12}^{34} \right\rangle_{\bar{\theta}_1}, \quad (2.4.20)$$

only the $\sin \left[\Delta \theta_{12}^{34(0)} \right]$ has dependence on the initial phases $\bar{\theta}_k$ so the average only acts on the sine and integrates to zero:

$$\frac{d\langle I_1^{(1)} \rangle_{\bar{\theta}_1}}{dt} = 0, \quad (2.4.21)$$

For the second order term $\mathcal{O}(\nu^2)$:

$$\begin{aligned} \frac{d\langle I_1^{(2)} \rangle_{\bar{\theta}_1}}{dt} = & 2 \left\langle \sum'_{234} \sum'_{567} \sum_{q=1}^4 \frac{1}{\sqrt{\bar{I}_q}} \sqrt{\bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \bar{I}_5 \bar{I}_6 \bar{I}_7} \delta_{12}^{34} \delta_{q5}^{67} \times \right. \\ & \left[\frac{\sin \left[\sigma_q \Delta \bar{\theta}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34} + (\sigma_q \Delta \bar{\Omega}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34}) t \right] + \sin \left[\Delta \bar{\theta}_{12}^{34} - \sigma_q \Delta \bar{\theta}_{q,5}^{6,7} + \Delta \bar{\Omega}_{12}^{34} t \right]}{\Delta \bar{\Omega}_{q5}^{67}} \right] \Bigg\rangle_{\bar{\theta}_1}, \end{aligned} \quad (2.4.22)$$

the average acts only on the trigonometric terms. To average equation (2.4.22), it is instructive to do so for each value of $q = 1, 2, 3, 4$ individually whilst keeping in mind the restrictions imposed by the extraction of the trivial resonances. An example of this is given in appendix (A.6). Now equation (2.4.22) can be rewritten

as:

$$\frac{d\langle I_1^{(2)} \rangle_{\bar{\theta}_1}}{dt} = 4\nu^2 \sum_{234} \bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \left[\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4} \right] \frac{\sin [\Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{12}^{34}} \delta_{12}^{34} \quad (2.4.23)$$

omitting the ' above the sum as the trivial resonances now cancel from the + + - - signs in the square brackets. As this is the only non-zero term remaining from averaging over the initial phases, this gives the discrete wave kinetic equation for the DNLS:

$$\frac{d\langle I_1 \rangle_{\bar{\theta}_1}}{d\tau} = 4 \sum_{234} \bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \left[\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4} \right] \frac{\sin [\Delta \bar{\Omega}_{12}^{34} \frac{\tau}{\nu^2}]}{\Delta \bar{\Omega}_{12}^{34}} \delta_{12}^{34}, \quad (2.4.24)$$

where $\tau = \nu^2 t$ is nonlinear time.

For the DNLS the term $\Delta \Omega_{12}^{34}$ reduces to $\Delta \omega_{12}^{34}$ since the nonlinear frequency shift is independent of k simply given by the Norm $\sum_i I_i$, ensuring that all shifts cancel. Hence we can replace all $\Delta \Omega_{12}^{34}$ with $\Delta \omega_{12}^{34}$ in the discrete wave kinetic equation. Then in the limit that $\nu \rightarrow 0$, we can form a Dirac delta denoted function as $\delta^{(D)}$ in ω :

$$\lim_{\nu \rightarrow 0} \frac{\sin (\Delta \omega_{12}^{34})}{\Delta \omega_{12}^{34}} = \pi \delta^{(D)}(\Delta \omega_{12}^{34}) \quad (2.4.25)$$

This yields the discrete-space version of the WKE, with both the criteria for resonant wave interactions present in the Dirac and Kronecker deltas:

$$\frac{d\langle I_1 \rangle_{\bar{\theta}_1}}{d\tau} = 4\pi \sum_{234} \bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \left[\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4} \right] \delta^{(D)}(\Delta \omega_{12}^{34}) \delta_{12}^{34}. \quad (2.4.26)$$

2.4.4 The thermodynamic limit of the discrete wave-kinetic equation for the DNLS

Now that we have arrived at the discrete-space equivalent of the WKE, we can show that through the thermodynamic limit we recover the standard continuous-space WKE for the NLS.

The thermodynamic limit is obtained by assuming that the physical space

becomes infinitely large, $L \rightarrow \infty$ sending the number of oscillators $N \rightarrow \infty$ at the same rate such that $L/N = a = 1$ remains constant. By going into this limit, the discrete Fourier space k become extremely dense, $\Delta k \rightarrow 0$, and approaches a continuum whilst keeping physical space discrete. This enables the existence of a resonant manifold in the WKE, in which resonant interactions are connected. Through the resonant manifold, we expect wave action to freely move between all the modes and thus reach a state of equilibrium.

In the limit of an infinitely dense k space, our action I_k is redefined to be the action density,

$$\mathcal{I}_k = \mathcal{I}(k, t) = \frac{I_k}{\Delta k}.$$

Taking such a limit also sends each sum to an integral,

$$\sum_k \rightarrow \int \frac{1}{\Delta k} dk,$$

and the Kronecker delta $\delta^{(K)}$ is now the continuous counter part, the Dirac delta $\delta^{(D)}$ function,

$$\delta^{(K)} \rightarrow \Delta k \delta^{(D)}.$$

The discrete Fourier space is now replaced by its thermodynamic analogue p ,

$$p = \Delta k \cdot k = \frac{2\pi k}{L},$$

where $p \in [0, 2\pi]$. The dispersion relation can then be expressed in terms of p as,

$$\omega_p = 4 \sin^2 \left(\frac{L}{N} \frac{\pi p}{2\pi} \right) = 4 \sin^2 \left(\frac{p}{2} \right),$$

retaining the shape of the dispersion relation for all wave numbers in the thermodynamic limit. A similar re-labelling procedure is also done for the FPUT chain in [44]. Applying the thermodynamic limit to equation (2.4.24)

therefore gives,

$$\frac{d\langle \mathcal{I}_1 \rangle_{\bar{\theta}_1}}{d\tau} = 4 \int dp_2 dp_3 dp_4 \bar{\mathcal{I}}_1 \bar{\mathcal{I}}_2 \bar{\mathcal{I}}_3 \bar{\mathcal{I}}_4 \left[\frac{1}{\bar{\mathcal{I}}_1} + \frac{1}{\bar{\mathcal{I}}_2} - \frac{1}{\bar{\mathcal{I}}_3} - \frac{1}{\bar{\mathcal{I}}_4} \right] \frac{\sin \left[\Delta \bar{\Omega}_{1,2}^{3,4} \frac{\tau}{\epsilon^2} \right]}{\Delta \bar{\Omega}_{1,2}^{3,4}} \delta_{1,2}^{3,4(D)}. \quad (2.4.27)$$

Lastly, assuming very weak nonlinearity, taking the limit of $\nu \rightarrow 0$, with the restriction that $\nu \rightarrow 0$ slower than $\Delta k \rightarrow 0$:

$$\lim_{(\Delta k, \nu) \rightarrow (0,0)} \frac{\sin \left[\omega_1 \frac{\tau}{\nu^2} + (2\Delta k \sum_2 \bar{I}_2 - \Delta k \bar{I}_1) \frac{\tau}{\nu} \right]}{\omega_1 + 2\nu \Delta k \sum_2 \bar{I}_2 - \nu \Delta k \bar{I}_1} = \frac{\sin \left[\omega_1 \frac{\tau}{\nu^2} \right]}{\omega_1} = \pi \delta(\Delta \omega_{1,2}^{3,4})^{(D)}, \quad (2.4.28)$$

results in the same wave kinetic equation for the NLS:

$$\frac{d\langle \mathcal{I}_1 \rangle_{\bar{\theta}_1}}{d\tau} = 4\pi \int dp_2 dp_3 dp_4 \bar{\mathcal{I}}_1 \bar{\mathcal{I}}_2 \bar{\mathcal{I}}_3 \bar{\mathcal{I}}_4 \left[\frac{1}{\bar{\mathcal{I}}_1} + \frac{1}{\bar{\mathcal{I}}_2} - \frac{1}{\bar{\mathcal{I}}_3} - \frac{1}{\bar{\mathcal{I}}_4} \right] \delta(\Delta \omega_{1,2}^{3,4})^{(D)} \delta_{1,2}^{3,4(D)}. \quad (2.4.29)$$

Noting that the linear dispersion relation here is different than that seen for NLS.

A unique aspect of this derivation is that so far no assumptions have been made about the action density $\mathcal{I}_p$. Typically in the derivation of the WKE, we make assumptions that the phases are statistically independent and randomly distributed on the interval $[0, 2\pi)$, and that the wave fields of the modes are quasi-Gaussian statistical quantities. It is through both of these assumptions in tandem that we can ensure statistical closure, reducing higher order correlators into second order ones and weak cumulants. Here however, the WKE has been derived only under the assumptions of weak nonlinearity and randomly distributed phases with a deterministic action density, achieving the same closure.

We will now make the assumption that $\mathcal{I}_p$ is a statistically independent random variable defining the wave action spectrum $n(p, t)$ with respect to the average of the initial action density distribution,

$$n(p, t) = \langle \mathcal{I}_p \rangle_{\bar{\theta}, \bar{I}}. \quad (2.4.30)$$

The subscript $\bar{I}$ here denotes that we are averaging in the action now. As these are statistically independent we therefore have the property:

$$\langle \bar{\mathcal{I}}_i \bar{\mathcal{I}}_j \bar{\mathcal{I}}_k \rangle_{\bar{\theta}_1, \bar{I}_1} = \langle \bar{\mathcal{I}}_i \rangle_{\bar{\theta}_1, \bar{I}_1} \langle \bar{\mathcal{I}}_j \rangle_{\bar{\theta}_1, \bar{I}_1} \langle \bar{\mathcal{I}}_k \rangle_{\bar{\theta}_1, \bar{I}_1} = \bar{n}_i \bar{n}_j \bar{n}_k,$$

where $i \neq j \neq k$ with $\bar{n}_p = n_p(t = 0)$. Therefore, equation (2.4.29) can be written as:

$$\frac{dn_1}{d\tau} = 4\pi \int dp_2 dp_3 dp_4 \bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 \left[\frac{1}{\bar{n}_1} + \frac{1}{\bar{n}_2} - \frac{1}{\bar{n}_3} - \frac{1}{\bar{n}_4} \right] \delta(\Delta\omega_{1,2}^{3,4})^{(D)} \delta_{1,2}^{3,4(D)}. \quad (2.4.31)$$

Equation (2.4.31) is in the expected form of the WKE with the caveat that the right-hand side is dependent on the initial wave action spectrum $\bar{n}_1 = n_1(t = 0)$, and is only valid at time $t = 0$.

To interpret this result as a WKE describing the slow evolution of the wave action spectrum, we make the additional assumption that the random phases and weak correlations persist on the nonlinear timescale τ . Under this assumption, we write the following:

$$\frac{dn_1}{d\tau} = 4\pi \int dp_2 dp_3 dp_4 n_1 n_2 n_3 n_4 \left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right] \delta(\Delta\omega_{1,2}^{3,4})^{(D)} \delta_{1,2}^{3,4(D)}. \quad (2.4.32)$$

It is important to recognise that this final step is not rigorously justified in the derivation presented and should be regarded as a purely heuristic argument. There are several works that provide a broader discussion of the assumptions underlying WWT, including the persistence of random phases and weak correlation [68, 69], however this is beyond the scope of this reproduction.

The derivation of the WKE presented here is only one possible approach and does not rely on the typical statistical treatment of achieving closure through the systematic factorization of higher-order correlators into second order ones. Instead, the form of the WKE is obtained by phase averaging the deterministic Hamiltonian dynamics written in action-angle variables, thereby removing the

explicit dependence on the fast phase variables at leading order. Only afterwards is the resulting wave action spectrum interpreted as a statistical quantity. In this thesis, we do not aim to compare or classify the various formulations of WWT theory. The purpose of this derivation is to illustrate the general formalism in a discrete-space setting and to introduce key physical concepts that will be used in chapter (3).

2.5 Thermodynamic solutions, equilibrium states and resonant interactions

The final section to this chapter will now discuss various properties of equation (2.4.32) and the stationary solutions that denote thermal equilibrium.

2.5.1 Extending conservation laws to the wave kinetic equation

We now re-write equation (2.4.32) in the following compact form,

$$\begin{aligned} \frac{dn_1}{d\tau} &= 4\pi \int Q_{12}^{34} dp_2 dp_3 dp_4, \\ Q_{12}^{34} &= n_1 n_2 n_3 n_4 \left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right] \delta(\Delta\omega_{1,2}^{3,4})^{(D)} \delta_{1,2}^{3,4(D)}. \end{aligned}$$

When obtaining the WKE, it is required that $\nu \rightarrow 0$. This implies that the nonlinear contributions from the DNLS Hamiltonian,

$$H = \sum_1 |\psi_1|^2 \omega_1 + \frac{1}{2} \nu \sum_{1,2,3,4} \psi_1^* \psi_2^* \psi_3 \psi_4 \delta_{1,2}^{3,4},$$

are vanishingly small. The Hamiltonian can therefore be approximated with the linear energy,

$$H \approx \sum_1 |\psi_1|^2 \omega_1. \quad (2.5.1)$$

Equation (2.5.1) can be re-written using the action angle transformation, averaging over the initial phases and taking the thermodynamic limit, carrying

all the same assumptions as before,

$$H \approx \int n_1 \omega_1 dp_1. \quad (2.5.2)$$

Taking the derivative in time τ denoted by a dot,

$$\dot{H} \approx \int \frac{dn_1}{d\tau} \omega_1 dp_1. \quad (2.5.3)$$

A similar argument can be made for the norm N_{DNLS} ,

$$\begin{aligned} \frac{N_{DNLS}}{N} &= \sum_1 |\psi_1|^2, \\ &= \int n_1 dp_1, \\ \dot{N}_{DNLS} &= \int \frac{dn_1}{d\tau} dp_1. \end{aligned} \quad (2.5.4)$$

Both of the conserved quantities in equations (2.5.3) and (2.5.4) possess the general structure,

$$\begin{aligned} \dot{\mathcal{C}} &= \int \frac{dn_1}{d\tau} F(p_1) dp_1, \\ &= 4\pi \int Q_{12}^{34} F(p_1) dp_1 dp_2 dp_3 dp_4, \end{aligned} \quad (2.5.5)$$

where $\mathcal{C}$ is some conserved quantity. $\mathcal{C}$ can be proven to be conservative by making use of the symmetry properties of Q_{12}^{34} ,

$$Q_{12}^{34} = Q_{21}^{34} = -Q_{34}^{12} = -Q_{43}^{21}.$$

Each relabelling of Q_{12}^{34} above shows four equivalent permutations that have a different initial index. If the above is substituted into equation (2.5.5),

$$\begin{aligned} \dot{\mathcal{C}} &= 4\pi \int \frac{1}{4} [Q_{12}^{34} + Q_{21}^{34} - Q_{34}^{12} - Q_{43}^{21}] F(p_1) dp_1 dp_2 dp_3 dp_4, \\ &= \pi \int Q_{12}^{34} [F(p_1) + F(p_2) - F(p_3) - F(p_4)] dp_1 dp_2 dp_3 dp_4. \end{aligned}$$

Now, the index permutations have been carried into the arbitrary function $F(p)$. Choosing $F(p)$ to be ω_p ,

$$\dot{H} = \pi \int Q_{12}^{34} [\omega_1 + \omega_2 - \omega_3 - \omega_4] dp_1 dp_2 dp_3 dp_4 = 0, \quad (2.5.6)$$

which is always zero for non-zero solutions to the wave kinetic equation, as a consequence of the Dirac delta in ω_p . Equivalently, if $F(p) = 1$,

$$\dot{N}_{DNLS} = \pi \int Q_{12}^{34} [1 + 1 - 1 - 1] dp_1 dp_2 dp_3 dp_4 = 0. \quad (2.5.7)$$

Hence both the norm and the linear energy are conserved quantities.

2.5.2 H-Theorem

Borrowing from Boltzmann's study of the kinetic theory of gases, a functional definition of entropy can be given as [70],

$$S[n(p_1, \tau)] = \int \ln[n(p_1, \tau)] dp_1. \quad (2.5.8)$$

Within H-theorem, it is stated that the entropy S of a system always increases in time until the system has reached equilibrium, for which it is then constant.

$$\frac{dS}{d\tau} \geq 0 \quad (2.5.9)$$

More so, entropy can never decrease. Taking the derivative of the functional form of S in equation (2.5.8),

$$\frac{dS}{d\tau} = \int \frac{1}{n_1} \frac{dn_1}{d\tau} dp_1. \quad (2.5.10)$$

Making use of equation (2.5.5) and setting $F(p) = 1/n_p$,

$$\frac{dS}{d\tau} = \pi \int n_1 n_2 n_3 n_4 \left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} \right]^2 \delta(\Delta\omega_{1,2}^{3,4})^{(D)} \delta_{1,2}^{3,4(D)} dp_1 dp_2 dp_3 dp_4 \geq 0. \quad (2.5.11)$$

Equation (2.5.11) will therefore always satisfy equation (2.5.9). This is apparent as n_p is a probability distribution and must be positive requiring, $n_1 n_2 n_3 n_4 \geq 0$. The term in the bracket is always ≥ 0 because of the square, and the Dirac deltas are either ∞ or 0 by definition. With this, equation (2.5.8) provides the expression for the entropy of the WKE.

2.5.3 Equilibrium state of the wave kinetic equation

To obtain the equilibrium state of the WKE, we maximise the entropy functional in equation (2.5.8) with respect to the other conserved quantities present. For the entropy the functional derivative is

$$\delta S(n_1) = \frac{1}{n_1} \delta n_1. \quad (2.5.12)$$

The first law of thermodynamics states that,

$$dH = TdS - \mu dN, \quad (2.5.13)$$

where T is the absolute temperature of the system and μ is the chemical potential.

From the Norm in equation (2.5.4),

$$\delta N_{DNLs}(n_1) = \delta n_1,$$

and from the Hamiltonian in equation (2.5.2),

$$\delta H(n_1) = \omega_1 \delta n_1.$$

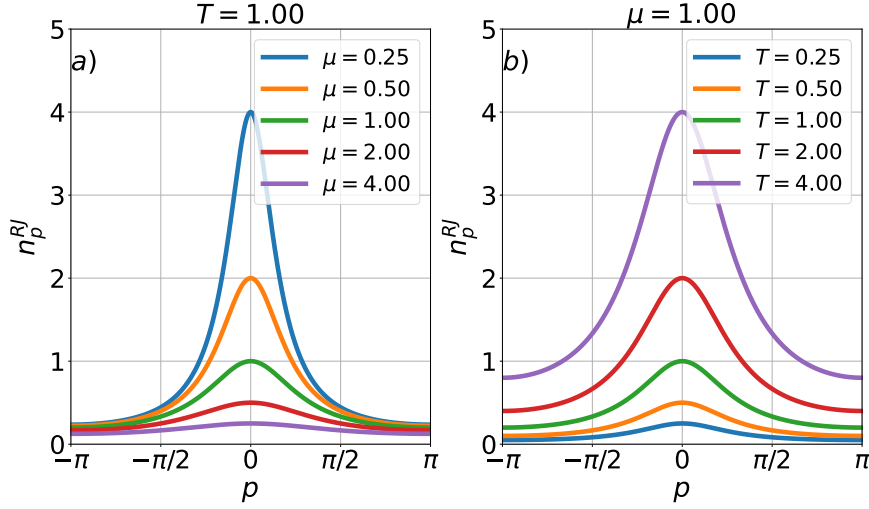


Figure 2.5.1: The derived Rayleigh-Jeans distribution for the DNLS for different choices of (T, μ) . As $\mu \rightarrow 0$ the wave action spectrum becomes more peaked around $p = 0$ and as $T \rightarrow 0$, the spectrum is zero everywhere.

Through equation (2.5.13) the Rayleigh-Jean distribution can be derived,

$$\begin{aligned}\delta H &= T\delta S - \mu\delta N_{DNLS}, \\ \omega_1\delta n_1 &= T\frac{1}{n_1}\delta n_1 - \mu\delta n_1, \\ n^{RJ}(k_1, t) &= \frac{T}{\omega_1 + \mu},\end{aligned}\tag{2.5.14}$$

and is plotted in Fig. (2.5.1). The Rayleigh-Jeans distribution describes the state of thermal equilibrium and is a stationary solution of the WKE. It is the distribution of the wave action spectrum once the entropy has maximised, and is expected to be invariant in time for a closed system. In chapter (3), much of the results will be concerned with the emergence of this thermal state and understanding how it is modified through the presence of discrete time.

2.5.4 Finding the resonant manifold for the DNLS

The thermal states of the WKE are reached solely through resonant interactions. These interactions enable the uninhibited transfer of wave action between different normal modes and through this redistribution, an equilibrium

state emerges.

To seek out the resonant manifold for equation (2.4.32), both Dirac delta functions must be satisfied simultaneously. Solutions of the resonant manifold exist when,

$$p_1 + p_2 = p_3 + p_4 \bmod 2\pi. \quad (2.5.15)$$

The aim now is to find a relation for $\delta(\omega_{1,2}^{3,4})^{(D)} = 0$. Writing out this condition in ω_p :

$$\begin{aligned} \omega(p_1) + \omega(p_2) &= \omega(p_3) + \omega(p_4), \\ 4 \sin^2 \left(\frac{p_1}{2} \right) + 4 \sin^2 \left(\frac{p_2}{2} \right) &= 4 \sin^2 \left(\frac{p_3}{2} \right) + 4 \sin^2 \left(\frac{p_4}{2} \right). \end{aligned} \quad (2.5.16)$$

Using equation (2.5.15), p_4 can be rewritten in terms of p_1, p_2 and p_3 ,

$$4 \sin^2 \left(\frac{p_1}{2} \right) + 4 \sin^2 \left(\frac{p_2}{2} \right) = 4 \sin^2 \left(\frac{p_3}{2} \right) + 4 \sin^2 \left(\frac{p_1 + p_2 - p_3}{2} \right)$$

This will now be solved for p_2 . Making the substitution for $\sin^2(x) = 1/2 - 1/2 \cos(2x)$ and $p = p + L2\pi$ where $L \in \mathbb{Z}$ (since $p \bmod 2\pi$),

$$\frac{1}{2} [\cos(p_1 + 2L\pi) - \cos(p_3 + 2L\pi)] = \frac{1}{2} [\cos(p_1 + p_2 - p_3 + 2L\pi) - \cos(p_2 + 2L\pi)]. \quad (2.5.17)$$

Taking the right hand (R.H.) of equation (2.5.17), the cosines can be expressed

in exponential form and factorised to give sine terms,

$$\begin{aligned}
\text{R.H.} &= \frac{e^{i(p_1+p_2-p_3+2L\pi)} + e^{-i(p_1+p_2-p_3+2L\pi)} - e^{i(p_2+2L\pi)} - e^{-i(p_2+2L\pi)}}{4}, \\
&= \frac{e^{i(p_1+p_2-p_3)}e^{i2L\pi} + e^{-i(p_1+p_2-p_3)}e^{-i2L\pi} - e^{ip_2}e^{i2L\pi} - e^{-ip_2}e^{-i2L\pi}}{4}, \\
&= \frac{\left(e^{i(\frac{2p_2+p_1-p_3}{2})} - e^{-i(\frac{2p_2+p_1-p_3}{2})}\right) \left(e^{i(\frac{p_1-p_3}{2})} - e^{-i(\frac{p_1-p_3}{2})}\right)}{4}, \\
&= -(i^2) \frac{\left(e^{i(\frac{2p_2+p_1-p_3}{2})} - e^{-i(\frac{2p_2+p_1-p_3}{2})}\right)}{2} \frac{\left(e^{i(\frac{p_1-p_3}{2})} - e^{-i(\frac{p_1-p_3}{2})}\right)}{2}, \\
&= -\sin\left(\frac{2p_2+p_1-p_3}{2}\right) \sin\left(\frac{p_1-p_3}{2}\right).
\end{aligned}$$

Nothing that $e^{\pm i2L\pi} = 1 \forall L$. Equation (2.5.17) can now be used to solve for p_2 ,

$$\begin{aligned}
\frac{1}{2} [\cos(p_1 + 2L\pi) - \cos(p_3 + 2L\pi)] &= -\sin\left(\frac{2p_2+p_1-p_3}{2}\right) \sin\left(\frac{p_1-p_3}{2}\right), \\
\arcsin\left[\frac{-\frac{1}{2} [\cos(p_1) - \cos(p_3)]}{\sin\left(\frac{p_1-p_3}{2}\right)}\right] &= p_2 + \frac{p_1-p_3}{2}, \\
p_2 &= \frac{p_3-p_1}{2} + \arcsin\left[\sin\left(\frac{p_1+p_3}{2}\right)\right].
\end{aligned}$$

The $\arcsin(\sin(x))$ in general has two possible solutions, x and $\pi - x$. Considering the solution branch $\arcsin(\sin(x)) = x$:

$$p_2 = \frac{p_3-p_1}{2} + \frac{p_1+p_3}{2} = p_3, \quad (2.5.18)$$

we find trivial solutions, that is solutions that cannot excite new wave numbers.

The other solution branch - $\arcsin(\sin(x)) = \pi - x$ yields:

$$p_2 = \frac{p_3-p_1}{2} - \frac{p_1+p_3}{2} + \pi = \pi - p_1. \quad (2.5.19)$$

which are non-trivial resonances, that can continuously excite new wave numbers. For example, we can take any incoming pair of wave numbers that satisfy $p_1 + p_2 = \pi \bmod 2\pi$, and excite any pair of outgoing waves that satisfy $p_3 + p_4 = \pi \bmod 2\pi$. As consequence, every mode is able to be excited through this four wave resonance and can therefore reach a thermal equilibrium.

To conclude, this chapter provides a summary of the WWT theory, from the perspective of a discrete-space continuous-time setting. The underlying assumptions, predictions of resonances and thermal states will be utilised in the chapter (3), and will underpin the main results of this thesis.

Photonic mesh lattice

The remainder of this thesis focuses solely on a discrete space-time optical lattice, here referred to as the photonic mesh lattice, experimentally realised through the coupling of two fibre loops via a beam splitter. Early descriptions of this type of optical lattice were given in the work of Bouwmeester et.al [71], describing the lattice as the optical analogue of a Galton board, a device often used to describe random walks, sketched in Fig. (3.0.1) panel a). This device is a vertical board with a pyramid of pegs mounted across the top half and a series of parallel bins across the bottom half. Beads are then dropped from the top of the board onto the pegs, bouncing between the pegs as they fall. Eventually the beads are collected into the bins at the bottom, with the spread of beads between the bins resembling that of a normal distribution.

The optical analogue of a Galton board, in Fig. (3.0.1) panel b), describes a pyramid of beam splitters connected by optical fibres, and instead of beads being dropped from the top of the board, pulses of light are injected into the lattice at the top of the pyramid. As a pulse traverses down the lattice, it splits left and right simultaneously at each beam splitter, ballistically spreading across the lattice reaching the next row of beam splitter. This splitting process repeats ad-nauseam as the pulses propagate further down the lattice.

An experimental draw back of the optical Galton board is the demand for an enormous amount of optical apparatus. That is, each successive layer of the pyramid requires a larger number of beam splitters than the previous layer,

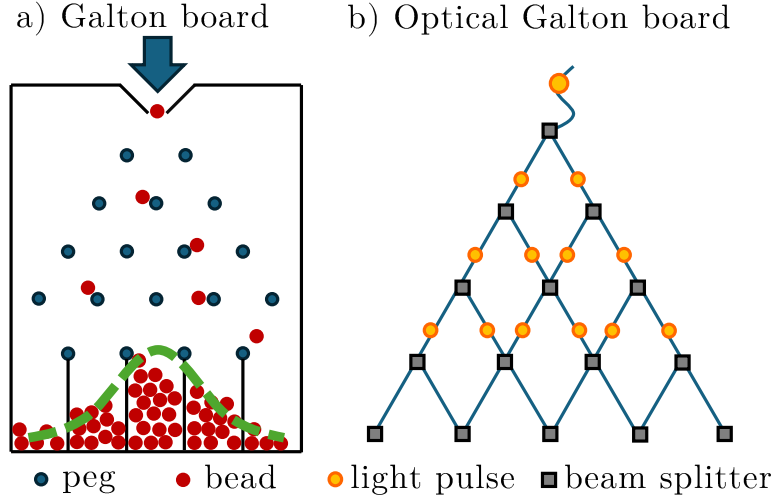


Figure 3.0.1: a) shows a Galton board, b) the optical Galton board. The beads in the bins of a) are distributed according to a normal distribution sketched by the green dashed line.

making the original idea experimentally impractical for larger lattices. Then in 2011 Regensburger et.al[72] experimentally realised the optical Galton board via the coupling of two different length optical fibre loops with a beam splitter. The core principles behind this remarkable setup lies within the lengths of the two fibre loops. By choosing one loop to be longer than the other, it ensures that any pulse split between the two loop will not simply recombine after a single transit, but rather stagger the return times of the split pulses. As pulses return to the beam splitter at staggered times, they split again, eventually forming trains of short optical pulses in both loops simultaneously. By considering the transit times of the two loops, that is the time it takes for a pulse of light to return to the beam splitter, two synthetic discrete space-time coordinates can be constructed, and the dynamics of the coupled fibre loops can be mapped into the optical Galton board. This mapping, which will be explicitly shown in the section following the introduction, is what we will refer to as the photonic mesh lattice.

Since then, the photonic mesh lattice has proven to be a versatile platform to study physics, with the implementation of Parity-Time symmetry through

gain-loss mechanisms and phase modulation [73, 74, 75, 76], studies of the topological properties of the unique dispersion bands [77, 78, 79, 80, 81, 82], photonics based neural networks [83, 84], and even studies into hydrodynamic phenomena in relation to superfluidity of light [85, 86]. Discrete time optical lattice models can also serve as building blocks in photonics based computing [87] and have been implemented in experiments of photonic quantum random walks [88, 89, 90, 91], providing a foundation for scalable quantum computers.

Alongside the vast applications of the photonic mesh lattice, recent works have explored thermodynamic processes among the linear normal modes of the lattice [92, 93], which are of particular interest in relation to WWT theory. The work of Muniz et.al [92] demonstrated experimentally the existence of a stable thermal state for a single dispersion band in the ensemble averaged dynamics, described by a Rayleigh-Jeans distribution. The existence of a thermal equilibrium described by a Rayleigh-Jeans distribution implies that the nonlinear wave interactions can be predicted by the theory of WWT. The implication of WWT theory, and the systems innate discrete-time evolution makes the photonic mesh lattice a very interesting setting, as WWT has yet to be explored for waves governed by discrete-time evolution, hence motivating the study of the photonic mesh lattice in this thesis.

3.1 The photonic mesh lattice and the linear theory

3.1.1 Obtaining the photonic mesh lattice from the two coupled fibre loops

To begin, we consider two different length fibre loops coupled by a beam splitter with the time evolution of a single pulse starting in the short loop depicted in panels a) - f) in Fig. (3.1.1). The shorter loop is denoted by wave-field u while the longer loop is given by the wave-field v where $u, v \in \mathbb{C}$. As observed in Fig. (3.1.1), by choosing a difference in path length between the loops, the return

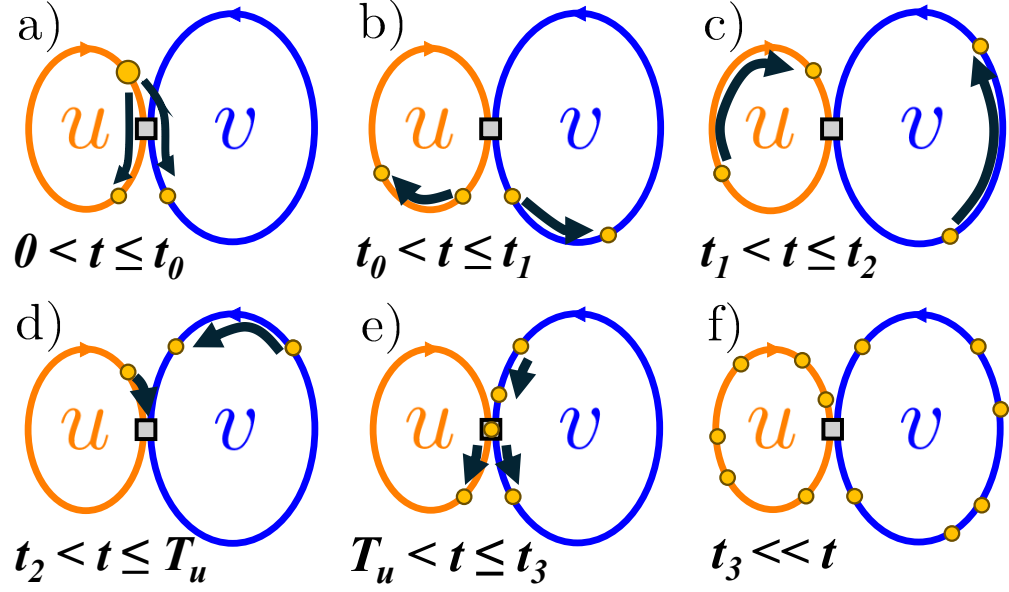


Figure 3.1.1: Panels a)-e) show the evolution in time of a pulse of light starting in the short loop given by the wave-field u . The grey square denotes a beam splitter. Due to the difference in the length of the fibre loops, the split pulse in the short loop returns to the beam splitter after time $t = T_u$ shown in panel d), where T_u is the time it takes to do one transit of the short loop. As consequence the split pulse in the short loop splits sooner as shown in panel e). As the pulses of light propagate for longer times, this process repeats, eventually forming trains of short optical pulses in both loops simultaneously, shown in panel f).

times to the beam splitter of any split pulses are staggered, such that a train of short optical pulses emerge in both loops simultaneously given enough time. In order to map this experiment into the photonic mesh lattice the transit times of either loop must be considered. The short loop transit time is given by T_u and the long loop transit time T_v , such that $T_u < T_v$. We now define an average transit time,

$$\bar{T} = \frac{T_u + T_v}{2}, \quad (3.1.1)$$

and a difference in transit time,

$$\Delta T = T_v - T_u. \quad (3.1.2)$$

Using equations (3.1.1) and (3.1.2), both T_u and T_v can be written as follows,

$$T_u = \bar{T} - \frac{\Delta T}{2}, \quad (3.1.3)$$

$$T_v = \bar{T} + \frac{\Delta T}{2}. \quad (3.1.4)$$

With these definitions, it is convenient to re-imagine the time evolution shown in Fig. (3.1.1) as a series of m snapshots in time, in which each snap shot is taken after integer multiples of the transit time $m\bar{T}$ where $m \in \mathbb{Z}^+$. This is illustrated in Fig. (3.1.2). Moving into the snapshot framework, we see that as the pulses of light are split between the two loops, the total propagation time t of any pulse is always shifted by $\pm\Delta T/2$, depending on what loop a pulse is propagating.

Therefore, the total propagation time of any pulse that arrives at the beam splitter is given by,

$$t = m\bar{T} + n\frac{\Delta T}{2}, \quad (3.1.5)$$

where $n \in \mathbb{Z}$. Through equation (3.1.5) is how we formulate the photonic mesh lattice. using the snapshot representation of the dynamics given in Fig. (3.1.2) and re-writing the horizontal time axis in terms of the integer n , the photonic mesh lattice can obtained as shown in Fig. (3.1.3). Now the notation of the wave fields changes to include the discrete coordinates (n, m) such that $(u, v) \rightarrow (u_n^m, v_n^m)$. Further, we define (n, m) to be the synthetic discrete space-time coordinates, respectively. Pulses of light propagating the short loop given by u_n^m , are now equivalent to pulses shifting to the left in space n with each step in time m , while pulses propagating the long loop given by v_n^m is the same as shifting to the right in space n with each step in time m .

To derive the equations of motion for this discrete space-time lattice, we first recognise that in Fig. (3.1.3) the evolution of u_n^m and v_n^m at each successive step in time $(n, m+1)$ is dependent on the information at sites $(n \pm 1, m)$ and can be

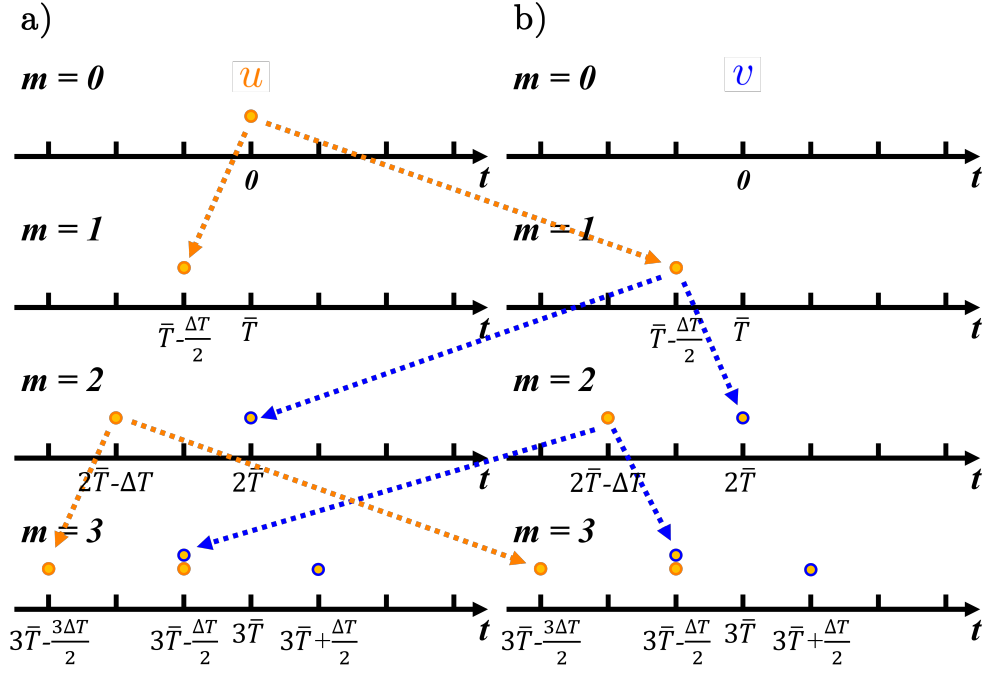


Figure 3.1.2: Panel a) and b) depict snapshots of what pulses are present in either loop after integer multiples of the average transit time $m\bar{T}$. The horizontal axis represents time t . The pulse outline indicates what loop the pulse arrived from, and the colour of the arrows shows how the pulses of light are split from that particular loop. As m increases, we see that more pulses emerge in either loop, shifted by an integer multiple of the difference in transit time $n\Delta T/2$. Pulses of light in the v loop always shift forwards in time, while pulses in the u loop shift backwards

given by some general matrix problem,

$$\begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} u_{n+1}^m \\ v_{n-1}^m \end{pmatrix}.$$

The matrix $\mathbf{M}$ is determined solely by the choice of splitting of the beam splitter and using the matrix representation for a generic splitting [94], the time evolution of both loops is given by,

$$\begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \begin{bmatrix} t & ir \\ ir & t \end{bmatrix} \begin{pmatrix} u_{n+1}^m \\ v_{n-1}^m \end{pmatrix}. \quad (3.1.6)$$

$t, r \in \mathbb{R}$ are constants that determine the transmission and reflection coefficients of the beam splitter, and are restricted by the relation $t^2 + r^2 = 1$. These parameters

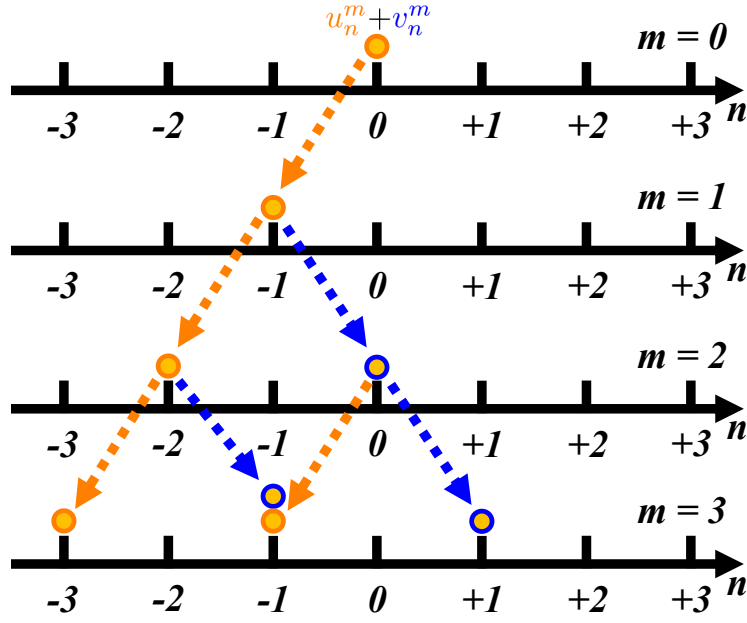


Figure 3.1.3: In this figure, we have taken the schematic of figure (3.1.2), swapped t for the integer n and superimposed the snapshots of the u_n^m and v_n^m loops. This forms the desired photonic mesh lattice.

are completely controllable experimentally, and determine the ratio of how the pulses of light are split between the short and long loop. From equation (3.1.6) two coupled linear recursive maps describing the evolution of u_n^m and v_n^m are obtained,

$$u_n^{m+1} = [tu_{n+1}^m + irv_{n-1}^m], \quad (3.1.7)$$

$$v_n^{m+1} = [ir u_{n+1}^m + tv_{n-1}^m]. \quad (3.1.8)$$

The commonplace convention used throughout the literature is that the index n is shifted for all the v_n^m fields so that $v_n^m \rightarrow v_{n+2}^m$ is substituted into equations (3.1.7) and (3.1.8),

$$u_n^{m+1} = [tu_{n+1}^m + irv_{n+1}^m], \quad (3.1.9)$$

$$v_{n+2}^{m+1} = [ir u_{n+1}^m + tv_{n+1}^m]. \quad (3.1.10)$$

Then, for equation (3.1.10) only, every n index is shifted by $n+2 \rightarrow n$ to arrive

at the typical linear equations of motion that model the photonic mesh lattice,

$$u_n^{m+1} = [tu_{n+1}^m + irv_{n+1}^m], \quad (3.1.11)$$

$$v_n^{m+1} = [iru_{n-1}^m + tv_{n-1}^m]. \quad (3.1.12)$$

This shifting ensures that all the n indices are the same on the right hand side of equations (3.1.11) and (3.1.12) and is merely a mathematical convenience.

3.1.2 Reverse time equations and the continuum limit

To derive an equivalent set of equations in reverse time given by $m-1$ we invert the matrix problem in equation (3.1.6),

$$\frac{1}{t^2 - (-ir)^2} \begin{bmatrix} t & -ir \\ -ir & t \end{bmatrix} \begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^m \\ v_{n-1}^m \end{pmatrix},$$

$$\frac{1}{t^2 + r^2} \begin{bmatrix} t & -ir \\ -ir & t \end{bmatrix} \begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^m \\ v_{n-1}^m \end{pmatrix}.$$

Since $t^2 + r^2 = 1$, the denominator on the left hand side simplifies to 1. Then we shift the indices $(n, m) \rightarrow (n-1, m-1)$ for the u_n^m fields and $(n, m) \rightarrow (n+1, m-1)$ for the v_n^m fields. With this we obtain the reverse time matrix problem,

$$\begin{bmatrix} t & -ir \\ -ir & t \end{bmatrix} \begin{pmatrix} u_{n-1}^m \\ v_{n+1}^m \end{pmatrix} = \begin{pmatrix} u_n^{m-1} \\ v_n^{m-1} \end{pmatrix}. \quad (3.1.13)$$

The reverse time equations of motion can then be written as,

$$u_n^{m-1} = [tu_{n-1}^m - irv_{n+1}^m], \quad (3.1.14)$$

$$v_n^{m-1} = [-iru_{n-1}^m + tv_{n+1}^m]. \quad (3.1.15)$$

To reach the continuum limit, as shown in [86], the reverse time equations (3.1.14) (3.1.15) can be subtracted from the respective forward time equations (3.1.11)

(3.1.12) to form central finite differences in both space and time,

$$\begin{aligned} u_n^{m+1} - u_n^{m-1} &= \frac{1}{\sqrt{2}} [u_{n+1}^m + iv_{n+1}^m - u_{n-1}^m + iv_{n-1}^m], \\ &= \frac{1}{\sqrt{2}} [u_{n+1}^m - u_{n-1}^m + i2v_{n+1}^m], \end{aligned} \quad (3.1.16)$$

$$\begin{aligned} v_n^{m+1} - v_n^{m-1} &= \frac{1}{\sqrt{2}} [iu_{n-1}^m + v_{n-1}^m + iu_{n+1}^m - v_{n+1}^m], \\ &= \frac{1}{\sqrt{2}} [v_{n-1}^m - v_{n+1}^m + i2u_{n-1}^m]. \end{aligned} \quad (3.1.17)$$

Note here that we choose $t = r = 1/\sqrt{2}$ as to simplify the equations, which is the case when the splitting between loops is 50 : 50. Now, define a new variable $V_n^m = v_{n+1}^m$. By shifting $n \rightarrow n + 1$ in equation (3.1.17), we obtain,

$$u_n^{m+1} - u_n^{m-1} = \frac{1}{\sqrt{2}} [u_{n+1}^m - u_{n-1}^m + i2V_n^m], \quad (3.1.18)$$

$$V_n^{m+1} - V_n^{m-1} = \frac{1}{\sqrt{2}} [V_{n-1}^m - V_{n+1}^m + i2u_n^m]. \quad (3.1.19)$$

A central finite differences in time can be taken on the left of equations and in space on the right of equations (3.1.18) (3.1.19) bringing the system into a continuum limit in both space and time,

$$\sqrt{2}\Delta t \partial_t u(x, t) = +\Delta x \partial_x u(x, t) + 2iV(x, t), \quad (3.1.20)$$

$$\sqrt{2}\Delta t \partial_t V(x, t) = -\Delta x \partial_x V(x, t) + 2iu(x, t), \quad (3.1.21)$$

where Δx and Δt are some arbitrary spacing in space and time. Then we define the speed of light of the mesh lattice to be $c = 1/\sqrt{2}$, similar to the convention of natural units ubiquitous in particle and atomic physics. From this we obtain the (1+1D) Dirac equation:

$$\partial_t u(x, t) = +\frac{\Delta x}{\Delta t} c \partial_x u(x, t) + \frac{2\sqrt{2}i}{\Delta t} c^2 V(x, t), \quad (3.1.22)$$

$$\partial_t V(x, t) = -\frac{\Delta x}{\Delta t} c \partial_x V(x, t) + \frac{2\sqrt{2}i}{\Delta t} c^2 u(x, t). \quad (3.1.23)$$

By taking linear combinations of equations (3.1.22) and (3.1.23) and defining $\psi_{\pm} = u \pm V$ to be the Dirac spinors, we obtain the final version of the (1+1D) Dirac equation,

$$\partial_t \psi_+ = +\frac{\Delta x}{\Delta t} c \partial_x \psi_- + \frac{2\sqrt{2}i}{\Delta t} c^2 \psi_+, \quad (3.1.24)$$

$$\partial_t \psi_- = +\frac{\Delta x}{\Delta t} c \partial_x \psi_+ - \frac{2\sqrt{2}i}{\Delta t} c^2 \psi_-. \quad (3.1.25)$$

From Hamilton's equations we can obtain a Hamiltonian for equations (3.1.24) (3.1.25),

$$i\partial_t \psi_{\pm} = \frac{\partial H_{\pm}}{\partial \psi_{\pm}^*}. \quad (3.1.26)$$

Hence by multiplying equations (3.1.24) (3.1.25) by $\psi_{\pm}^*$ we can obtain $H_{\pm}$ and the total Hamiltonian H ,

$$\begin{aligned} H_+ &= \int +i\alpha \psi_+^* \partial_x \psi_- + \mu |\psi_+|^2 dx, \\ H_- &= \int +i\alpha \psi_-^* \partial_x \psi_+ - \mu |\psi_-|^2 dx, \\ H &= \int i\alpha (\psi_+^* \partial_x \psi_- + \psi_-^* \partial_x \psi_+) + \mu (-|\psi_+|^2 + |\psi_-|^2) dx, \end{aligned} \quad (3.1.27)$$

that is consistent with the operator typically seen in literature [95]. For convenience we set $\alpha = \frac{\Delta x}{\Delta t} c$ and $\mu = \frac{2\sqrt{2}}{\Delta t} c^2$. Panel a-b) in Fig. (3.1.4) demonstrate numerically that this Hamiltonian is a real time invariant quantity, up to numerical precision, an expected property of Hamiltonian systems.

3.1.3 Conserved quantities in the linear theory

In the instance such that the photonic mesh lattice introduces no losses and is not externally forced or dissipated, it possess one known conserved quantity given by the optical power P :

$$P = \sum_{n=-N/2}^{N/2-1} |u_n^m|^2 + |v_n^m|^2, \quad (3.1.28)$$

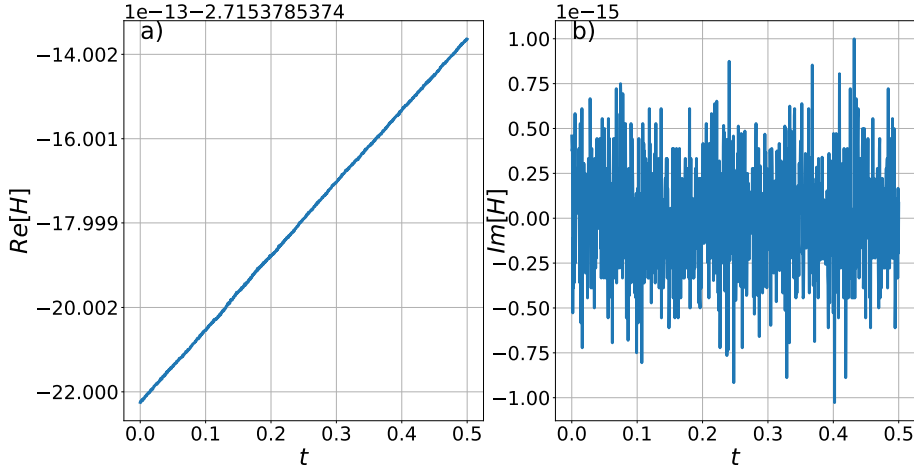


Figure 3.1.4: a) and b) shows the real and imaginary parts of the derived Hamiltonian evolved according to equations (3.1.24) (3.1.25). We set $\Delta x = 0.0625$, $\Delta t = 0.0005$, $\alpha = 1$ such that $c = 0.008$ and $\mu = 0.362$. An initial condition given by random noise was used for $\psi(x, t = 0)_{\pm}$. Panel a) shows that the real part of H is time invariant, up to some precision and b) shows that the imaginary part is computationally zero.

where $N \in \mathbb{Z}$ determines the total width of the lattice in space n . P is time invariant, such that at each step in time it remains constant. To show invariance in the discrete-time setting, we consider the difference of P^m at subsequent time steps where we include the superscript m for demonstration,

$$P^{m+1} - P^m = \sum_{n=-N/2}^{N/2-1} (|u_n^{m+1}|^2 + |v_n^{m+1}|^2) - (|u_n^m|^2 + |v_n^m|^2). \quad (3.1.29)$$

By substituting the forward time equations (3.1.11) (3.1.12) into equation (3.1.29):

$$\begin{aligned} P^{m+1} - P^m = \sum_{n=-N/2}^{N/2-1} & \left(|tu_{n+1}^m + irv_{n+1}^m|^2 + |tv_{n-1}^m + iru_{n-1}^m|^2 \right) \\ & - (|u_n^m|^2 + |v_n^m|^2). \end{aligned} \quad (3.1.30)$$

Expanding out equation (3.1.30),

$$\begin{aligned}
P^{m+1} - P^m &= \sum_{n=-N/2}^{N/2-1} t^2 |u_{n+1}^m|^2 - itr u_{n+1}^m v_{n+1}^{m*} + itr v_{n+1}^m u_{n+1}^{m*} + r^2 |v_{n+1}^m|^2 \\
&\quad + t^2 |v_{n-1}^m|^2 - itr v_{n-1}^m u_{n-1}^{m*} + itr u_{n-1}^m v_{n-1}^{m*} + r^2 |u_{n-1}^m|^2 - (|u_n^m|^2 + |v_n^m|^2).
\end{aligned} \tag{3.1.31}$$

Since n is a dummy index, we are free to re-label the indices as $n+1 \rightarrow j$ $n-1 \rightarrow k$ and express equation (3.1.31) as three separate sums over the indices j, k, n ,

$$\begin{aligned}
P^{m+1} - P^m &= \sum_{j=-N/2}^{N/2-1} t^2 |u_j^m|^2 - itr u_j^m v_j^{m*} + itr v_j^m u_j^{m*} + r^2 |v_j^m|^2 \\
&\quad + \sum_{k=-N/2}^{N/2-1} t^2 |v_k^m|^2 - itr v_k^m u_k^{m*} + itr u_k^m v_k^{m*} + r^2 |u_k^m|^2 - \sum_{n=-N/2}^{N/2-1} (|u_n^m|^2 + |v_n^m|^2).
\end{aligned} \tag{3.1.32}$$

Using the property $t^2 + r^2 = 1$ from the beam splitter, the terms with index n can be written as,

$$\begin{aligned}
P^{m+1} - P^m &= \sum_{j=-N/2}^{N/2-1} t^2 |u_j^m|^2 - itr u_j^m v_j^{m*} + itr v_j^m u_j^{m*} + r^2 |v_j^m|^2 \\
&\quad + \sum_{k=-N/2}^{N/2-1} t^2 |v_k^m|^2 - itr v_k^m u_k^{m*} + itr u_k^m v_k^{m*} + r^2 |u_k^m|^2 \\
&\quad - \sum_{n=-N/2}^{N/2-1} (t^2 + r^2)(|u_n^m|^2 + |v_n^m|^2).
\end{aligned} \tag{3.1.33}$$

Since j, k, n are dummy indices, they can all be re-labelled as n , cancelling all terms in equation (3.1.33) shown by the colours below,

$$\begin{aligned}
P^{m+1} - P^m &= \sum_{n=-N/2}^{N/2-1} t^2 |u_n^m|^2 - itr u_n^m v_n^{m*} + itr v_n^m u_n^{m*} + r^2 |v_n^m|^2 \\
&\quad + \sum_{n=-N/2}^{N/2-1} t^2 |v_n^m|^2 - itr v_n^m u_n^{m*} + itr u_n^m v_n^{m*} + r^2 |u_n^m|^2 \\
&\quad - \sum_{n=-N/2}^{N/2-1} (t^2 + r^2)(|u_n^m|^2 + |v_n^m|^2) = 0.
\end{aligned} \tag{3.1.34}$$

This conservation holds for periodic boundary conditions, reflective boundary conditions, where light propagating in the u_n^m path switches to the v_n^m path at the left boundary edge at $-N/2$ and vice versa for the right edge at $N/2 - 1$, and an infinitely wide system.

As currently understood, the optical power is the only conserved quantity present in the photonic mesh lattice. The absence of other quantities such as a Hamiltonian is suspected to be consequence of the discrete-time evolution. By discretising in time, the symplectic form of the Hamiltonian is no longer preserved. This is demonstrated in section (1.2) for the simple harmonic oscillator. Hence we are not free to simply discretise in space and time the (1+1)D Dirac Hamiltonian and obtain a new conserved quantity. For example if we discretise in space and time using central finite differences for the derivatives equation (3.1.27) becomes,

$$H_n^m = \sum_n \frac{i\alpha}{2\Delta x} (\psi_{n+}^{m*} (\psi_{n+1-}^m - \psi_{n-1-}^m) + \psi_{n-}^{m*} (\psi_{n+1+}^m - \psi_{n-1+}^m) + \mu(-|\psi_{n+}^m|^2 + |\psi_{n-}^m|^2)). \quad (3.1.35)$$

We can evolve this quantity according to the lattice equations (3.1.7) (3.1.8) when the transmission and reflection coefficients $t = r = 1/\sqrt{2}$, recognising that $\psi_{n\pm}^m = u_n^m \pm v_{n+1}^m$ and see that it is not time invariant, see Fig. (3.1.5).

One possible way to obtain a form for the Hamiltonian could be to guess a Hamiltonian, such that equations (3.1.7) (3.1.8) are obtained from a discretisation method that is symplectic, see chapter (2) for more information. This idea will not be explored in this thesis however.

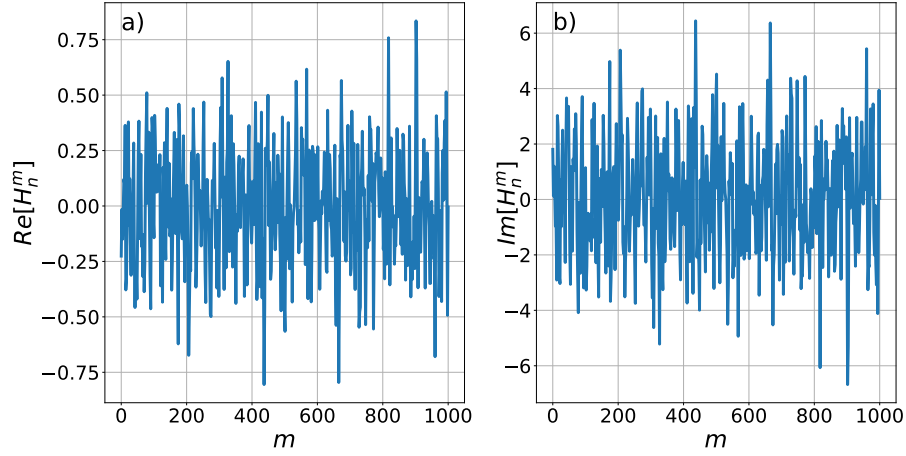


Figure 3.1.5: a) and b) shows the real and imaginary parts of the discretised Hamiltonian evolved according to equations (3.1.7) (3.1.8). We set $\Delta x = 0.0625$, $\Delta t = 0.0005$, $\alpha = 1$ such that $c = 0.008$ and $\mu = 0.362$. $t = r = 1/\sqrt{2}$ with $N = 320$, evolved for 1,000 time steps. The same initial condition used in Fig. (3.1.4) was used to create an initial condition in u_n^m v_n^m . We see that H_n^m is no longer time invariant or real.

3.2 Linear dispersion relation and normal modes in the stroboscopic evolution

Next we want to derive the dispersion relation and respective normal modes from the linear theory. The normal modes are the standard setting in WWT, and are where we shall observe the thermal equilibrium processes to occur. Here we will discuss the stroboscopic evolution of the lattice and derive the linear dispersion relation, using Bloch's theorem and Floquet theory. Then we will derive the linear normal modes and show some of the mathematical nuances in the derivation. Lastly we will test these modes numerically to ensure that everything is consistent.

3.2.1 Stroboscopic evolution

To start, let us re-introduce the equations of motion in forward time but now including a time dependent phase shift $\phi^m \in \mathbb{R}$, applied only to the short loop

wave field:

$$u_n^{m+1} = [tu_{n+1}^m + irv_{n+1}^m] e^{i\phi^m}, \quad (3.2.1)$$

$$v_n^{m+1} = [irv_{n-1}^m + tv_{n-1}^m]. \quad (3.2.2)$$

The equivalent schematic of equations (3.2.1) and (3.2.2) is given in Fig. (3.2.1) panel a) including the phase modulator needed to apply the phase shift. The introduction of the time dependent phase shift is something that is done experimentally to alter the geometry of the dispersion relation and will be discussed in more detail in the following subsection.

The new representation of the lattice highlights the atypical diamond connectivity present. This has important but rather subtle consequences in how we treat the system mathematically, as what spatial sites n we can access is dependent on time m . For example at $m = 0$, the only accessible spatial sites are $n = 0, \pm 2, \pm 4, \dots$, and at $m = 1$ only $n = \pm 1, \pm 3, \pm 5, \dots$ are accessible, losing access to the sites at

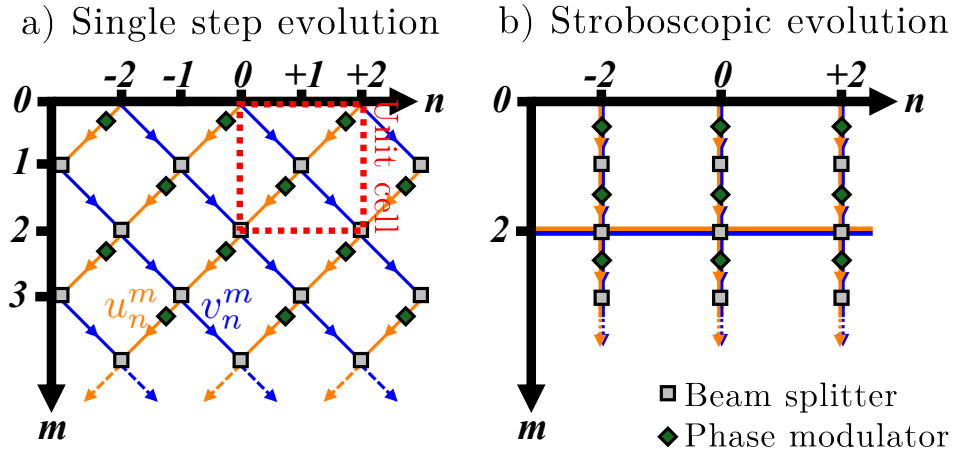


Figure 3.2.1: a) A new schematic of the photonic mesh lattice, with the n and m axes sketched. Arrows moving to the left are given in orange to represent the short loop u_n^m , while the right arrows are in blue representing the long loop v_n^m . A green diamond has also been included in the short loop, denoting the phase modulating device that applies a phase shift ϕ^m to the short loop wave field only. The red dashed square represents the unit cell—the smallest repeating unit used to build the lattice. b) Shows the same lattice, this time in the stroboscopic step. Now it is recast from a diamond to a square connected lattice that only accesses even coordinates in space-time (n, m) .

the previous step in time. However as shown by the red unit cell in Fig. (3.2.1) panel a), if we evolve the system two steps in time, we always pass information to the same n with each successive m . The two-step evolution is what we refer to as the stroboscopic evolution, and will be the framework in which we consider going forward.

By moving into the stroboscopic evolution, we are effectively working on a sublattice of typical square connectivity, see Fig. (3.2.1) panel b). The consequence of working in the stroboscopic step is that all theory must be considered in the two step and that we only initialise and sample at even sites in (n, m) . For example if the lattice has width N in space and length $M \in 2\mathbb{Z}^+$ in time then the stroboscopic lattice has width $N/2$ and length $M/2$, but is still evolved for the same number of single steps in time. This also introduces a further requirement, that the width N and length M of the lattice in space-time must be restricted to even integers, $N, M \in 2\mathbb{Z}$.

3.2.2 Linear dispersion relation

To obtain the linear dispersion relation, we follow the derivation found in the supplemental material of [77]. To start, we write the equations (3.2.1) and (3.2.2) in terms of the stroboscopic evolution,

$$\begin{aligned}
 u_n^{m+2} &= [tu_{n+1}^{m+1} + irv_{n+1}^{m+1}] e^{-i\phi_0} \\
 &= \left[t [tu_{n+2}^m + irv_{n+2}^m] e^{+i\phi_0} + ir [iru_n^m + tv_n^m] \right] e^{-i\phi_0} \\
 &= \left[t^2 u_{n+2}^m e^{+i\phi_0} + itr v_{n+2}^m e^{+i\phi_0} - r^2 u_n^m + itr v_n^m \right] e^{-i\phi_0}, \quad (3.2.3)
 \end{aligned}$$

$$\begin{aligned}
 v_n^{m+2} &= [iru_{n-1}^{m+1} + tv_{n-1}^{m+1}] \\
 &= \left[ir [tu_n^m + irv_n^m] e^{+i\phi_0} + t [iru_{n-2}^m + tv_{n-2}^m] \right] \\
 &= \left[itr u_n^m e^{+i\phi_0} - r^2 v_n^m e^{+i\phi_0} + itr u_{n-2}^m + t^2 v_{n-2}^m \right]. \quad (3.2.4)
 \end{aligned}$$

Here an alternating phase modulation has been chosen such that its magnitude is constant, but its sign changes with odd/even steps in time,

$$\phi^m = \begin{cases} +\phi_0 & \text{if } m+1 \text{ is odd,} \\ -\phi_0 & \text{if } m+1 \text{ is even.} \end{cases} \quad (3.2.5)$$

where $\phi_0 \in \mathbb{R}$. The choice of phase modulation is particularly relevant to the geometry of the dispersion relation, and will be discussed shortly. For now we proceed onwards and recognise that in order to obtain the linear dispersion relation, we must transform equations (3.2.3) and (3.2.4) into discrete Fourier space k and discrete frequency space ω_k . Since the system is discrete in both space and in time, we choose the following Bloch-Floquet ansatz,

$$\begin{pmatrix} u_n^m \\ v_n^m \end{pmatrix} = \sum_{k=-\pi}^{\pi-4\pi/N} \begin{pmatrix} \tilde{u}_k^m \\ \tilde{v}_k^m \end{pmatrix} e^{+i\frac{kn}{2}} = \sum_{k=-\pi}^{\pi-4\pi/N} \begin{pmatrix} U \\ V \end{pmatrix} e^{+i\frac{kn}{2}} e^{-i\frac{\omega_k m}{2}}. \quad (3.2.6)$$

k and ω_k are defined as $k = \left\{ \frac{4\pi}{N}n \mid n = -N/4, -N/4 + 1, \dots, N/4 - 1 \right\} \bmod 2\pi$ and $\omega_k = \left\{ \frac{4\pi}{M}m \mid m = 0, 1, \dots, M/2 - 1 \right\} \bmod 2\pi$ and both match the size of the stroboscopic lattice in n and m . Since we are sampling half the points over the same amount of space and time, we must double the spacing in both the k and ω_k spaces to accommodate, hence the factors $4\pi/N$ and $4\pi/M$. This also ensures consistency numerically when dealing with fast Fourier transforms. The wave fields $(\tilde{u}_k^m, \tilde{v}_k^m)^T$ are the discrete Fourier space analogues of $(u_n^m, v_n^m)^T$ and $(U, V)^T$ are some complex amplitudes. Further, we assume periodic boundary conditions in n such that lattice site $N/2 = -N/2$.

Now, substitute equation (3.2.6) into equations (3.2.3) and (3.2.4):

$$\begin{aligned}
 Ue^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m+2)}{2}} &= \left[t^2 Ue^{+\frac{ik(n+2)}{2}}e^{-\frac{i\omega_k(m)}{2}}e^{+i\phi_0} + itrVe^{+\frac{ik(n+2)}{2}}e^{-\frac{i\omega_k(m)}{2}}e^{+i\phi_0} \right. \\
 &\quad \left. - r^2 Ue^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m)}{2}} + itrVe^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m)}{2}} \right] e^{-i\phi_0} \\
 Ue^{-i\omega_k} &= \left[t^2 Ue^{+ik}e^{+i\phi_0} + itrVe^{+ik}e^{+i\phi_0} - r^2 U + itrV \right] e^{-i\phi_0}, \tag{3.2.7}
 \end{aligned}$$

$$\begin{aligned}
 Ve^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m+2)}{2}} &= \left[itrUe^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m)}{2}}e^{+i\phi_0} - r^2 Ve^{+\frac{ikn}{2}}e^{-\frac{i\omega_k(m)}{2}}e^{+i\phi_0} \right. \\
 &\quad \left. + itrUe^{+\frac{ik(n-2)}{2}}e^{-\frac{i\omega_k(m)}{2}} + t^2 Ve^{+\frac{ik(n-2)}{2}}e^{-\frac{i\omega_k(m)}{2}} \right] \\
 Ve^{-i\omega_k} &= \left[itrUe^{+i\phi_0} - r^2 Ve^{+i\phi_0} + itrUe^{-ik} + t^2 Ve^{-ik} \right], \tag{3.2.8}
 \end{aligned}$$

for which we can write the following matrix problem in $(U, V)^T$:

$$\underbrace{\begin{bmatrix} +t^2e^{+ik} - r^2e^{-i\phi_0} - e^{-i\omega_k} & +itre^{+ik} + itre^{-i\phi_0} \\ +itre^{+i\phi_0} + itre^{-ik} & -r^2e^{+i\phi_0} + t^2e^{-ik} - e^{-i\omega_k} \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2.9}$$

Thus non-trivial solutions are found when the determinant of $\mathbf{M}$ is equal to zero,

$$\begin{aligned}
 \det[M] &= (+t^2e^{+ik} - r^2e^{-i\phi_0} - e^{-i\omega_k})(-r^2e^{+i\phi_0} + t^2e^{-ik} - e^{-i\omega_k}) \\
 &\quad - (+itre^{+ik} + itre^{-i\phi_0})(+itre^{+i\phi_0} + itre^{-ik}) = 0, \\
 &= e^{-2i\omega_k} + e^{-i\omega_k} \left[-t^2e^{ik} + r^2e^{-i\phi_0} + r^2e^{+i\phi_0} - t^2e^{-ik} \right] \\
 &\quad - t^2r^2e^{+ik}e^{i\phi_0} + t^4 + r^4 - r^2t^2e^{-i\phi_0}e^{-ik} \\
 &\quad + t^2r^2(e^{+ik}e^{+i\phi_0} + 1 + 1 + e^{-i\phi_0}e^{-ik}) = 0.
 \end{aligned}$$

The above can be simplified through cancellations, trigonometric expressions and

$$t^2 + r^2 = 1,$$

$$\begin{aligned}
 \det[\mathbf{M}] &= e^{-2i\omega_k} + e^{-i\omega_k} \left[-t^2 2 \cos(k) + 2r^2 \cos(\phi_0) \right] \\
 &\quad - t^2 r^2 \cos(k + \phi_0) + 1 - 2t^2 r^2 + t^2 r^2 \cos(k + \phi_0) + 2t^2 r^2 = 0, \\
 &= e^{-2i\omega_k} + e^{-i\omega_k} \left[-t^2 2 \cos(k) + 2r^2 \cos(\phi_0) \right] + 1 = 0 \tag{3.2.10}
 \end{aligned}$$

Let $z = [-t^2 2 \cos(k) + 2r^2 \cos(\phi_0)]$, where we know that $-1 \leq z \leq 1$, since in general $-1 \leq \cos(x) \leq 1$ and $0 \leq t^2 \leq 1$ with $t^2 + r^2 = 1$. We now solve the quadratic in $e^{-i\omega_k}$,

$$e^{-i\omega_k} = \frac{-z \pm \sqrt{z^2 - 4}}{2} \quad (3.2.11)$$

Extracting a factor of -1 from the square root and taking the natural logarithms of both sides,

$$\omega_k = i \ln \left[\frac{-z}{2} \pm i \sqrt{1 - \left(\frac{-z}{2}\right)^2} \right] \quad (3.2.12)$$

We can then associate either branch of ω_k to the inverse cosine of z ,

$$\begin{aligned} \omega_k^+ &= i \ln \left[\frac{-z}{2} - i \sqrt{1 - \left(\frac{-z}{2}\right)^2} \right] = + \arccos \left(\frac{-z}{2} \right) \mod 2\pi, \\ \omega_k^- &= i \ln \left[\frac{-z}{2} + i \sqrt{1 - \left(\frac{-z}{2}\right)^2} \right] = - \arccos \left(\frac{-z}{2} \right) \mod 2\pi. \end{aligned}$$

Therefore upon substituting $z = [-t^2 2 \cos(k) + 2r^2 \cos(\phi_0)]$ we derive the linear dispersion relation $\omega_k^\pm$,

$$\begin{aligned} \omega_k^\pm &= \pm \arccos \left(\frac{-[-t^2 2 \cos(k) + 2r^2 \cos(\phi_0)]}{2} \right) \mod 2\pi, \\ &= \pm \arccos ([t^2 \cos(k) - r^2 \cos(\phi_0)]) \mod 2\pi. \end{aligned} \quad (3.2.13)$$

Figs. (3.2.2) and (3.2.3) show the dispersion relation for different choices of (t^2, ϕ_0) .

Equation (3.2.2) has two interesting properties. The first is that it is periodic in both k and ω_k because of discrete-space and discrete-time respectively. The periodicity in k allows for Umklapp wave scattering events, folding wave numbers back into k via the $\mod 2\pi$. The periodicity in ω_k allows for a similar property in the vertical direction of the $(k, \omega_k^\pm)$ space, introducing an infinite number of replica bands outside the first Brillouin-zone. The first Brillouin-zone is highlighted by the red shaded regions of each panel in Figs. (3.2.2) and (3.2.3) and the first set of duplicate bands are also plotted for demonstration.

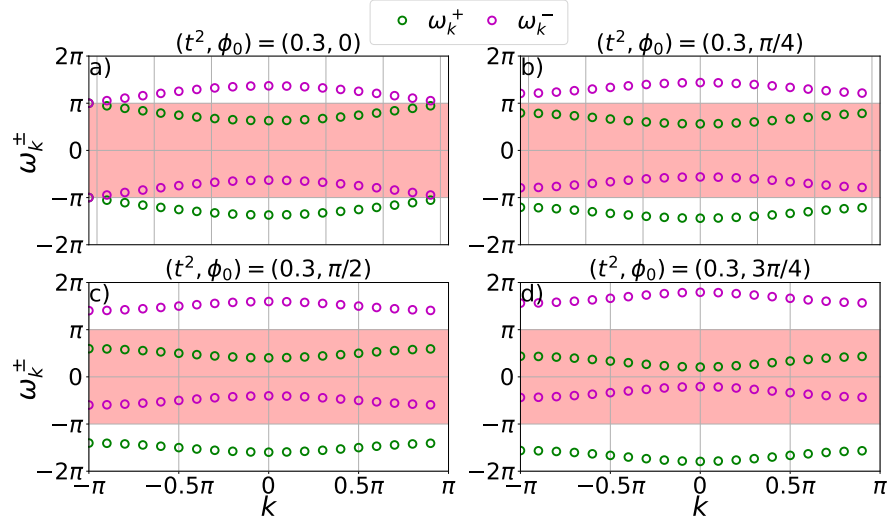


Figure 3.2.2: In panels a-d) we plot the discrete dispersion relation for fixed $t^2 = 0.3$ and varied ϕ_0 with $N = 40$.

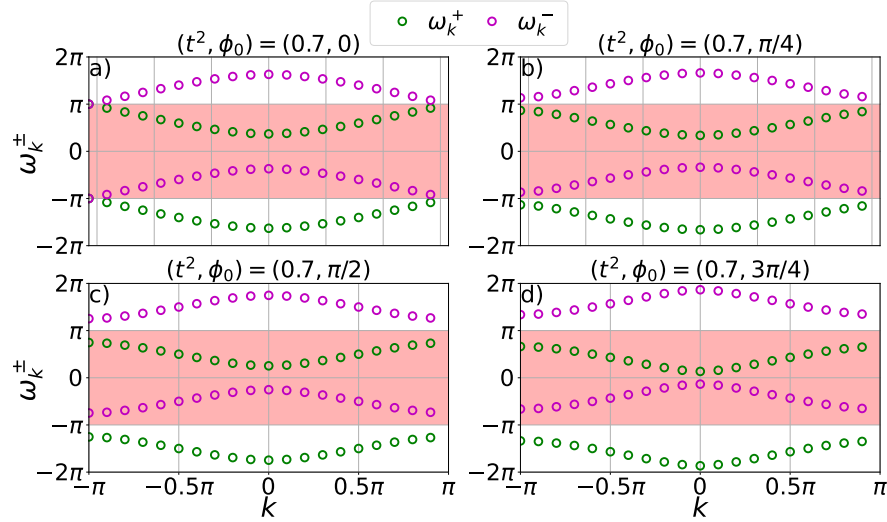


Figure 3.2.3: In panels a-d) we plot the discrete dispersion relation for fixed $t^2 = 0.7$ and varied ϕ_0 with $N = 40$.

The second interesting property is that the dispersion relation is comprised of two bands, for which the separation of these bands is controlled entirely by (t^2, ϕ_0) . Since the bands are periodic in $\omega_k^\pm$, this results in two distinct band gaps. The first gap is between the bands in the first Brillouin-zone at $k = 0$, and the second is between the first and second set of bands at the edge mode $k = -\pi$. In the instance that $\phi_0 = 0$, the edge mode gap closes for any choice of t^2 , resulting in a single band connected through the $\omega_k^\pm$ periodicity.

One such way to achieve band separation is through the choice of alternating phase modulation, given in equation (3.2.5). If the phase modulation did not alternate in sign with each single time step, then the bands are connected. It is worth noting that the choice of equation (3.2.5) is not the only way to separate bands. Isolation of the two bands can also be achieved through other means such as introducing auxiliary lattice sites at the inaccessible points of the single step lattice, and also by considering the lattice in a moving frame [86]. We will choose to keep the band separated, as this has been the choice for previous thermalisation studies in the lattice [92].

3.2.3 Linear normal modes

Associated to each band of the dispersion relation where ω_k^+ is the upper band and ω_k^- the lower band, are a set of unique normal modes which we will now derive in the case of alternating phase modulation. Once again, we substitute equation (3.2.6) into equations (3.2.3) (3.2.4), normalising $U = 1$,

$$e^{-i\omega_k^\pm} = \left[t^2 e^{+ik} e^{+i\phi_0} + itr V e^{+ik} e^{+i\phi_0} - r^2 + itr V \right] e^{-i\phi_0}, \quad (3.2.14)$$

$$V e^{-i\omega_k^\pm} = \left[itr e^{+i\phi_0} - r^2 V e^{+i\phi_0} + itr e^{-ik} + t^2 V e^{-ik} \right]. \quad (3.2.15)$$

Multiplying equation (3.2.14) by V and equating to equation (3.2.15),

$$\begin{aligned} \left[V t^2 e^{+ik} e^{+i\phi_0} + itr V^2 e^{+ik} e^{+i\phi_0} - r^2 V + itr V^2 \right] e^{-i\phi_0} = \\ \left[itr e^{+i\phi_0} - r^2 V e^{+i\phi_0} + itr e^{-ik} + t^2 V e^{-ik} \right]. \end{aligned} \quad (3.2.16)$$

We then want to solve for the amplitude V ,

$$\begin{aligned} V^2 \left[itr e^{+ik} + itr e^{-i\phi_0} \right] + V \left[t^2 e^{+ik} - r^2 e^{-i\phi_0} + r^2 e^{+i\phi_0} - t^2 e^{-ik} \right] \\ - itr e^{+i\phi_0} - itr e^{-ik} = 0. \end{aligned} \quad (3.2.17)$$

Using the trigonometric identities $2 \cos(x) = e^{ix} + e^{-ix}$ and $2i \sin(x) = e^{ix} - e^{-ix}$ we can simplify each term of equation (3.2.17),

$$V^2 2itr \cos\left(\frac{k + \phi_0}{2}\right) e^{\frac{ik}{2}} e^{\frac{-i\phi_0}{2}} + V \left[t^2 2i \sin(k) + r^2 2i \sin(\phi_0) \right] - itr \left[2 \cos\left(\frac{k + \phi_0}{2}\right) \right] e^{\frac{i\phi_0}{2}} e^{\frac{-ik}{2}} = 0, \quad (3.2.18)$$

and dividing equation (3.2.18) by $2itr \cos\left(\frac{k + \phi_0}{2}\right) e^{i\phi_0/2}$,

$$V^2 e^{\frac{ik}{2}} e^{-i\phi_0} + V \underbrace{\frac{t^2 \sin(k) + r^2 \sin(\phi_0)}{tr \cos\left(\frac{k + \phi_0}{2}\right)}}_F e^{\frac{-i\phi_0}{2}} - e^{\frac{-ik}{2}} = 0. \quad (3.2.19)$$

Equation (3.2.19) can now be solved for the two solution branches $V^\pm$,

$$\begin{aligned} V^\pm &= \frac{-F e^{\frac{-i\phi_0}{2}} \pm \sqrt{(F e^{\frac{-i\phi_0}{2}})^2 - 4 e^{\frac{ik}{2}} e^{-i\phi_0} (-e^{\frac{-ik}{2}})}}{2 e^{\frac{ik}{2}} e^{-i\phi_0}} \\ &= \frac{-F e^{\frac{-i\phi_0}{2}} e^{-\frac{ik}{2}} e^{+i\phi_0}}{2} \pm \sqrt{\left(\frac{F e^{\frac{-i\phi_0}{2}} e^{-\frac{ik}{2}} e^{+i\phi_0}}{2} \right)^2 + e^{-\frac{ik}{2}} e^{+i\phi_0}} \\ &= \frac{-F e^{-\frac{ik}{2}} e^{+i\phi_0}}{2} \pm \sqrt{(-1)^2 \left(\frac{F e^{-\frac{ik}{2}} e^{+i\phi_0}}{2} \right)^2 + e^{-\frac{ik}{2}} e^{+i\phi_0}} \\ &= \left[\frac{-F}{2} \pm \sqrt{\left(-\frac{F}{2} \right)^2 + 1} \right] e^{-\frac{ik}{2}} e^{+i\phi_0} \end{aligned} \quad (3.2.20)$$

We can then use the natural logarithm form of the inverse hyperbolic function $\operatorname{arcsinh}(x)$,

$$\begin{aligned} \ln \left[\frac{-F}{2} + \sqrt{\left(-\frac{F}{2} \right)^2 + 1} \right] &= -\operatorname{arcsinh} \left(\frac{F}{2} \right), \\ \ln \left[\frac{-F}{2} - \sqrt{\left(\frac{F}{2} \right)^2 + 1} \right] &= \operatorname{arcsinh} \left(-\frac{F}{2} \right) + i\pi. \end{aligned}$$

Hence the two solution branches for the amplitudes $V^\pm$ are,

$$V^+ = e^{-\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \quad (3.2.21)$$

$$V^- = e^{\lambda_k + i\pi} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} = -e^{\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \quad (3.2.22)$$

where,

$$\lambda_k = \operatorname{arcsinh} \left(\frac{t^2 \sin(k) + r^2 \sin(\phi_0)}{2tr \cos\left(\frac{k+\phi_0}{2}\right)} \right). \quad (3.2.23)$$

In order to correctly associate the amplitudes in equations (3.2.21) and (3.2.22), we can back substitute them into equation (3.2.14) and solve for when this is zero, e.g.:

$$\left[t^2 e^{+ik} e^{+i\phi_0} + itr V^\pm e^{+ik} e^{+i\phi_0} - r^2 + itr V^\pm \right] e^{-i\phi_0} - e^{-i\omega_k^\pm} = 0$$

This was checked numerically as shown in Fig. (3.2.4). Rather interestingly, we see that the definitions of $V^\pm$ are not solely associated to the upper or the lower band, but instead swap after any wave number $k \geq \pi - \phi_0$. The definition swap only occurs in the presence of non-zero phase modulation. The association of amplitudes $V^\pm$ given by equations (3.2.21) and (3.2.22) can be formally written as,

$$RH[(3.2.14)|_{V=V^+}] = \begin{cases} e^{-i\omega_k^+} & \text{for } k \geq \pi - \phi_0 \\ e^{-i\omega_k^-} & \text{for } k < \pi - \phi_0 \end{cases},$$

$$RH[(3.2.14)|_{V=V^-}] = \begin{cases} e^{-i\omega_k^+} & \text{for } k < \pi - \phi_0 \\ e^{-i\omega_k^-} & \text{for } k \geq \pi - \phi_0 \end{cases},$$

where $RH[(3.2.14)|_{V=V^\pm}]$ refers to the right hand side of equation (3.2.14) when $V = V^\pm$. Thus accounting for the definition swapping, the set of two normalised

eigenvector amplitudes can be written as,

$$|UB\rangle = \begin{cases} \frac{1}{\sqrt{1+e^{+2\lambda_k}}} \begin{pmatrix} 1 \\ -e^{+\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \end{pmatrix} & \text{for } k < \pi - \phi_0 \\ \frac{1}{\sqrt{1+e^{-2\lambda_k}}} \begin{pmatrix} 1 \\ +e^{-\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \end{pmatrix} & \text{for } k \geq \pi - \phi_0 \end{cases} \quad (3.2.24)$$

$$|LB\rangle = \begin{cases} \frac{1}{\sqrt{1+e^{-2\lambda_k}}} \begin{pmatrix} 1 \\ +e^{-\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \end{pmatrix} & \text{for } k < \pi - \phi_0 \\ \frac{1}{\sqrt{1+e^{+2\lambda_k}}} \begin{pmatrix} 1 \\ -e^{+\lambda_k} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \end{pmatrix} & \text{for } k \geq \pi - \phi_0 \end{cases} \quad (3.2.25)$$

obtaining the same form as previous derivations done without an alternating phase modulation [79]. Note that the intervals in which we associate these eigenvector amplitudes depends on the choice of ϕ_0 . If $\phi_0/(4\pi/N)$ is not an integer, then $\pi - \phi \notin k$, with the matching in Fig. (3.2.4) misaligning at that value.

Equations (3.2.24) and (3.2.25) are orthogonal in the linear evolution, possessing

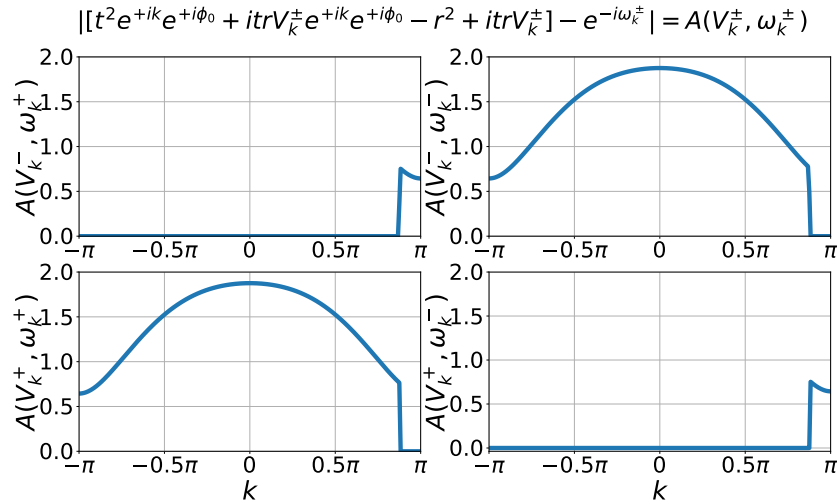


Figure 3.2.4: This figure shows what combinations of $V_k^\pm$ and $\omega_k^\pm$ match for different wave numbers k . Whenever $A(V_k^\pm, \omega_k^\pm) = 0$, the amplitude is correctly associated to the band of the dispersion relation.

the following properties:

$$\langle UB|UB\rangle = \langle LB|LB\rangle = 1,$$

$$\langle LB|UB\rangle = \langle UB|LB\rangle = 0.$$

We can use these properties to derive the linear normal modes associated to either band. This is done by considering a vector given by the linear superposition of equations (3.2.24) and (3.2.25) with some arbitrary coefficients,

$$|\tilde{u}_k^m \tilde{v}_k^m\rangle = a_k^{m+} |UB\rangle + a_k^{m-} |LB\rangle.$$

Solving for a_k^{m+} using $\langle UB|UB\rangle = 1$ and $\langle UB|LB\rangle = 0$:

$$\begin{aligned} \langle UB|\tilde{u}_k^m \tilde{v}_k^m\rangle &= a_k^{m+} \underbrace{\langle UB|UB\rangle}_{=1} + a_k^{m-} \underbrace{\langle UB|LB\rangle}_{=0}, \\ a_k^{m+} &= \begin{cases} \frac{\tilde{u}_k^m - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases} \end{aligned} \quad (3.2.26)$$

Equation (3.2.26) prescribe the linear normal modes associated to the upper band of the dispersion relation in terms of the Fourier space wave-fields $(\tilde{u}_k^m, \tilde{v}_k^m)$. We can do the exact same thing to solve for the linear normal modes associated to the lower band a_k^{m-} ,

$$\begin{aligned} \langle LB|\tilde{u}_k^m \tilde{v}_k^m\rangle &= a_k^{m+} \underbrace{\langle LB|UB\rangle}_{=0} + a_k^{m-} \underbrace{\langle LB|LB\rangle}_{=1}, \\ a_k^{m-} &= \begin{cases} \frac{\tilde{u}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^m - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases} \end{aligned} \quad (3.2.27)$$

Equations (3.2.26) and (3.2.27) are the linear normal modes of the photonic mesh lattice, and will be the primary means of studying nonlinear wave phenomena. This is because these modes are non-interacting in the linear evolution, and any interactions these modes have in the nonlinear evolution are completely a response

of the nonlinear effects. We know this from Floquet theory, in which the time evolution of the normal modes is given by,

$$a_k^{m+2\pm} = a_k^{m\pm} e^{-i\omega_k^\pm}, \quad (3.2.28)$$

describing the evolution of the modes in time by simply applying a phase shift of $\exp(-i\omega_k/2)$ at every single time step. Since the phase is complex, this implies that the intensity of every mode is constant in time,

$$|a_k^{m+2\pm}|^2 = a_k^{m\pm} e^{-i\omega_k^\pm} a_k^{m\pm*} e^{+i\omega_k^\pm} = |a_k^{m\pm}|^2, \quad (3.2.29)$$

hence making the modal intensity invariant in time for the linear evolution.

We can then re-arrange the expressions in equations (3.2.26) and (3.2.27) to write out the $\tilde{u}_k^m$ and $\tilde{v}_k^m$ wave fields completely in terms of the normal modes. Starting from equation (3.2.26),

$$\tilde{u}_k^m = a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + e^{\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m$$

which when substituted into equation (3.2.27),

$$\begin{aligned} a_k^{m-} &= \frac{a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + e^{\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1 + e^{-2\lambda_k}}} \\ -a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + a_k^{m-} \sqrt{1 + e^{-2\lambda_k}} &= \tilde{v}_k^m \left(e^{\lambda_k} + e^{-\lambda_k} \right) e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \\ \tilde{v}_k^m &= \frac{-a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + a_k^{m-} \sqrt{1 + e^{-2\lambda_k}}}{(e^{\lambda_k} + e^{-\lambda_k})} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} \end{aligned}$$

We simplify the square roots by,

$$\begin{aligned} \sqrt{1 + e^{2\lambda_k}} &= \sqrt{e^{\lambda_k} + e^{-\lambda_k}} e^{\frac{\lambda_k}{2}}, \\ \sqrt{1 + e^{-2\lambda_k}} &= \sqrt{e^{\lambda_k} + e^{-\lambda_k}} e^{-\frac{\lambda_k}{2}}, \end{aligned}$$

this reduces down the equation to,

$$\tilde{v}_k^m = \frac{-a_k^{m+} e^{+\frac{\lambda_k}{2}} + a_k^{m-} e^{-\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}}. \quad (3.2.30)$$

To get the expression for $\tilde{u}_k^m$ rearrange equation, (3.2.26),

$$\tilde{v}_k^m = \frac{-a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + \tilde{u}_k^m}{e^{\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}}}$$

and substitute into equation (3.2.27):

$$\begin{aligned} a_k^{m-} &= \frac{\tilde{u}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} - a_k^{m+} \sqrt{1 + e^{2\lambda_k}} + \tilde{u}_k^m}{e^{\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}}} \\ (1 + e^{-2\lambda_k}) \tilde{u}_k^m &= a_k^{m-} \sqrt{1 + e^{-2\lambda_k}} + a_k^{m+} e^{-2\lambda_k} \sqrt{1 + e^{+2\lambda_k}} \\ (e^{\lambda_k} + e^{-\lambda_k}) e^{-\lambda_k} \tilde{u}_k^m &= a_k^{m-} \sqrt{e^{\lambda_k} + e^{-\lambda_k}} e^{-\frac{\lambda_k}{2}} + a_k^{m+} \sqrt{e^{\lambda_k} + e^{-\lambda_k}} e^{-\frac{3\lambda_k}{2}} \\ \tilde{u}_k^m &= \frac{a_k^{m+} e^{-\frac{\lambda_k}{2}} + a_k^{m-} e^{+\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} \end{aligned} \quad (3.2.31)$$

Accounting for the definition swapping in equations (3.2.30) and (3.2.31) we obtain:

$$\tilde{u}_k^m = \begin{cases} \frac{a_k^{m+} e^{-\frac{\lambda_k}{2}} + a_k^{m-} e^{+\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{a_k^{m-} e^{-\frac{\lambda_k}{2}} + a_k^{m+} e^{+\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases} \quad (3.2.32)$$

$$\tilde{v}_k^m = \begin{cases} \frac{-a_k^{m+} e^{+\frac{\lambda_k}{2}} + a_k^{m-} e^{-\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} & \text{for } k < \pi - \phi_0 \\ \frac{-a_k^{m-} e^{+\frac{\lambda_k}{2}} + a_k^{m+} e^{-\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} & \text{for } k \geq \pi - \phi_0 \end{cases} \quad (3.2.33)$$

By deriving expression for the normal modes in equations (3.2.26) and (3.2.27), and the expression in equations (3.2.32) and (3.2.33), we now can select specific modes as an initial condition and evolve them in time using the single step equations (3.2.1) and (3.2.2). Fig. (3.2.5) illustrates this, by showing the linear evolution over 500 steps starting from different initial modal excitation for a variety of system parameters.

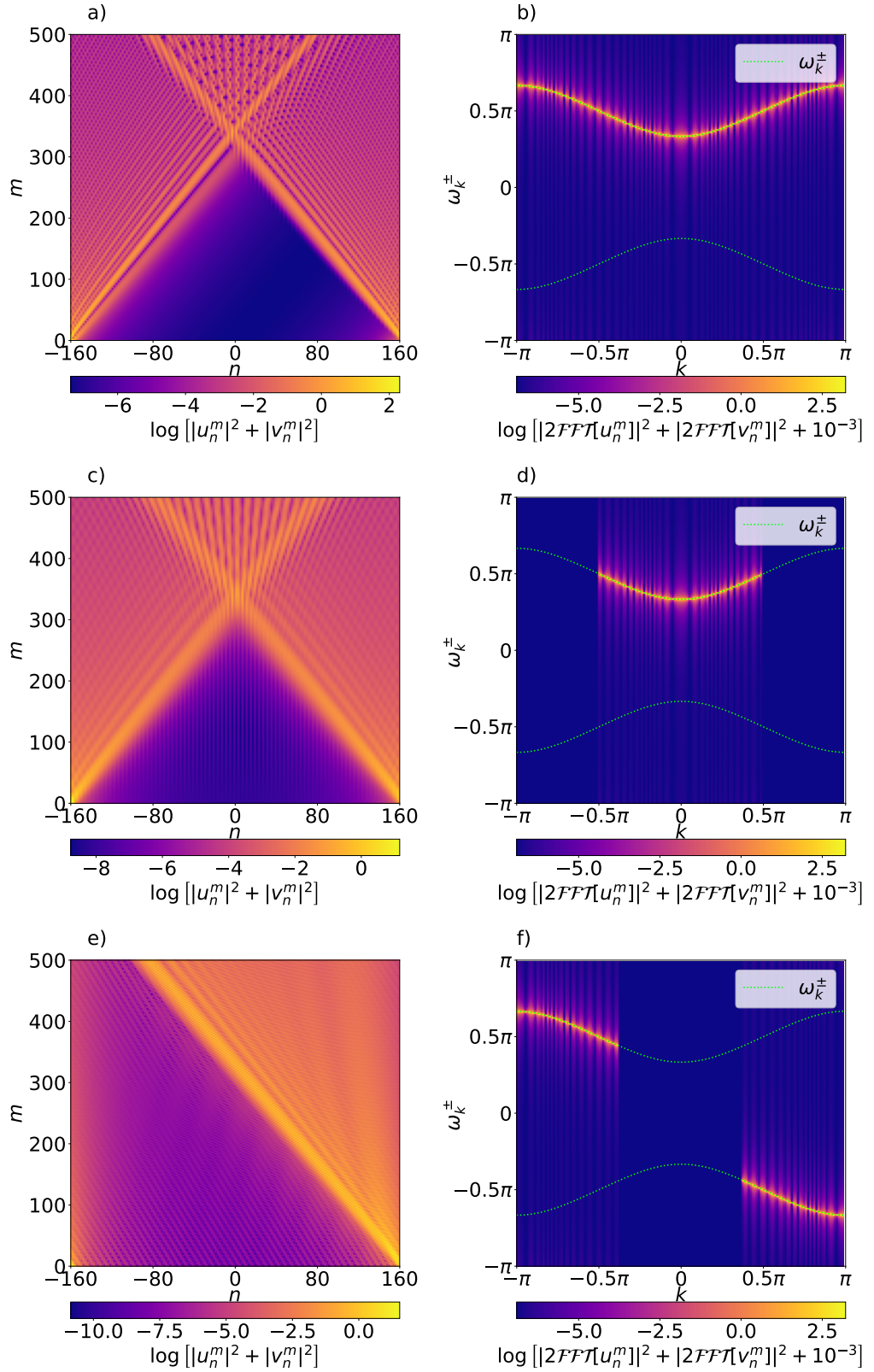


Figure 3.2.5: For lattice of spatial width $N = 320$, evolved for 500 steps in time, we show the linear evolution of different initially excited modes for different choices of parameters. Panels a), c) and e) show the real space evolution, and b), d) and f) show the $(k, \omega_k^\pm)$ spectrum recovered from the real space evolution. The factor of 10^{-3} in panels b), d) and f) is added for visual clarity of the spectrum when taking the logarithm.

In each case specific modes were initialised with an intensity of $|a_k^{m\pm}|^2 = 0.1$ and identical phases and were transformed into $(\tilde{u}_k^m, \tilde{v}_k^m)$ using equations (3.2.32) and (3.2.33). These were then evolved in time according to equations (3.2.1) and (3.2.2). The resulting spatio-temporal evolution was sampled stroboscopically, as shown in panels a), c) and e) in Fig. (3.2.5). This sampling produced a two-dimensional data set for the real space wave evolution in space and time which was then discrete Fourier transformed in space, and inverse discrete Fourier transformed in time to recover the (k, ω_k) spectrum as shown in the panels of b), d) and f) of Fig. (3.2.5). As expected, from the orthogonality of the modes, the initially excited modes, do not change intensity throughout the entire evolution.

3.3 Nonlinear theory

Now that we have obtained the linear dispersion relation and associated normal modes, we extend the photonic mesh lattice to include nonlinearity. In the nonlinear regime modal interactions are now possible and as such thermalisation can occur. We will now discuss how to implement nonlinearity, the effect on conservation of power and linear normal mode interactions. This will be then followed by the novelties of this thesis, concerned with thermalisation processes.

3.3.1 The fully nonlinear equations in forward time

Extending the evolution to include nonlinear effects, the equations of motion are derived from an analogous matrix problem to equation (3.1.6), but now incorporating Kerr-type phase shifts in both the u_n^m and v_n^m wave fields,

$$\begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \begin{bmatrix} te^{i\phi^m} & ire^{i\phi^m} \\ ir & t \end{bmatrix} \begin{pmatrix} u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} \\ v_{n-1}^m e^{i\chi|v_{n-1}^m|^2} \end{pmatrix}. \quad (3.3.1)$$

Here, the short loop phase modulation is absorbed into the matrix elements.

Kerr effects in optics arise from media with third order electric susceptibilities, typically written as $\chi^{(3)}$, where the intensity of the propagating pulse induces a small change in the refractive index [96]. The intensity dependent refractive index leads to a phase shift proportional to the local intensity of the wave, giving rise to self-phase modulation [97]. As such, we include the nonlinearity in our equations as a complete exponential term whose argument depends on the intensity of the pulse.

By applying the same index shifting as for the linear evolution, we can arrive at the fully nonlinear equations of motion,

$$u_n^{m+1} = \left[tu_{n+1}^m e^{i\chi|u_{n+1}^m|^2} + irv_{n+1}^m e^{i\chi|v_{n+1}^m|^2} \right] e^{i\phi^m}, \quad (3.3.2)$$

$$v_n^{m+1} = \left[tv_{n-1}^m e^{i\chi|v_{n-1}^m|^2} + iru_{n-1}^m e^{i\chi|u_{n-1}^m|^2} \right]. \quad (3.3.3)$$

The parameter χ denotes the third order electric susceptibility, for which we drop the superscript (3) for simplicity. Experimentally, the nonlinear effects are introduced into the system by extending the path lengths of both loops, such that a significant nonlinear phase shift can be accumulated via Kerr effects. Fig (3.3.1) shows a naive depiction of how the double loop experiment has to be altered to include nonlinear effects.

A very unusual feature of equations (3.3.2) and (3.3.3) is that the nonlinearity enters in terms of an exponential function. Since the exponential function can be written in terms of a power series,

$$e^{ix} = \sum_{j=0}^{\infty} \frac{(ix)^j}{j!}, \quad (3.3.4)$$

the nonlinear equations contain an infinite hierarchy of even wave interactions.

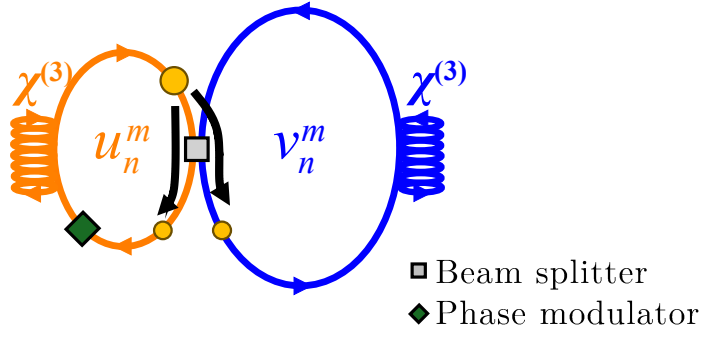


Figure 3.3.1: Here path length in both loops is extended by the same amount, represented by the coiled optical fibres. This allows for significant accumulation of a nonlinear phase shift for pulses in either loop brought about through the third order electric susceptibility $\chi^{(3)}$ of the fibre.

For example the following term of equation (3.3.2) can be expanded as,

$$u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} = \underbrace{u_{n+1}^m}_{1 \leftrightarrow 1} + \underbrace{i\chi|u_{n+1}^m|^2 u_{n+1}^m}_{2 \leftrightarrow 2} - \underbrace{\frac{\chi^2}{2}|u_{n+1}^m|^4 u_{n+1}^m}_{3 \leftrightarrow 3} - \underbrace{\frac{i\chi^3}{6}|u_{n+1}^m|^6 u_{n+1}^m}_{4 \leftrightarrow 4} + \dots \quad (3.3.5)$$

where $1 \leftrightarrow 1$ represents one incoming and outgoing wave, $2 \leftrightarrow 2$ represents two incoming and outgoing waves and so on. From the perspective of WWT, this implies the possibility of resonances occurring at every order of nonlinearity. This will be of particular relevance in the results later on in this chapter.

3.3.2 The fully nonlinear equations in reverse time

Obtaining the equations in reverse time is recovered by inverting the forward time matrix problem:

$$\frac{e^{-i\phi^m}}{t^2 - (ir)^2} \begin{bmatrix} t & -ir e^{i\phi^m} \\ -ir & t e^{i\phi^m} \end{bmatrix} \begin{pmatrix} u_n^{m+1} \\ v_n^{m+1} \end{pmatrix} = \begin{pmatrix} u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} \\ v_{n-1}^m e^{i\chi|v_{n-1}^m|^2} \end{pmatrix}. \quad (3.3.6)$$

We then shift $(n, m) \rightarrow (n-1, m-1)$ in the u_n^m field and $(n, m) \rightarrow (n+1, m-1)$

in the v_n^m field,

$$\begin{pmatrix} u_{n-1}^m \\ v_{n+1}^m \end{pmatrix} \begin{bmatrix} te^{-i\phi^m} & -ir \\ -ire^{-i\phi^m} & t \end{bmatrix} = \begin{pmatrix} u_n^{m-1} e^{i\chi|u_n^{m-1}|^2} \\ v_n^{m-1} e^{i\chi|v_n^{m-1}|^2} \end{pmatrix}. \quad (3.3.7)$$

Equation (3.3.2) yields the following reverse time evolution,

$$u_n^{m-1} = \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] e^{-i\chi|u_n^{m-1}|^2} \quad (3.3.8)$$

$$v_n^{m-1} = \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] e^{-i\chi|v_n^{m-1}|^2} \quad (3.3.9)$$

To ensure that all of the $m-1$ indices are on the left and the m indices are on the right of equations (3.3.8) (3.3.9), we recognise that,

$$\begin{aligned} |u_n^{m-1}|^2 &= \left| \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] e^{-i\chi|u_n^{m-1}|^2} \right|^2 = \left| \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] \right|^2 \\ |v_n^{m-1}|^2 &= \left| \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] e^{-i\chi|v_n^{m-1}|^2} \right|^2 = \left| \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] \right|^2 \end{aligned}$$

Hence we can substitute in the expression for $|u_n^{m-1}|^2$ and $|v_n^{m-1}|^2$ into equations (3.3.8) (3.3.9) to obtain the fully nonlinear reverse time equations,

$$u_n^{m-1} = \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] e^{-i\chi \left| \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] \right|^2} \quad (3.3.10)$$

$$v_n^{m-1} = \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] e^{-i\chi \left| \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] \right|^2} \quad (3.3.11)$$

The reverse time equations will be used later on to measure numerical error (see appendix (A.9)).

3.3.3 Conserved quantities in the nonlinear theory

We now return to the notion of conserved quantities and re-visit the optical power P ,

$$P^m = \sum_{n=-N/2}^{N/2-1} |u_n^m|^2 + |v_n^m|^2,$$

To reiterate, conservation in time is given by,

$$\dot{P} = P^{m+1} - P^m = \sum_{n=-N/2}^{N/2-1} (|u_n^{m+1}|^2 + |v_n^{m+1}|^2) - (|u_n^m|^2 + |v_n^m|^2). \quad (3.3.12)$$

By substituting fully nonlinear equations (3.3.2) (3.3.3) into equation (3.3.12):

$$\begin{aligned} \dot{P} = P^{m+1} - P^m = \sum_{n=-N/2}^{N/2-1} & \left(\left| \left[t u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} + i r v_{n+1}^m e^{i\chi|v_{n+1}^m|^2} \right] e^{i\phi(m)} \right|^2 \right. \\ & \left. + \left| \left[t v_{n-1}^m e^{i\chi|v_{n-1}^m|^2} + i r u_{n-1}^m e^{i\chi|u_{n-1}^m|^2} \right] \right|^2 \right) - (|u_n^m|^2 + |v_n^m|^2). \end{aligned} \quad (3.3.13)$$

The following substitution can be made to drastically simplify things,

$$\begin{aligned} A_n^m &= u_n^m e^{i\chi|u_n^m|^2}, \quad |A_n^m|^2 = |u_n^m|^2 \\ B_n^m &= v_n^m e^{i\chi|v_n^m|^2}, \quad |B_n^m|^2 = |v_n^m|^2. \end{aligned}$$

We can now expand equation (3.3.13) to be,

$$\begin{aligned} \dot{P} = P^{m+1} - P^m = \sum_{n=-N/2}^{N/2-1} & t^2 |A_{n+1}^m|^2 - i t r A_{n+1}^m B_{n+1}^{m*} + i t r B_{n+1}^m A_{n+1}^{m*} + r^2 |B_{n+1}^m|^2 \\ & + t^2 |B_{n-1}^m|^2 - i t r B_{n-1}^m A_{n-1}^{m*} + i t r A_{n-1}^m B_{n-1}^{m*} + r^2 |A_{n-1}^m|^2 - (|A_n^m|^2 + |B_n^m|^2). \end{aligned} \quad (3.3.14)$$

This is exactly the same form as equation (3.1.32) with u_n^m replaced by A_n^m and v_n^m replaced by B_n^m , and will therefore cancel to zero in the same way.

However, if we make the substitution for weak nonlinearity instead given by truncating the exponential up to $\mathcal{O}(\chi)$ for example,

$$\begin{aligned} A_n^m &\approx u_n^m + i\chi u_n^m |u_n^m|^2, \quad |A_n^m|^2 \neq |u_n^m|^2 \\ B_n^m &\approx v_n^m + i\chi v_n^m |v_n^m|^2, \quad |B_n^m|^2 \neq |v_n^m|^2, \end{aligned}$$

The optical power is no longer conserved in time because $|A_n^m|^2 \neq |u_n^m|^2$ and

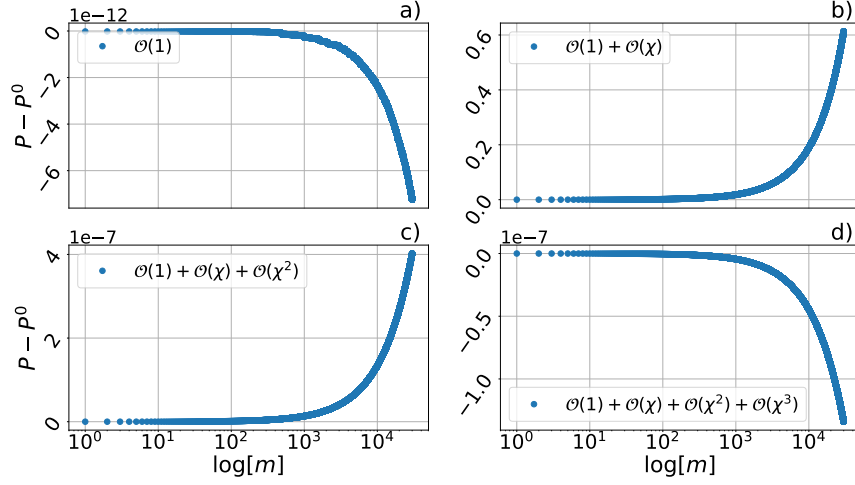


Figure 3.3.2: For an initial power $P^0 = 8$, $N = 320$, $\chi = 0.04$, $t^2 = 0.3$ and $\phi_0 = \pi/8$, the power evolution is plotted for varying degrees of truncation in panels a)-d) over 30,000 steps in m . a) is the linear and power is conserved up to the order of 10^{-12} . b) is the order χ case, in which the power deviates by approximately 10% of the initial value P^0 . c) and d) show that including the order χ^2 and order χ^3 terms significantly improves the power conservation, with deviations of the order 10^{-7} . Through deviations in the total power, we can understand what timescales are valid for different orders of truncation.

$|B_n^m|^2 \neq |v_n^m|^2$. As a consequence the dynamics become unstable, with the total power P exhibiting unbounded growth in time. Fig. (3.3.2) illustrates this by showing the temporal evolution of the power for the same initial condition under different orders of truncation.

Interestingly, the instability diminishes with the addition of higher order terms in χ , as these contributions partially balance one another. Therefore, care must be taken when truncating the nonlinear terms, with the validity of the dynamics generally limited to short timescales for lower-order truncations.

3.3.4 Linear normal mode interactions and nonlinear strength

Under the influence of nonlinearity the linear normal modes $a_k^{m\pm}$ can interact and in this section we shall demonstrate this numerically, and discuss the effects of the strength of the nonlinear wave interaction. To begin, we will first show the effect on the dynamics of starting from randomly distributed mode phases, one

of the assumptions made in WWT.

Panels a)–b) in Fig. (3.3.3) show the nonlinear evolution, starting from an initial condition equally occupying every upper band mode when all the initial phases are equal. In the real space evolution in panel a), two well-defined solitary waves emerge that propagate down the lattice for the full 500 time steps. The emergence of these waves is a direct consequence of phase coherence with the identical phases leading to constructive interference, producing large, localized intensity peaks. In the presence of nonlinearity, these high-amplitude regions experience enhanced self-phase modulation that balances the waves dispersion through chirping, enabling the formation and persistence of long-lived solitary structures [97]. These structures are also reflected in panel b), with the bright horizontal fringe between the two bands and the slightly tilted fringe below the lower band. The tilt of this fringe corresponds to the transverse velocity component of the solitary wave [98]. Since the fringes are approximately horizontal, the transverse components are very small and hence the solitary waves travel straight down the lattice.

By contrast panels c)-d) starts from the same upper band occupation but with randomised phases, drastically altering the wave dynamics. Since the phases are randomised the phase coherence is very weak and the modes no longer constructively interfere. This leaves the real-space evolution in panel c) to be a sea of weakly interacting waves. Equivalently panel d) no longer possesses the bright fringes, and only shows a high area of intensity slightly shifted off the linear dispersion relation. Notably now, the upper band modes occupy a broadened shape of the linear dispersion relation, with a fuzz of intensity surrounding the band. The slight shift is consequence of $\chi > 0$ shifting the band downwards in $\omega_k^\pm$ through the self-phase modulation; for $\chi < 0$ the shift would be upwards. The broadening effect, also referred to as nonlinear broadening is also a consequence of these Kerr effects, with the modes evolving over a wider frequency distribution. We also note that there is slight modal occupation in the lower band, demonstrating that the modes interact through nonlinearity.

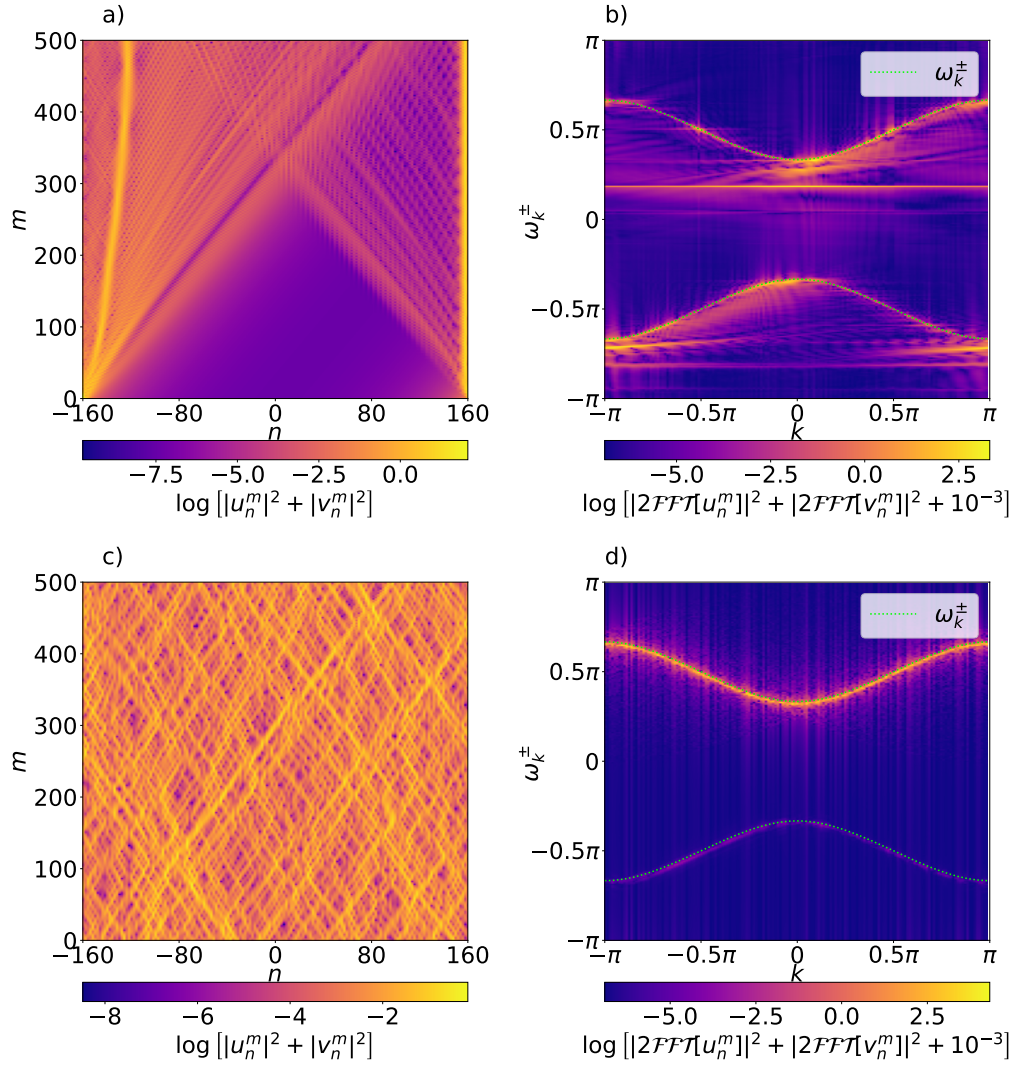


Figure 3.3.3: Here we show two sets of simulations, where $N = 320$, $(t^2\phi_0) = (0.5, \pi/2)$, $P = 16$ and $\chi = 0.15$. Both simulations initially excite every upper band mode. Panels a)-b) do not have random phases while c)-d) do have random phases.

Starting from random phases, as required in WWT theory, helps ensure that modes do not interact too strongly forming coherent structures. In general, the formation of coherent structures is undesirable in the WWT context, as they can inhibit the free movement of wave action by trapping wave intensity within a limited set of modes.

Now that we have seen the effect of phase coherence on the strength of wave dynamics, we will next consider contributors that effect the strength of the nonlinearity. The first contribution is the parameter χ . It is obvious that for

larger χ the strength of the wave interactions will increase, since the nonlinear contributions at each order of interaction j scale proportionally to χ^j :

$$u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} = u_{n+1}^m + i\chi u_{n+1}^m |u_{n+1}^m|^2 - \frac{\chi^2}{2} u_{n+1}^m |u_{n+1}^m|^4 + \dots$$

Another property of the system that will influence the strength of nonlinearity is the mode intensities $|a_k^{m\pm}|^2$, with greater intensities enabling stronger wave interactions. The initial mode intensity is of course determined by the initial condition and evolves with time, making it difficult to parametrise the instantaneous nonlinear strength per mode. As such, we consider an average measure of mode strength given by the power density P/N . We therefore choose to measure the strength of nonlinearity as:

$$\frac{\chi P}{N}. \quad (3.3.15)$$

In Fig. (3.3.4) we show that by changing χ , P and N , we can achieve qualitatively similar dynamics provided that $\chi P/N$ is the same.

As $\chi P/N \propto \chi |u_n^m|^2, \chi |v_n^m|^2$, which determine the strength of nonlinearity, we can use equation (3.3.15) to ensure that the wave dynamics are weakly nonlinear fitting into the framework of WWT theory. More specifically we require that:

$$\frac{\chi P}{N} < \mathcal{O}(1) \quad (3.3.16)$$

ensuring that the nonlinear wave dynamics are weak compared to the linear waves.

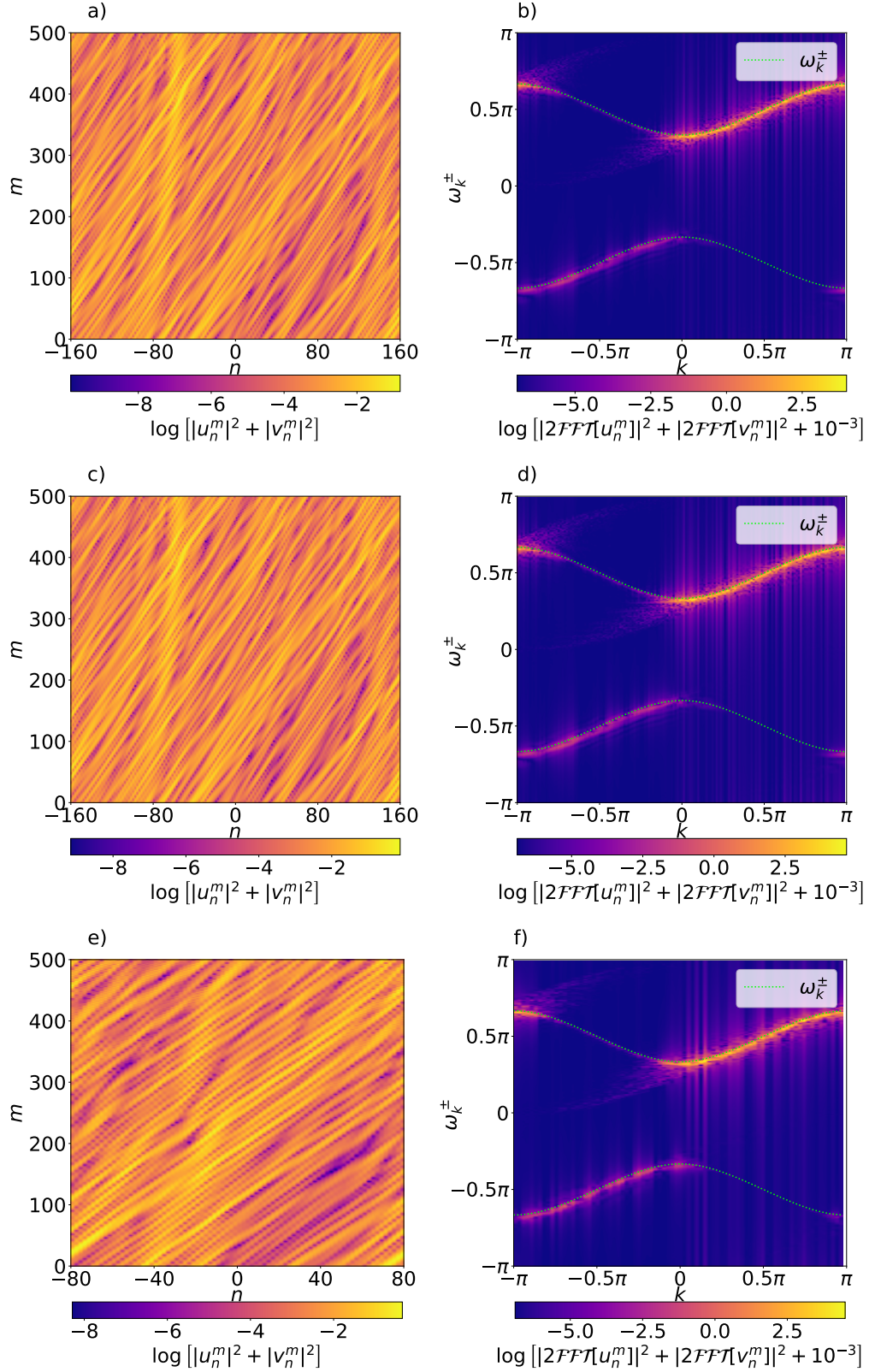


Figure 3.3.4: Here we perform three sets of simulations, starting from excitation of the upper band modes where $k \geq 0$ for $(t^2, \phi_0) = (0.5, \pi/2)$. In each case, $\chi P/N = 0.0075$. Panels a)-b) $\chi = 0.3$, $P = 8$, $N = 320$. c)-d) $\chi = 0.15$, $P = 16$, $N = 320$. e)-f) $\chi = 0.15$, $P = 8$, $N = 160$.

3.4 Thermalisation processes in a photonic mesh lattice

In this section, we will discuss the main results of this thesis, that pertain to thermalisation processes and observations of discrete time effects on these equilibrium processes.

3.4.1 Single band thermal equilibrium and power leakage

Prior to this thesis, experimental work had already been performed to explicitly show that thermal equilibrium processes occur in a single band, described by a Rayleigh-Jeans distribution [92]. The work observed that, the distribution had either a positive or negative temperature, depending on the initial modal occupation of the experiment. By exciting the middle modes of the upper band, a positive temperature Rayleigh-Jeans emerged in that band only, and by occupying the outer modes of the upper band, a negative temperature distribution followed. The Rayleigh-Jeans distribution was observed in the wave action spectrum, given by $n_k^{m\pm} = \langle |a_k^{m\pm}|^2 \rangle$, where $\langle \cdot \rangle$ is the ensemble average over many realisations, for which each realisation starts from an initial state where the phases are randomly distributed across the interval $[0, 2\pi)$.

To obtain the observed Rayleigh-Jeans, the authors assumed that in either band there was an optical power,

$$P^\pm = \sum_k |a_k^{m\pm}|^2, \quad (3.4.1)$$

which is time invariant in the absence of power transfer between bands. A linear energy in either band,

$$E_{lin}^\pm = \sum_k \omega_k^\pm n_k^{m\pm} \quad (3.4.2)$$

that is also time invariant assuming that the nonlinear waves are weakly interacting and that there is no power transfer between bands. Also a band

entropy, obtained by taking a Stirling approximation of the Boltzmann entropy formula is given by [99],

$$S^\pm = \sum_k \ln(n_k^{m^\pm}) \quad (3.4.3)$$

To obtain the Rayleigh-Jeans distribution observed experimentally, we maximise the entropy function with respect to the conserved quantities present. We do this by taking the functional derivative with respect to the wave action at thermal equilibrium, denoted as $n_k^{RJ\pm}$, and constrain equation (3.4.3) with respect to Lagrange multipliers $T^\pm$ and $\mu^\pm$, that are introduced from the conserved quantities in equations (3.4.1) (3.4.2):

$$\begin{aligned} \frac{\delta}{\delta n_k^{RJ\pm}} (T^\pm S^\pm - E_{lin}^\pm + \mu^\pm P^\pm) &= 0 \\ \Rightarrow n_k^{RJ\pm} &= \frac{T^\pm}{\omega_k^\pm - \mu^\pm}. \end{aligned} \quad (3.4.4)$$

$T^\pm, \mu^\pm \in \mathbb{R}$ are the temperature and chemical potential in the upper and lower band respectively. What is interesting here is that, because we possess two distinct bands for non-zero ϕ_0 , it is predicted that either band will have its own distinct $(T^\pm, \mu^\pm)$.

To begin the investigation into thermalisation processes, we repeat numerically the case of single band thermalisation, choosing the same parameters used experimentally, $t^2 = 0.3$ and $\phi_0 = \pi/8$, with a much weaker nonlinearity to ensure the wave dynamics are weakly nonlinear and that linear energy is conserved. An explanation of how the numerical simulations for many realisation are performed is given in appendix (A.7).

To visualise the evolution of the modes in either band, we now define the wave action of each mode in terms of the wave action at each frequency $\omega_k^\pm$ to match the typical conventions used within the literature,

$$n_\omega^\pm = n_k^\pm + n_{-k}^\pm. \quad (3.4.5)$$

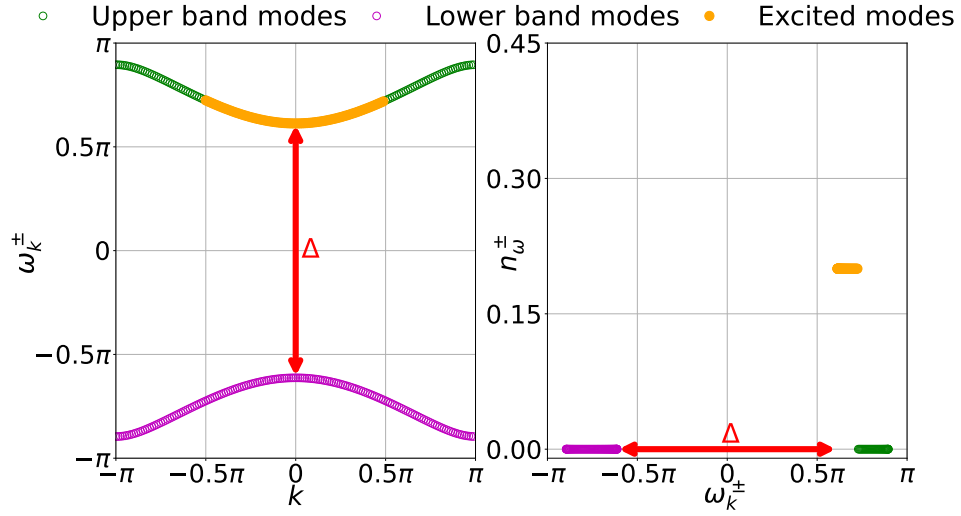


Figure 3.4.1: The left panel shows the typical two banded dispersion relation, and as an example, the upper band middle modes have been excited given by the orange points. We then plot this same state in terms of $n_{\omega}^{\pm}$ in the right panel, highlighting the band gap Δ in the dispersion relation for reference.

In this form we add the wave action at some mode k^* to the mode at $-k^*$ as these occupy the same frequency due to the symmetry of the bands about $k = 0$. Fig. (3.4.1) serves as a guide on how to interpret this new representation.

We now evolve the lattice for 1,500,000 steps in time which is shown over the snapshots in Fig. (3.4.2). It is noted that this far exceeds experimental lattice sizes and the timescale of the observed Rayleigh-Jeans in [92] ($m = 500$). However, we have intentionally made the nonlinearity much smaller than experiment in these preliminary simulations, to ensure that we are in the regime of WWT.

In each panel the initial condition is given by the shaded region. As the lattice evolves the upper band reorganises its occupation into a Rayleigh-Jeans distribution of positive temperature fitted by the red curve, matching the previous experimental observations [92] and more recent computational works [93]. An explanation of how the fitting is obtained is given in appendix (A.8). The single band thermal equilibrium can be characterised by time invariant (T^+, μ^+) , as this implies that the measured distribution is stationary in time. We plot how $(T^{\pm}, \mu^{\pm})$ evolve in Fig. (3.4.3) panels a)-b).

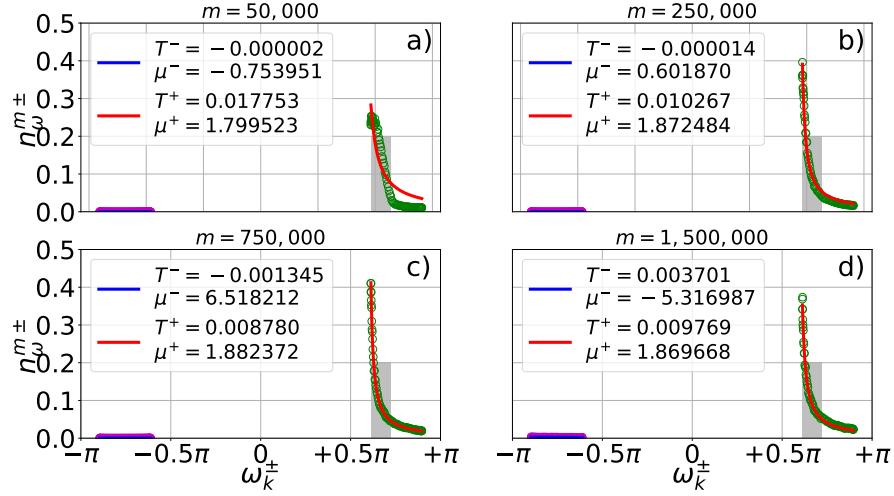


Figure 3.4.2: Panels a)-d) show snapshots of the wave action spectrum at different points throughout the evolution. Here $(t^2, \phi_0) = (0.3, \pi/8)$, $P = 8$, $N = 320$ and $\chi \approx 0.1767$ such that $\chi P/N \approx 0.0044$. The upper modes are given by the green circles and the lower modes the magenta circles. The blue curve fits the lower band Rayleigh-Jeans spectrum and the red fits the upper band spectrum. The initial condition is given by the grey shaded region of each plot.

Interestingly, once (T^+, μ^+) reach a steady value at $\sim 500,000$ time steps, a significant non-zero temperature is measured in the lower band. From this point onwards the upper band temperature begins to slowly increase. The non-zero temperature measurement T^- occurs because a small amount of power is transferred from the upper band to the lower band as evident in panels c)-d) in Fig.(3.4.3). Effectively, once the upper band reaches a Rayleigh-Jeans distribution, a power leakage effect takes hold between bands, destabilising the single band thermal state. This observation was not previously reported in [92] and raises the question of whether there is a full thermal state between two bands.

Before addressing the idea of a full thermal equilibrium, the cause of the power leakage must be explained physically, to ensure that it is not an artefact of numerical error. Numerical errors are discussed in more detail in appendix (A.9). In order to explain the leakage process, we will now derive the weakly nonlinear equations of the linear normal modes $a_k^{m\pm}$ and search for non-zero modal interactions that could enable power transfer between bands. These

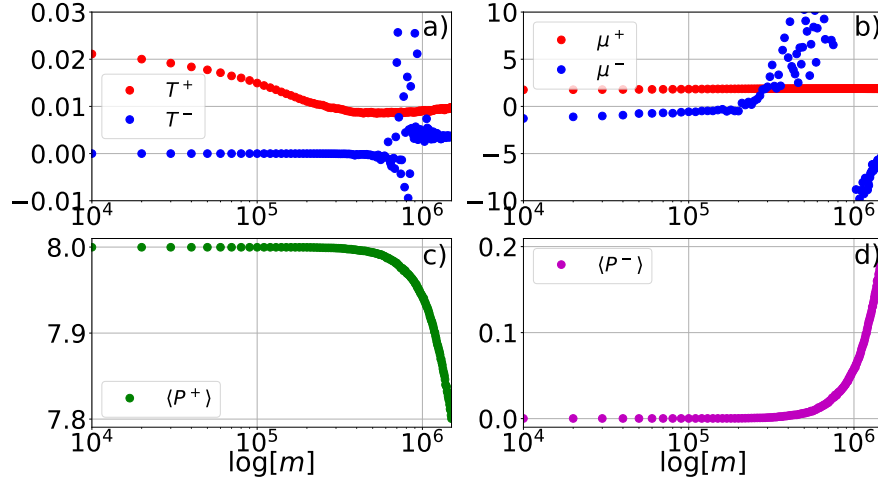


Figure 3.4.3: Panel a) is the evolution of $T^\pm$ in time. Panel b) is the evolution of $\mu^\pm$ in time. In panel a) we see that in the region of 400,000 to 600,000 time steps, T^+ is approximately constant in time and so is μ^+ in b). Around this time scale, we see the lower band fitting reading non-zero temperature, signifying that the distribution in the lower band is no longer approximately zero; power has begun to leak into the lower band evident in c) and d). Once the slow power leakage takes hold, it can be seen that the T^+ starts to slowly increase in time for the rest of the time window.

terms would be of the form,

$$a_1^{m+2-} \sim \chi \sum_{234} T_{12}^{34} a_2^{m*+} a_3^{m*+} a_4^{m*+} \delta_{12}^{23}. \quad (3.4.6)$$

Here we have used the same short hand notation as in chapter (2) with $a_{k_1}^{m+} \rightarrow a_1^{m+}$, $\sum_{k_2, k_3, k_4} \rightarrow \sum_{234}$ and the Kronecker delta $\delta_{k_1+k_2, k_3+k_4} = \delta_{12}^{34}$. Equation (3.4.6) is a wave interaction in which three modes from the upper band interact to excite a single mode in the lower band. The coefficient $T_{12}^{34} \in \mathbb{C}$ is the respective interaction coefficient of this four wave scattering process, and in order for the leakage to be physical, this must be non-zero.

3.4.2 Deriving power leakage terms for the weakly nonlinear equations of motion of the normal modes

To begin the derivation, we consider the single step equations of motion up to $\mathcal{O}(\chi)$ for the usual alternating phase modulation,

$$u_n^{m+1} = [tu_{n+1}^m + irv_{n+1}^m + \chi(itu_{n+1}^m|u_{n+1}^m|^2 - rv_{n+1}^m|v_{n+1}^m|^2)]e^{+i\phi_0}, \quad (3.4.7)$$

$$v_n^{m+1} = [tv_{n-1}^m + iru_{n-1}^m + \chi(itv_{n-1}^m|v_{n-1}^m|^2 - ru_{n-1}^m|u_{n-1}^m|^2)], \quad (3.4.8)$$

in which we explicitly encode the phase modulation for each step in time. Next, we move into the stroboscopic evolution:

$$u_n^{m+2} = [tu_{n+1}^{m+1} + irv_{n+1}^{m+1} + \chi(itu_{n+1}^{m+1}|u_{n+1}^{m+1}|^2 - rv_{n+1}^{m+1}|v_{n+1}^{m+1}|^2)]e^{-i\phi_0}, \quad (3.4.9)$$

$$v_n^{m+2} = [tv_{n-1}^{m+1} + iru_{n-1}^{m+1} + \chi(itv_{n-1}^{m+1}|v_{n-1}^{m+1}|^2 - ru_{n-1}^{m+1}|u_{n-1}^{m+1}|^2)]. \quad (3.4.10)$$

For every u_n^m and v_n^m term in equations (3.4.9), (3.4.10) we substitute,

$$u_{n+1}^{m+1} = [tu_{n+2}^m + irv_{n+2}^m + \chi(itu_{n+2}^m|u_{n+2}^m|^2 - rv_{n+2}^m|v_{n+2}^m|^2)]e^{+i\phi_0}, \quad (3.4.11)$$

$$v_{n+1}^{m+1} = [tv_n^m + iru_n^m + \chi(itv_n^m|v_n^m|^2 - ru_n^m|u_n^m|^2)], \quad (3.4.12)$$

$$u_{n-1}^{m+1} = [tu_n^m + irv_n^m + \chi(itu_n^m|u_n^m|^2 - rv_n^m|v_n^m|^2)]e^{+i\phi_0}, \quad (3.4.13)$$

$$v_{n-1}^{m+1} = [tv_{n-2}^m + iru_{n-2}^m + \chi(itv_{n-2}^m|v_{n-2}^m|^2 - ru_{n-2}^m|u_{n-2}^m|^2)]. \quad (3.4.14)$$

For the cubic terms at time in equations (3.4.9), (3.4.10) we again truncate to order χ neglecting terms that emerge at higher orders. For example, consider the orders of χ in the following term,

$$\begin{aligned} it\chi u_{n+1}^{m+1}|u_{n+1}^{m+1}|^2 &\approx it\mathcal{O}(\chi)[\mathcal{O}(1) + \mathcal{O}(\chi)][\mathcal{O}(1) + \mathcal{O}(\chi)][\mathcal{O}(1)^* + \mathcal{O}(\chi)^*]e^{i\phi_0}e^{i\phi_0}e^{-i\phi_0} \\ &\approx it\mathcal{O}(\chi)\mathcal{O}(1)\mathcal{O}(1)\mathcal{O}(1)^*e^{i\phi_0} \end{aligned}$$

Upon expanding out the brackets, only a single term appears at $\mathcal{O}(\chi)$, with the remaining terms appearing at higher orders. As a consequence, the orders of wave interaction don't occur at different orders of χ in the two step evolution,

but instead span several orders: $\mathcal{O}(\chi)$, $\mathcal{O}(\chi^2)$, $\mathcal{O}(\chi^3)$, $\mathcal{O}(\chi^4)$.

So upon substituting equations (3.4.11) - (3.4.14) into equations (3.4.9) and (3.4.10) keeping only the $\mathcal{O}(\chi)$ contributions from the cubic terms, we arrive at the weakly nonlinear stroboscopic evolution equations,

$$\begin{aligned} u_n^{m+2} = & \left[t^2 u_{n+2}^m + itr v_{n+2}^m + \chi (it^2 u_{n+2}^m |u_{n+2}^m|^2 - rt v_{n+2}^m |v_{n+2}^m|^2) \right] e^{i\phi_0} \\ & + itr v_n^m - r^2 u_n^m + \chi (-tr v_n^m |v_n^m|^2 - ir^2 u_n^m |u_n^m|^2) \\ & + it\chi [tu_{n+2}^m + ir v_{n+2}^m] [tu_{n+2}^m + ir v_{n+2}^m] [tu_{n+2}^{m*} - ir v_{n+2}^{m*}] e^{i\phi_0} \\ & - r\chi [tv_n^m + ir u_n^m] [tv_n^m + ir u_n^m] [tv_n^{m*} - ir u_n^{m*}] \Big] e^{-i\phi_0} \end{aligned} \quad (3.4.15)$$

$$\begin{aligned} v_n^{m+2} = & \left[t^2 v_{n-2}^m + itr u_{n-2}^m + \chi (it^2 v_{n-2}^m |v_{n-2}^m|^2 - rt u_{n-2}^m |u_{n-2}^m|^2) \right] e^{i\phi_0} \\ & + (itr u_n^m - r^2 v_n^m + \chi (-rt u_n^m |u_n^m|^2 - ir^2 v_n^m |v_n^m|^2)) e^{+i\phi_0} \\ & + it\chi [tv_{n-2}^m + ir u_{n-2}^m] [tv_{n-2}^m + ir u_{n-2}^m] [tv_{n-2}^{m*} - ir u_{n-2}^{m*}] \\ & - r\chi [tu_n^m + ir v_n^m] [tu_n^m + ir v_n^m] [tu_n^{m*} - ir v_n^{m*}] \Big] e^{i\phi_0} \end{aligned} \quad (3.4.16)$$

These equations are dynamically unstable due to the truncation of the nonlinear terms and can therefore not be compared to the fully nonlinear equations (3.3.2) (3.3.3) for long time scales. However, at short time scales when the instability is small, we can make comparisons to the fully nonlinear equations. As such our aim now is to derive the four wave term evaluated at the first stroboscopic step in time and numerically compare this to the fully nonlinear evolution. We will also use these real space equations (3.4.15) (3.4.16) as a reference point to ensure that the remainder of the derivation stays consistent, as these weakly nonlinear equations should diverge in the same way as their normal mode analogue.

Next we substitute the Bloch component of the ansatz in equation (3.2.6),

$$\begin{pmatrix} u_n^m \\ v_n^m \end{pmatrix} = \sum_{k=-\pi}^{\pi-4\pi/N} \begin{pmatrix} \tilde{u}_k^m \\ \tilde{v}_k^m \end{pmatrix} e^{+i\frac{kn}{2}},$$

into equations (3.4.15) (3.4.16) after expanding out all the terms. This gives the

two weakly nonlinear maps in discrete Fourier space,

$$\tilde{u}_1^{m+2} = \left[t^2 \tilde{u}_1^m e^{i\phi_0} e^{ik_1} + itr \tilde{v}_1^m e^{i\phi_0} e^{ik_1} + itr \tilde{v}_1^m - r^2 \tilde{u}_1^m + \chi \sum_{234} \mathcal{F}_u^m \delta_{12}^{34} \right] e^{-i\phi_0}, \quad (3.4.17)$$

$$\tilde{v}_1^{m+2} = \left[t^2 \tilde{v}_1^m e^{-ik_1} + itr \tilde{u}_1^m e^{-ik_1} + itr \tilde{u}_1^m e^{i\phi_0} - r^2 \tilde{v}_1^m e^{i\phi_0} + \chi \sum_{234} \mathcal{F}_v^m \delta_{12}^{34} \right], \quad (3.4.18)$$

The terms $\mathcal{F}_u^m$ and $\mathcal{F}_v^m$ in equations (3.4.17) and (3.4.18) are the nonlinear contributions of either equation and can be written in full as,

$$\begin{aligned} \mathcal{F}_u^m = & \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m \left[(it^2 e^{i\phi_0} + it^4 e^{i\phi_0}) e^{i(k_3+k_4-k_2)} + (-ir^2 - ir^4) \right] \\ & + \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \left[(-rte^{i\phi_0} - tr^3 e^{i\phi_0}) e^{i(k_3+k_4-k_2)} + (-tr - rt^3) \right] \\ & - \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m \left[2t^3 r e^{i\phi_0} e^{i(k_3+k_4-k_2)} + 2tr^3 \right] \\ & + \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m \left[2it^2 r^2 e^{i\phi_0} e^{i(k_3+k_4-k_2)} - 2it^2 r^2 \right] \\ & + \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m \left[t^3 r e^{i\phi_0} e^{i(k_3+k_4-k_2)} + tr^3 \right] \\ & + \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \left[-it^2 r^2 e^{i\phi_0} e^{i(k_3+k_4-k_2)} + it^2 r^2 \right] \end{aligned} \quad (3.4.19)$$

$$\begin{aligned} \mathcal{F}_v^m = & \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \left[(it^2 + it^4) e^{-i(k_3+k_4-k_2)} + (-ir^2 e^{+i\phi_0} - ir^4 e^{+i\phi_0}) \right] \\ & + \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m \left[(-rt - tr^3) e^{-i(k_3+k_4-k_2)} + (-rte^{+i\phi_0} - rt^3 e^{+i\phi_0}) \right] \\ & - \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{u}_4^m \left[2t^3 r e^{-i(k_3+k_4-k_2)} + 2tr^3 e^{i\phi_0} \right] \\ & + \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{u}_4^m \left[2it^2 r^2 e^{-i(k_3+k_4-k_2)} - 2it^2 r^2 e^{i\phi_0} \right] \\ & + \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \left[t^3 r e^{-i(k_3+k_4-k_2)} + tr^3 e^{i\phi_0} \right] \\ & + \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m \left[-it^2 r^2 e^{-i(k_3+k_4-k_2)} + it^2 r^2 e^{i\phi_0} \right] \end{aligned} \quad (3.4.20)$$

From our definitions of the linear normal modes in equations (3.2.26) and (3.2.27)

we know that the stroboscopic evolution of either band is given as,

$$a_k^{m+2+} = \begin{cases} \frac{\tilde{u}_k^{m+2} - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^{m+2}}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^{m+2} + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^{m+2}}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases}$$

$$a_k^{m+2-} = \begin{cases} \frac{\tilde{u}_k^{m+2} + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^{m+2}}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^{m+2} - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^{m+2}}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases}$$

By the addition of equations (3.4.17) and (3.4.18), the weakly evolution equations of the linear normal modes are,

$$a_1^{m+2+} = \begin{cases} a_1^{m+} e^{-i\omega_1^+} + \frac{\chi}{\sqrt{1+e^{2\lambda_1}}} \sum_{234} \left[\mathcal{F}_u^m e^{-i\phi_0} - e^{\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m \right] \delta_{12}^{34} & \text{for } k < \pi - \phi_0 \\ a_1^{m+} e^{-i\omega_1^+} + \frac{\chi}{\sqrt{1+e^{-2\lambda_1}}} \sum_{234} \left[\mathcal{F}_u^m e^{-i\phi_0} + e^{-\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m \right] \delta_{12}^{34} & \text{for } k \geq \pi - \phi_0, \end{cases} \quad (3.4.21)$$

$$a_1^{m+2-} = \begin{cases} a_1^{m-} e^{-i\omega_1^-} + \frac{\chi}{\sqrt{1+e^{-2\lambda_1}}} \sum_{234} \left[\mathcal{F}_u^m e^{-i\phi_0} + e^{-\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m \right] \delta_{12}^{34} & \text{for } k < \pi - \phi_0, \\ a_1^{m-} e^{-i\omega_1^-} + \frac{\chi}{\sqrt{1+e^{2\lambda_1}}} \sum_{234} \left[\mathcal{F}_u^m e^{-i\phi_0} - e^{\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m \right] \delta_{12}^{34} & \text{for } k \geq \pi - \phi_0 \end{cases} \quad (3.4.22)$$

with the linear contributions factoring down to the phases $\exp(-i\omega_1^\pm)$ as per the Floquet theory. The nonlinear terms of equations (3.4.21) and (3.4.22) can then be factorised,

$$\begin{aligned} \mathcal{F}_u^m e^{-i\phi_0} - e^{\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m &= A^+ \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m + B^+ \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m + C^+ \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m \\ &\quad + D^+ \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m + E^+ \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m + F^+ \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \end{aligned} \quad (3.4.23)$$

$$\begin{aligned} \mathcal{F}_u^m e^{-i\phi_0} + e^{-\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \mathcal{F}_v^m &= A^- \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m + B^- \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m + C^- \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m \\ &\quad + D^- \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m + E^- \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m + F^- \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m \end{aligned} \quad (3.4.24)$$

with each coefficient written as,

$$\begin{aligned} A^\pm = & + \left[(it^2 e^{i\phi_0} + it^4 e^{i\phi_0}) e^{i(k_3+k_4-k_2)} + (-ir^2 - ir^4) \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \left[(-rt - tr^3) e^{-i(k_3+k_4-k_2)} + (-rte^{+i\phi_0} - rt^3 e^{+i\phi_0}) \right] \end{aligned} \quad (3.4.25)$$

$$\begin{aligned} B^\pm = & + \left[(-rte^{i\phi_0} - tr^3 e^{i\phi_0}) e^{i(k_3+k_4-k_2)} + (-tr - rt^3) \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \left[(it^2 + it^4) e^{-i(k_3+k_4-k_2)} + (-ir^2 e^{+i\phi_0} - ir^4 e^{+i\phi_0}) \right] \end{aligned} \quad (3.4.26)$$

$$\begin{aligned} C^\pm = & - \left[2t^3 r e^{i\phi_0} e^{i(k_3+k_4-k_2)} + 2tr^3 \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \left[2it^2 r^2 e^{-i(k_3+k_4-k_2)} - 2it^2 r^2 e^{i\phi_0} \right] \end{aligned} \quad (3.4.27)$$

$$\begin{aligned} D^\pm = & + \left[2it^2 r^2 e^{i\phi_0} e^{i(k_3+k_4-k_2)} - 2it^2 r^2 \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} (-1) \left[2t^3 r e^{-i(k_3+k_4-k_2)} + 2tr^3 e^{i\phi_0} \right] \end{aligned} \quad (3.4.28)$$

$$\begin{aligned} E^\pm = & + \left[t^3 r e^{i\phi_0} e^{i(k_3+k_4-k_2)} + tr^3 \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \left[-it^2 r^2 e^{-i(k_3+k_4-k_2)} + it^2 r^2 e^{i\phi_0} \right] \end{aligned} \quad (3.4.29)$$

$$\begin{aligned} F^\pm = & + \left[-it^2 r^2 e^{i\phi_0} e^{i(k_3+k_4-k_2)} + it^2 r^2 \right] e^{-i\phi_0} \\ & \mp e^{\pm\lambda_1} e^{\frac{ik_1}{2}} e^{-\frac{i\phi_0}{2}} \left[t^3 r e^{-i(k_3+k_4-k_2)} + tr^3 e^{i\phi_0} \right] \end{aligned} \quad (3.4.30)$$

To express the nonlinear components in equations (3.4.23) and (3.4.24) in terms of the normal modes we use equations (3.2.32) and (3.2.33),

$$\begin{aligned} \tilde{u}_k^m = & \begin{cases} \frac{a_k^{m+} e^{-\frac{\lambda_k}{2}} + a_k^{m-} e^{+\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{a_k^{m-} e^{-\frac{\lambda_k}{2}} + a_k^{m+} e^{+\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases} \\ \tilde{v}_k^m = & \begin{cases} \frac{-a_k^{m+} e^{+\frac{\lambda_k}{2}} + a_k^{m-} e^{-\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} & \text{for } k < \pi - \phi_0 \\ \frac{-a_k^{m-} e^{+\frac{\lambda_k}{2}} + a_k^{m+} e^{-\frac{\lambda_k}{2}}}{\sqrt{e^{+\lambda_k} + e^{-\lambda_k}}} e^{-\frac{ik}{2}} e^{+\frac{i\phi_0}{2}} & \text{for } k \geq \pi - \phi_0 \end{cases} \end{aligned}$$

Therefore every product of three wave fields present in (3.4.23) and (3.4.24) can

be generically written as,

$$[Q_a^* + P_a^*][Q_b + P_b][Q_c + P_c] = Q_a^*Q_bQ_c + Q_a^*Q_bP_c + Q_a^*P_bQ_c + Q_a^*P_bP_c + \\ P_a^*Q_bQ_c + P_a^*Q_bP_c + P_a^*P_bQ_c + P_a^*P_bP_c,$$

where Q_a represents the upper band normal mode with some coefficient at mode a and P_a the lower band normal mode with some coefficient at mode a . In the case of single band excitation of just the upper band, any term containing P_a^* , P_b or P_c is zero for $m = 0$ as the lower band is unoccupied. Therefore each triple product of waves greatly simplifies,

$$[Q_a^* + P_a^*][Q_b + P_b][Q_c + P_c] = Q_a^*Q_bQ_c.$$

Hence if only the upper band is excited, all triple products of waves on the interval $k < \pi - \phi_0$ can be written as,

$$\begin{aligned} \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m &= + \frac{e^{-\frac{(\lambda_2 + \lambda_3 + \lambda_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \\ \tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m &= - \frac{e^{\frac{i(k_2 - k_3 - k_4)}{2}} e^{\frac{i\phi_0}{2}} e^{\frac{(\lambda_2 + \lambda_3 + \lambda_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \\ \tilde{u}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m &= - \frac{e^{\frac{-\lambda_2 - \lambda_3 + \lambda_4}{2}} e^{-\frac{ik_4}{2}} e^{\frac{i\phi_0}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \\ \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{+\lambda_2 - \lambda_3 + \lambda_4}{2}} e^{\frac{i(k_2 - k_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \\ \tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m &= - \frac{e^{\frac{+\lambda_2 - \lambda_3 - \lambda_4}{2}} e^{\frac{ik_2}{2}} e^{-\frac{i\phi_0}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \\ \tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{-\lambda_2 + \lambda_3 + \lambda_4}{2}} e^{-\frac{i(k_3 + k_4)}{2}} e^{i\phi_0}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{\lambda_3} + e^{-\lambda_3}} \sqrt{e^{\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^m a_4^{m+}] \end{aligned}$$

and for the interval $k \geq \pi - \phi_0$,

$$\begin{aligned}
\tilde{u}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m &= + \frac{e^{+\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}] \\
\tilde{v}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{i(k_2-k_3-k_4)}{2}} e^{+\frac{i\phi_0}{2}} e^{-\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}] \\
\tilde{u}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{+\lambda_2+\lambda_3-\lambda_4}{2}} e^{-\frac{ik_4}{2}} e^{+\frac{i\phi_0}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}] \\
\tilde{v}_2^{m*} \tilde{u}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{-\lambda_2+\lambda_3-\lambda_4}{2}} e^{\frac{i(k_2-k_4)}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}] \\
\tilde{v}_2^{m*} \tilde{u}_3^m \tilde{u}_4^m &= + \frac{e^{\frac{-\lambda_2+\lambda_3+\lambda_4}{2}} e^{+\frac{ik_2}{2}} e^{-\frac{i\phi_0}{2}}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}] \\
\tilde{u}_2^{m*} \tilde{v}_3^m \tilde{v}_4^m &= + \frac{e^{\frac{+\lambda_2-\lambda_3-\lambda_4}{2}} e^{-\frac{i(k_3+k_4)}{2}} e^{+i\phi_0}}{\sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} [a_2^{m+*} a_3^{m+} a_4^{m+}]
\end{aligned}$$

Therefore, in the event of upper band excitation at $m = 0$, the lower band evolves according to,

$$\begin{aligned}
a_1^{(m=2)-} &= \chi \sum_{2,3,4} \frac{1}{\sqrt{1 + e^{-2\lambda_1}} \sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} \left[\right. \\
&\quad A^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{-\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}} - B^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{i(k_2-k_3-k_4)}{2}} e^{+\frac{i\phi_0}{2}} e^{+\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}} \\
&\quad - C^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{-\lambda_2-\lambda_3+\lambda_4}{2}} e^{-\frac{ik_4}{2}} e^{+\frac{i\phi_0}{2}} + D^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{+\lambda_2-\lambda_3+\lambda_4}{2}} e^{\frac{i(k_2-k_4)}{2}} \\
&\quad - E^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{+\lambda_2-\lambda_3-\lambda_4}{2}} e^{+\frac{ik_2}{2}} e^{-\frac{i\phi_0}{2}} \\
&\quad \left. + F^- [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{-\lambda_2+\lambda_3+\lambda_4}{2}} e^{-\frac{i(k_3+k_4)}{2}} e^{+i\phi_0} \right] \delta_{1+2}^{3+4} \\
&= \chi \sum_{2,3,4} T_{12}^{34-} a_2^{0+*} a_3^{0+} a_4^{0+} \delta_{1+2}^{3+4} \text{ for } k < \pi - \phi_0 \tag{3.4.31}
\end{aligned}$$

$$\begin{aligned}
a_1^{(m=2)-} &= \chi \sum_{2,3,4} \frac{1}{\sqrt{1 + e^{+2\lambda_1}} \sqrt{e^{\lambda_2} + e^{-\lambda_2}} \sqrt{e^{+\lambda_3} + e^{-\lambda_3}} \sqrt{e^{+\lambda_4} + e^{-\lambda_4}}} \left[\right. \\
&\quad A^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{+\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}} + B^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{i(k_2-k_3-k_4)}{2}} e^{+\frac{i\phi_0}{2}} e^{-\frac{(\lambda_2+\lambda_3+\lambda_4)}{2}} \\
&\quad + C^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{+\lambda_2+\lambda_3-\lambda_4}{2}} e^{-\frac{ik_4}{2}} e^{+\frac{i\phi_0}{2}} + D^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{-\lambda_2+\lambda_3-\lambda_4}{2}} e^{\frac{i(k_2-k_4)}{2}} \\
&\quad + E^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{-\lambda_2+\lambda_3+\lambda_4}{2}} e^{+\frac{ik_2}{2}} e^{-\frac{i\phi_0}{2}} \\
&\quad \left. + F^+ [a_2^{0+*} a_3^{0+} a_4^{0+}] e^{\frac{+\lambda_2-\lambda_3-\lambda_4}{2}} e^{-\frac{i(k_3+k_4)}{2}} e^{+i\phi_0} \right] \delta_{1+2}^{3+4} \\
&= \chi \sum_{2,3,4} T_{12}^{34-} a_2^{0+*} a_3^{0+} a_4^{0+} \delta_{1+2}^{3+4} \text{ for } k \geq \pi - \phi_0 \tag{3.4.32}
\end{aligned}$$

Equations (3.4.31) and (3.4.32) explicitly show that starting from a populated upper band and an empty lower band, after a single stroboscopic step the lower band is excited, provided that scattering coefficient $T_{12}^{34} \neq 0$. This can be tested numerically using a simple initial condition such as the excitation of a single upper band mode. In this case $k_1 = k_2 = k_3 = k_4$ and we expect that there will be a non-zero amount of wave field in the lower band after this step. Fig. (3.4.4) explicitly shows the transfer. By initially exciting the $k_* \approx 0.314159$ mode of the upper band we see that after one stroboscopic step the k_* of the lower band is excited.

We can then compute the magnitude of the scattering coefficient T_{12}^{34} for every k in the case of single mode excitation, shown in Fig. (3.4.5).

Whilst this leakage is physical for a single realisation, it is important to consider whether it is also resonant. As previously discussed, WWT theory shows what wave interactions are resonant, under the criteria set by conservation of

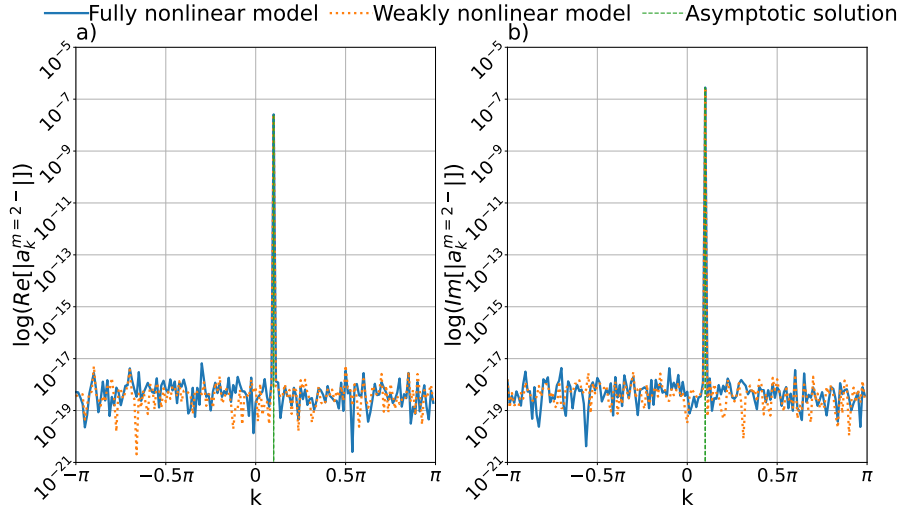


Figure 3.4.4: Here $N = 320$, $t^2 = 0.3$, $\phi_0 = \pi/8$ and $\chi = 0.05$. We initially excite a mode given by $k_* \approx 0.314159$ in the upper band and evolve forward a stroboscopic step in time. Panels a) and b) is a semilog plot of the real and imaginary parts of the lower band field after this stroboscopic step. For comparison the wave fields of the fully nonlinear model given by equations (3.3.2) (3.3.3), the weakly nonlinear model from (3.4.15) (3.4.16) and the derived asymptotic solution in equations (3.4.31) (3.4.32). As can be seen for both the real and imaginary part, every model peaks at the mode k_* to the same order of magnitude, and is many orders of magnitude above the computational noise populating all the other modes.

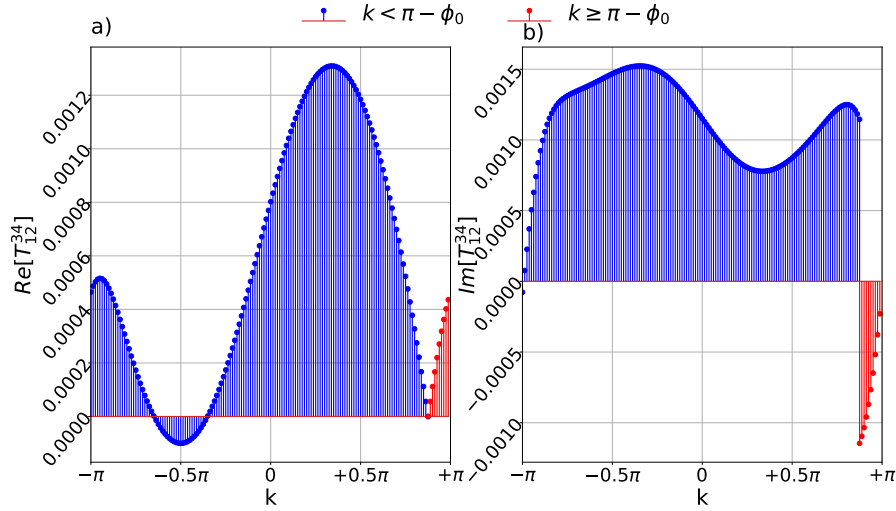


Figure 3.4.5: Here $N = 320$, $t^2 = 0.3$, $\phi_0 = \pi/8$. Panel a) and b) are the real and imaginary parts of T_{12}^{34} respectively. The scattering matrix for the case of single mode excitation is found to be non-zero for every mode, showing that there is a physical mechanism that can leak wave-field from one band into another everywhere in k . The blue points are the interval $k < \pi - \phi_0$ and the red are when $k \geq \pi - \phi_0$. Despite appearing discontinuous in shape, the red points align with the curve from $k < -\pi$, with the interval being shifted by $-\phi_0$ and being periodic in k .

momentum and linear energy. For the photonic mesh lattice, conservation of momentum at a four wave interaction demands that,

$$k_1 + k_2 = k_3 + k_4 \bmod 2\pi, \quad (3.4.33)$$

and the linear energy requires,

$$\omega_1^\pm + \omega_2^\pm = \omega_3^\pm + \omega_4^\pm \bmod 2\pi. \quad (3.4.34)$$

In the case of single band, single mode excitation such that $k_1 = k_2 = k_3 = k_4$, equation (3.4.2) is satisfied. The type of interaction necessary for the leakage from a single mode, requires a specific choice of manifold from equation (3.4.2) which is,

$$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- \bmod 2\pi, \quad (3.4.35)$$

which is found to be completely empty numerically and therefore not resonant.

Despite this interaction not being resonant, it could still explain in part how the leakage process takes hold. The small amount of wave field passed into the lower enabled by this non-resonant interaction may trigger other resonant interactions to take place among the bands that are very weakly populated. Over time these weakly populated resonances gradually accumulate more wave field and eventually contribute significantly to the dynamics causing the power leakage to occur. This would align well with the long timescale needed for leakage to be non-negligible.

Moreover, the above derivation has only explored the $\mathcal{O}(\chi)$ terms, with an infinite hierarchy of more complex wave interactions not yet explored. It is entirely possible that at higher orders of wave interaction there are inter-band resonant processes which transfer wave field to the lower band starting from single band excitation. For example consider the following six wave resonance, that scatters three upper band modes into two new upper band modes and one lower band mode

$$\omega_1^+ + \omega_2^+ + \omega_3^+ = \omega_4^+ + \omega_5^+ + \omega_6^- - 2\pi. \quad (3.4.36)$$

The manifold in equation (3.4.36) is shown to be non-empty in the thermodynamic limit with an example of one such manifold given in Fig. (3.4.6). The fact that there are no four wave resonances that facilitate power leakage but there does exist resonances at a six wave aligns well with what is observed in Fig. (3.4.2). The onset of power leakage only occurs at the longer time scales once a single band thermal state is reached. As the nonlinearity is weak, the six wave processes will become relevant to the dynamics on a much longer time scale than the four wave, possibly explaining the delay in the leakage process. What is particularly interesting about equation (3.4.36) is that it is only possible through the discrete-time effects allowing the use of Floquet theory and would otherwise be absent in a continuous-time setting. The factor of 2π is effectively implying that we are scattering into a mode on the first set of duplicate bands.

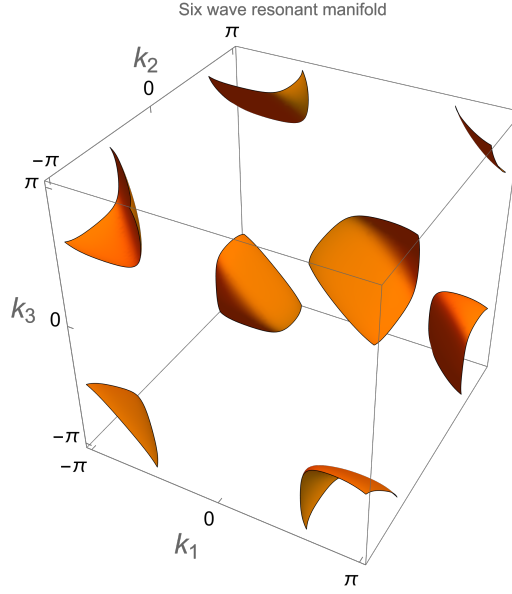


Figure 3.4.6: Here we fix $k_4 = 12\pi/40$ and $k_5 = -7\pi/40$ in order to visualise the manifold in 3 dimensional space.

However, the non-empty manifold in equation (3.4.36) also requires that the respective interaction coefficient is non-zero, and is therefore not enough proof alone to say for certain that this resonance explains the leakage. We will return to the physical origin of the power leakage in section (3.5), providing more evidence that the leakage process occurs through resonant interactions only possible in discrete time systems.

3.4.3 Thermalisation timescales of a single band Rayleigh-Jeans distribution.

In this final subsection on single band thermalisation processes, we investigate how the time required to reach a single band Rayleigh-Jeans distribution depends on the nonlinear parameter $\chi P/N$.

It is expected that as $\chi P/N$ decreases, the nonlinear interactions become weaker, thus increasing timescale needed to thermalise. As the nonlinearity enters through exponential terms, the evolution equations (3.3.2), (3.3.3) can be

expressed as an infinite series of terms,

$$\begin{aligned}
 u_n^{m+1} = & \left[tu_{n+1}^m + irv_{n+1}^m + \chi (itu_{n+1}^m |u_{n+1}^m|^2 - rv_{n+1}^m |v_{n+1}^m|^2) \right. \\
 & \left. + \frac{\chi^2}{2} (-tu_{n+1}^m |u_{n+1}^m|^4 - irv_{n+1}^m |v_{n+1}^m|^4) - \frac{\chi^3}{6} (itu_{n+1}^m |u_{n+1}^m|^5 - rv_{n+1}^m |v_{n+1}^m|^5) + \dots \right] e^{i\phi^m}
 \end{aligned}
 \tag{3.4.37}$$

$$\begin{aligned}
 v_n^{m+1} = & \left[tv_{n-1}^m + iru_{n-1}^m + \chi (itv_{n-1}^m |v_{n-1}^m|^2 - ru_{n-1}^m |u_{n-1}^m|^2) \right. \\
 & \left. + \frac{\chi^2}{2} (-tv_{n-1}^m |v_{n-1}^m|^4 - iru_{n-1}^m |u_{n-1}^m|^4) - \frac{\chi^3}{6} (itv_{n-1}^m |v_{n-1}^m|^5 - ru_{n-1}^m |u_{n-1}^m|^5) + \dots \right]
 \end{aligned}
 \tag{3.4.38}$$

with each order of wave interaction entering at a different power of χ . At each order of wave interaction, there is the potential for resonant wave processes to exist, subject to restrictions given by the momentum and linear energy. According to WWT theory, these resonances drive the system to a thermal state, observed in the averaged wave dynamics, provided that the resonances are interconnected, i.e. are able to scatter waves into unique wave numbers, enabling the free movement of wave action between modes.

Since each order of wave interaction enters at a different power of χ^α in equations (3.4.37), (3.4.38), where $\alpha \in \mathbb{Z}^+$, we expect that each order of wave interaction contributes to the dynamics at different time scales proportional to $1/\chi^\alpha$. Therefore, the timescale to reach thermal equilibrium can be estimated to be proportional to $1/\chi^\alpha$, in which the value of α corresponds to the term that has interconnected resonances.

Moreover, thermal equilibrium is observed in the wave action spectrum $n_k^{m\pm} = \langle |a_k^{m\pm}|^2 \rangle$ and not the short and long loop wave fields u_n^m and v_n^m . By considering

equations (3.2.26) and (3.2.27),

$$a_k^{m+} = \begin{cases} \frac{\tilde{u}_k^m - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases}$$

$$a_k^{m-} = \begin{cases} \frac{\tilde{u}_k^m + e^{-\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{-2\lambda_k}}} & \text{for } k < \pi - \phi_0 \\ \frac{\tilde{u}_k^m - e^{+\lambda_k} e^{+\frac{ik}{2}} e^{-\frac{i\phi_0}{2}} \tilde{v}_k^m}{\sqrt{1+e^{+2\lambda_k}}} & \text{for } k \geq \pi - \phi_0 \end{cases}$$

it is clear to see that $|a_k^{m\pm}|^2 \propto (|\tilde{u}_k^m|^2, |\tilde{v}_k^m|^2, \tilde{u}_k^m \tilde{v}_k^{m*}, \tilde{u}_k^{m*} \tilde{v}_k^m)$. The discrete Fourier space evolution equations possess the same property of wave interactions entering at different power of χ^α . Hence from this simple dimensional argument, the resonant order of χ^α driving thermal equilibrium is assumed to enter on a time scale proportional to $\chi^{2\alpha}$. In the derivation of the WKE for the DNLS in chapter (2), this scaling emerged naturally from the weak expansion of the action-angle coordinates. Here for the photonic mesh lattice, we aim to find an equivalent scaling numerically.

Next to estimate the value of 2α , we must perform numerical simulations for a

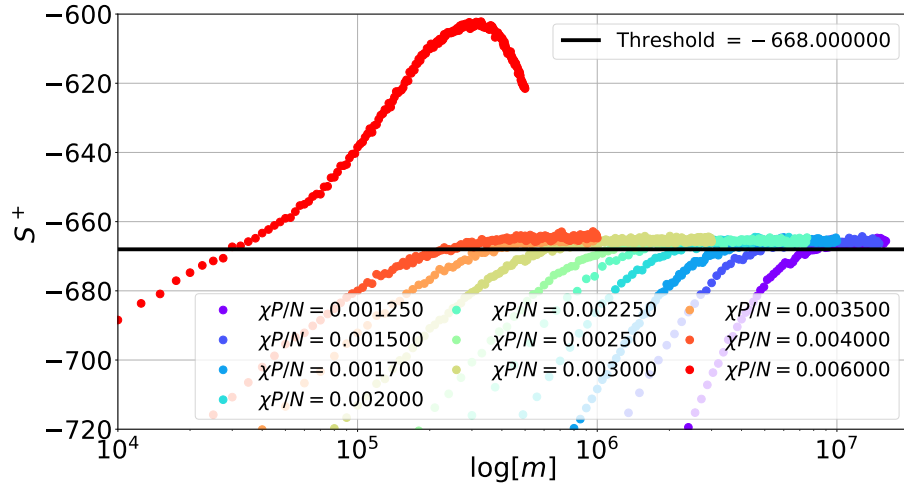


Figure 3.4.7: Numerical simulations averaged over 500 realisations were performed with $t^2 = 0.3$, $\phi_0 = \pi/8$, $N = 320$ and $P = 8$ for different values of χ , initially exciting the middle upper modes only as done in Fig. (3.4.2). For the choice of $\chi P/N = 0.006$, we see that the entropy does not plateau. This is because inter-band processes are non-negligible.

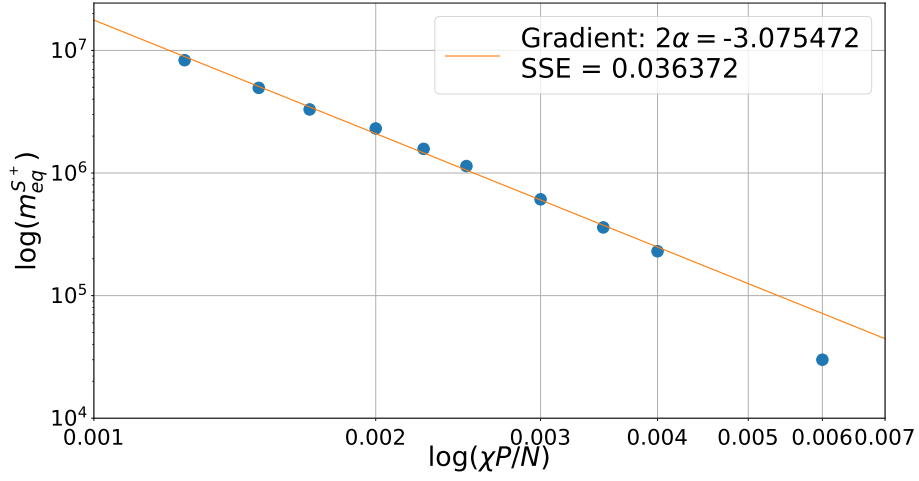


Figure 3.4.8: Here we take a log-log plot of m_{eq}^{S+} measured in figure (3.4.7) vs the choices of $\chi P/N$. A linear function can be fitted using the polyfit package from numpy, showing a power law relation between $\chi P/N$ and m_{eq}^{S+} . SSE is the sum of squared errors measured from the fitting. For the values of $\chi P/N = 0.006$, we see that that it does not obey the measured power law. The power law was measured neglecting the data at $\chi P/N = 0.006$.

range of $\chi P/N$ and define a measure for thermal equilibrium. Previous work has done exactly this for the 1D FPUT chain, by measuring thermal equilibrium with an arbitrary entropy cut off, chosen just before the entropy reaches an extremum[40]. In Fig. (3.4.7) we do exactly this, defining a threshold for thermalisation in a single band. The thermalisation timescale m_{eq}^{S+} is defined here as the first value of S^+ over the threshold. Fig. (3.4.8) shows a log-log plot of m_{eq}^{S+} vs nonlinearity $\chi P/N$ obtained from the entropy cut-off measurements, finding a power law relation of $2\alpha \approx -3.075$.

The validity of defining an entropy cut-off for equation (3.4.3) is limited by the strength of nonlinearity. As evident in Fig. (3.4.7), when $\chi P/N = 0.006$ the entropy does not plateau just above the estimated threshold, but instead continues to evolve in time. Further, when we obtain the power law in Fig. (3.4.8), this data point does not match the prediction given by the fitting. The discrepancy in the $\chi P/N = 0.006$ simulation arises from non-negligible inter-band processes transferring wave action to the lower band, as clearly shown in the linear energy in Fig. (3.4.9). These processes significantly

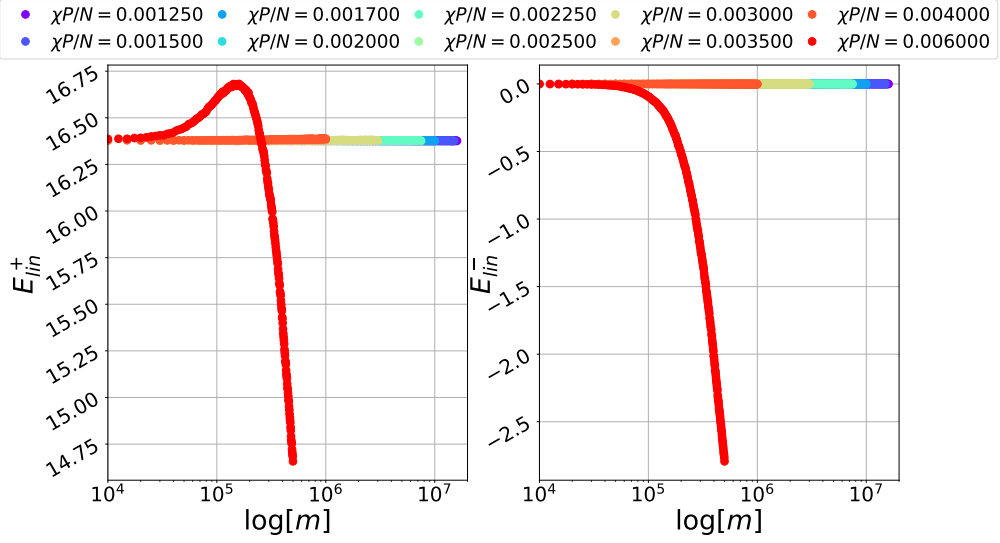


Figure 3.4.9: The evolution of the upper and lower linear energies in time. As can be seen, in every simulation besides $\chi P/N = 0.006$, the linear energy is approximately time invariant in both bands.

influence the dynamics, making the definition of the single band entropy in equation (3.4.3) no longer suitable. This is because when inter-band interactions are significant, the entropy in a single band is no longer band independent. In this case, a more appropriate definition for the single band entropy would include a cross term,

$$S^+ = \sum_k \ln(n_k^{m+}) + f(n_k^{m+}, n_k^{m-}), \quad (3.4.39)$$

where $f(n_k^{m+}, n_k^{m-})$ is some generic function that depends on the wave action in both bands, and would tend to zero when the lower band wave action tends to zero.

From the measurement of the gradient, the timescales for thermalisation for every choice of nonlinearity can then be re-scaled, excluding $\chi P/N = 0.006$, as show in Fig. (3.4.10), demonstrating that the entropy evolution for every simulation collapses on top of one another, verifying the choice of the arbitrary threshold when inter-band processes are negligible.

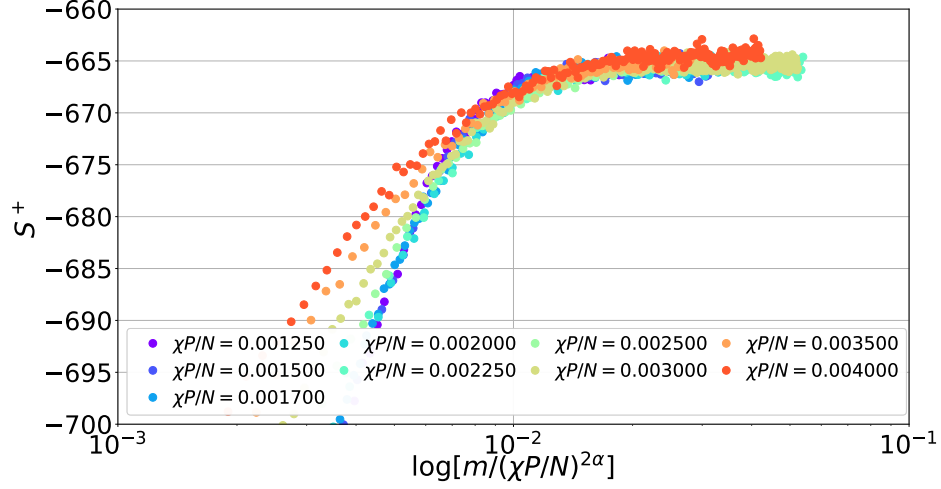


Figure 3.4.10: The entropy evolution with time re-scaled according to the measurement of 2α in Fig. (3.4.8).

Before discussing the meaning of 2α , we perform an alternative thermalisation measure, by defining an upper band temperature cut-off, as a consistency check. Here we choose a temperature cut off based on the final observed Rayleigh-Jeans distribution, with the reason being that as $\chi P/N$ gets increasingly small, the parameters for the single band Rayleigh-Jeans (T^+, μ^+) will approach some asymptotic values at thermal equilibrium. Since every simulation starts from the same parameters and initial conditions, we expect that the measured (T^+, μ^+) will tend to the same values asymptotically for every choice of χ . Fig. (3.4.11) shows the evolution of (T^+, μ^+) for each choice of $\chi P/N$, with a temperature threshold defined as a measure for thermalisation.

Measurements of the thermalisation timescale $m_{eq}^{T^+}$ are then taken, once measured temperatures are within $\pm 5\%$ of the cut-off. A log-log plot of $m_{eq}^{T^+}$ vs $\chi P/N$ is then plotted in Fig. (3.4.12), finding a gradient of $2\alpha \approx -2.745$, similar to the entropy prediction. Comparisons of the wave action profiles is made in Fig. (3.4.13), showing that the entropy cut-off typically under predicts the single band thermal state. Despite this, both the entropy and temperature cut off appear to be reliable predictors, with both values of 2α being of similar magnitude.

To understand what terms have coefficients of $\chi^{2\alpha}$, we can make a simple

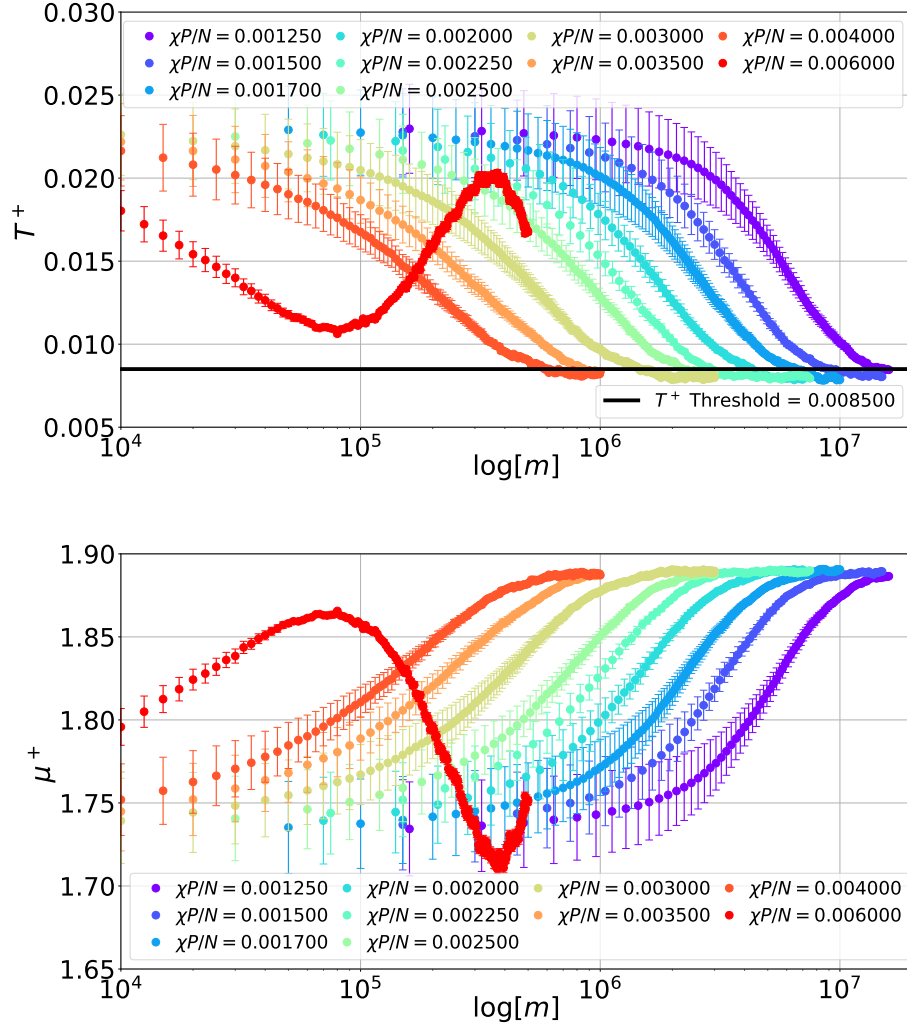


Figure 3.4.11: Temperature T^+ and chemical potential μ^+ evolution of the upper band, with error bars given by the standard deviation outputted from the curve_fit function of the SciPy library. See appendix (A.8) for more details on how these parameters are measured. For $\chi P/N = 0.006$, we see that it does not tend to the same temperature and chemical potential, due to the leakage process.

dimensional argument about the evolution equation of the modal intensity $|a_k^{m+}|^2$ by considering the modulus squared of equation (3.4.37). To begin, we generically express equation (3.4.37) as a series of functions entering at each power of χ ,

$$u_n^{m+1} \approx \left[\underbrace{L_n^m}_{1 \leftrightarrow 1} + \chi \underbrace{F_n^m}_{2 \leftrightarrow 2} + \chi^2 \underbrace{G_n^m}_{3 \leftrightarrow 3} + \chi^3 \underbrace{H_n^m}_{4 \leftrightarrow 4} \right] \quad (3.4.40)$$

where $L_n^m, F_n^m, G_n^m, H_n^m$ are functions representing the linear, four, six and eight

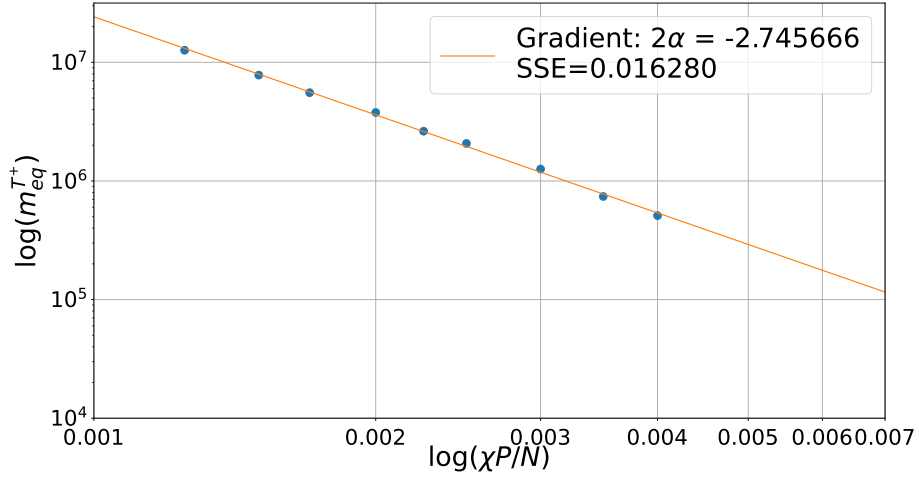


Figure 3.4.12: Estimation of thermalisation timescale based on temperature cut-off. A power law is observed of gradient -2.7 .

wave interactions respectively. Taking the modulus square of both sides we obtain,

$$\begin{aligned}
 |u_n^{m+1}|^2 &\approx [L_n^m + \chi F_n^m + \chi^2 G_n^m + \chi^3 H_n^m][L_n^m + \chi F_n^m + \chi^2 G_n^m + \chi^3 H_n^m]^* \\
 &\approx |L_n^m|^2 + \chi (L_n^m F_n^{m*} + L_n^{m*} F_n^m) + \chi^2 (L_n^m G_n^{m*} + L_n^{m*} G_n^m + |F_n^m|^2) \\
 &\quad + \chi^3 (L_n^m H_n^{m*} + L_n^{m*} H_n^m + F_n^m G_n^{m*} + F_n^{m*} G_n^m) + \dots
 \end{aligned} \tag{3.4.41}$$

As is evident in equation (3.4.41), terms with coefficient χ^3 enter as products of linear and eight wave interactions, and four and six wave interactions, which according to the measurements of 2α are the terms that govern the single band thermalisation processes. We note that these products don't combine to higher order wave interactions because in the Fourier representation, these interactions are already selected by the Kronecker deltas.

We now test the prediction numerically by using the $\mathcal{O}(\chi^3)$ truncated equations given by the first three terms in equations (3.4.37) and (3.4.38) to evolve the lattice in time. For sufficiently weak $\chi P/N = 0.004$, the truncated dynamics are stable, with P approximately conserved as shown in Fig. (3.4.14) panel a). When comparing the truncated dynamics to the full dynamics for the same

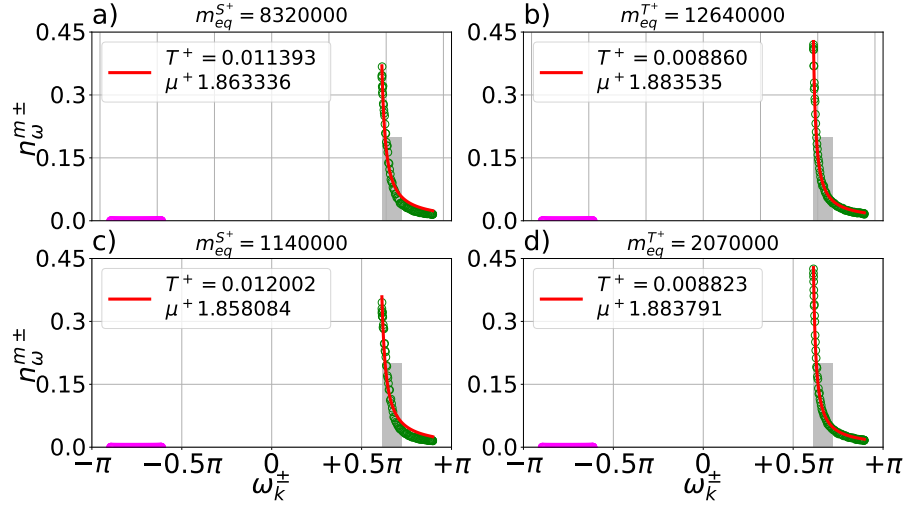


Figure 3.4.13: Comparisons of the predicted thermalised for two different choices of $\chi P/N$: $\chi P/N = 0.00125$ for a) and b) and $\chi P/N = 0.0025$ for c) and d). The left hand panels a) and c) are the spectrum at the time predicted from the entropy cut-off m_{eq}^{S+} . The right hand panels b) and d) show the spectrum at the time predicted from the temperature cutoff m_{eq}^{T+} . In both cases the entropy under-predicts the timescale.

choice of parameters and initial condition, as in Fig. (3.4.14) panels b) - d), both evolve qualitatively the same way and thermalise to the approximately the same (T^+, μ^+) , with the entropy maximising to the same value on the same timescale. The equations truncated at $\mathcal{O}(\chi^3)$ find good agreement with the fully nonlinear numerics, showing that thermalisation processes in a single band only requires the order eight wave interactions and below, in support of the dimensional argument above.

We will now seek out the different resonant structures present in the $\mathcal{O}(\chi^3)$ equations. The terms present enter through combinations of linear and eight wave interactions and four and six wave interactions. The criteria for resonance for each order is given by equalities in k and $\omega_k^{\pm}$. At a four wave these restrictions are,

$$k_1 + k_2 = k_3 + k_4 \bmod 2\pi \quad (3.4.42)$$

$$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+ \bmod 2\pi, \quad (3.4.43)$$

and at a six wave,

$$k_1 + k_2 + k_3 = k_4 + k_5 + k_6 \bmod 2\pi, \quad (3.4.44)$$

$$\omega_1^+ + \omega_2^+ + \omega_3^+ = \omega_4^+ + \omega_5^+ + \omega_6^+ \bmod 2\pi, \quad (3.4.45)$$

and at an eight wave interaction,

$$k_1 + k_2 + k_3 + k_4 = k_5 + k_6 + k_7 + k_8 \bmod 2\pi \quad (3.4.46)$$

$$\omega_1^+ + \omega_2^+ + \omega_3^+ + \omega_4^+ = \omega_5^+ + \omega_6^+ + \omega_7^+ + \omega_8^+ \bmod 2\pi. \quad (3.4.47)$$

For single band resonances, we consider resonances that only involve the positive branch ω_k^+ . By computing each of these manifolds, we can observe whether they possess solutions and whether these solutions are non-trivially related. In general, a non-trivial manifold is one that allows for scattering events to excite unique wave numbers, eventually exciting all the modes simultaneously. These type of wave interactions are understood to be what enables thermal equilibrium

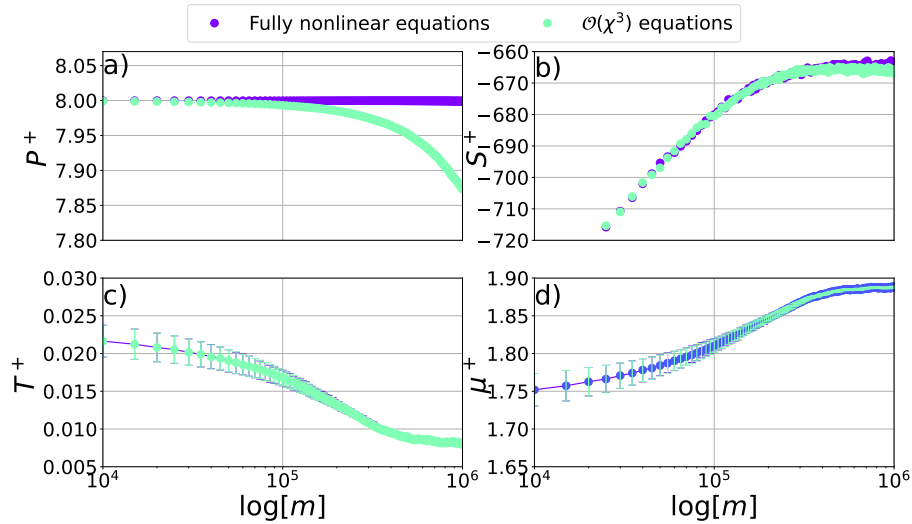


Figure 3.4.14: Panels a)-d) show a comparison of the time evolution of different quantities for the fully nonlinear equations and the truncated $\mathcal{O}(\chi^3)$ equations for $\chi P/N = 0.004$, $(t^2\phi_0) = (0.3, \pi/8)$ initially exciting middle upper bands only. In panel a) the power for the truncated equations is approximately conserved for the full evolution, decreasing by $\sim 1.6\%$ of the initial power. Panels b)-d) show the entropy, upper band temperature and chemical potential respectively for both simulations. Good agreement is found between the evolution of these quantities, with similar errors measured in the fitting of (T^+, μ^+) .

processes among all the linear normal modes according to WWT theory. In contrast there can also exist trivial-manifolds, which are wave interactions that do not excite unique wave numbers.

Further, because of the finite discrete k space many resonances are inaccessible, and are only present in the thermodynamic limit. Hence most resonant interactions are accessed via nonlinear broadening effects in the discrete setting. The broadening effect, observed in Fig. (3.3.3) panel d) for example, relaxes the criteria of resonance in $\omega_k^\pm$. At a four wave the relaxed criteria would be,

$$k_1 + k_2 = k_3 + k_4 \bmod 2\pi \quad (3.4.48)$$

$$\omega_1^+ + \omega_2^+ \leq \omega_3^+ + \omega_4^+ + \Delta \bmod 2\pi, \quad (3.4.49)$$

where Δ is a measure of how much the linear spectrum broadens with nonlinearity. Δ can be approximated by taking a cross section of the $(k, \omega_k^\pm)$ spectrum recovered from the evolution. Fig. (3.4.15) measures the nonlinear broadening of the averaged spectrum to be $\Delta \approx 0.001676$ for a weaker nonlinearity $\chi P/N = 0.00125$.

Figs. (3.4.16), (3.4.17) and (3.4.18) show examples of resonant manifolds for the four, six and eight wave resonances respectively, computed using Mathematica. In each figure the thermodynamic limit is plotted in the left panels, while the discrete limit is shown in the right panels. In the discrete case, the blue points denote exact resonances whereas red points indicate resonances accessed via broadening effects, using the measured Δ from Fig. (3.4.15).

The broadened resonances are defined by an interval that is bounded above by $\Delta \approx 0.001676$ and below by 0.0001, ensuring exact resonances are not double counted. In the discrete limit, the k space is shown for when $N = 80$, due to computational limitations. In simulations where $N = 320$, we expect these manifolds to be much denser. Moreover since the measured Δ is for a weaker nonlinearity $\chi P/N = 0.00125$, the number of resonances accessed via broadening

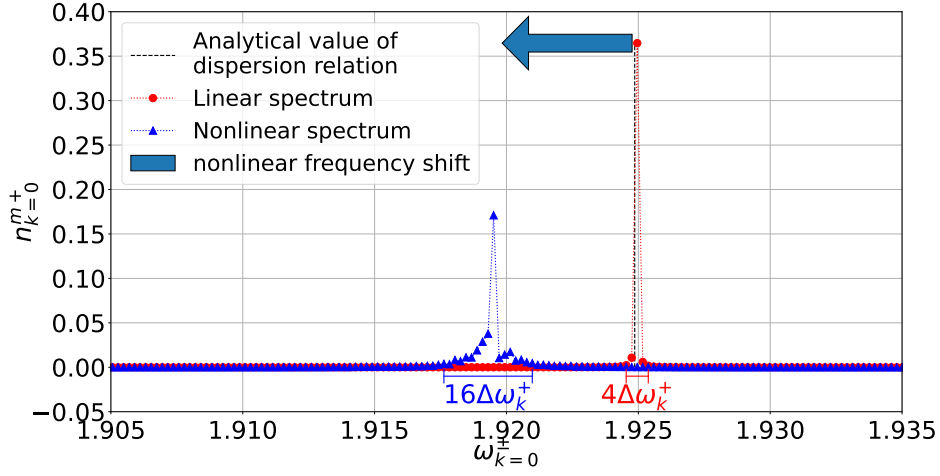


Figure 3.4.15: Here we take the final time step of the $\chi P/N = 0.00125$ simulation of each realisation and evolve further in time for $M = 60,000$ steps with and without nonlinear effects. Then for each realisation, we extract the $(k, \omega_k^\pm)$ spectrum along the $k = 0$ mode to obtain a cross section. The cross section of each realisation is then averaged to give the linear spectrum in red and the nonlinear spectrum in blue. For the linear spectrum, we see that it is peaked approximately near to where the analytical value of the dispersion relation should lie, with some very slight broadening effects occurring from numerical discretisation. We approximate these numerical broadening effects to be $4\Delta\omega_k^\pm$, with $\Delta\omega_k^\pm = 4\pi/M$. The nonlinear spectrum is shifted downwards from the linear spectrum due to $\chi > 0$. The broadening of the nonlinear spectrum is measured to be $16\Delta\omega_k^\pm \approx 0.001676$, larger than the effects of numerical discretisation seen for the linear case.

is also likely under-predicted; consequently we expect that these manifolds to also be denser in simulation.

In the four wave resonance, the manifolds are visualised in a three dimensional space by reducing the dimensions from conservation of wave number: $k_4 = k_1 + k_2 - k_3 \bmod 2\pi$. For the six wave resonance k_4 and k_5 have been selected at random and fixed so that the plot can be visualised in three dimensions. Equivalently for an eight wave resonance k_4, k_5, k_6, k_7 are also randomly selected and fixed. This means that Figs. (3.4.17) and (3.4.18) are representing one possible projection of the manifold.

For every manifold, it is clear that the exact resonances in blue are only able to access trivial resonances, typically given by flat planes or lines in the (k_1, k_2, k_3)

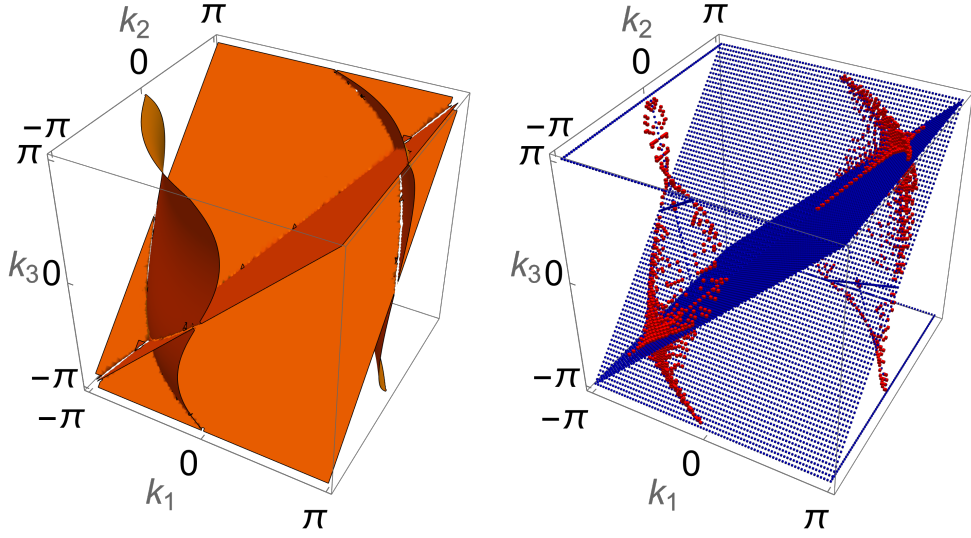


Figure 3.4.16: Four wave resonant manifold satisfying $\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+$.

space. The non-trivial manifolds that possess more complex shapes and thus can excite unique wave numbers, are accessed entirely through the broadening effects, given by the red points.

With Figs. (3.4.16), (3.4.17), (3.4.18) we have shown that for $\mathcal{O}(\chi)$, $\mathcal{O}(\chi^2)$ and $\mathcal{O}(\chi^3)$ there exist resonant manifolds, with non-trivial geometry. Through nonlinear broadening the non-trivial manifolds are accessed and could explain

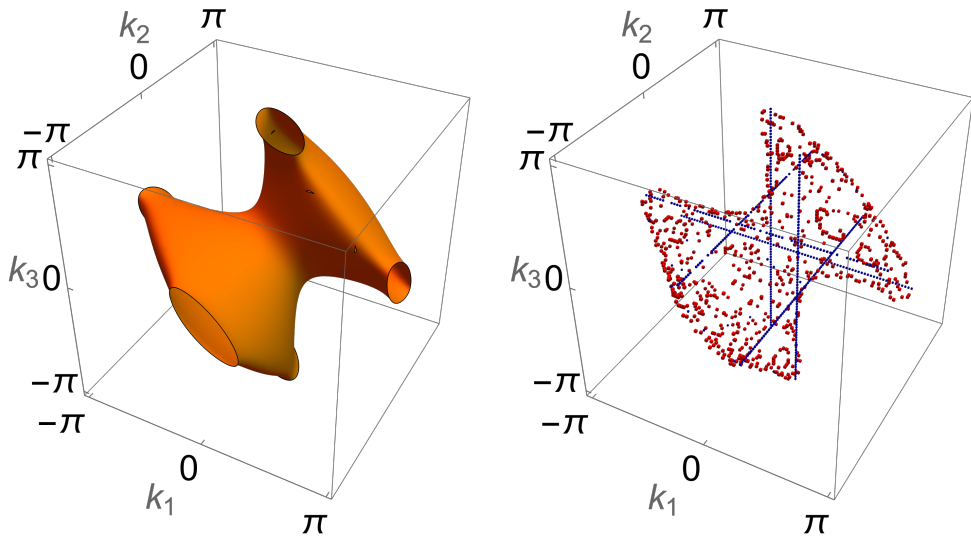


Figure 3.4.17: Six wave resonant manifold with $k_4 = 12\pi/40$ and $k_5 = -7\pi/40$ satisfying $\omega_1^+ + \omega_2^+ + \omega_3^+ = \omega_4^+ + \omega_5^+ + \omega_6^+$.

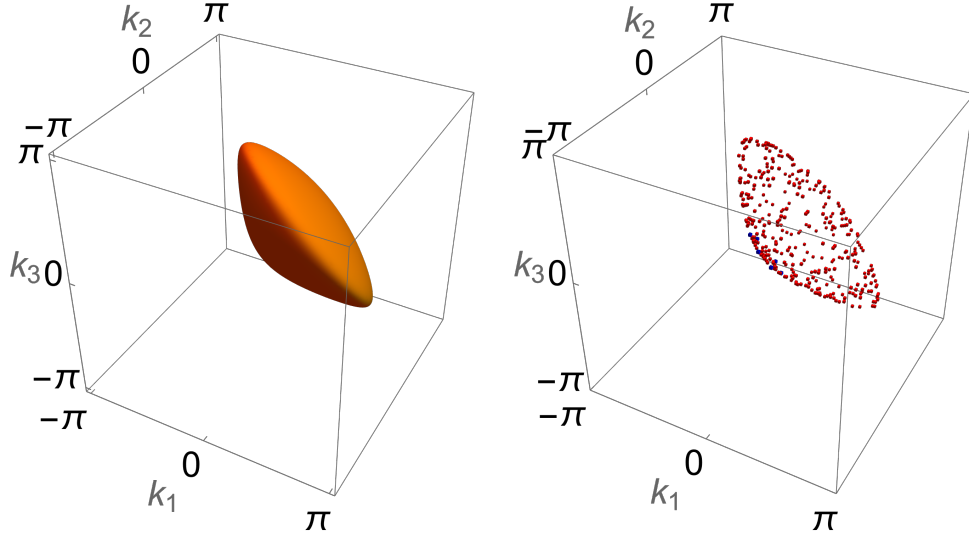


Figure 3.4.18: Eight wave resonant manifold with $k_4 = 12\pi/40$, $k_5 = -7\pi/40$, $k_6 = 3\pi/40$ and $k_7 = -\pi/40$ satisfying $\omega_1^+ + \omega_2^+ + \omega_3^+ + \omega_4^+ = \omega_5^+ + \omega_6^+ + \omega_7^+ + \omega_8^+$.

the single-band thermal state. While none of the interaction coefficients have been derived for these terms, the observations of thermalisation in the $\mathcal{O}(\chi^3)$ truncated dynamics shows that only these terms are needed for this process to occur, implying that each order of interaction could be contributing to the thermalisation through resonant processes presented above.

3.5 Thermalisation processes between both bands

In the final section of this chapter, we explore the full thermal equilibrium between bands, starting from single band occupation enabled by the power leakage observed in Section (3.4.1). The results presented in this section are currently submitted to *Physical Review Letters* under review, and a pre-print of the letter can be found on the arXiv: [100].

To begin, we recount the previous simulation of single band thermal equilibrium shown in Fig. (3.4.2). After the single band Rayleigh-Jeans is reached, a slow power leakages process takes hold destabilising the measured single band Rayleigh-Jeans distribution (panels c)-d) in Fig. (3.4.3)). For this particular

simulation after 1,500,000 steps in time, approximately 2.5% of the total power has passed into the lower band, making the leakage processes extremely weak for the choice of nonlinearity $\chi P/N \approx 0.0044$. As consequence, the timescale to observe a full thermal state between bands would be exceedingly large, requiring extensive computational times and drastically increasing the error accumulated over the evolution. Therefore, in order to accelerate the power leakage process, we will now choose a larger nonlinearity given by $\chi P/N = 0.04$, with $N = 320$, $P = 8$ and $\chi = 1.6$. While this choice of χ is considerably larger than that used for all the previous simulations, it is more representative of the nonlinearities used in experimental works [101].

To observe the full thermal equilibrium between bands, we now consider two different simulations. The first simulation has the same parameters as the previous sections $(t^2, \phi_0) = (0.3, \pi/8)$, while the second simulation has $(t^2, \phi_0) = (0.75, \pi/3)$, with the dispersion relation plotted for both in Fig. (3.5.1) for comparison.

For both simulations, we start from seeding the middle upper modes of either band, with the same choice of nonlinearity $\chi P/N = 0.04$, with $N = 320$, $P = 8$

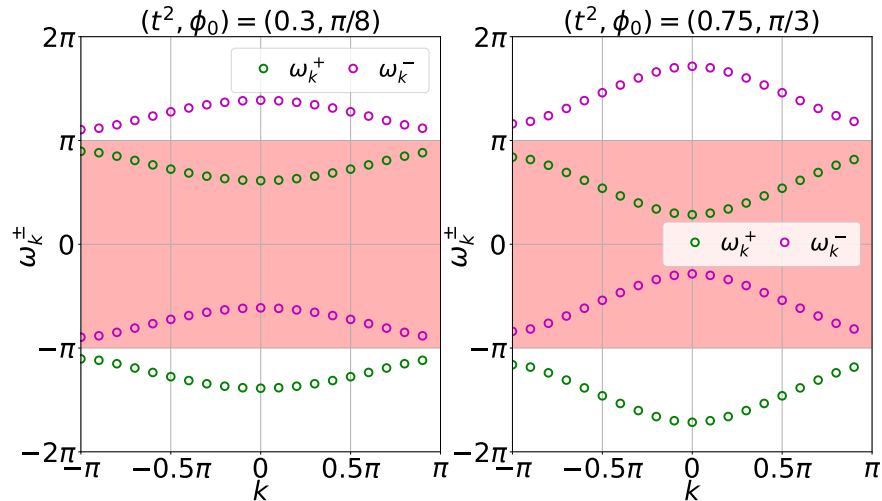


Figure 3.5.1: The dispersion relations for the different choice of parameters. By changing to $t^2 = 0.75$ and $\phi_0 = \pi/3$, we narrow the $k = 0$ band gap, allowing the bands to span larger intervals over the $\omega_k^\pm$ domain.

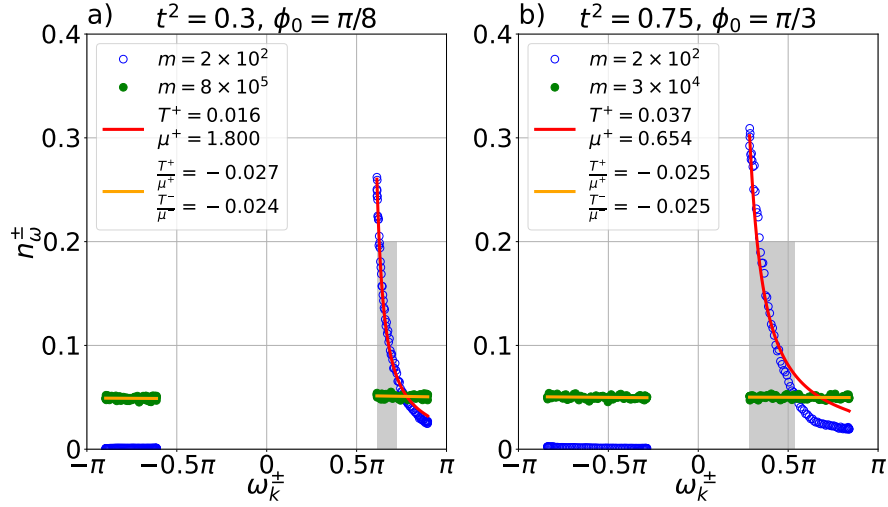


Figure 3.5.2: Here we show the evolution of the wave action spectrum for $(t^2, \phi_0) = (0.3, \pi/8)$, starting from exciting the middle upper band modes only (grey shaded regions in both panels).

and $\chi = 1.6$ and evolve for 3,000,000 steps in time, averaging over 500 realisations. Snapshots for both simulations are plotted in Fig. (3.5.2). The error of these simulations is discussed in appendix (A.9) in particular for the lattice $(t^2, \phi_0) = (0.3, \pi/8)$, as to further justify the physical origin of the power leakage.

At the early times in both panels $m = 200$ (blue circles), the single band Rayleigh-Jeans state is observed (red line), discussed in the previous section. However this state is short lived, with the power leakage process taking hold and destabilising the Rayleigh-Jeans distribution. At longer time scales in both simulations (green points), the power equally partitions and every mode possesses equal occupation of P/N . In this case, the Rayleigh-Jeans spectra of both bands (orange lines) describes a flat featureless state independent of the mode labelling given by $n_k^{RJ\pm} = P/N$.

To better understand the behaviour, we plot the full evolution of the power, linear energy and the ratio of the $T^\pm/\mu^\pm$ for both simulations in Fig. (3.5.3). Panels a), c), e) are for the $(t^2, \phi_0) = (0.3, \pi/8)$ simulation and b), d), f) for the $(t^2, \phi_0) = (0.75, \pi/3)$ simulation. At the early times when the single band

thermal state is reached (the grey shaded regions in Fig. (3.5.3)), we observe that the power is almost entirely contained in the upper band (panels a) and b)), dominated by four wave processes. At this stage the interband processes are weakly acting, with small lower band occupation enabled through non-resonant wave processes. At longer times, the interband coupling becomes significant and destabilises the single band thermal state, mediated through non-resonant seeding that triggers interband resonances, and higher order resonances only possible

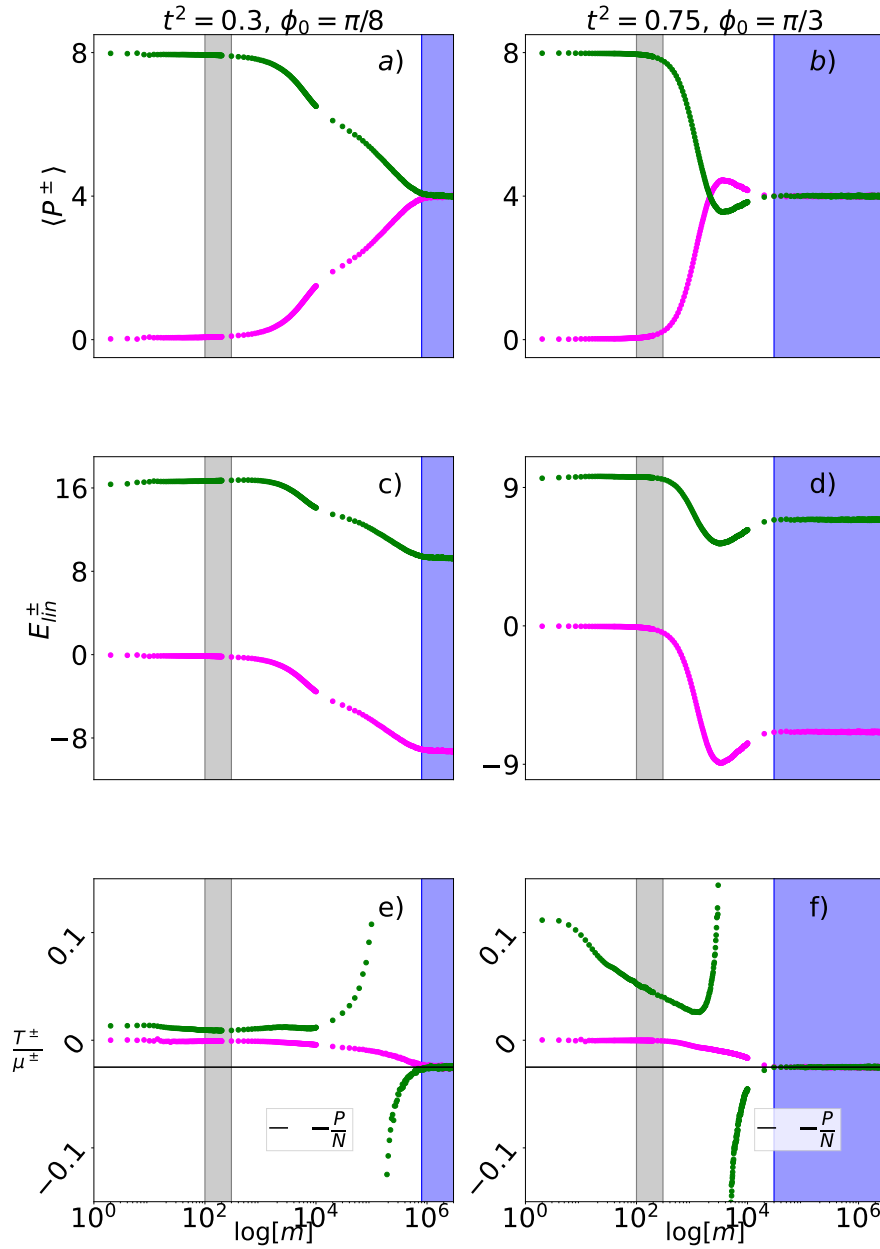


Figure 3.5.3: The full evolution of various quantities for both simulations.

through discrete time effects (see the end of section (3.4.2) for an example). Ultimately the power is partitioned equally between bands, (blue shaded regions in Fig.(3.5.3)), resulting in a universal featureless thermal state, independent of initial condition and lattice parameters (t^2, ϕ_0) . Several more simulations are shown in appendix (A.10), to verify this independence.

The power equipartition between all modes can be explained through the periodicity in $\omega_k^\pm$. For example we consider the six wave resonance discussed briefly at the end of section (3.4.2) given by equation (3.4.36):

$$\omega_1^+ + \omega_2^+ + \omega_3^+ = \omega_4^+ + \omega_5^+ + \omega_6^- - 2\pi.$$

This resonance is admissible purely through the Floquet dynamics, since the 2π is re-absorbed via the modulo 2π in $\omega_k^\pm$. However this wave process violates the conservation of linear energy with the extra factor of -2π . At early times, these resonances are weakly populated, with the linear energy remaining approximately constant per band evident in panels b) and d) in Fig. (3.5.3). This allows for the short lived single band thermal state described by the typical Rayleigh-Jeans distribution:

$$n_k^{RJ\pm} = \frac{T^\pm}{\omega_k^\pm - \mu^\pm}.$$

At long times, when these resonances are more populated, they contribute strongly to the dynamics and invalidate the conservation of linear energy. Now the entropy maximisation used to obtain the Rayleigh-Jeans must be performed without the linear energy constraint, yielding a modified distribution:

$$\begin{aligned} \frac{\delta}{\delta n_k^{RJ\pm}} (T^\pm S^\pm + \mu^\pm P^\pm) &= 0 \\ \Rightarrow n_k^{RJ\pm} &= -\frac{T^\pm}{\mu^\pm}. \end{aligned} \tag{3.5.1}$$

In panels c) and f) of Fig. (3.5.3) we plot the ratio of the measured $T^\pm/\mu^\pm$ from the fitting of the typical Rayleigh- distribution. It is observed that for both

bands in both simulations, this ratio converges to a value of $-P/N$ indicating the system has reached a full thermal state (blue shaded region).

We can further verify this ratio by considering the definition of the power at the full thermal state. As power is equally partitioned between bands we can write the following equality:

$$\langle P^\pm \rangle = \frac{P}{2} = \sum_k n_k^{RJ} = -\frac{N}{2} \left(\frac{T^\pm}{\mu^\pm} \right) \Rightarrow \frac{P}{N} = -\frac{T^\pm}{\mu^\pm}. \quad (3.5.2)$$

Therefore, the full thermal state of any photonic mesh lattice is determined entirely by the initial input power P and the number of modes N , provided there are no losses. Here we have observed the first effect of discrete-time in thermalisation. Extra resonant channels brought about through Floquet dynamics violate linear energy conservation, ensuring a flat featureless thermal state between bands.

Another observed effect of discrete-time evolution is that it can also accelerate thermalisation processes. For example, consider the times at which the full thermal state is reached in panels a) and b) in Fig. (3.5.2). For $(t^2, \phi_0) = (0.3, \pi/8)$ the full equilibrium emerges around 800,000 steps. In comparison when $(t^2, \phi_0) = (0.75, \pi/3)$ this time is drastically reduced to 30,000 steps. Yet again, this can be explained through these extra resonant processes. For $(t^2, \phi_0) = (0.75, \pi/3)$, we see that the bands in Fig. (3.5.1) span a much larger interval in the $\omega_k^\pm$ direction than $(t^2, \phi_0) = (0.3, \pi/8)$. As such, $(t^2, \phi_0) = (0.75, \pi/3)$ can access an extra four wave resonant manifold:

$$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^- + 2\pi \quad (3.5.3)$$

that is otherwise inaccessible for the narrower bands of $(t^2, \phi_0) = (0.3, \pi/8)$. We plot the manifold in equation (3.5.3) in Fig. (3.5.4). Since this extra manifold arises from a four wave resonance, it is believed to contribute on shorter time scales and therefore contribute significantly to the dynamics. The scattering

event is also uni-directional between bands, i.e. two upper band modes excite two lower band modes, the resonant process is very efficient at transferring power to the lower band, further explaining the large discrepancy in time scales.

In table (3.1) we list all the resonances at a four wave interaction up to the first two sets of duplicate bands for both choices of parameters, checking numerically with Mathematica whether they exist for the two choices of lattice parameters. The only extra manifold corresponding to a four wave resonance between the two lattices is given by equation (3.5.3).

We now perform a full sweep of the (t^2, ϕ_0) parameter space, checking whether the manifold in equation (3.5.3) exists in the thermodynamic limit, shown in Fig. (3.5.5). We can see that there are two distinct regions of the parameter space where the extra four wave manifold is either present or absent. From these observations, we expect that choices of parameters with the extra manifold

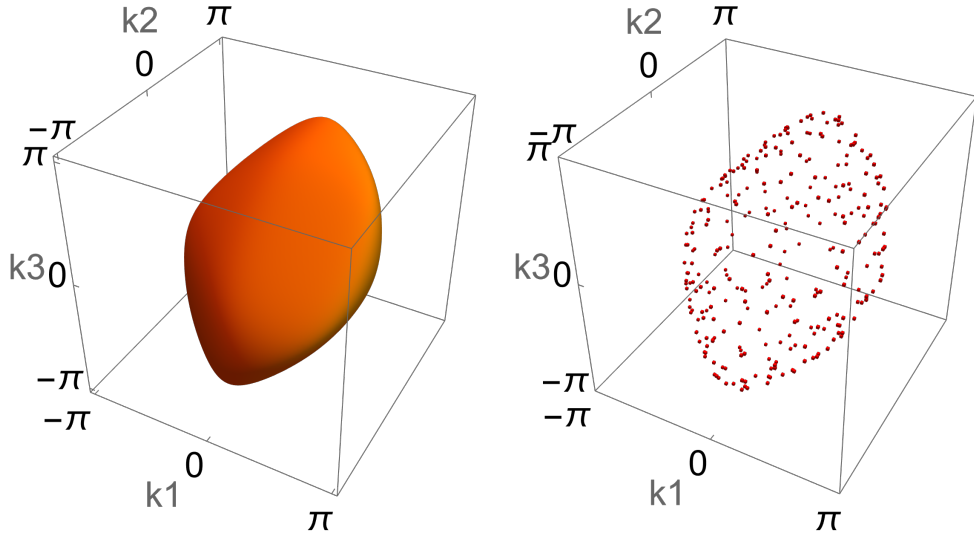


Figure 3.5.4: A plot of the manifold given in equation (3.5.3) with the thermodynamic limit shown on the left and the discrete manifold on the right for $N = 80$. This manifold is entirely accessed through nonlinear broadening effects in the discrete k limit. Here we have taken the value of the broadening measured in figure (3.4.15) to calculate what points in the manifold are accessed. Since the nonlinearity is much higher for these simulations, using this value of the broadening is most likely an underestimate, meaning the manifold is likely to be more densely populated.

Sweep for non-empty four wave manifolds		
Manifold	$t^2 = 0.3, \phi_0 = \pi/8$	$t^2 = 0.75, \phi_0 = \pi/3$
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+$	✓	✓
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+ + 2\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+ - 2\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+ + 4\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^+ - 4\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^-$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- + 2\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- - 2\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- + 4\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- - 4\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^-$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^- + 2\pi$	×	✓
$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^- - 2\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^- + 4\pi$	×	×
$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^- - 4\pi$	×	×
$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^-$	✓	✓
$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^- + 2\pi$	×	×
$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^- - 2\pi$	×	×
$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^- + 4\pi$	×	×
$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^- - 4\pi$	×	×

Table 3.1: Here we have tabulated all the different classes of four wave resonant manifolds that are possible, including the first two sets of duplicate bands for each class. These are checked in the thermodynamic limit. By changing the parameters (t^2, ϕ_0) we gain access to one extra four wave manifold, given in equation (3.5.3). In appendix (A.11) we discuss how we reduce the possible manifolds into different classes.

should thermalise on shorter timescales. To verify this we consider a contour of this manifold, as highlighted by the blue horizontal line in Fig. (3.5.5), and perform nine sets of simulations across this contour, measuring the thermalisation times for each simulation. To measure the thermalisation timescale we evolve the lattice until the ratio of $T^\pm/\mu^\pm = -P/N \pm 1\%$ in both bands simultaneously.

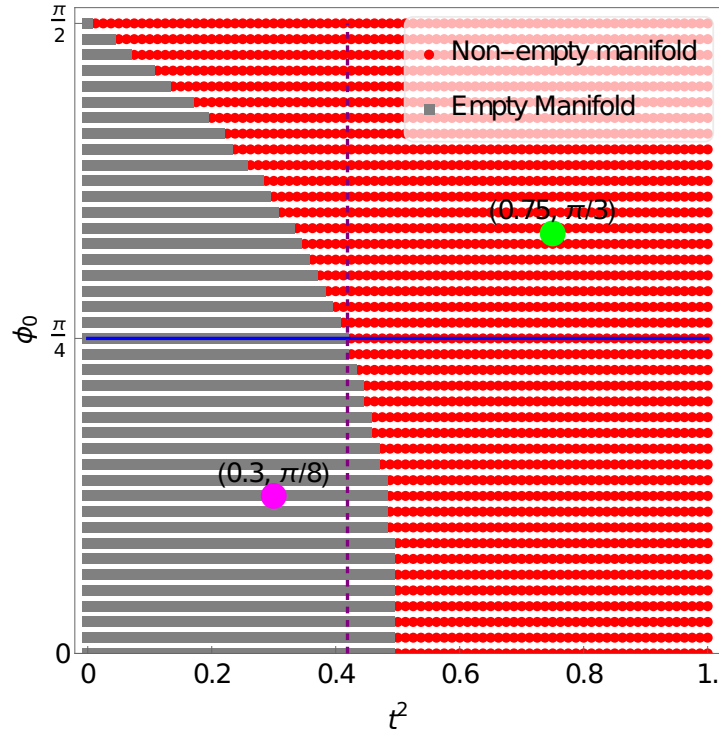


Figure 3.5.5: We sweep across the full parameter space of $t^2 \in [0, 1]$ and $\phi_0 \in [0, \pi/2]$ in a 81×81 grid computing the manifold in equation (3.5.3) in the thermodynamic limit for every pair of (t^2, ϕ_0) . Grey squares indicate an empty manifold and red circles a non-empty manifold. The purple and green points are the parameters for the two set of simulations discussed above.

The measured timescales are presented in Fig. (3.5.6). What we observe is a clear decrease in the timescale required for full thermalisation as we cross into the range of t^2 that possesses the extra manifold. This is strong evidence to support the notion that the manifold in equation (3.5.3) is accelerating the time taken to reach a full thermal state.

Further, we can check the size of the manifold across the horizontal blue line, by calculating the manifold in the thermodynamic limit for every choice of (t^2, ϕ_0) in Fig. (3.5.5), visualised in the Fig. (3.5.7). Interestingly, the manifold becomes larger as t^2 increases, allowing for more resonant interactions. The growing manifold coincides with the bands occupying a larger range of the $\omega_k^\pm$ domain, which increase the number of possible resonances. This would explain why the measurements of thermalisation time scale in Fig. (3.5.5) decrease smoothly across the manifold boundary.

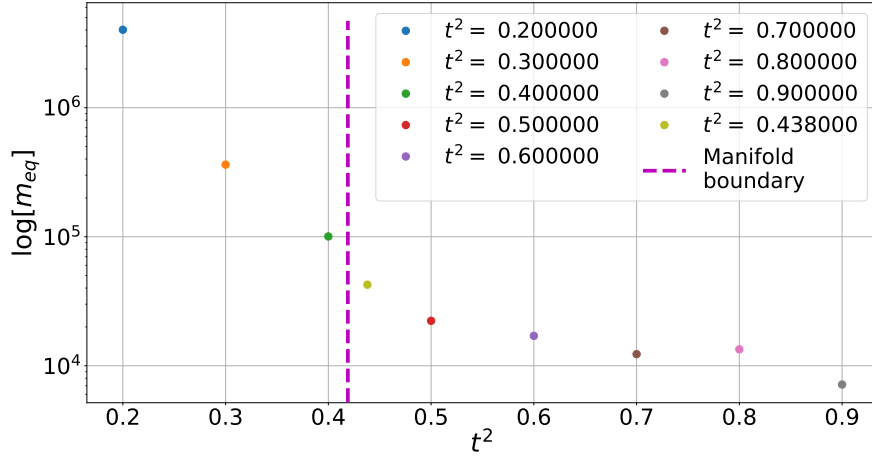


Figure 3.5.6: Measurement of the time taken to reach a full thermal state between bands for fixed $\phi_0 = \pi/4$ with t^2 varied. We omit the choices of $t^2 = 0, 1.0$ since these are edge cases with dispersionless bands. $t^2 = 0.1$ was also omitted, as the time scale for simulations was becoming exceedingly large. Here we have sketched a vertical dotted purple line to indicate the boundary of empty and non-empty manifold. This same line is sketched in Fig. (3.5.5).

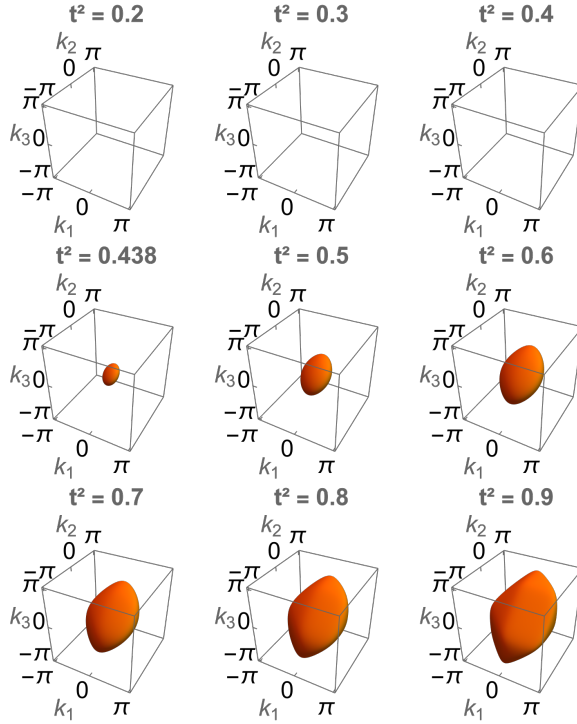


Figure 3.5.7: Plots of every manifold in the thermodynamic limit, for all the simulations performed along the blue horizontal line when $\phi_0 = \pi/4$.

In this section, we provide the first numerical evidence that discreteness in time has a non-trivial impact on thermalisation processes, allowing resonant

interactions that are otherwise inaccessible in continuous-time systems. Through these resonances we found that: (i) the full thermal state is a universal distribution independent of (t^2, ϕ_0) and initial modal occupation described by a flat featureless state, (ii) the process of thermal equilibrium can be greatly accelerated through the choice of (t^2, ϕ_0) .

To conclude this section, possible avenues of experimental and theoretical works will be discussed that could further verify the observed phenomena.

3.5.1 A possible avenue for experimental verification of discrete time effects

The numerical observation of discrete-time effects shows stark changes in system behaviour, evident in the long-time dynamics once the full thermal state between bands is reached. The long time scales to reach full thermal equilibrium are currently not achievable with current double loop arrangements, which can achieve several thousand steps in time at most. Further experimental complications include the size of the lattice which has been reported to have values of up to $N = 42$ [92]. In contrast, the numerics performed here have used $N = 320$ far exceeding what is currently possible experimentally.

Instead, we propose a single realisation experiment considering a reduced mode model of $N = 8$ such that there are two modes of unique $\omega_k^\pm$ in the upper and lower band. Then through careful selection of (t^2, ϕ_0) we can ensure that the four modes are resonant according to equation (3.5.3). We give a numerical example of this experiment in Fig. (3.5.8) showing the single realisation evolution of $m = 500$. Here $\chi = 1.6$, $P = 0.2$ with each upper band mode starting from an occupation of 0.1 intensity and random phases. When $(t^2, \phi_0) = (0.3, \pi/8)$, we see that the two upper band modes exchange occupation periodically with no lower band excitation. This is because the extra discrete-time manifold in equation (3.5.3) is absent here. However for $(t^2, \phi_0) = (0.5, \pi/2)$ we see that the lower band modes are now occupied through the interaction described by

$$(t^2, \phi_0) = (0.3, \pi/8)$$

$$(t^2, \phi_0) = (0.5, \pi/2)$$

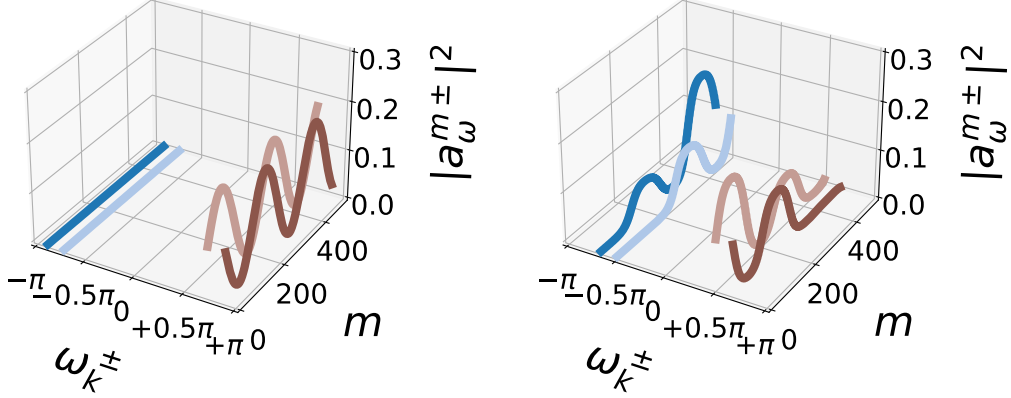


Figure 3.5.8: The evolution of the modal occupation for each value of $\omega_k^{\pm}$ in either band for a single realisation of two different lattices. Here $|a_{\omega}^{m\pm}|^2 = |a_k^{m\pm}|^2 + |a_{-k}^{m\pm}|^2$.

equation (3.5.3).

3.5.2 Theoretical verification through a wave kinetic equation and the difficulties that follow

As was shown in chapter (2), WWT theory is governed by the resonant processes that emerge from the derivation of the WKE. In the above work, we have assumed their existence for the photonic mesh lattice and shown numerically that this assumptions appear to be valid. One concrete way to exactly verify the existence of the discrete-time resonances would be through a WKE formulated from equations (3.3.2) and (3.3.3):

$$\begin{aligned} u_n^{m+1} &= \left[t u_{n+1}^m e^{i\chi|u_{n+1}^m|^2} + i r v_{n+1}^m e^{i\chi|v_{n+1}^m|^2} \right] e^{i\phi^m}, \\ v_n^{m+1} &= \left[t v_{n-1}^m e^{i\chi|v_{n-1}^m|^2} + i r u_{n-1}^m e^{i\chi|u_{n-1}^m|^2} \right]. \end{aligned}$$

There are several difficulties in doing this however which will now be discussed.

One difficulty is the way that the nonlinearity enters in the system. As it emerges in these exponential functions we possess infinitely many terms making the equations very hard to work with algebraically. Truncation of these terms

also leads to unstable dynamics requiring that all terms up to $\mathcal{O}(\chi^3)$ be included for stable dynamics for weak nonlinearity as shown in section (3.4.3). Keeping all the terms up to $\mathcal{O}(\chi^3)$, despite being simpler, is still extremely non-trivial to work with yielding hundreds of terms in the evolution equation. To further complicate things, the evolution is described by two coupled wave fields that must be written in terms of the normal mode amplitudes.

Another problem is that the WWT formulation often requires the separation of time scales. Here as time enters through a discrete index, we cannot simply re-scale the index m by χ in a meaningful way. One possible approach could be to bring the system into a continuum much like what was done in section (3.1.2) and derive the in the time continuum WKE using the standard approaches. Once the equation is derived, then one may be able to re-discretise time which may reveal some of the interesting effects of discrete-time. If the continuum WKE is derived using symplectic methods, this could reveal some Hamiltonian structure present in the lattice that is currently not known. Deriving a continuum WKE is not straightforward however, again because the equations are coupled and possess an exponential nonlinearity.

It should be noted however, that simplifications can be made that will make any theoretical works easier while retaining the interesting discrete-time phenomena. One such simplification relates to the choice (t^2, ϕ_0) . Since the final thermal state is independent of the lattice parameters, we can choose them such that it simplifies the equations of motion. The best choice for this would be $(t^2, \phi_0) = (0.5, 0)$, as in this case $t = r$ and the phase modulation term vanishes. In appendix (A.10) we show this exact case numerically, and observe the universal flat thermal state. Through these parameter choices, this would allow one to formulate the evolution in the single step instead of the stroboscopic step while also removing the definition swapping of the normal modes.

Driven/damped lattice dynamics

In the final part of our results, we transition the photonic mesh lattice from a thermodynamically closed to an open system. By introducing both driving and dissipation, we seek to investigate how the system reaches steady states within the upper and lower dispersion bands. While previous experimental work has examined out of equilibrium photonic lattice through controlled losses [72], there has been no exploration into the emergence of steady states mediated jointly by forcing and dissipation. Accordingly, this final section aims to probe this idea numerically, highlighting its potential as a direction for future research.

4.1 The implementation of forcing and dissipation in the photonic mesh lattice

In order to study the out of equilibrium dynamics, it is first important to explain and justify the implementation of the forcing and damping. For all simulations in this chapter we force every upper band mode while simultaneously dissipating every lower band mode in the stroboscopic step. The forcing and dissipation is parametrised by a factor $\gamma \in \mathbb{R}$ which scales the amplitude of each mode

accordingly:

$$\alpha_k^{m+} = a_k^{m+} \left(1 + \frac{\gamma}{2}\right), \quad (4.1.1)$$

$$\alpha_k^{m-} = a_k^{m-} (1 - \gamma). \quad (4.1.2)$$

$\alpha_k^{m\pm}$ denote the wave fields after forcing and damping. A factor of 1/2 is included in the damping for several reasons. Since the power leakage between bands does not occur instantaneously, a saturation period in the upper band is required for the resonant channels to become populated, enabling power transfer between bands. If the forcing is applied too strongly at the onset, the dynamics blow up becoming unstable, see panels a) and c) in Fig. (4.1.1). On the contrary, if the forcing is applied too weakly compared to the dissipation, the lower band wave field is depleted and as consequence, slows down inter-band processes, increasing the timescale of the steady state, see panels b) and d) in Fig. (4.1.1). Whilst other factors than 1/2 are shown to be stable, we opt for the 1:2 ratio of forcing and dissipation for numerical convenience. This ratio will be used throughout the remainder of this section.

Next we will define some new quantities relevant to the driven/damped dynamics. These quantities will be useful in examining steady state behaviour, and determining the time scales at which the steady state is reached. We define the forcing per k in the intensity field as,

$$F_k^m = |\alpha_k^{m+}|^2 - |a_k^{m+}|^2, \quad (4.1.3)$$

and the dissipation per k mode,

$$D_k^m = |a_k^{m-}|^2 - |\alpha_k^{m-}|^2. \quad (4.1.4)$$

Then the external flux per k mode is,

$$\Pi_k^m = F_k^m - D_k^m. \quad (4.1.5)$$

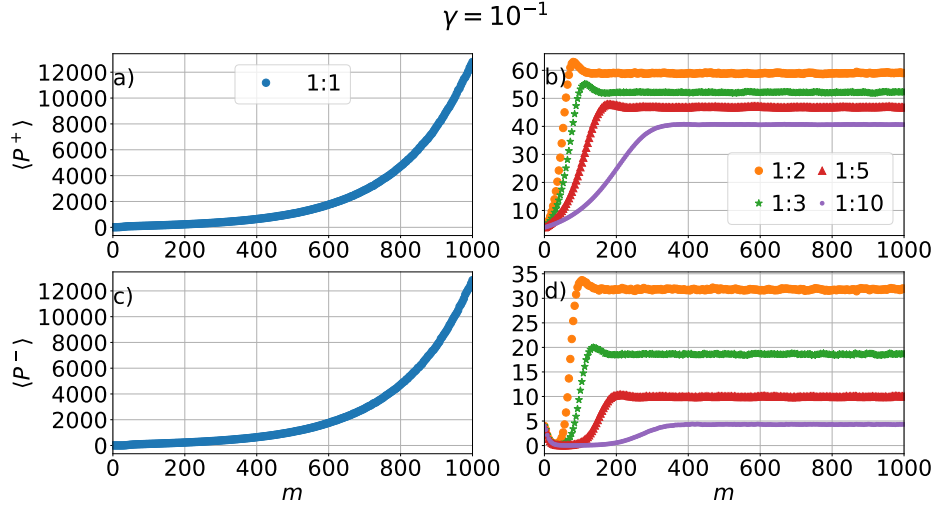


Figure 4.1.1: The evolution of the band power for different ratios of forcing and dissipation. Each simulation starts from an initial seeding given by the full thermal state of the closed system. Initial power $P^0 = 8$, $\chi = 1.6$ $N = 320$, $(t^2, \phi_0) = (0.5, \pi/4)$, averaged over 500 simulations. $\gamma = 10^{-1}$, the largest value considered throughout this section. a) and c) show how the power evolves if the ratio of forcing to dissipation is 1:1. In this case we observe numerical blow up. Panels b) and d) show that if the ratio is not 1:1, the dynamics are stable, with smaller choices of forcing taking longer to reach a steady state in $\langle P^\pm \rangle$. Interestingly, the ratio of forcing to dissipation, corresponds near exactly to the ratio of power in either band, providing a robust way to control the band power.

It is important to acknowledge that equation (4.1.5) is computing the flux between like k modes in the upper and lower band only and not between different values of k . It therefore won't completely capture all the physics of how efficient transfer is between individual modes. From equation (4.1.5) we can define the total flux over all k modes at time step m as,

$$\Pi^m = \sum_k \Pi_k^m \quad (4.1.6)$$

4.1.1 The effect of γ on the steady state

With the choice of forcing and dissipation established, we now examine the effect of γ on the dynamics. This will serve primarily as a numerical diagnostic, identifying the range of γ values that yield stable and reliable steady states without unphysical blow-up or exceedingly long computation times.

To proceed we perform nine set of simulations for $\chi = 1.6$, an initial power $P^0 = 8$, $N = 320$ and $(t^2, \phi_0) = (0.5, \pi/4)$, averaged over 500 realisations and evolved for 1,000,000 time steps or 3,000,000 depending on the size of γ , starting from an initial seeding given by the full thermal state observed between bands in section (3.5).

Fig. (4.1.2) shows how the power in each band evolves for each choice of γ over the 1,000,000 steps. We span an interval of $\gamma \in [10^{-5}, 10^{-1}]$, trying different values arbitrarily. For $\gamma < 10^{-5}$, exceedingly long time scales are needed to reach a steady state, while for $\gamma > 10^{-1}$ the dynamics become unstable. By increasing γ , the power occupation in each band increases and the timescale to reach a steady state decreases significantly. Time averaged power per band, given by $\langle \overline{P^\pm} \rangle$, is taken over steps 800,000 – 1,000,000 and is plotted in Fig. (4.1.3). In this time interval, every simulation possesses a stable power in both bands, and is therefore assumed to be within the steady state. We then fit exponential curves for $\langle \overline{P^\pm} \rangle$ vs $\log[\gamma]$ of the form,

$$\langle \overline{P^\pm} \rangle \approx Ae^{B \log[\gamma]} + C,$$

which simplifies to give a power law scaling between $\langle \overline{P^\pm} \rangle$ and γ ,

$$\langle \overline{P^\pm} \rangle \approx A\gamma^B + C,$$

where A, B, C are given for either band in Fig. (4.1.3).

The effect of γ is also evident in the single realisation evolution of the steady state. Fig. (4.1.4) plots the real space evolution and the recovered $(k, \omega_k^\pm)$ spectrum for different choices of γ , evolved for a further 1,000 steps starting from $m = 1,000,000$. For the largest γ in Fig. (4.1.4), the real space evolution in panel a) interacts very strongly, appearing as large intensity background noise. Panel b) shows the recovered spectrum is extremely broadened shifting the upper band downwards into the middle of the frame. As γ becomes smaller,

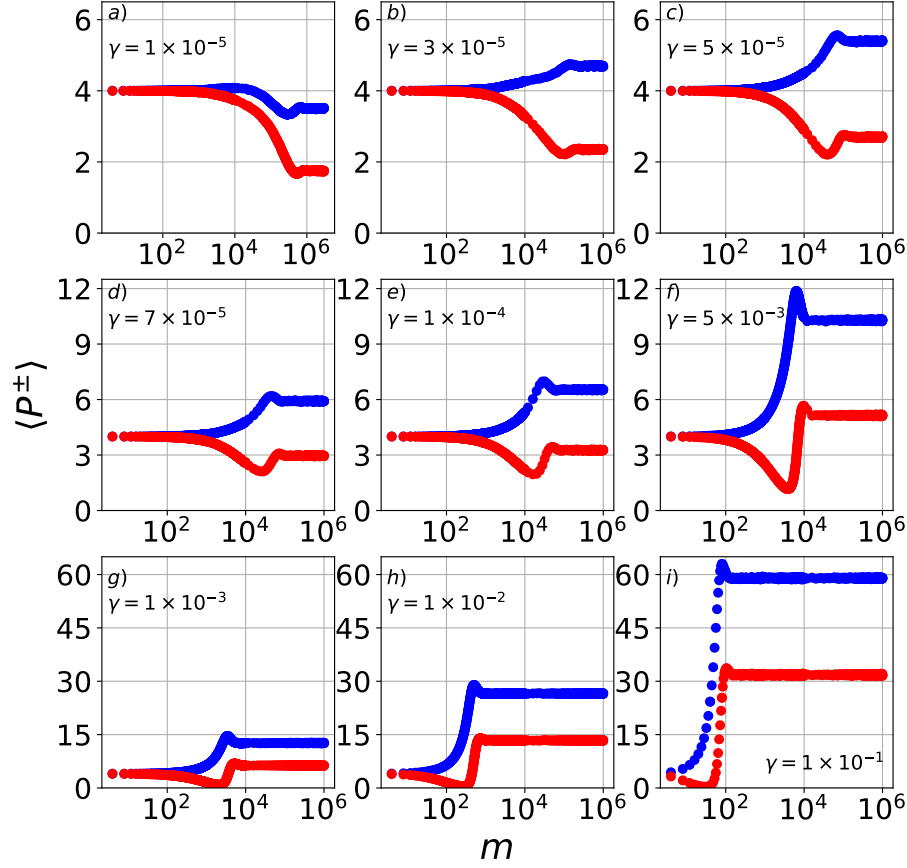


Figure 4.1.2: Power evolution of the upper band in blue and lower band in red for each choice of γ . For each simulation the power plateaus in either band, serving as an indicator that the system has reached a steady state. For panel a) we evolve up to 3,000,000 steps in time, to ensure that the steady state is reached.

the real space evolution has more clear structure in wave propagation in panels c) and e), with the respective spectra in d) and f) remaining sharper with significantly less broadening, and a smaller shift downwards as the self phase modulation is weaker.

Lastly we aim to determine numerically when the steady state is reached. This is more challenging compared to the closed system due to the presence of external forcing and dissipation introducing greater fluctuations in thermodynamic quantities. To add further complications, these fluctuations scale with the magnitude of γ , requiring extra care when finding the steady

state. To determine the onset of the steady state numerically, we adopt the following procedure:

1. As the flux typically reaches a maximum value before approaching zero, we first truncate the flux data to values after Π^m has maximised. This data is further constrained to values bounded by $|\Pi^m| < \gamma$, since once the forcing and dissipation balance we expect the the flux to tend to zero. By bounding $|\Pi^m| < \gamma$ we consider fluctuations around zero that scale with the amount forced/dissipated.
2. Among these data points, we select those where instantaneous (T^+, μ^+) lie within 5% of their time averaged values over the final 200,000 steps - the time interval for which the power in every simulation for both bands is stable in time.
3. The earliest time step satisfying these conditions simultaneously is defined as the onset of the steady state.

Note that we only use (T^+, μ^+) to determine the steady state, as restricting the data by (T^-, μ^-) as well is found to over constrain the data. Additionally,

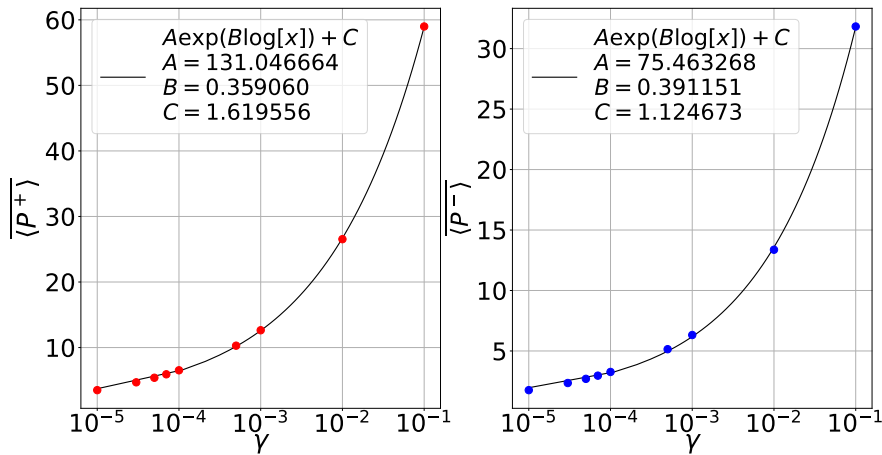


Figure 4.1.3: Semi-log x plot of time averaged power vs γ . We take the time average over steps 800,000 to 1,000,000 as in this time frame, the band power in panels a)-i) of Fig. (4.1.2) have all plateaued, and therefore are assumed to be the steady state power.

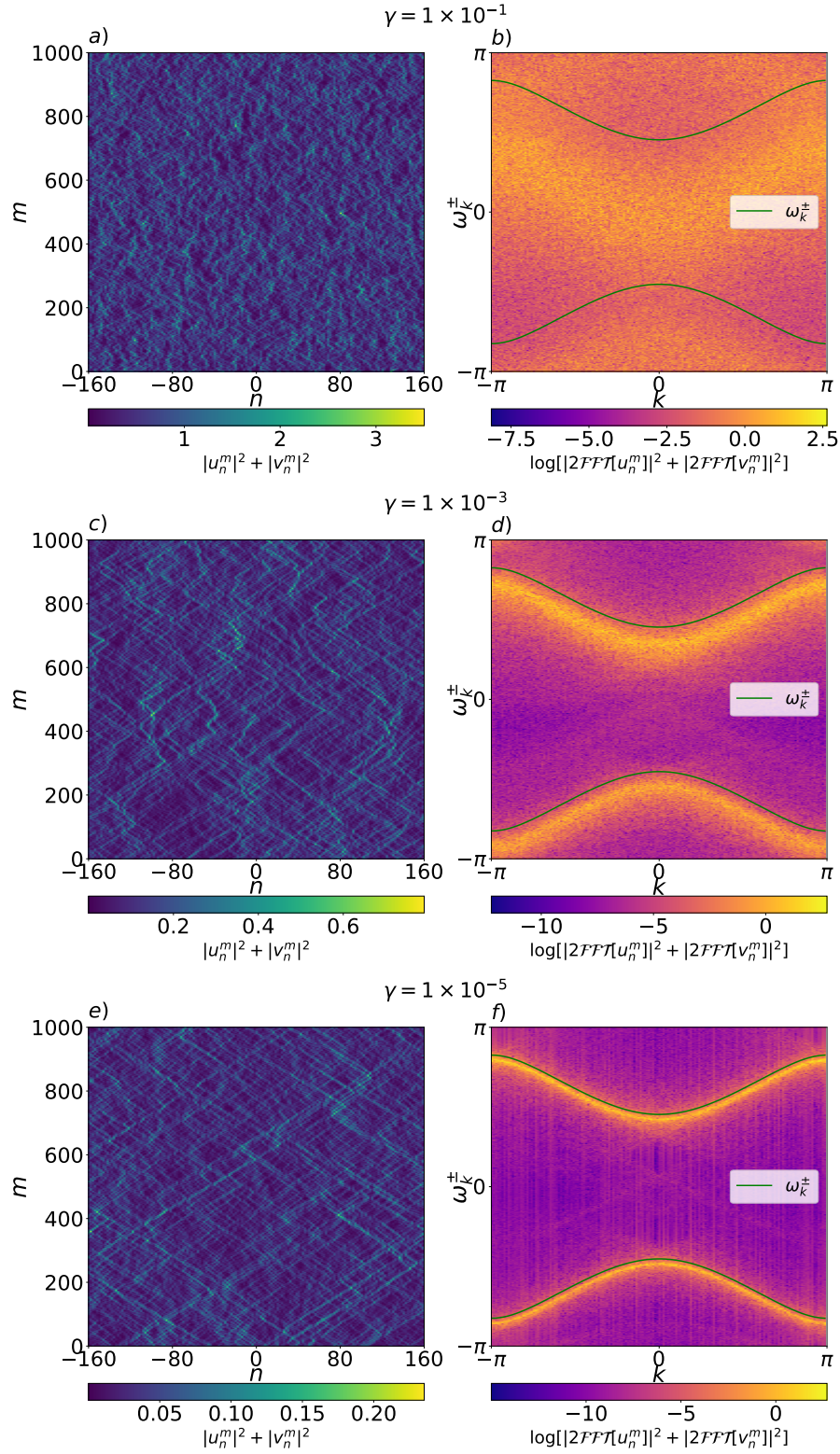


Figure 4.1.4: Here we show the single realisation evolution starting from $m = 1,000,000$ evolved for 1,000 more steps, for different choices of γ . Panels a), c), e) show the real space evolution and b), d), f) the recovered $(k, \omega_k^\pm)$ spectrum.

the above procedure assumes aprior that the steady state can be described by a Rayleigh-Jeans distribution of the form,

$$n_k^{RJ\pm} = \frac{T^\pm}{\omega_k^\pm - \mu^\pm}. \quad (4.1.7)$$

Fig. (4.1.5) illustrates what time steps are identified to be in the steady state, marked with the red crosses for the simulation $\gamma = 7 \times 10^{-5}$.

Repeating this procedure for all choices of γ yields the snapshots of the predicted onset of the steady states in Fig. (4.1.6). The time evolution for Π^m , $T^\pm$ and $\mu^\pm$ of each simulation is included in appendix (A.12). By driving and dissipating the lattice, the steady-states for every γ agree well with the Rayleigh-Jeans distribution. For smaller $\gamma < 5 \times 10^{-4}$ the temperatures of both bands tend towards the same value, whilst the chemical potentials remain different, a finding previously reported in continuous-time multi-band closed systems [102]. For larger γ , the instantaneous $(T^\pm, \mu^\pm)$ fluctuate more greatly deviating from this observation because the increased total power flattens the spectra, reducing the reliability of the Rayleigh-Jeans fitting, see Fig. (4.1.7) for a comparison between $\gamma = 1 \times 10^{-3}$ and $\gamma = 7 \times 10^{-5}$.

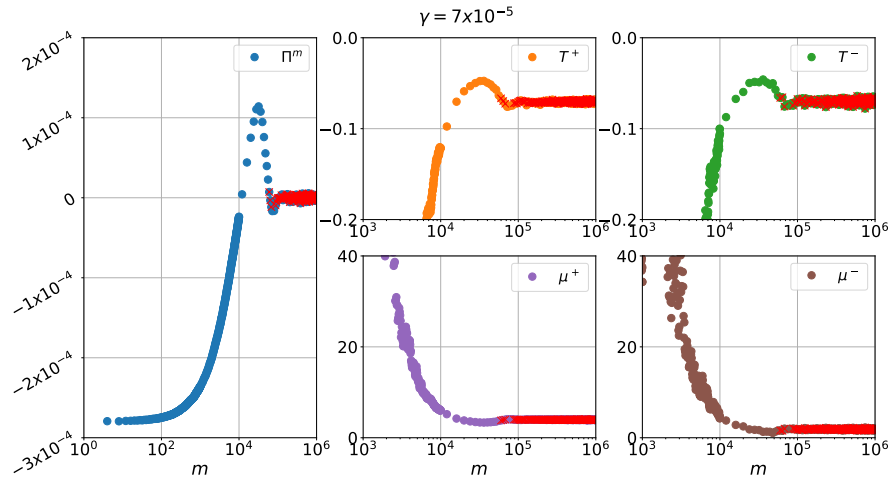


Figure 4.1.5: An example of what points are determined to be within the steady state defined by the total flux Π^m and the measured upper band quantities (T^+, μ^+) .

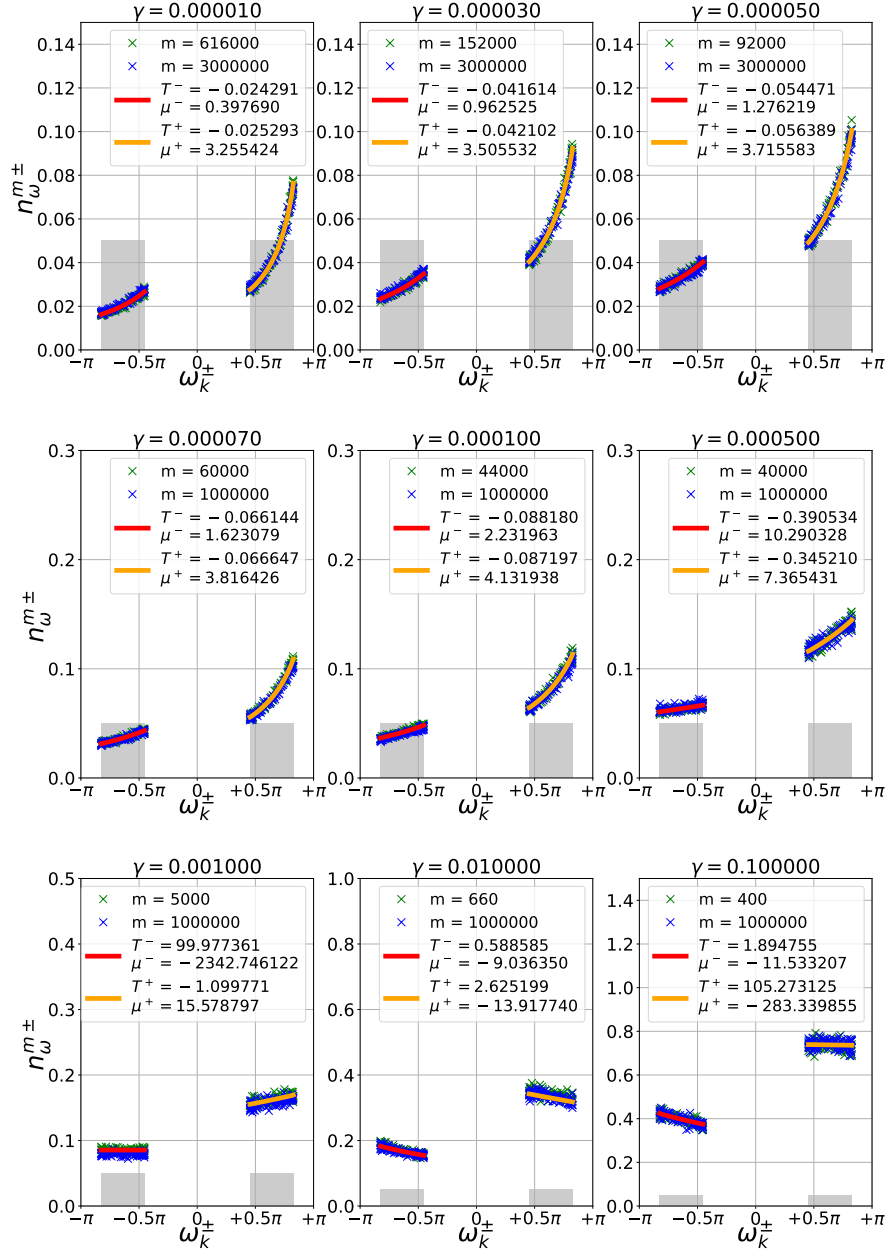


Figure 4.1.6: The nine panels show snapshots of the out-of-equilibrium driven/damped evolution for different choices of γ ordered from smallest to largest. The grey shaded region indicates the initial state, green crosses is the spectrum once it first reaches a steady state and the blue crosses are the final spectra at the end of each simulation. The red and orange lines are the fitted Rayleigh-Jeans distributions to the green crosses for the lower and upper band respectively.

From analysis of band power evolution, single realisation dynamics, and measurements of the onset of the steady state, we find that all values of γ that we simulated are numerically stable. However, larger values $\gamma \geq 1 \times 10^{-3}$ lead

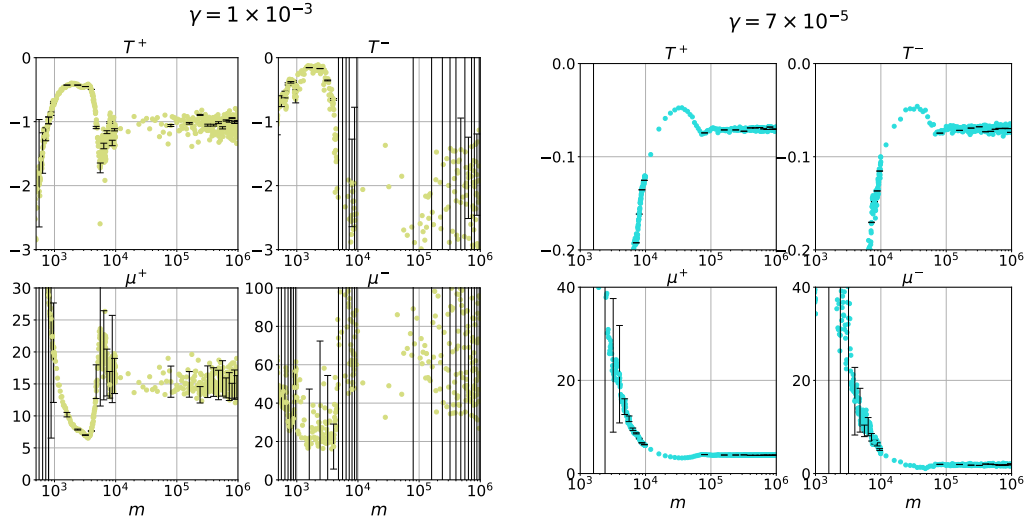


Figure 4.1.7: The left hand subplot shows the evolution of $(T^\pm, \mu^\pm)$ for $\gamma = 1 \times 10^{-3}$ and the right subplot for $\gamma = 7 \times 10^{-5}$ at the onset of the steady state. Error bars are plotted in black. As can be seen, the error is much larger for $\gamma = 1 \times 10^{-3}$, particularly in the lower band because the lower band spectrum is almost flat.

to ambiguous thermodynamic characterization as the spectrum in either band flattens and fluctuations measurements increase. Conversely, smaller values such as $\gamma = 1 \times 10^{-5}$, have steady states occur at longer times. Therefore we conclude that viable choices lie within the interval $1 \times 10^{-5} < \gamma < 1 \times 10^{-3}$. This range provides a practical balance between computational efficiency, stability, and clarity in steady state behaviour.

4.1.2 The effect of band separation on steady states

With an appropriate implementation of forcing and dissipation established, we now explore the effect of the shape of the dispersion relation for when $\gamma = 1 \times 10^{-4}$. For fixed $\phi_0 = \pi/4$ we vary t^2 in increments of 0.1, as shown in Fig. (4.1.8) to control the width of both band gaps at $k = 0$ and $k = -\pi$. Smaller values of t^2 have much narrower bands, spanning a smaller interval in the $\omega_k^\pm$ direction with larger band gaps. As understood from section (3.5), narrower bands have access to fewer frequencies and as consequence do not possess the extra four wave manifold brought about through discrete time, that the wider band counterparts possess.

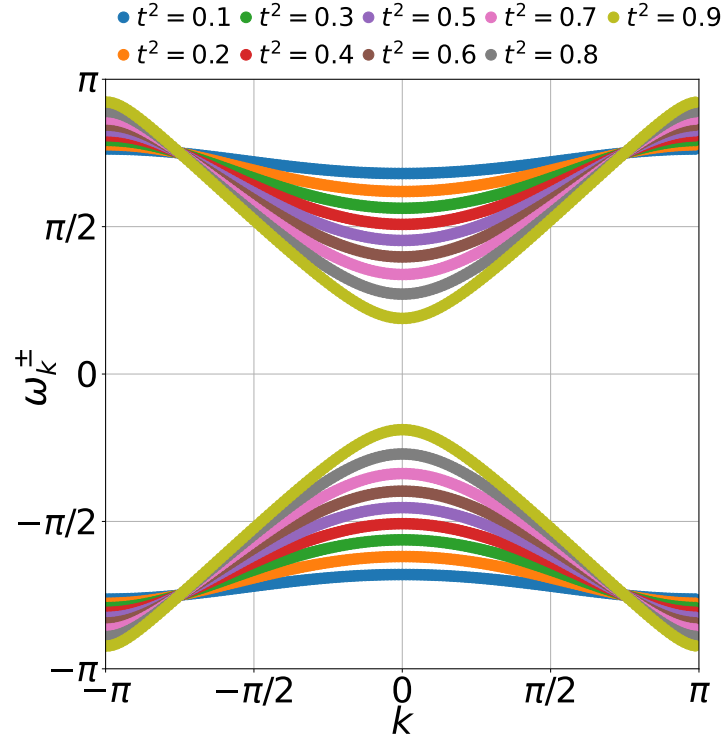


Figure 4.1.8: Dispersion relation for $\phi_0 = \pi/4$ for varied t^2 .

In this section, we aim to understand what role this plays in the driven/damped setting.

To begin we first measure the flux Π^m in Fig. (4.1.9) panel a) to determine the onset of the steady state. Here we measure the steady state from when the total flux in time approaches zero, with no further criteria on (T^+, μ^+) unlike the previous section. This is because in general the dynamics are more stable and do not vary too quickly in time as γ is fixed across all simulations, thus making the thermodynamic characterisation much easier.

In Fig. (4.1.10) we plot the evolution of $(T^\pm, \mu^\pm)$ for every choice of t^2 with the steady state highlighted by the green shaded region. For all simulations, we see that each measured quantity tends to a certain value; we note that for smaller t^2 (e.g. $t^2 = 0.1$) the fluctuations are much greater because of the larger increase in total power evident in panel b) Fig.(4.1.9). This makes the steady state harder to precisely pin point in time. In general we see through Fig. (4.1.10) that by defining the onset of the steady-state solely through the flux, we get a good

estimate.

In the measurements of Π^m , we can immediately see an effect of the the widened bands. Smaller t^2 has a larger flux before the onset of the steady state as the interband processes are weak due to narrower bands, causing a pile up effect in the upper band. Then there is a sudden transfer of wave action to the lower band and the system equilibrates. In contrast lattices with wider band do not have this pile up effect as there are more interband processes that can facilitate efficient transfer through the extra discrete-time resonant channels. These effects are also mirrored in the measurements of the total flux in panel b) of Fig. (4.1.9), with narrower bands having larger positive net changes in power.

However, unlike in the closed system, we observe that the onset of the steady state is independent of the lattice parameters, with all lattices equilibrating on similar time scales. One possible explanation is that lattices with more efficient interband processes (e.g. $t^2 = 0.9$) dissipate power much more rapidly in the lower band while forcing the upper band more slowly, since both forcing and dissipation are proportional to the instantaneous band occupation. Conversely, lattices with less efficient interband processes (e.g. $t^2 = 0.1$) experience much stronger forcing in the upper band and much weaker dissipation in the lower band for the same reasons. In both cases, these opposing effects compensate in similar ways, leaving the steady state time scale purely determined by γ .

Next we look at the the distributions in the steady state shown for each t^2 in Fig. (4.1.11). For every simulation, we observe the standard Rayleigh-Jeans distribution in both bands constrained by $\omega_k^\pm$. As we increase t^2 we also observe that the $T^+ \approx T^-$ with $\mu^+ \neq \mu^-$, again matching previous findings in continuous-time multiband closed systems [102]. In Fig. (4.1.12), we plot the time average of $(T^\pm, \mu^\pm)$ and also their absolute difference between bands to further demonstrate this property.

By considering the net dissipation and forcing per mode in Fig. (4.1.13) we can better understand how the shape of the stationary state changes with wider

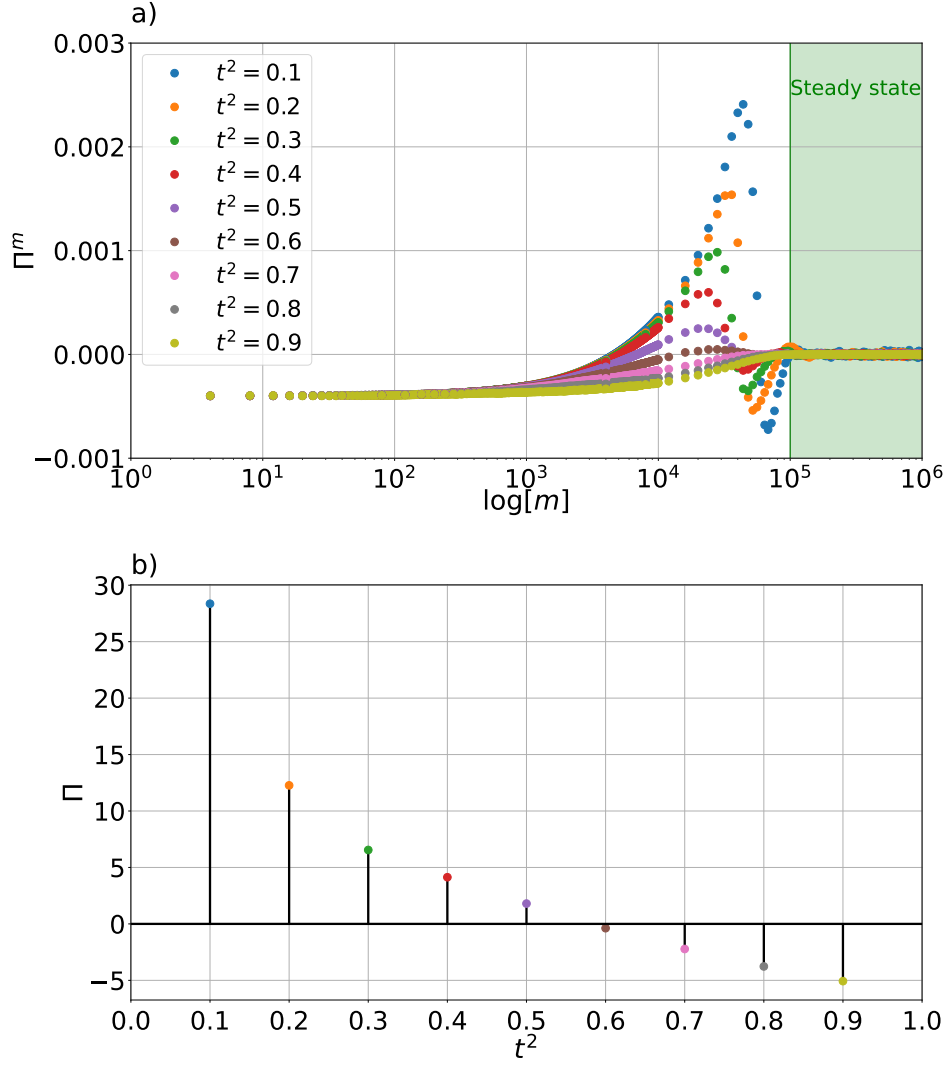


Figure 4.1.9: a) shows the evolution of the flux in term for each simulation. b) shows the net flux for each choice of t^2 .

band lattices. Modes that possess greater net dissipation can be perceived to be involved in more interband processes and are therefore more efficient at band transfer. The rationale is that the dissipation is proportional to the occupation, and lower band modes with greater occupation will dissipate greater amounts. Since the lower band modes are not forced, and rely on a source of power from the upper band, there must be more efficient resonant channels to this mode enabling a greater leakage. Conversely if a mode has greater net forcing (which is proportional to the occupation of that particular mode), this would indicate that power is less free to move from this mode into other modes, and is therefore less efficient at interband transfer.

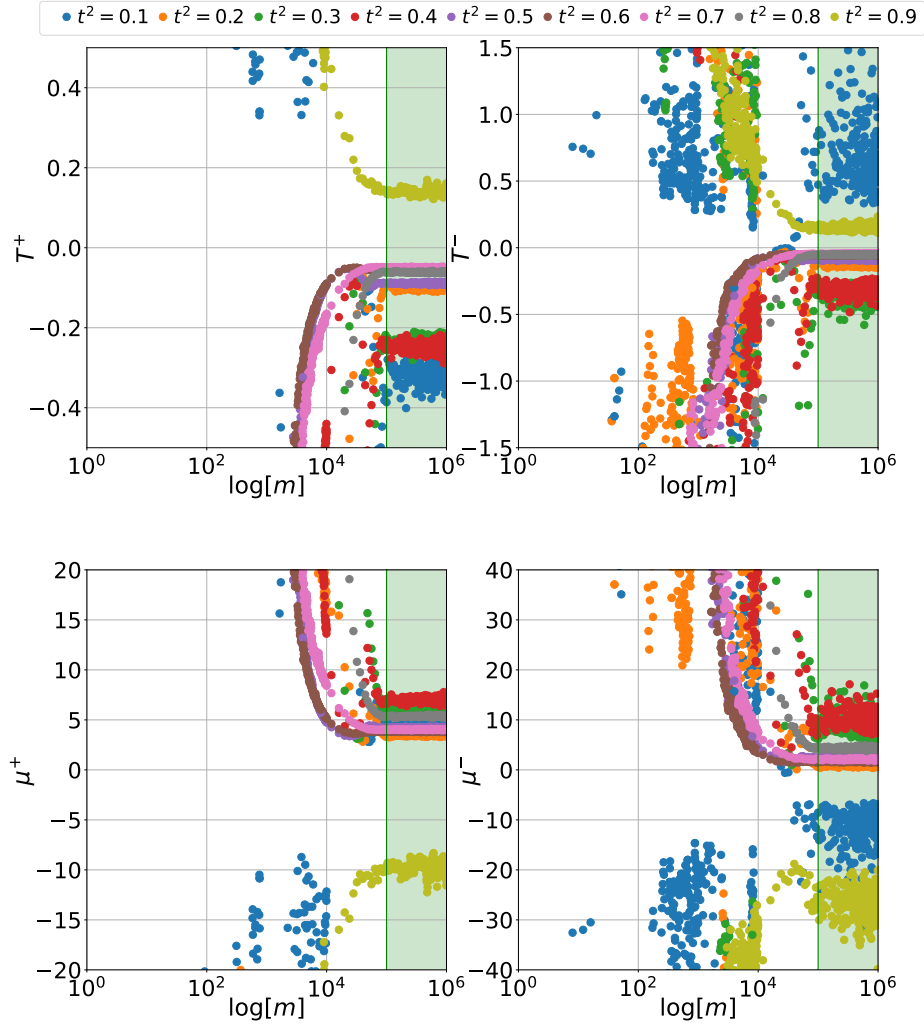


Figure 4.1.10: Here we see the $T^\pm, \mu^\pm$ begin to reach a steady state after 10^5 steps. Some simulations fluctuate more than other because the power is much higher. As it is out of equilibrium there is expected to be some noise.

As the bands widen with greater t^2 we observe that every mode in both bands possess similar forcing and dissipation. This aligns well with the current understanding that there are more possible resonant channels between modes for larger t^2 , since we occupy a greater interval on $\omega_k^\pm$.

By driving and dissipating the lattice we appear to lose many of the properties brought about through the discrete-time effects. While we find more efficient interband processes for lattice parameters that possess the extra resonant manifold, these no longer accelerate the equilibrium process, nor do they alter the final thermal state to be independent of $\omega_k^\pm$. The resulting observed

behaviour is more akin to continuous-time multi-band systems [102], as shown in the steady-states in both bands possessing the same temperature and different chemical potential.

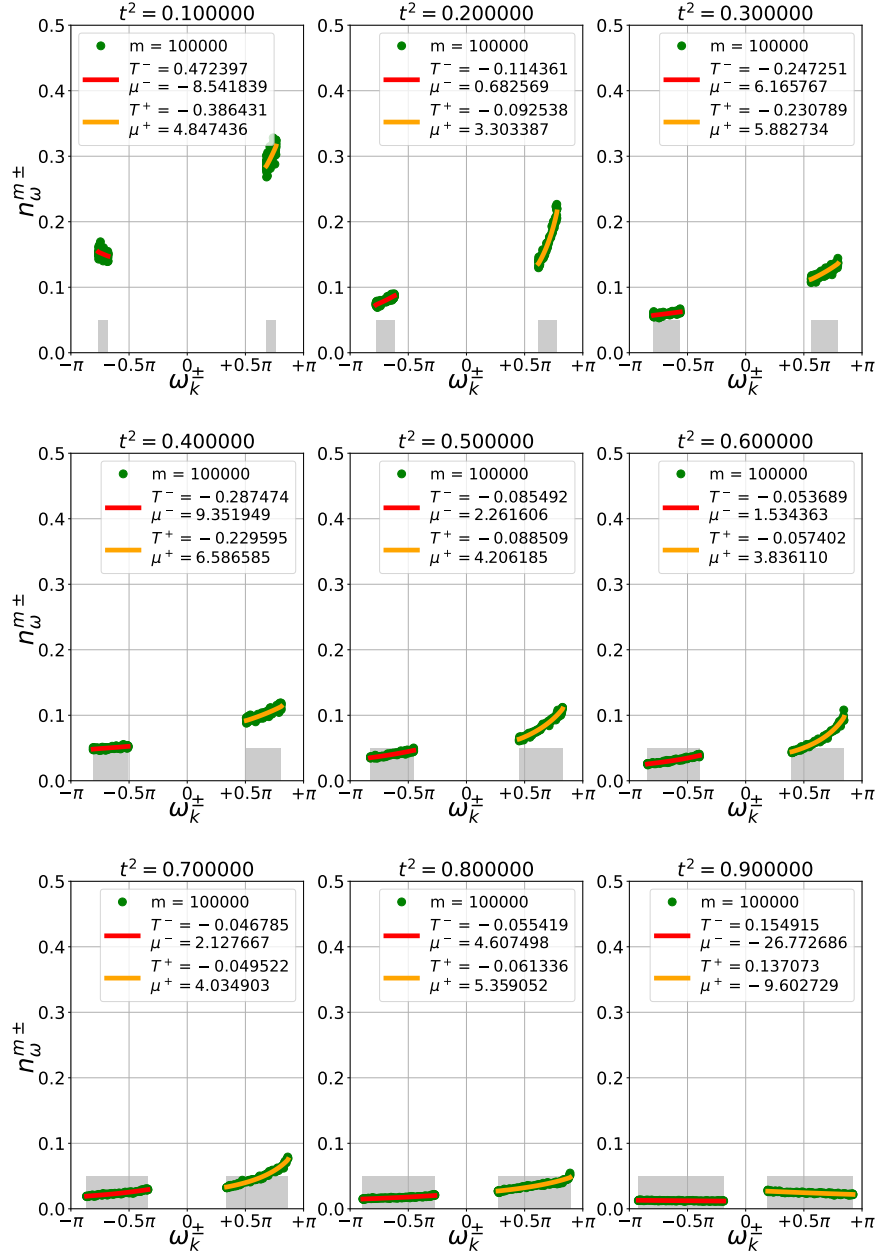


Figure 4.1.11: Here we plot the spectra of the steady state at $m = 100,000$ and fit a Rayleigh-Jeans distribution to both bands. The initial condition is given by the grey shaded region.

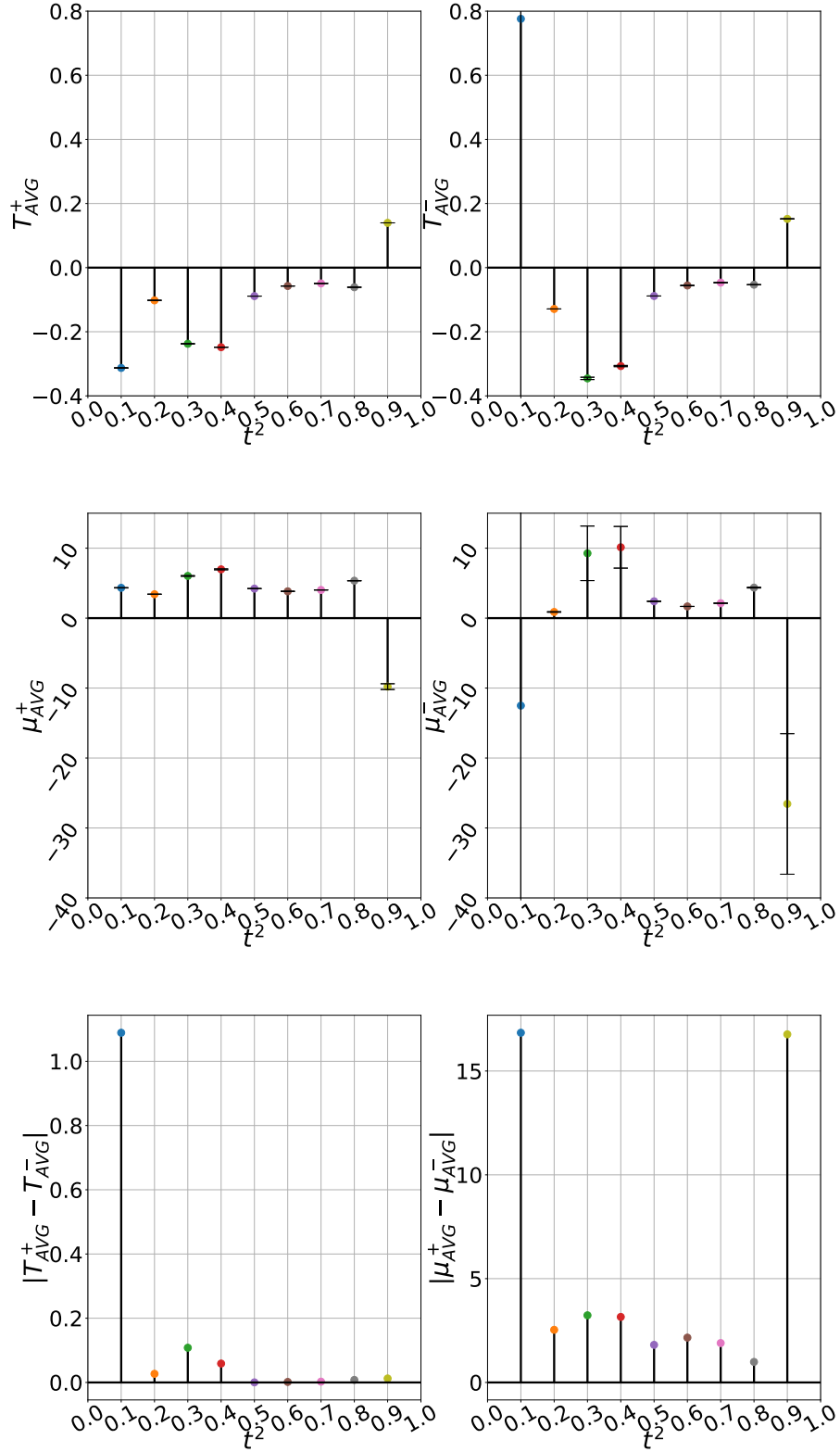
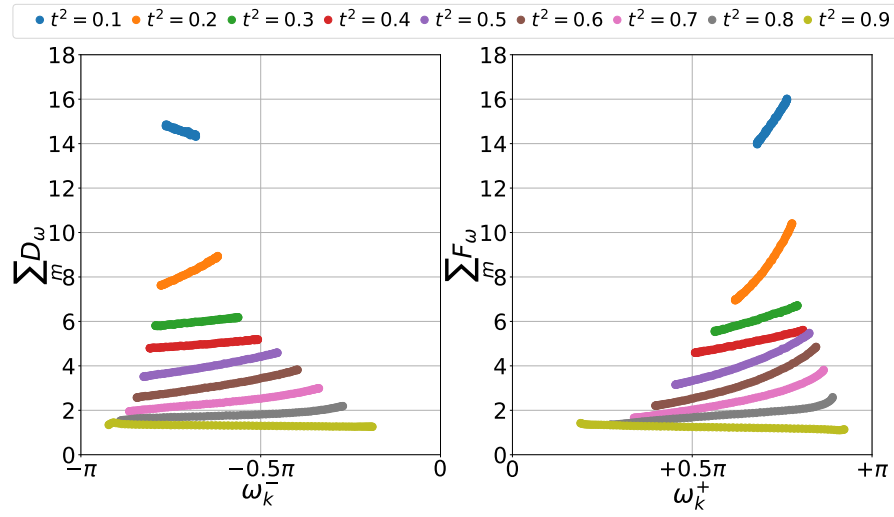


Figure 4.1.12: The time averaged measurements of $(T^\pm, \mu^\pm)$ with the averaged error, and the differences in $T^\pm$ and $\mu^\pm$ over the steady state from $m = 100,000 - 1,000,000$. The discrepancy in T^- for $t^2 = 0.1$ compared to the snapshot in Fig. (4.1.11) is due to larger fluctuations in the measurement because of the large amount of system power accumulated.

Figure 4.1.13: Total damping and forcing of each mode in $\omega_k^\pm$.

Conclusions

In this thesis, nonlinear wave dynamics of discrete space-time systems have been investigated for the first time using weak wave turbulence theory. Through the context of the photonic mesh lattice, various properties of the single band thermal state and the full thermal state between bands have been shown, alongside a preliminary investigation into steady states brought about through forcing and dissipation.

In the single band setting, a power leakage process between bands was observed not previously reported in the literature. This behaviour was justified through careful asymptotic analysis of the four wave interaction, numerical explorations of higher order resonances and error analysis. The single band thermalisation process was understood to arise from the cumulative effect of two, four, six and eight wave processes, determined via a power law scaling of thermalisation time scale with the nonlinear coefficient. Excellent agreement was found between the dynamics of the full evolution and the truncated nonlinear equations, with both exhibiting a single band thermal state on the same timescales, reaching similar temperature and chemical potential. Each order of nonlinear wave interaction was shown to possess non-trivial resonances that are believed to drive the thermal state. Importantly, these resonances were accessed exclusively through nonlinear spectral broadening of the dispersion bands, demonstrating that nonlinear broadening is an essential mechanism for thermalisation in discrete space-time systems.

The full thermal state involving both bands was then investigated numerically, revealing unusual thermodynamic behaviour not previously reported. This behaviour was explained through the existence of a unique class of resonant interactions brought about entirely through discrete-time effects. These resonances guarantee a universal, featureless thermal state for any choice of lattice parameters and initial modal occupation. The universality arises because the resonances violate linear energy conservation, thereby removing the linear energy constraint in the entropy maximisation procedure used to derive the Rayleigh-Jeans distribution. Further, these resonances were shown to also accelerate thermalisation processes through the existence of an extra four wave resonant channel that facilitates efficient uni-directional power leakage. This manifold was found to be absent in lattices with narrow dispersion bands, and present for wider bands, with the manifold size scaling with band size, which in turn further accelerated thermalisation. These findings agreed well with measurements of thermal equilibrium time scales for different dispersion bands finding wider bands reaching a thermal state at shorter times.

Finally, a preliminary investigation into driven/damped lattice dynamics was conducted by forcing the upper band and dissipating the lower band, with the aim of identifying potential avenues for future research. Here we investigated the role of discrete-time on the emergent steady state by varying the size of the dispersion bands. Interestingly, we found that while wider bands did increase the rate of inter-band transfer, the thermalisation timescale was independent of this and seemed to be solely determined by the magnitude of forcing and dissipation. Moreover, the universal thermal state of equal modal occupation was no longer observed. Instead, the bands tended to the typical Rayleigh-Jeans distribution constrained by the linear energy. Notably, the temperatures of the two bands were found to be similar, reminiscent of previous works in continuous-time multi-band systems [102]. These results indicate that driving/damping the system suppresses many of the distinctive thermalisation effects associated with discrete-time dynamics observed previously.

In conclusion, this thesis has demonstrated that nonlinear systems governed by discrete space–time evolution provide a novel platform for the study of thermalisation and weak wave turbulence. These systems exhibit unique thermodynamic behaviour that has no direct analogue in continuous-time settings. To finish, some future directions of research will be presented.

5.1 Future work

1. Experimental verification of discrete-time effects would provide a concrete means of validating the numerical observations presented in this thesis. As shown in section (3.5.1), numerical results suggest that a reduced mode system offer a viable experimental route for observing discrete-time effects.
2. Deriving a wave kinetic equation for the photonic mesh lattice would further validate the existence of the discrete-time resonances. This however is a highly non-trivial task as it requires the development of multiple-scale expansions in discrete-time and tackling the complicated form of the nonlinearity in the lattice (see section (3.5.2) for a more in-depth discussion.).
3. In a similar scope to suggestion 2., obtaining the continuous space-time analogue of the fully nonlinear evolution would be of significant interest. The continuum limit may reveal additional conserved quantities, depending on the specific approach to undo the space-time discretisation. Moreover a wave kinetic equation derived in the continuum setting could provide insights into the form of the discrete space-time wave kinetic equation analogue in the thermodynamic limit, as done for the DNLS in chapter (2).
4. Developing improved quantitative measures of nonlinear spectral broadening would provide deeper insight into the number and nature of resonant interactions contributing to thermalisation. In this thesis,

broadening was measured for a single, relatively weak nonlinearity to provide an initial modest estimate. Repeating this analysis across a wider range of nonlinear strengths could establish a systematic relationship between nonlinearity and spectral broadening.

5. The driven/damped setting could also be used to better model experiments. Often amplifiers are used in both optical loops to compensate for losses [92]. These compensators may be suppressing the observed power leakage process evident in numerics but absent in experiment. By designing an out-of-equilibrium scheme that mimics experimental compensation, this discrepancy may be better explained.
6. More complicate forcing/dissipation could be considered, such as nonlinear forcing and linear dissipation across bands or individual modes. Through this, more complicated non-equilibrium steady states could be observed among modes with non-zero flux.

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A

Appendix

A.1 Conservation laws for the DNLS: analytical method

To demonstrate how we show quantities are conserved analytically, we consider the Norm of the DNLS in equation (1.1.17):

$$N_{\text{DNLS}} = \sum_n |\psi_n|^2,$$

for which we have not specified what interval n spans. Conservation laws require invariance in time, so to begin, we take the derivative of N_{DNLS} with respect to time, t , indicated for brevity by the dot:

$$\begin{aligned}\dot{N} &= \sum_n \frac{d}{dt} |\psi_n|^2, \\ &= \sum_n \left[\dot{\psi}_n \psi_n^* + \psi_n \dot{\psi}_n^* \right].\end{aligned}$$

Now using equation (1.1.14),

$$i \frac{d\psi_n(t)}{dt} = c(2\psi_n - \psi_{n+1} - \psi_{n-1}) + \nu |\psi_n|^2 \psi_n.$$

expressions can be given for $\dot{\psi}_n \psi_n^*$ and $\psi_n \dot{\psi}_n^*$:

$$\dot{\psi}_n \psi_n^* = ic (\psi_{n+1} \psi_n^* + \psi_{n-1} \psi_n^* - 2|\psi_n|^2) - i\nu |\psi_n|^4,$$

$$\psi_n \dot{\psi}_n^* = -ic (\psi_{n+1}^* \psi_n + \psi_{n-1}^* \psi_n - 2|\psi_n|^2) + i\nu |\psi_n|^4,$$

These then simplifies to the following expression for $\dot{N}$:

$$\dot{N} = \sum_n ic (\psi_{n+1} \psi_n^* - \psi_{n+1}^* \psi_n + \psi_{n-1} \psi_n^* - \psi_{n-1}^* \psi_n). \quad (\text{A.1.1})$$

Now let us split equation (A.1.1) into two sums,

$$\dot{N} = \sum_n ic (\psi_{n+1} \psi_n^* - \psi_{n+1}^* \psi_n) + \sum_n ic (\psi_{n-1} \psi_n^* - \psi_{n-1}^* \psi_n),$$

and relabel the right hand sum dummy index to $m = n - 1$,

$$\dot{N} = \sum_n ic (\psi_{n+1} \psi_n^* - \psi_{n+1}^* \psi_n) + \sum_m ic (\psi_m \psi_{m+1}^* - \psi_m^* \psi_{m+1}).$$

Since n and m are both dummy indices, we are free to let $m = n$, cancelling every term in the sum:

$$\dot{N} = 0. \quad (\text{A.1.2})$$

This holds true for any choice of boundary condition and an infinitely large system with discrete-space. A similar method can be applied to H_{DNLS} to prove conservation.

A.2 Modified Hamiltonian and phase-space area for the Symplectic Euler method

To obtain the modified Hamiltonian conserved under the symplectic Euler evolution $\hat{H}_{SE}^m$, we assume a general quadratic ansatz,

$$\hat{H}_{SE}^m(x^m, p^m) = \frac{1}{2} (\alpha(x^m)^2 + 2\beta x^m p^m + \gamma(p^m)^2), \quad (\text{A.2.1})$$

where α, β, γ are unknown constants. This generic form is natural choice because the discrete equations are linear maps, which correspond to quadratic

Hamiltonians. Including higher order terms in (x^m, p^m) would fundamentally change the dynamics by introducing nonlinearity, which are not present in the original linear system. We can express the quadratic ansatz as a matrix,

$$\hat{H}_{SE}^m(X^m) = \frac{1}{2}(X^m)^T Q X^m \quad (\text{A.2.2})$$

where,

$$X^m = \begin{pmatrix} x^m \\ p^m \end{pmatrix}, Q = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}. \quad (\text{A.2.3})$$

The modified Hamiltonian $\hat{H}$ is conserved under discrete-time evolution if,

$$\hat{H}_{SE}^m(X^{m+1}) = \hat{H}_{SE}^m(X^m) \quad (\text{A.2.4})$$

Expanding the left hand side of equation (A.2.4) using equation (1.2.22),

$$\begin{pmatrix} x^{m+1} \\ p^{m+1} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 - \delta t^2 & \delta t \\ -\delta t & 1 \end{bmatrix}}_{M_{SE}} \begin{pmatrix} x^m \\ p^m \end{pmatrix}.$$

Through this substitution we can obtain the values for α, β, γ :

$$(M_{SE})^T (X^m)^T Q A_{SE} X^m = (X^m)^T Q X^m. \quad (\text{A.2.5})$$

Solving the matrix problem (A.2.5) requires a long derivation, so for convenience we solve using symbolic coding language Mathematica, finding $\alpha = \gamma, -\beta = \gamma\delta t/2$ with γ being a free parameter. Hence the modified Hamiltonian takes the form,

$$\hat{H}_{SE}^m(x^m, p^m) = \frac{1}{2} (\gamma(x^m)^2 - \gamma\delta t x^m p^m + \gamma(p^m)^2). \quad (\text{A.2.6})$$

In the limit that $\delta t \rightarrow 0$ we expect to recover the continuous time Hamiltonian in equation (1.2.2), hence we choose $\gamma = 1$ giving the modified Hamiltonian that

is conserved for the symplectic Euler method,

$$\hat{H}_{SE}^m(x^m, p^m) = \frac{1}{2} \left((x^m)^2 - \delta t x^m p^m + (p^m)^2 \right), \quad (\text{A.2.7})$$

describing the tilted ellipse observed in the phase-space shown in Fig. (1.2.3).

To obtain the area swept by tilted elliptical contour we must (i) rotate equation (A.2.7) so that (x^m, p^m) lie along the eigenvectors of the matrix problem $Qv = \Lambda v$ and (ii) compute the area after this rotation of coordinates using the standard form of the ellipse centered at the origin. To rotate the ellipse, we first solve the eigenvalue problem:

$$\begin{bmatrix} 1 - \Lambda & -\delta t \\ -\delta t & 1 - \Lambda \end{bmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (\text{A.2.8})$$

where v is the eigenvector and Λ are the eigenvalues. From equation (A.2.8) these are,

$$\Lambda^\pm = 1 \pm \frac{\delta t}{2}, \quad (\text{A.2.9})$$

and when substituted into equation (A.2.8) gives the two normalised eigenvectors,

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} +1 \\ +1 \end{pmatrix}. \quad (\text{A.2.10})$$

Defining new coordinates along these eigenvectors,

$$\bar{x} = \frac{x^m - p^m}{\sqrt{2}}, \quad \bar{p} = \frac{x^m + p^m}{\sqrt{2}}, \quad (\text{A.2.11})$$

allow us to change coordinates via $(x^m, p^m) \rightarrow (\bar{x}, \bar{p})$,

$$x^m = \frac{\bar{x} + \bar{p}}{\sqrt{2}}, \quad p^m = \frac{\bar{x} - \bar{p}}{\sqrt{2}}. \quad (\text{A.2.12})$$

Equation (A.2.7) in the rotated coordinates $(\bar{x}, \bar{p})$ is therefore,

$$\begin{aligned} H_{SE}^m &= \frac{1}{2} \left[\left(\frac{\bar{x} + \bar{p}}{\sqrt{2}} \right)^2 - \delta t \frac{\bar{x} + \bar{p}}{\sqrt{2}} \frac{\bar{x} - \bar{p}}{\sqrt{2}} + \left(\frac{\bar{x} - \bar{p}}{\sqrt{2}} \right)^2 \right], \\ &= \frac{1}{2} \left[\left(1 - \frac{\delta t}{2} \right) \bar{x}^2 + \left(1 + \frac{\delta t}{2} \right) \bar{p}^2 \right], \end{aligned} \quad (\text{A.2.13})$$

removing the cross term $-\delta t x^m p^m$ in equation (A.2.7) responsible for tilting the ellipse. Assuming we start from the initial data $(x^0, p^0) = (x_0, p_0)$ then the Hamiltonian is given by the constant $\hat{H}$ in equation (1.2.26):

$$\hat{H}(x_0, p_0) = \hat{H} = \frac{1}{2} [x_0^2 - \delta t x_0 p_0 + p_0^2].$$

Using $\hat{H}$ we can then write equation (A.2.13) as the standard equation of an ellipse,

$$\frac{x}{a^2} + \frac{y}{b^2} = \frac{(1 - \frac{\delta t}{2}) \bar{x}^2}{2\hat{H}} + \frac{(1 + \frac{\delta t}{2}) \bar{p}^2}{2\hat{H}} = 1. \quad (\text{A.2.14})$$

where $a, b \in \mathbb{R}$ and the semi-major and semi-minor axes respectively. The area is then,

$$A_{SE} = \pi ab = \pi \sqrt{\frac{2\hat{H}}{(1 - \frac{\delta t}{2})}} \sqrt{\frac{2\hat{H}}{(1 + \frac{\delta t}{2})}} = 2\pi \frac{\hat{H}}{\sqrt{1 - (\frac{\delta t}{2})^2}}. \quad (\text{A.2.15})$$

A.3 Action angle canonical decomposition

To see that $I_k(t)$ and $\theta_k(t)$ are also canonical coordinates of the Hamiltonian, we need to satisfy the following criteria for the Poisson brackets:

$$\{I_k, I_k\} = i \left(\frac{\partial I_k}{\partial \psi_k} \frac{\partial I_k}{\partial \psi_k^*} - \frac{\partial I_k}{\partial \psi_k^*} \frac{\partial I_k}{\partial \psi_k} \right) = 0, \quad (\text{A.3.1})$$

$$\{\theta_k, \theta_k\} = i \left(\frac{\partial \theta_k}{\partial \psi_k} \frac{\partial \theta_k}{\partial \psi_k^*} - \frac{\partial \theta_k}{\partial \psi_k^*} \frac{\partial \theta_k}{\partial \psi_k} \right) = 0, \quad (\text{A.3.2})$$

$$\{I_k, \theta_k\} = i \left(\frac{\partial I_k}{\partial \psi_k} \frac{\partial \theta_k}{\partial \psi_k^*} - \frac{\partial I_k}{\partial \psi_k^*} \frac{\partial \theta_k}{\partial \psi_k} \right) = 1, \quad (\text{A.3.3})$$

where the factor of i emerges because $(\psi_k, \psi_k^*) \in \mathbb{C}$. From the action angle decomposition we can find the following expressions:

$$I_k = |\psi_k|^2, \quad (\text{A.3.4})$$

$$\theta_k = \frac{i}{2} \ln \left[\frac{\psi_k}{\psi_k^*} \right]. \quad (\text{A.3.5})$$

Taking partial derivatives we obtain:

$$\begin{aligned} \frac{\partial I_k}{\partial \psi_k} &= \psi_k^* \\ \frac{\partial I_k}{\partial \psi_k^*} &= \psi_k \\ \frac{\partial \theta_k}{\partial \psi_k} &= \frac{i}{2} \frac{1}{\psi_k} \\ \frac{\partial \theta_k}{\partial \psi_k^*} &= -\frac{i}{2} \frac{1}{\psi_k^*} \end{aligned}$$

From the above, it is clear that equations (A.3.1) and (A.3.2) are zero. Upon substituting the derivatives for equation (A.3.3):

$$\{I_k, \theta_k\} = i \left(\psi_k^* \left[-\frac{i}{2} \frac{1}{\psi_k^*} \right] - \psi_k \left[\frac{i}{2} \frac{1}{\psi_k} \right] \right) = i \cdot (-i) = 1 \quad (\text{A.3.6})$$

as required.

A.4 Hamiltonian derivative factor

To demonstrate the emergence of a factor of 4 in the coefficient, consider one of Hamilton's equation:

$$\frac{d\theta_1}{dt} = \frac{\partial H}{\partial I_1}, \quad (\text{A.4.1})$$

and the Hamiltonian,

$$H = \sum_1 \omega_1 I_1 + \frac{\nu}{2} \sum_{1234} \sqrt{I_1 I_2 I_3 I_4} \cos(\Delta\theta_{12}^{34}) \delta_{12}^{34}$$

The derivative is taken with respect to some I_k that is prescribed by a dummy variable k . Since k is just a label, there are actually four equivalent derivatives existing in the summation,

$$\begin{aligned} \frac{\delta H}{\delta I_1} &= \omega_1 + \frac{\nu}{4} \sum_{234} \sqrt{\frac{I_2 I_3 I_4}{I_1}} \cos [\Delta \theta_{12}^{34}] \delta_{12}^{34} + \frac{\nu}{4} \sum_{134} \sqrt{\frac{I_1 I_3 I_4}{I_2}} \cos [\Delta \theta_{21}^{34}] \delta_{21}^{34} \\ &+ \frac{\nu}{4} \sum_{124} \sqrt{\frac{I_1 I_2 I_4}{I_3}} \cos [\Delta \theta_{32}^{14}] \delta_{32}^{14} + \frac{\nu}{4} \sum_{123} \sqrt{\frac{I_1 I_2 I_3}{I_4}} \cos [\Delta \theta_{42}^{31}] \delta_{42}^{31}, \\ &= \omega_1 + \nu \sum_{2,3,4} \sqrt{\frac{I_2 I_3 I_4}{I_1}} \cos [\Delta \theta_{12}^{34}] \delta_{12}^{34}. \end{aligned}$$

For the evolution of $I_1(t)$, the factor is 2 because when differentiating w.r.t θ_1 , we do not get a factor of 1/2 from the square root like we do when differentiating w.r.t I_1 .

A.5 Obtaining the action angle evolution for the equation of motion

In action angle coordinates the equation of motion for the DNLS is:

$$i \left[\frac{d\sqrt{I_1}}{dt} - i \frac{d\theta_1}{dt} \sqrt{I_1} \right] e^{-i\theta_1} = \omega_1 I_1 e^{-i\theta_1} + \nu \sum_{234} \sqrt{I_2 I_3 I_4} e^{i(\theta_2 - \theta_3 - \theta_4)} \delta_{12}^{34}. \quad (\text{A.5.1})$$

This can be simplified as follows:

$$i \frac{d\sqrt{I_1}}{dt} + \frac{d\theta_1}{dt} \sqrt{I_1} = \omega_1 \sqrt{I_1} + \nu \sum_{234} \sqrt{I_2 I_3 I_4} e^{i\Delta \theta_{1,2}^{3,4}} \delta_{12}^{34}. \quad (\text{A.5.2})$$

Equating the real components of the left and right hand side we get the angle evolution:

$$\begin{aligned} \frac{d\theta_1}{dt} \sqrt{I_1} &= \omega_1 \sqrt{I_1} + \nu \sum_{234} \sqrt{I_2 I_3 I_4} \cos (\Delta \theta_{12}^{34}) \delta_{12}^{34} \\ \frac{d\theta_1}{dt} &= \omega_1 + \nu \sum_{234} \sqrt{\frac{I_2 I_3 I_4}{I_1}} \cos (\Delta \theta_{12}^{34}) \delta_{12}^{34} \end{aligned} \quad (\text{A.5.3})$$

And from the imaginary components we get the action evolution:

$$\frac{d\sqrt{I_1}}{dt} = \nu \sum_{234} \sqrt{I_2 I_3 I_4} \sin(\Delta\theta_{12}^{34}) \delta_{12}^{34}$$

To construct the derivative in the action we use a product rule:

$$\frac{dI_1}{dt} = 2\sqrt{I_1} \frac{d\sqrt{I_1}}{dt},$$

yielding the desired equation:

$$\frac{dI_1}{dt} = 2\nu \sum_{234} \sqrt{I_1 I_2 I_3 I_4} \sin(\Delta\theta_{12}^{34}) \delta_{12}^{34} \quad (\text{A.5.4})$$

A.6 Random phase averaging of the DNLS at $\mathcal{O}(\nu^2)$

The equation being averaged is,

$$\begin{aligned} \left\langle \frac{dI_1^{(2)}}{dt} \right\rangle &= 2 \left\langle \sum_{234}' \sum_{567}' \sum_{q=1}^4 \frac{1}{\sqrt{I_q}} \sqrt{\bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \bar{I}_5 \bar{I}_6 \bar{I}_7} \delta_{12}^{34} \delta_{q5}^{67} \times \right. \\ &\quad \left[\frac{\sin[\sigma_q \Delta \bar{\theta}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34} + (\sigma_q \Delta \bar{\Omega}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34}) t] + \sin[\Delta \bar{\theta}_{12}^{34} - \sigma_q \Delta \bar{\theta}_{q5}^{67} + \Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{q5}^{67}} \right] \Bigg\rangle. \end{aligned} \quad (\text{A.6.1})$$

The averaging procedure only effects the trigonometric terms in the square bracket, therefore averaging over the square bracket averages equation (A.6.1):

$$\sum_{q=1}^4 \delta_{q5}^{67} \left\langle \left[\frac{\sin[\sigma_q \Delta \bar{\theta}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34} + (\sigma_q \Delta \bar{\Omega}_{q5}^{67} - \Delta \bar{\theta}_{12}^{34}) t] + \sin[\Delta \bar{\theta}_{12}^{34} - \sigma_q \Delta \bar{\theta}_{q5}^{67} + \Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{q5}^{67}} \right] \right\rangle.$$

Substitute the case for when $q = 1$,

$$\delta_{15}^{67} \left\langle \left[\frac{\sin[\sigma_1 \Delta \bar{\theta}_{15}^{67} - \Delta \bar{\theta}_{12}^{34} + (\sigma_1 \Delta \bar{\Omega}_{15}^{67} - \Delta \bar{\theta}_{12}^{34}) t] + \sin[\Delta \bar{\theta}_{12}^{34} - \sigma_1 \Delta \bar{\theta}_{15}^{67} + \Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{15}^{67}} \right] \right\rangle.$$

When $\delta_{15}^{67} = 1$ implies that $1 = 6 + 7 - 5$. Substituting this into the kronecker delta δ_{12}^{34} , $2 + 6 + 7 = 3 + 4 + 5$. Since the trivial resonances $1 = 3$, $2 = 4$ and

$1 = 4, 2 = 3$ have been omitted, this places restrictions on $2 + 6 + 7 = 3 + 4 + 5$. That is only two solutions are allowed, $2 = 5, 3 = 6, 4 = 7$ or $2 = 5, 3 = 7, 4 = 6$. For both of these cases this implies that $\delta_{15}^{67} = \delta_{12}^{34}$. Therefore the term in the bracket when $\delta_{15}^{67} = 1$ is,

$$\delta_{15}^{67} \left\langle \left[\frac{\sin [\Delta \bar{\theta}_{12}^{34} - \Delta \bar{\theta}_{12}^{34} + (\Delta \bar{\Omega}_{12}^{34} - \Delta \bar{\theta}_{12}^{34}) t] + \sin [\Delta \bar{\theta}_{12}^{34} - \Delta \bar{\theta}_{12}^{34} + \Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{12}^{34}} \right] \right\rangle = 2 \frac{\sin [\Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{12}^{34} t}$$

The exact same procedure is followed for $q = 2, 3, 4$ with the only difference being a negative sign in the denominator for $q = 3, 4$. Substituting each case of q into equation (A.6.1):

$$\left\langle \frac{dI_1^{(2)}}{dt} \right\rangle = 4 \sum_{234} \bar{I}_1 \bar{I}_2 \bar{I}_3 \bar{I}_4 \left[\frac{1}{\bar{I}_1} + \frac{1}{\bar{I}_2} - \frac{1}{\bar{I}_3} - \frac{1}{\bar{I}_4} \right] \frac{\sin [\Delta \bar{\Omega}_{12}^{34} t]}{\Delta \bar{\Omega}_{12}^{34} t} \delta_{12}^{34}$$

A.7 Work flow of many realisation numerical simulations

In order to perform such large scale simulations of many realisations the following work flow was performed for every set of averaged dynamics.

A variables.npy file was created for the specific choice of parameters, length of the simulation and resolution in time, and a set of initial 500 unique conditions was created using the random phase approximation, that possess the same intensity profile, but different phase information. Then the 500 initial conditions is partitioned into 10 sets of 50 simulations, which are each saved as 20 .npy files for the real and imaginary components of the initial conditions.

The initial conditions and variable files are then uploaded onto the HPC environment into uniquely named directories. In each unique directory, 10 python scripts are uploaded that call the variables and 1 of the 10 sets of 50 simulations. A .sub script is then ran 10 times to run each python script,

parallelising the 500 simulations. All of the data is then exported to the unique directory for that run, which is then downloaded locally, and analysed post evolution.

Parallelisation is absolutely necessary in order to perform so many separate realisations, otherwise computation times would be in the hundreds of hours. Further the choice of 500 realisations used throughout was chosen after several test simulations of different sizes, providing a good balance of computational time, and clean averaged dynamics.

A.8 Fitting the Rayleigh-Jeans distributions

A nonlinear least squares method is used to fit the Rayleigh-Jeans distribution, utilizing the built-in `curve_fit` function from the SciPy library. At each time step, we test 100 initial parameter estimates with $T_{est}^+ = 1$ and $\mu_{est}^+ \in [-10, 10]$. Each initial guess is passed to `curve_fit`, which adjusts the parameters to best fit the Rayleigh-Jeans distribution and estimates the variance of the fitted parameters. For each resulting parameter pair (T_{out}^+, μ_{out}^+) , we measure the difference between the fitted Rayleigh-Jeans distribution with the distribution of wave action in the normal modes of the upper band at each time step using a sum of squared errors (SSE),

$$SSE = \sum_k \left(\langle |a_k^{m+}|^2 \rangle - \frac{T_{out}^+}{\omega_k^+ - \mu_{out}^+} \right)^2. \quad (\text{A.8.1})$$

Equation (A.8.1) is computed for each of the 100 initial guesses and are all compared to find out which outputted value minimizes equation (A.8.1), giving the measured (T^+, μ^+) . A sum of square errors is used as an error metric because it penalizes larger deviations between the data and fitting more harshly than smaller deviations due to the squaring. This is appropriate, as small deviations are expected since we are studying the ensemble averaged dynamics.

A similar algorithm is done to fit a Rayleigh-Jeans distribution in the lower band, given by the orange curve, except the initial guess of temperature for every step in

time is the T_{out}^+ that minimizes equation (A.8.1) in the upper band distribution.

A.9 Numerical error in the photonic mesh lattice

In order to ensure that the power leakage between bands is a physical process, we must rule out the notion of numerical error.

The numerical simulations of the photonic mesh lattice requires no integration method because the dynamics are given by coupled discrete space-time maps. As a result, we expect that the numerical error entering from evolution is low as we just repeatedly iterate the maps to evolve in time. Instead, the error in these maps manifests from the exponential nonlinearities since the exponential is not computationally exact, incurring rounding errors with each iteration.

The first test of numerical error we will then perform is by repeating the evolution for a single realisation at different precisions. Here we denote the double precision wave fields as $a_k^{m\pm(D)}$ and the quadruple precision wave fields as $a_k^{m\pm(Q)}$. We then plot the magnitude of the difference of the intensity fields at different precisions $||a_k^{m\pm(D)}|^2 - |a_k^{m\pm(Q)}|^2|$, given by the intensity maps in Fig. (A.9.1). The difference in intensity fields after 200 steps is of the order 10^{-10} and 10^{-11} in the upper and lower band respectively, which is notably much smaller than the power leakage to the lower band after a single stroboscopic step, displayed in Fig. (A.9.2). This is a good indicator that the leakage is physical, since the dynamics are largely insensitive to the precision. Further, as the leakage is observed in a single step in both precisions, the error from map iteration is low because only a single iteration has been performed in the stroboscopic sense. We also consider a different norm in Fig. (A.9.3), showing a norm between the wave fields fields instead $|a_k^{m\pm(D)} - a_k^{m\pm(Q)}|$. This norm also finds tiny differences of 10^{-10} in both the upper and lower band waves fields, reinforcing that the leakage is physical. We note as well that, since we are concerned with the wave action $n_\omega^\pm \propto |a_k^{m\pm}|^2$, the difference in the phase

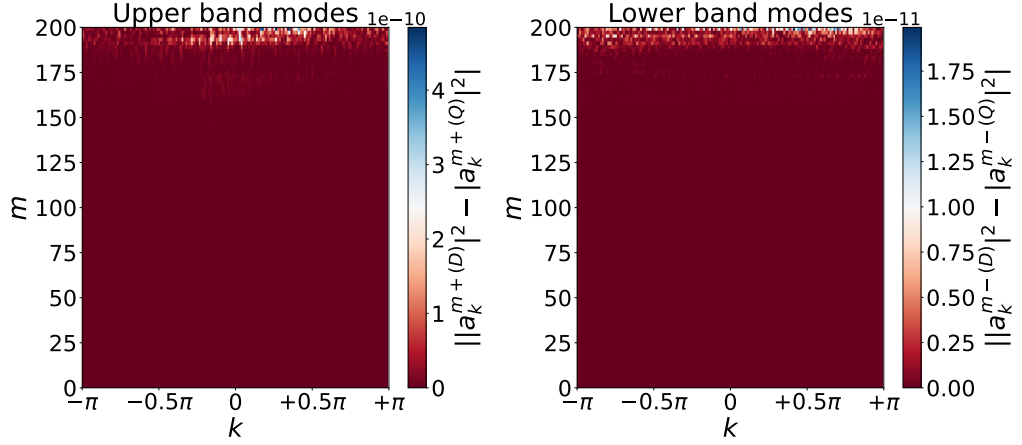


Figure A.9.1: Difference of intensity fields for double and quadruple precision. The left panel shows the difference in the upper band and the right the difference in the lower band.

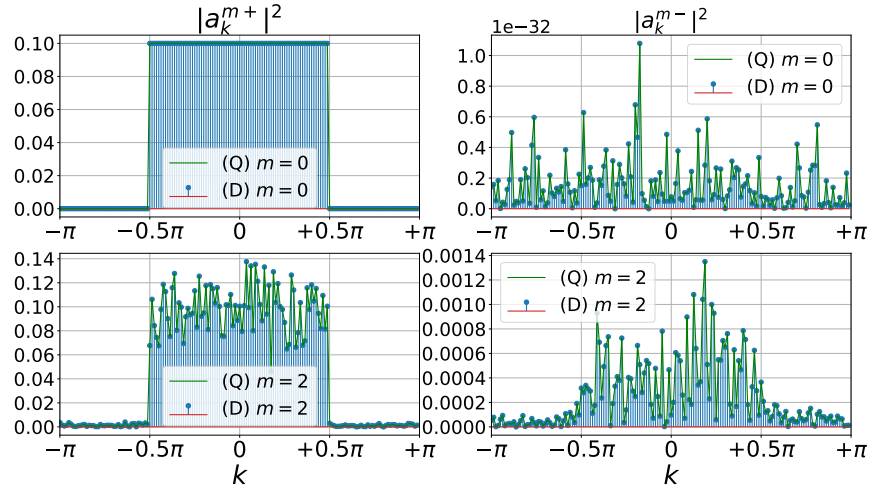


Figure A.9.2: Snapshots of the upper band are given at time $m = 0$ and $m = 2$ on the left and snapshots of the lower band at $m = 0$ and $m = 2$ are given on the right. The blue stem plot is double precision and the green line is quadruple. After a single step, both simulations show significant leakage, with discrepancies in their fields unresolvable on the order of the plots.

information of the wave fields is not as important for this work. Hence we will only concern ourselves with the intensity norm from now on. For longer time simulations, we find that the agreement between simulations rapidly diverges. Fig. (A.9.4) shows the same evolution as in Fig. (A.9.1), except it is evolved for 300 steps in time. Within an extra 100 steps in time, the error grows several orders of magnitude in both bands. We understand this error to emerge from

the chaotic nature of the map. Effectively, the difference in double and quadruple precision creates a small discrepancy through the approximation of the exponential. This slight deviation in wave field then grows as we evolve longer in time, eventually diverging. As the error enters through the exponential term, we can also observe a similar effect in the differences between double and quadruple precision, by increasing the size of the parameter χ , demonstrated in Fig. (A.9.5).

To demonstrate that the map is chaotic, we next compute the Lyapunov exponent $\Lambda_k^\pm$ for every mode in both bands. We define $\Lambda_k^\pm$ as [103],

$$\Lambda_k^\pm = \frac{1}{M} \ln \left[\frac{d_k^{m\pm}}{d_k^{2\pm}} \right], \quad (\text{A.9.1})$$

where M is the total time evolution and $d_k^{m\pm} = ||a_k^{m\pm(D)}|^2 - |a_k^{m\pm(Q)}|^2|$ is the difference in the intensity fields at every step in time. The intuition of the Lyapunov exponent is that we consider some initial condition, in our case we choose the wave fields in double precision at $m = 2$, $a_k^{2\pm(D)}$, (we choose $m = 2$ as we start from the same initial condition at both precisions). We then

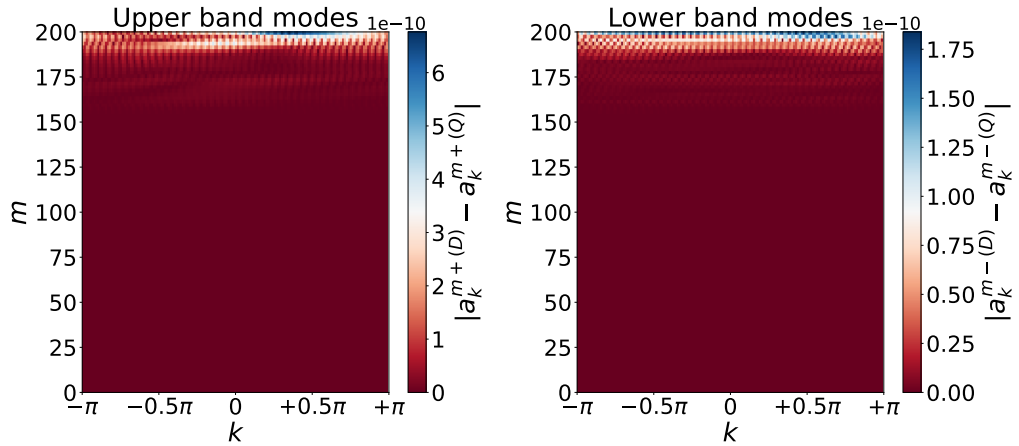


Figure A.9.3: Here we consider the difference of the wave fields in double and quadruple precision in the upper band on the left and lower band on the right. Since we are considering the wave fields, and not intensities, phase differences are also present, a feature not of interest when considering the wave action spectra $n_\omega^{m\pm}$

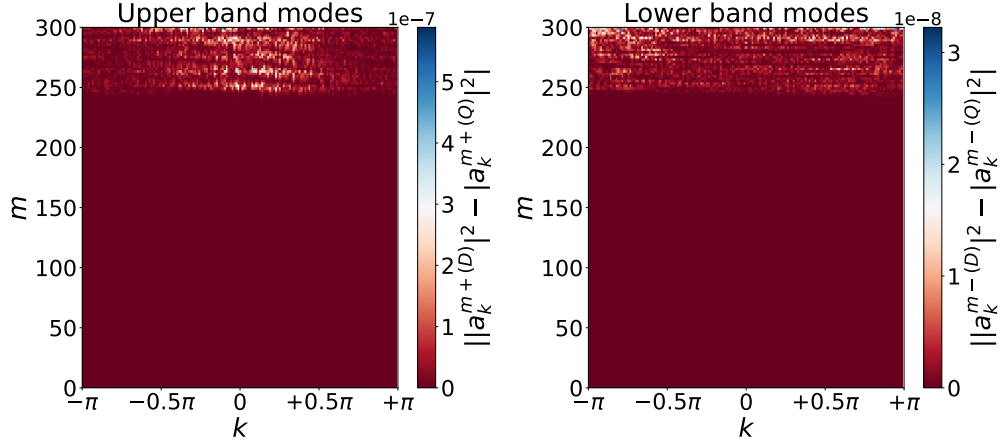


Figure A.9.4: Here measure the difference in intensity fields at double and quadruple precision for 300 steps instead of 200, with the error rapidly growing. This rapid growth can be attributed to chaos seeded by exponential error.

consider another initial condition that is extremely close in value $a_k^{2\pm(Q)}$, such that $d_k^{2\pm} = ||a_k^{2\pm(D)}||^2 - ||a_k^{2\pm(Q)}||^2$ is extremely small. Then we say that after M iterations the separation of these two fields in time is given by $d_k^{M\pm} \approx d_k^{2\pm} e^{M\Lambda_k^\pm}$. From this approximation we can arrive at equation (A.9). The sign of the Lyapunov exponent provides a strong indicator if the system is chaotic or not. If $\Lambda_k^\pm > 0$ and is constant, then that implies that the evolution of the double

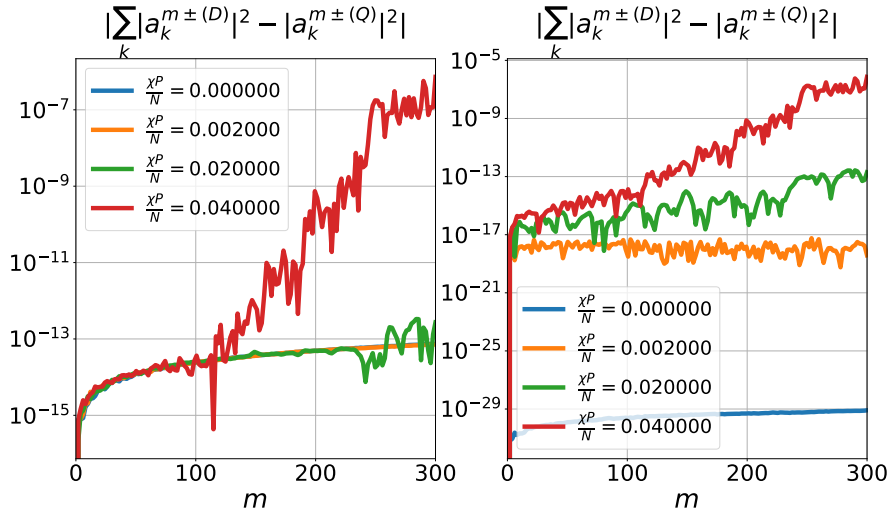


Figure A.9.5: Here we sum over k the intensity norm of the double vs quadruple precision for different choices of χ . As χ becomes larger the error once again rapidly grows.

and quadruple precisions are diverging at a constant rate, consistent with what is expected for chaotic systems. If $\Lambda_k^\pm < 0$, then the evolutions converge, and if $\Lambda_k^\pm = 0$ then they are staying separated by a fixed amount. Fig. (A.9.6) plots $\Lambda_k^\pm$ for every mode over 2,000 steps in time, showing that the exponent is positive and stable, saturating in every mode after ~ 500 steps in time. Notably, all the modes plateau to into a range of values between $\Lambda_k^\pm \in (0.25, 0.45)$, with modes diverging more rapidly than others.

Finally, we can show that the the averaged dynamics are qualitatively the same as in Fig. (A.9.7), with the same spectra in both bands as the power leakage takes hold. With all of these simulations at double vs quadruple simulations we are able to show that power leakage occurs in both simulations, in the same qualitative way. This either means that the power leakage is physical, or that the quadruple precision is not precise enough.

Instead of repeating the simulations to arbitrary precision, we now perform a different set of simulations to check error that further verify the observations of

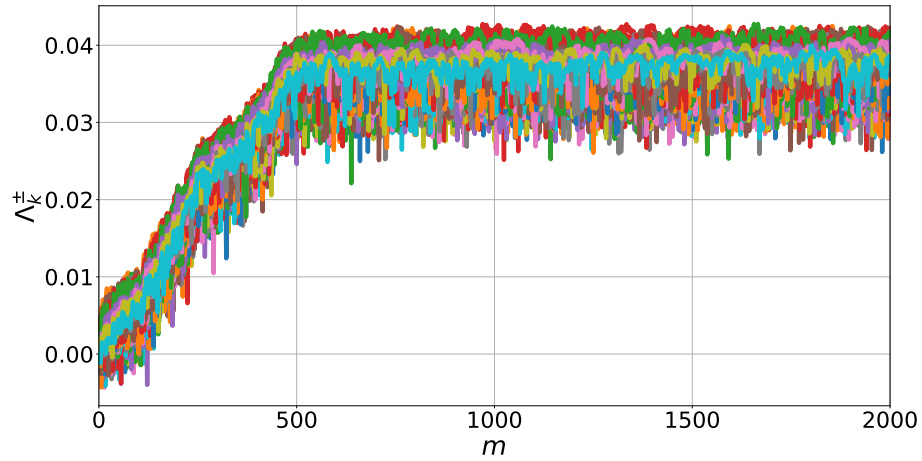


Figure A.9.6: The Lyapunov exponents of every mode in both both bands.

this section. Since the forward time equations (3.3.2) and (3.3.3),

$$\begin{aligned} u_n^{m+1} &= \left[tu_{n+1}^m e^{i\chi|u_{n+1}^m|^2} + irv_{n+1}^m e^{i\chi|v_{n+1}^m|^2} \right] e^{i\phi^m}, \\ v_n^{m+1} &= \left[tv_{n-1}^m e^{i\chi|v_{n-1}^m|^2} + iru_{n-1}^m e^{i\chi|u_{n-1}^m|^2} \right], \end{aligned}$$

are explicitly invertible in time, given by equations (3.3.10) and (3.3.11),

$$\begin{aligned} u_n^{m-1} &= \left[tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m \right] e^{-i\chi|tu_{n-1}^m e^{-i\phi^m} - irv_{n+1}^m|^2}, \\ v_n^{m-1} &= \left[-iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m \right] e^{-i\chi| -iru_{n-1}^m e^{-i\phi^m} + tv_{n+1}^m|^2}, \end{aligned}$$

a check we can perform is seeing if the power leakage reverses if we evolve

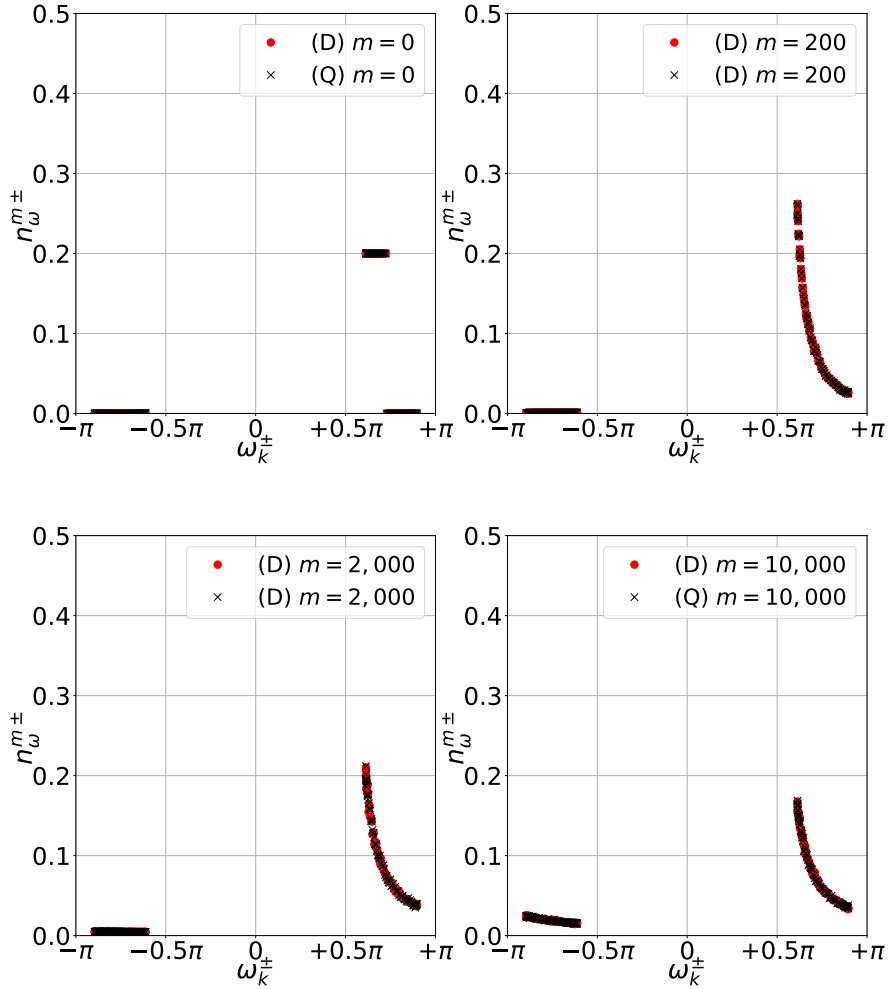


Figure A.9.7: Comparison of the wave action spectrum for double precision in red and quadruple precision in black for the first 10,000 steps in time.

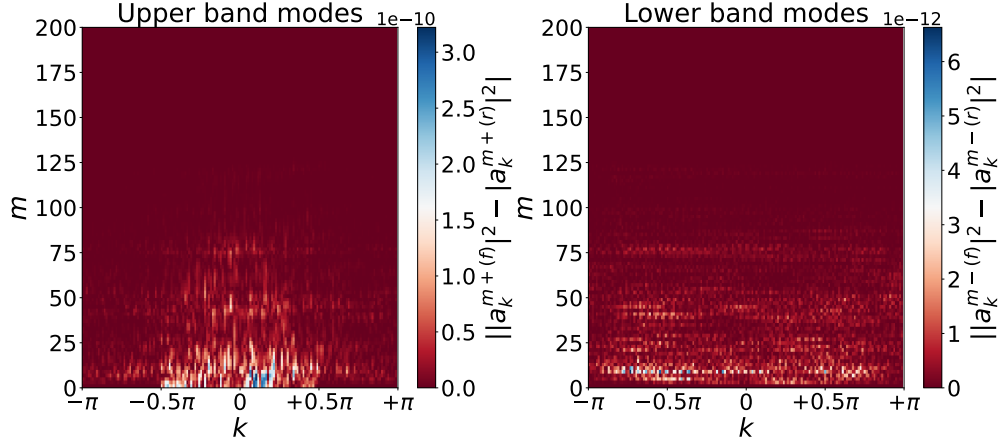


Figure A.9.8: Here we measure discrepancies in the two bands of order 10^{-10} and 10^{-12} in the upper and lower band respectively.

backwards in time (Thanks to Alberto Villois for the idea!).

The argument being that if the leakage occurs because of error in forward time, then reversing the dynamics should not undo the leakage. If the leakage is a physical interaction, then we expect majority of the wave field to return back to the initially occupied band, up to some precision. We now denote the forward time intensities as $|a_k^{m\pm(f)}|^2$ and the reverse time intensities $|a_k^{m\pm(r)}|^2$. We start in forward time from the usual initial condition used prior, and evolve forward in time until $m = 200$. Then we take the final state of the forward wave field and evolve this backwards in time until $m = 0$. We plot the intensity difference of these simulations using the norm $||a_k^{m\pm(f)}|^2 - |a_k^{m\pm(r)}|^2|$ in Fig. (A.9.8). As can be seen, the discrepancy in the intensity fields is much smaller than the leakage in a single stroboscopic step shown previously in Fig. (A.9.2). As with the double and quadruple precision, the forwards and reverse maps are susceptible to chaotic divergence, dependent on the timescale of evolution and χ because of rounding due to the exponential terms. Both the precision checks and the reverse time checks provide strong evidence that the leakage process is in fact physical.

A.10 Final thermal state for different initial conditions, no bandgap and smaller lattices

Here we consider various initial modal occupations, lattice parameters and lattice sizes, showing that all thermalise to the same distribution.

Different initial occupation

Here we consider the ensemble averaged dynamics over 500 realisations, for $(t^2, \phi_0) = (0.3, \pi/8)$ with $P = 8$, $N = 320$, $\chi = 1.6$ such that $\chi P/N = 1.6$, evolved for 3,000,000 time steps. Snapshots of different configurations are shown in Fig. (A.10.1). Fig. (A.10.2) shows the ratio of $T^\pm/\mu^\pm$ for each simulation. As expected, each simulation tends to $T^\pm/\mu^\pm = -P/N$, reaching the universal thermal state. Notably simulation a) in Fig. (A.10.1) thermalises much faster, despite having the same nonlinear strength. This is because power is already equally partitioned between bands, thereby accelerating the thermalisation process.

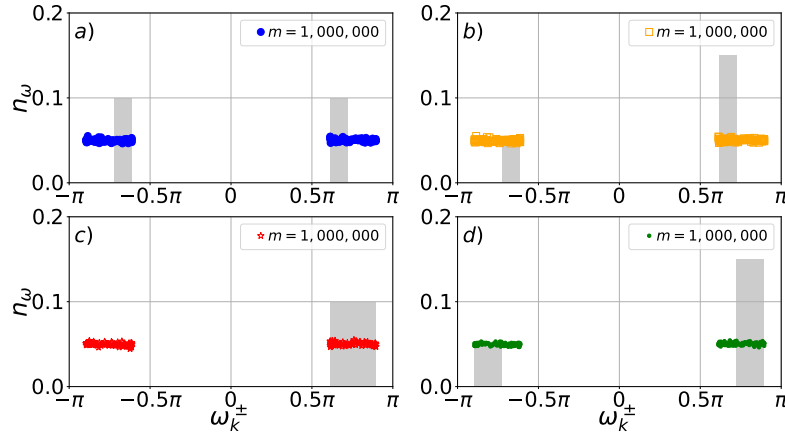


Figure A.10.1: Different initial modal occupations given by the grey shaded region. a) starts from exciting the middle band modes of each band splitting power equally, b) is the same but 25 : 75 ratio between upper and lower band power distribution. c) excited every mode of the upper band, and d) excites the outer modes of both bands, again with 25 : 75 power ratio. Snapshots at 1,000,000 steps in time shows every simulation reaches equal modal occupation at thermal equilibrium.

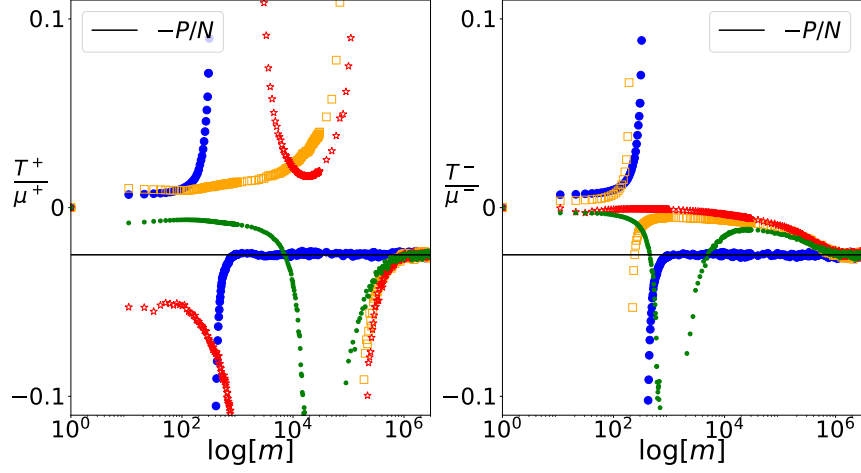


Figure A.10.2: $T^\pm/\mu^\pm$ for each simulation. The colour and shape of the markers match that used in Fig. (A.10.1).

No band gap

Here we show another simulation with $P = 8$, $N = 320$ and $\chi = 1.6$ but $(t^2, \phi_0) = (0.5, 0)$ thus removing the edge mode bandgap. Now we have a system of one dispersion band connected for discrete-time periodicity. We note that this system still reaches the same universal thermal state, as expected, see Fig. (A.10.3)

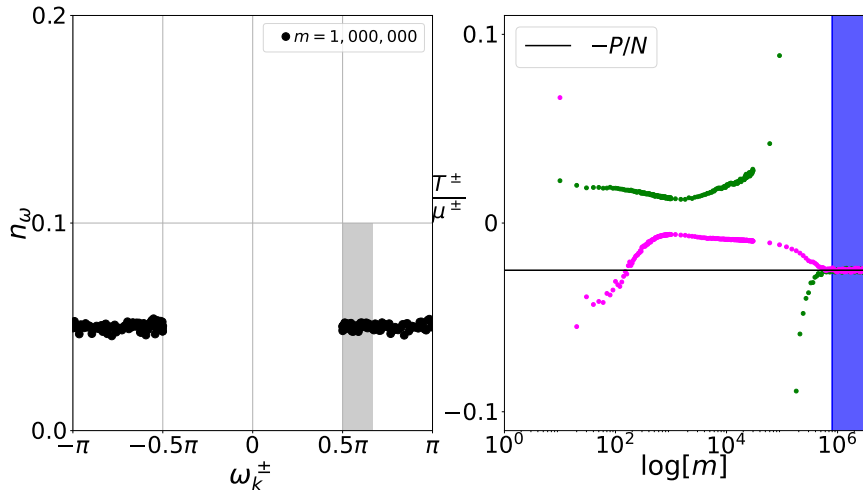


Figure A.10.3: Here we see the initial condition and snapshot of modal occupation on the left, and the measure $T^\pm/\mu^\pm$ evolution on the right. The green points correspond to the upper band measurement and the purple the lower band.

Different lattice sizes

Lastly, we show that for lattices of the same parameter and same nonlinearity $\chi P/N$ but different P and N thermalise in the same way to the universal thermal state, see Fig. (A.10.4).

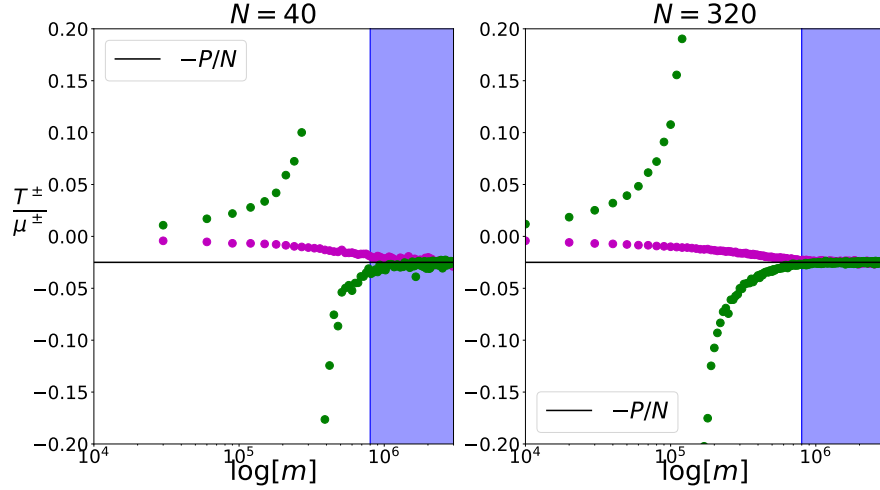


Figure A.10.4: Here we compare the simulation shown for the full thermalisation in section (3.5) for $(t^2, \phi_0) = (0.3, \pi/8)$ where $P = 8$, $N = 320$ and $\chi = 1.6$ to a equivalent simulation that has $(t^2, \phi_0) = (0.3, \pi/8)$ but $P = 1$, $N = 40$ and $\chi = 1.6$, starting from exciting the initial upper band modes in a like way. As can be seen, the evolution in both cases are qualitatively the same, exhibiting the same plateau in the final spectrum of $T^\pm/\mu^\pm = -P/N$. The green points denote the upper band and the magenta the lower band.

A.11 Combination reduction of manifolds

Here we show how we reduce the number of possible resonances into a handful of cases. For example let us consider:

$$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^- \quad (\text{A.11.1})$$

We note that any indexing involving three plus branches and one minus branch is equivalent to any indexing of three minus branches and one plus branch since

$$\omega_1^+ = -\omega_1^-. \text{ E.G.:}$$

$$\omega_1^+ + \omega_2^+ = \omega_3^+ + \omega_4^-$$

$$\omega_1^+ + \omega_2^+ = \omega_3^- + \omega_4^+$$

$$\omega_1^+ + \omega_2^- = \omega_3^+ + \omega_4^+$$

$$\omega_1^- + \omega_2^+ = \omega_3^+ + \omega_4^+$$

$$\omega_1^- + \omega_2^- = \omega_3^- + \omega_4^+$$

$$\omega_1^- + \omega_2^- = \omega_3^+ + \omega_4^-$$

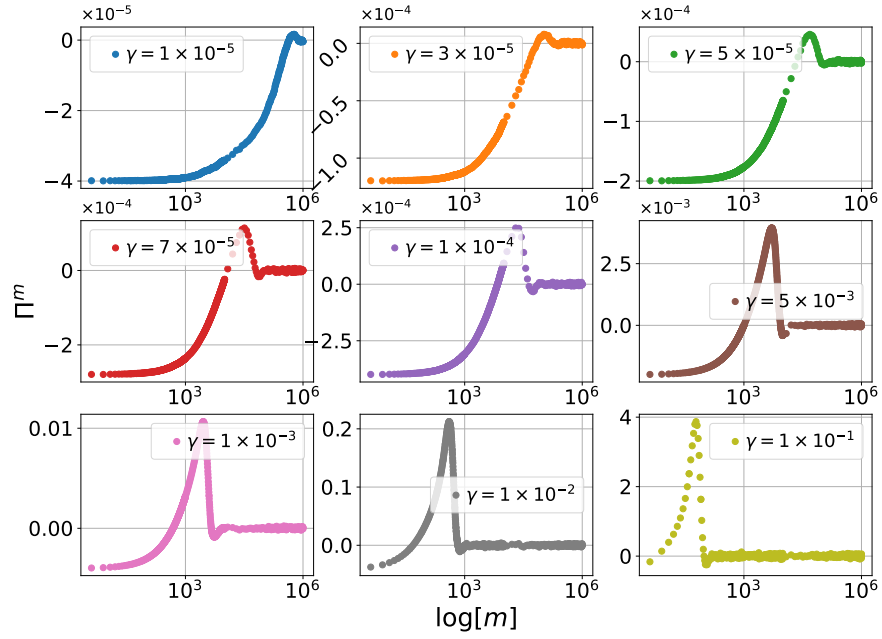
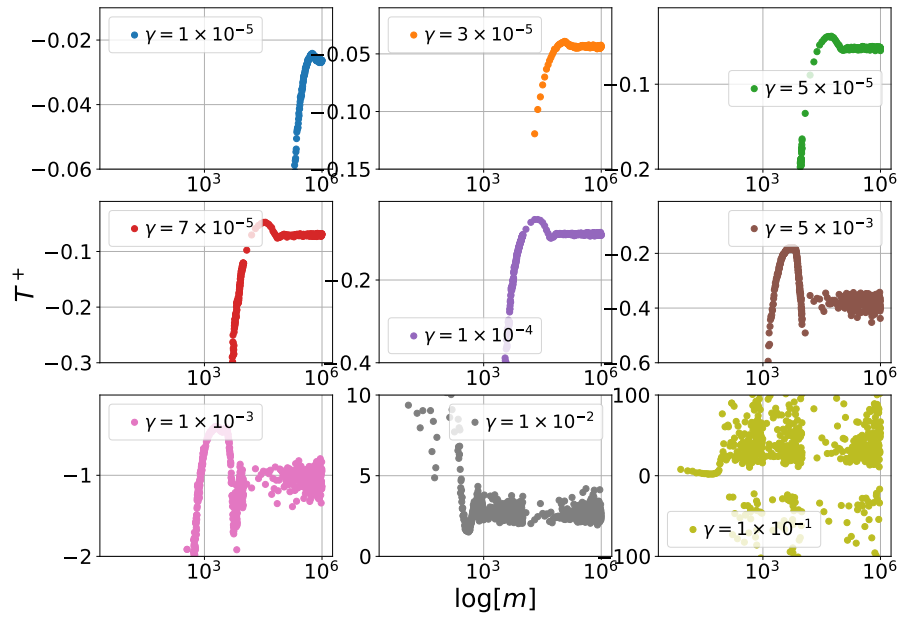
$$\omega_1^- + \omega_2^+ = \omega_3^- + \omega_4^-$$

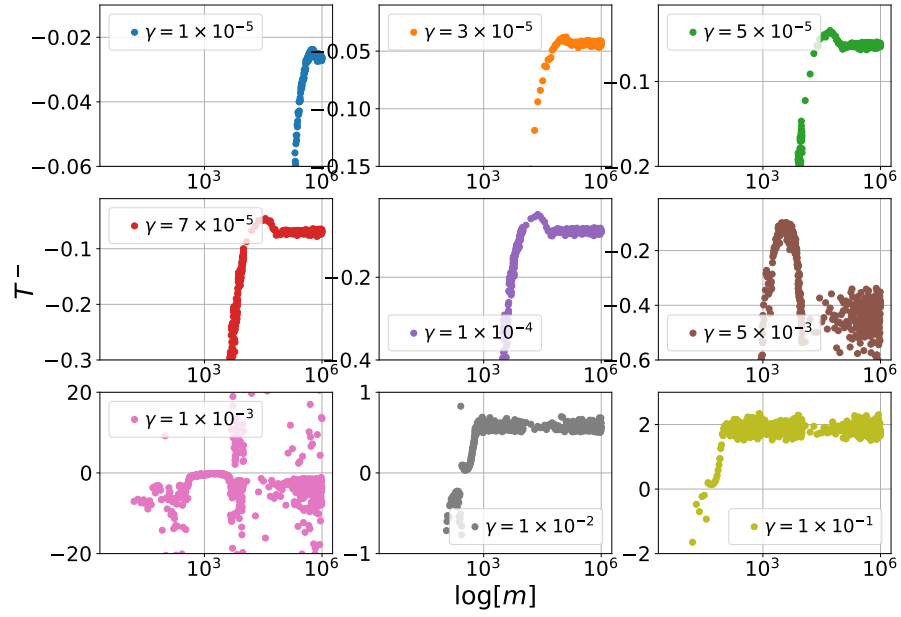
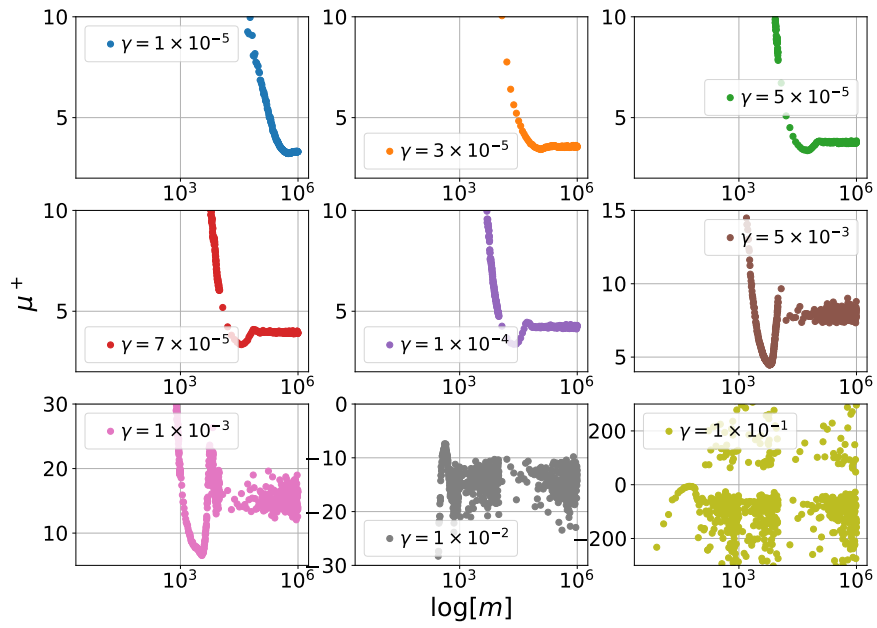
$$\omega_1^+ + \omega_2^- = \omega_3^- + \omega_4^-$$

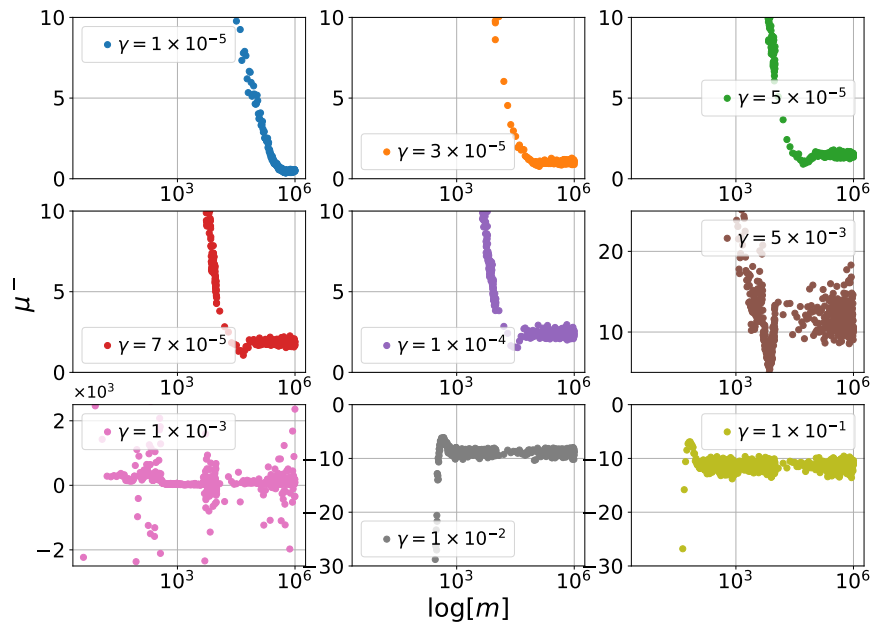
are all equivalent wave processes. Hence we only need check the resonance in equation (A.11.1). The same arguments can be made about the other tabulated resonances in table (3.1).

A.12 Evolution of different quantities for varied γ

In Figs. (A.12.1), (A.12.2), (A.12.3), (A.12.4), (A.12.5), we plot the total flux Π^m and $(T^\pm, \mu^\pm)$ for all the choices of γ used in each simulation. The plots of $(T^\pm, \mu^\pm)$ are magnified in the y axis to show that they do tend to a steady state. For larger γ , the measure $(T^\pm, \mu^\pm)$ are less reliable with greater fluctuations in the measured values.

Figure A.12.1: Total flux for each choice of γ .Figure A.12.2: Upper band temperature evolution for each choice of γ .

Figure A.12.3: Lower band temperature evolution for each choice of γ .Figure A.12.4: Upper band chemical potential evolution for each choice of γ .

Figure A.12.5: Lower band chemical potential evolution for each choice of γ .