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Author for correspondence:

Daniel J. Netherwood

e-mail: daniel.netherwood@adelaide.edu.au

Evolving wrinkles: time-dependent buckling of an elastic sheet on a liquid substrate

Daniel J. Netherwood¹, Ben S. Humphries²,

Connor Robbins³ and Doireann O'Kiely⁴

¹School of Computer and Mathematical Sciences, University of Adelaide, Adelaide, South Australia, 5000, Australia

²School of Engineering, Mathematics, and Physics, University of East Anglia, Norwich, NR4 7TJ, UK

³Climatic Research Unit, School of Environmental Sciences, University of East Anglia, Norwich, UK

⁴MACSI, Department of Mathematics & Statistics, University of Limerick, Limerick, Ireland

DJN, 0000-0002-3428-0165; BSH, 0000-0003-0767-6663; CR, 0000-0002-3762-4744; DOK, 0000-0001-8535-2395

We model the formation and evolution of wrinkles in a floating elastic sheet under uniaxial compression. This is a canonical set-up in the study of wrinkling, and while its static equilibrium configuration is well characterized, its dynamics are not. Here, we focus on modelling the transition from early-time, inertia-dominated wrinkle growth to late-time gravity-moderated equilibrium. For an initially flat sheet, an initial disturbance will first grow at the shortest available wavelengths, because this requires the least kinetic energy, but will subsequently transition to a longer preferred wavelength that minimizes potential energy. We observe that the evolving wave pattern must be a spectrum, as opposed to a fundamental wrinkle mode whose wavelength evolves in time. Our results demonstrate that changes in the dominant wrinkle wavelength are coupled to a decay in the compressive force, which is to be expected from equilibrium theory. As part of this study, we found that the system must have a dissipation mechanism in order to evolve into an equilibrium steady state.

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The dominant wrinkle wavelength and compressive force appear to be independent of the level of dissipation, although it does influence the shape of the wrinkle wave packet, with the spectrum narrowing as dissipation increases.

1. Introduction

Wrinkles occur in a plethora of real-world scenarios, both naturally and through human engineering. In the natural world, examples include gyrification of the human brain during fetal development [1–3], growth-induced buckling of epithelial cell layers [4–6] and the formation and expansion of biofilms [7–9]. In engineering, wrinkles can be used to redirect light [10,11], modify surface properties [12] or facilitate flexibility in electronic components [13]; these wrinkles can be formed by, for example, compressing a sheet that is attached to a substrate. In all cases, wrinkles form because they are an energy-efficient mechanism for a material to accommodate excess length or area, and the wrinkle formation occurs at an emergent wavelength that balances the competing forces within a given system. In particular, the bending stiffness of the sheet itself resists high-curvature (short-wavelength, small-amplitude) configurations, while substrate stiffness, tension and curvature along wrinkles, and confining walls, all resist large-displacement (long-wavelength, large-amplitude) configurations. More complex patterns can also be generated through biaxial confinement; for example, the shrinking associated with curing or drying [14] and the stresses induced by forcing a naturally curved surface to be flat [15] can both lead to visually striking multi-directional wrinkles.

The competing effects that determine wrinkle patterns offer routes for controlling pattern selection. For example, decreasing the stiffness of the sheet and increasing substrate stiffness will both act to decrease the characteristic size of the wrinkles, while modifying applied forces gives opportunities for tuning the wrinkle pattern in a given sheet-substrate system [16–18]. The general rule for wrinkle wavelength selection is neatly summarized in the literature [16–18] as

$$\lambda^* = 2\pi \left(\frac{D^*}{K^*} \right)^{1/4} \quad (1.1)$$

in equilibrium, where D^* is the bending stiffness of the material and K^* is a resisting stiffness, which is equal to density $\times$ gravity if the substrate is a hydrostatic fluid, and can be adapted for more complicated scenarios. Allowing for dynamics introduces an additional layer of questions: how does a wrinkled configuration evolve to its equilibrium state, and what are the factors that control this evolution? Experimental investigations have pointed to viscous dissipation, substrate inertia and sheet inertia as additional factors that can affect how wrinkles evolve towards their final state [14,19,20]. Processes like polymer curing motivate studies where wrinkling is dynamic owing to a viscous substrate [14]. For example, compression of a thin elastic sheet on a lubricated layer can be described mathematically and yields wrinkles whose wavelength grows slowly in time t^* according to [21]

$$\lambda^* \propto \left(\frac{t^*}{\log t^*} \right)^{1/6}, \quad (1.2)$$

while for an elastic sheet on a viscous half-space, Draux *et al.* [22] suggest

$$\lambda^* \propto (t^*)^{1/3}, \quad (1.3)$$

with a possibility of a logarithmic correction as in equation (1.2). Practically, a tri-layer system composed of a pre-stretched elastic base, a viscous layer and a thin elastic sheet gives a route for probing viscosity-mediated wrinkling experimentally [23]. When the pre-stretch in the base elastic

layer is released, liquid in the viscous layer is pulled inwards, which in turn drags the thin layer on top inwards, inducing wrinkles whose development must displace the viscous fluid further. Wrinkle localization and ridge formation can also occur [24,25].

While the scenarios mentioned in the paragraph above are viscosity-dominated, in other scenarios, the dynamics of wrinkle formation over a fluid are affected by inertia. Indentation of a floating sheet is one example of a system where this can be studied. Indentation of the sheet centre induces hoop stresses that in turn drive the formation of spoke-like wrinkles; for quasi-static indentation, the wavelength of these wrinkles decreases as the indentation increases [26–28]. At high indentation speeds, the inertia of the fluid delays the formation of large-wavelength wrinkles, and so at early times, the wrinkle wavelength grows from vanishingly small, with a trend close to [19,29]

$$\lambda^* \propto (t^*)^{2/5}, \quad (1.4)$$

again with a (possibly logarithmic) deviation associated with inextensibility. Energetically, this trend comes from a balance between bending and kinetic energy: the bending energy per unit length of sheet scales with $(D^*/2)(w^*)^2/(\lambda^*)^4$ for bending stiffness D^* and amplitude w^* , while the kinetic energy per unit length of sheet scales with $(\rho^*/2)(w^*/t^*)^2\lambda^*$, where ρ^* is density and the factor λ^* comes from integrating the fluid over the disturbed depth. However, modelling the dynamics beyond very early times is complicated by a variety of factors associated with the sheet shape and size as well as geometric and material nonlinearities; we attempt to elucidate some of these factors in this paper.

If the dynamics of the substrate is removed, then dynamic effects can come from the sheet itself, for example, from the inertia of the wrinkling elastic material. One way of studying this experimentally is to indent or contract an elastic sheet in free fall [20,30–33]. This opens an additional angle for wrinkles to behave dynamically—here, in the absence of a substrate, the inertia of the sheet itself dominates at early time, again leading to the generation of initially fine wrinkles which subsequently coarsen.

In this paper, we consider wrinkling of an elastic beam floating on a liquid substrate, accounting for both inertia and viscosity, with the goal of resolving the wrinkle evolution from initiation to equilibrium. We avoid the problems encountered previously for indentation of a circular sheet by moving to a more theoretically convenient scenario: a beam (or infinitely wide sheet) compressed uniaxially by prescribed displacement of its ends. This paper is organized as follows. In §2, we outline the mathematical model and present the various parameter regimes applicable to this study. In §3, we non-dimensionalize the problem before exploiting the parameter regime in order to derive a relatively simple mathematical description of a beam subject to hydrodynamic, hydrostatic and viscous forces. A solution of this reduced problem is then sought by performing a model decomposition by Fourier transform in space, which yields a system of evolution equations for the amplitudes of different wrinkling modes with different wrinkle wavelengths. This facilitates a quantification of the length, kinetic energy and potential energy associated with each mode. The evolution equations are coupled via a length constraint, forming a differential algebraic system which requires a careful numerical approach, as outlined in §4. We present numerical solutions and interpret our findings in §5 and draw conclusions in §6.

2. Mathematical model

(a) Problem description

We consider a long, thin and linearly elastic beam of length $2L^*$, which in its base-state configuration (see figure 1a) is flat and lies on the surface of a hydrostatic fluid occupying a semi-infinite domain $(x^*, z^*) \in (-L^*, L^*) \times (-\infty, 0)$. We denote the beam properties as Young's modulus E^* , Poisson

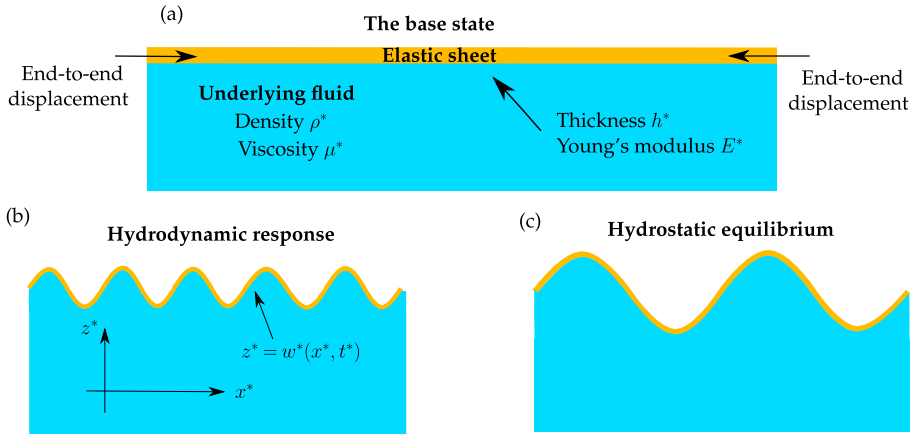


Figure 1. (a) The base-state configuration of an elastic beam, which at time $t^* = 0$ is subject to a prescribed end-to-end displacement of size Δ^* , buckles the beam out-of-plane, displacing the underlying viscous fluid leading to (b) a highly wrinkled beam configuration, induced by viscous and inertial restoring forces. As time evolves, the wrinkles coarsen, eventually leading to (c) the late-time steady state for which hydrostatic restoring forces overcome visco-inertial forces and combine to balance bending effects.

ratio ν , thickness h^* and bending stiffness $D^* = E^*h^{*3}/12(1 - \nu^2)$, and we denote the fluid properties as density ρ^* and dynamic viscosity μ^* . In its base-state configuration, the beam is located at $z^* = 0$ and occupies the domain $x^* \in (-L^*, L^*)$. We consider the case in which both endpoints of the beam are instantaneously displaced inwards by an amount Δ^* . In response, the beam buckles with out-of-plane displacement w^* , which in turn excites an unsteady viscous flow in the underlying fluid. The precise asymptotic regime under consideration, as well as the dimensional equations governing both the fluid flow and the beam's displacements, is presented in the following two sections.

(b) The asymptotic regime

We consider an asymptotic regime in which the beam's characteristic displacement amplitude W^* is small when compared with the characteristic axial length scale ℓ^* , which will later be chosen as the emergent late-time steady-state wrinkle wavelength. Hence, we define the wrinkle aspect ratio

$$\epsilon = \frac{W^*}{\ell^*} \ll 1. \quad (2.1)$$

Our next task is to quantify the relative importance of linear inertial effects, nonlinear inertial effects and viscous effects. In this work, we assume that linear inertial effects are important, but also that the effects associated with nonlinear inertia may be neglected and that viscous effects are weak but non-negligible. The importance of nonlinear versus linear inertial effects is quantified by the parameter $\epsilon \ll 1$. For viscous effects, in this problem, a naive Reynolds number $Re = \rho^* V^* \ell^* / \mu^*$ (with V^* defined to be the characteristic magnitude of the fluid velocity) quantifies the importance of *nonlinear* inertial effects relative to viscous effects, so it is helpful to introduce a time scale $T^* = W^* / V^*$ and define a modified Reynolds number Re/ϵ , which quantifies *linear* inertial effects relative to viscous effects. That is, the overall importance of viscous effects is measured by

$$\mu = \frac{\mu^* V^* / \ell^{*2}}{\rho^* V^* / T^*} = \frac{W^*}{\ell^*} \frac{\mu^*}{\rho^* \ell^* V^*} = \frac{\epsilon}{Re}. \quad (2.2)$$

Our regime choice then corresponds to the distinguished limit in which $\mu \rightarrow 0$ as $\epsilon, Re \rightarrow 0$, or alternatively

$$\epsilon \ll \mu \ll 1. \quad (2.3)$$

In the model development below, we extract a leading-order description in the limit $\epsilon \rightarrow 0$, and then afterwards, consider the size of μ and develop a description that can include terms up to $O(\mu)$. The scenario $\mu = 0$, when there are no viscous effects, then just corresponds to the single limit $\epsilon \rightarrow 0$.

(c) Fluid mechanics

The fluid velocity $\mathbf{v}^*$ and hydrodynamic pressure p^* are governed by the Navier–Stokes equations in the fluid bulk, with a dynamic boundary condition at the interface that equates stresses in the fluid with those in the beam. For the regime $\epsilon \ll \mu \ll 1$ specified in §2b, we follow the work of Dias *et al.* [34] to simplify the fluid-mechanical governing equations by resolving the fluid velocity in terms of scalar and vector potentials ϕ^* and $\boldsymbol{\psi}^* = \psi^* \mathbf{j}$ via a Helmholtz decomposition

$$\mathbf{v}^* = \nabla^* \phi^* + \nabla^* \times (\psi^* \mathbf{j}) = \left(\frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial z^*} \right) \mathbf{i} + \left(\frac{\partial \phi^*}{\partial z^*} + \frac{\partial \psi^*}{\partial x^*} \right) \mathbf{k}, \quad (2.4)$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the usual Cartesian unit vectors. Using this decomposition, it is shown in appendix A.2 that mass and momentum conservation are described by

$$\nabla^{*2} \phi^* = 0 \quad \text{and} \quad \mu^* \nabla^{*2} \psi^* = \rho^* \frac{\partial \psi^*}{\partial t^*}, \quad (2.5)$$

respectively, while continuity of normal stress at the fluid–beam interface is described by

$$\rho^* \frac{\partial \phi^*}{\partial t^*} + \rho^* g^* w^* + 2\mu^* \left(\frac{\partial^2 \phi^*}{\partial z^{*2}} + \frac{\partial^2 \psi^*}{\partial z^* \partial x^*} \right) = \sigma_{22}^*, \quad \text{at} \quad z^* = w^*, \quad (2.6)$$

where σ_{22}^* is the component of stress in the sheet oriented out of the plane, which at $z^* = w^*$ is equal to the pressure exerted on the beam by the fluid, and

$$2\mu^* \left(\frac{\partial^2 \phi^*}{\partial z^* \partial x^*} - \frac{\partial^2 \psi^*}{\partial z^{*2}} \right) + \rho^* \frac{\partial \psi^*}{\partial t^*} = \sigma_{12}^*, \quad \text{at} \quad z^* = w^*, \quad (2.7)$$

for the tangential stress component, where σ_{12}^* is the tangential shear at the fluid–beam interface. Finally, it is shown that the continuity of normal velocities at the fluid–beam interface gives rise to the kinematic condition

$$\frac{\partial \phi^*}{\partial z^*} + \frac{\partial \psi^*}{\partial x^*} = \frac{\partial w^*}{\partial t^*} + \left(\frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial z^*} \right) \frac{\partial w^*}{\partial x^*}, \quad \text{at} \quad z^* = w^*. \quad (2.8)$$

Our model also requires prescription of boundary and initial conditions. We impose an infinite-depth condition

$$\frac{\partial \phi^*}{\partial z^*} \rightarrow 0, \quad \text{as} \quad z^* \rightarrow -\infty \quad (2.9)$$

on the fluid velocity. We assume that the beam length is sufficiently large that finite-length boundary effects may be ignored, rendering the x^* -axis effectively infinite. Formally, the required condition is then that the fluid velocity remains bounded in the far field. We postpone discussion of the initial conditions to §4b.

(d) Beam mechanics

Turning our attention to the beam mechanics, under the small-strain assumption, the out-of-plane displacement and axial tension in the beam are governed by the Föppl–von Kármán equations:

$$\frac{\partial F^*}{\partial x^*} = 0, \quad (2.10)$$

$$D^* \frac{\partial^4 w^*}{\partial x^{*4}} - F^* \frac{\partial^2 w^*}{\partial x^{*2}} = -\sigma_{22}^*, \quad (2.11)$$

where $F^* = F^*(t^*)$ is the thickness-integrated time-dependent axial stress. In principle, there is an additional contribution, σ_{12}^* , to the right-hand side of [equation \(2.10\)](#) arising owing to viscous shear effects at the fluid–beam interface from [equation \(2.7\)](#). However, on comparing scales (see §3a), we find that the relative importance of these shear effects is $O(\epsilon\mu)$ and so they are negligible in this framework. In addition, we have assumed here that the beam is in stress equilibrium, which requires beam inertia be negligible. This is valid provided that the mass per unit thickness of the beam is small when compared with the mass per unit wavelength of the fluid [35].

Eliminating the out-of-plane stress component σ_{22}^* between [equations \(2.6\)](#) and [\(2.11\)](#) yields

$$\rho^* \frac{\partial \phi^*}{\partial t^*} + D^* \frac{\partial^4 w^*}{\partial x^{*4}} - F^*(t^*) \frac{\partial^2 w^*}{\partial x^{*2}} + \rho^* g^* w^* + 2\mu^* \left(\frac{\partial^2 \phi^*}{\partial z^{*2}} + \frac{\partial^2 \psi^*}{\partial z^* \partial x^*} \right) = 0, \quad (2.12)$$

at $z^* = w^*$. [Equation \(2.12\)](#) is to be appended to the governing [equation \(2.5\)](#) for the fluid flow in the bulk as a dynamic boundary condition that captures the fluid–structure interaction.

The final governing equation, which may be interpreted as a closure condition for the time-dependent eigenvalue, F^* , is given by

$$\frac{1}{2} \int_{-L^*}^{L^*} \left(\frac{\partial w^*}{\partial x^*} \right)^2 dx^* = 2\Delta^*. \quad (2.13)$$

[Equation \(2.13\)](#) is a geometrically nonlinear length constraint on the beam that normalizes the out-of-plane displacement, w^* , by ensuring that it absorbs all of the excess length available for wrinkling induced by the end-to-end displacement, and is valid provided $|F^*|/(E^*h^*) \ll \Delta^*/L^*$.

3. Mathematical analysis

(a) Scalings and non-dimensionalization

In this section, we discuss variable scalings and present the non-dimensionalization of [equations \(2.5\)–\(2.13\)](#). We begin by using [equation \(2.5\)](#) to motivate the introduction of a common horizontal and vertical length scale ($x^* \sim z^*$), denoted ℓ^* . Balancing the restoring force owing to bending with hydrostatic/buoyancy restoring forces in [equation \(2.12\)](#) motivates the choice

$$\ell^* = \left(\frac{D^*}{\rho^* g^*} \right)^{1/4}. \quad (3.1)$$

On this length scale, the terms in [equation \(2.12\)](#) associated with axial bending and tension–curvature balance when the scale for the tension is chosen to be

$$F^* \sim \sqrt{D^* \rho^* g^*}. \quad (3.2)$$

The scales ([equations \(3.1\)](#) and [\(3.2\)](#)), which form the basis of our choice of non-dimensionalization of [equations \(2.5\)–\(2.13\)](#), are classical results in static tension-field theory (e.g. [17,35,36]). Thus, we

scale

$$\begin{aligned} w^* &= W^* w, & (x^*, z^*) &= \ell^*(x, z), & (\phi^*, \psi^*) &= \frac{\ell^* W^*}{T^*} (\phi, \mu\psi), \\ p^* &= \rho^* g^* W^* p, & t^* &= T^* t, & F^* &= \sqrt{D^* \rho^* g^*} F, \end{aligned} \quad (3.3)$$

where the scale for the out-of-plane displacement $W^* = \ell^* \sqrt{\Delta^*/L^*}$ has been chosen on examination of [equation \(2.13\)](#), and where the explicit time scale $T^* = \sqrt{\ell^*/g^*}$ has been obtained by balancing linear fluid inertia with hydrostatic forces in [equation \(2.12\)](#). Our choice of scaling for ψ^* in [equation \(3.3\)](#) comes from analysing the tangential stress condition, [equation \(2.7\)](#), which suggests that it is natural to take $\psi^* \sim \epsilon Re^{-1} \phi^* = \mu \phi^*$.

The dimensionless parameter μ quantifying the effect of viscosity relative to inertia can now be cast in terms of experimentally relevant parameters as

$$\mu = \frac{\mu^*}{(\rho^*)^{\frac{5}{8}} (g^*)^{\frac{1}{8}} (D^*)^{\frac{3}{8}}} \ll 1, \quad (3.4)$$

while $\epsilon = \sqrt{\Delta^*/L^*} \ll 1$ is satisfied provided that the imposed fractional end-to-end displacement is small.

(b) Model reduction and dimensionless system of governing equations

We now substitute the non-dimensionalization [equation \(3.3\)](#) into the system (2.5)–(2.13) and simplify the resulting dimensionless system by retaining terms that are leading order with respect to the small parameter ϵ . We preserve weak dissipative effects by retaining contributions that are first order with respect to $\mu \ll 1$. The dimensionless governing equations to be solved in the fluid domain are

$$\nabla^2 \phi = 0, \quad \mu \nabla^2 \psi = \frac{\partial \psi}{\partial t}, \quad \text{on} \quad (x, z) \in (-L, L) \times (-\infty, 0), \quad (3.5)$$

where $L = L^*/\ell^*$, and we anticipate taking the limit $L \rightarrow \infty$ below. [Equations \(3.5\)](#) are to be solved subject to the far-field boundary condition

$$\frac{\partial \phi}{\partial z} \rightarrow 0, \quad \text{as} \quad z \rightarrow -\infty, \quad (3.6)$$

together with the requirement that the fluid velocity remains bounded at the beam's endpoints $|x| = L$. At the linearized fluid–beam interface, we have the leading-order kinematic condition

$$\frac{\partial w}{\partial t} = \frac{\partial \phi}{\partial z} + \mu \frac{\partial \psi}{\partial x}, \quad \text{at} \quad z = 0. \quad (3.7)$$

The normal stress condition, [equation \(2.6\)](#), is now

$$\frac{\partial \phi}{\partial t} + \frac{\partial^4 w}{\partial x^4} - F(t) \frac{\partial^2 w}{\partial x^2} + w + 2\mu \frac{\partial^2 \phi}{\partial z^2} = 0, \quad \text{at} \quad z = 0, \quad (3.8)$$

and we interpret it as a Föppl–von Kármán equation with hydrodynamic pressure and viscous damping. The tangential stress condition, [equation \(2.7\)](#), can in principle be used to infer the shear exerted on the sheet by the fluid, but this shear does not affect the dynamics at leading order and so is not investigated further here.

The amplitude of w is set by the geometric constraint

$$\frac{1}{2} \int_{-L}^L \left(\frac{\partial w}{\partial x} \right)^2 dx = 2L. \quad (3.9)$$

Note that in steady state, there is no fluid velocity, and [equation \(3.8\)](#) reduces to

$$\frac{\partial^4 w}{\partial x^4} - F \frac{\partial^2 w}{\partial x^2} + w = 0, \quad (3.10)$$

which is the Föppl–von Kármán equation for a beam with unit bending stiffness, on a substrate with unit restoring stiffness, subjected to an axial tension F . It is well-known that [equation \(3.10\)](#) has solutions of the form e^{ikx} with $k \in \mathbb{R}$ for compressive forces $F \leq -2$, and that the solution $k = 1$ with $F = -2$ minimizes the sum of bending and substrate energies subject to the constraint [equation \(3.9\)](#) [17,18].

(c) Modal decomposition

By linearity in the governing [equations \(3.5\)–\(3.8\)](#), it is natural to seek a solution by Fourier transform in x so that a separable solution takes the form:

$$\phi(x, z, t) = \sqrt{\frac{L}{\pi}} \int_{-\infty}^{\infty} \frac{1}{|k|} f(k, t) e^{ikx} e^{|k|z} dk, \quad (3.11)$$

$$\psi(x, z, t) = \sqrt{\frac{L}{\pi}} \int_{-\infty}^{\infty} \frac{1}{ik} g(k, t) e^{ikx} e^{|m|z} dk, \quad (3.12)$$

$$w(x, t) = \sqrt{\frac{L}{\pi}} \int_{-\infty}^{\infty} a(k, t) e^{ikx} dk, \quad (3.13)$$

where $k \in \mathbb{R}$ is the wrinkle wavenumber and $m = m(k)$. The inclusion of the prefactors within the integrands of [equations \(3.11\)](#) and [\(3.12\)](#) are motivated by the kinematic condition [equation \(3.7\)](#), and the geometric prefactors in each of [equations \(3.11\)–\(3.13\)](#) have been included for convenience when substituting the transforms into the length constraint [equation \(3.9\)](#). We note at this stage that while the beam's dimensionless length L is finite, it shall here and henceforth be taken as sufficiently large that finite-length effects at the beams endpoints may be ignored, effectively rendering the beam infinite.

Substituting the [transforms \(3.11\)–\(3.13\)](#) into the mass-conservation [equations \(3.5\)](#) and the kinematic condition [equation \(3.7\)](#), we find that a , f and g are related by

$$\frac{\partial g}{\partial t} = \mu (|m|^2 - k^2) g(k, t) \quad \text{and} \quad \frac{\partial a}{\partial t} = f(k, t) + \mu g(k, t). \quad (3.14)$$

Similarly, the Föppl–von Kármán [equation \(3.8\)](#) gives

$$\frac{1}{|k|} \left(\frac{\partial^2 a}{\partial t^2} - \mu \frac{\partial g}{\partial t} \right) + (k^4 + Fk^2 + 1) a + 2\mu |k| \left(\frac{\partial a}{\partial t} - \mu g \right) = 0, \quad (3.15)$$

where we have used the result [equation \(3.14\)](#) to eliminate f . Finally, noting that $\mu \partial g / \partial t = O(\mu^2)$ by [equation \(3.14\)](#) and eliminating all terms at this order, the governing [equation \(3.15\)](#) for a simplifies to

$$\frac{1}{|k|} \frac{\partial^2 a}{\partial t^2} + 2\mu |k| \frac{\partial a}{\partial t} + (k^4 + Fk^2 + 1) a = 0. \quad (3.16)$$

[Equation \(3.16\)](#) is a second-order partial differential equation governing the amplitude, $a(k, t)$, of the beam's out-of-plane displacements as a function of the wavenumber and time, as well as the axial tension, F , as a function of time. The closure condition enabling the determination of F at each moment in time is obtained by substituting [equation \(3.13\)](#) into the length constraint [equation \(3.9\)](#),

and is found to be

$$\int_{-\infty}^{\infty} \frac{1}{2} k^2 |a|^2 dk = 1, \quad (3.17)$$

where $|a|^2 = a\bar{a}$ with $\bar{a}$ denoting the complex conjugate of a . Equations (3.16) and (3.17) together with suitable initial conditions are investigated numerically below.

(d) Energetic considerations and steady states

Before equipping the system equations (3.16) and (3.17) with initial conditions and solving numerically, we first investigate the system's energetics to motivate the expected outcomes. In the absence of beam inertia, the only source of kinetic energy comes from the underlying fluid. By contrast, the system's potential energy has contributions from both the bending moment in the beam and the weight of the underlying fluid. The total dimensionless energy can be decomposed as $E = E_{\text{kin}} + E_{\text{pot}}$, where the components E_{kin} and E_{pot} are written in terms of the spectrum equation (3.13) as

$$E_{\text{kin}}(t) = \int_{-\infty}^{\infty} \mathcal{E}_{\text{kin}} dk \quad \text{and} \quad E_{\text{pot}}(t) = \int_{-\infty}^{\infty} \mathcal{E}_{\text{pot}} dk, \quad (3.18)$$

respectively, and

$$\mathcal{E}_{\text{kin}}(k, t) = \frac{1}{2|k|} \left| \frac{\partial a}{\partial t} \right|^2 \quad \text{and} \quad \mathcal{E}_{\text{pot}}(k, t) = \frac{1}{2} (k^4 + 1) |a|^2 \quad (3.19)$$

are the respective kinetic and potential energy densities, respectively, which, along with their global counterparts, have been scaled with a factor L^2/π for convenience. These expressions can be derived from physical quantities or inferred from equations (3.16) and (3.17) (see appendix B, for details).

Differentiating the total energy E in time and making use of the governing equation (3.16), its complex conjugate, and the time-derivative of equation (3.17), yields

$$\frac{dE}{dt} = -2\mu \int_{-\infty}^{\infty} |k| \left| \frac{\partial a}{\partial t} \right|^2 dk \leq 0. \quad (3.20)$$

Comparing equations (3.18) and (3.20), we see that the total energy is positive and non-increasing. If $\mu = 0$, the total energy will remain at its initial value. If $\mu \neq 0$, then, by application of the monotone convergence theorem, it will decrease to some other positive constant. In the latter case (dissipative, $\mu \neq 0$), dE/dt decays to zero and so too must the integrand on the right-hand side of equation (3.20); this requires that $\partial a/\partial t \rightarrow 0$ and so the system eventually reaches a steady state.

From the evolution equation (3.16), the only possible steady states are those where a has isolated peaks at the roots of $k^4 + Fk^2 + 1 = 0$, and such real roots exist if $F \leq -2$. From the static literature [17,18], minimizing the potential energy density with respect to k , subject to the length constraint equation (3.17), yields

$$k^2 = 1, \quad \text{with} \quad F = -2, \quad (3.21)$$

as expected. It is natural to wonder whether a system initiated close to a different steady state (i.e. with $F < -2$ and a peak at $k^2 \neq 1$) might stay close to that steady state. From a physical standpoint, any perturbation that excites modes that do not satisfy $k^4 + Fk^2 + 1 = 0$ will induce velocities and hence dissipation, driving it towards equation (3.21) provided $\mu \neq 0$.

More generally, from a mathematical viewpoint, we can ask if any steady state with $F < -2$ is a local minimizer of total energy. To answer this question, we study the energetic stability of the

system by calling on variational calculus. Following classical theory, one formulates the appropriate energy functional (with constraints), before computing first variations to obtain the governing Euler–Lagrange equations. Second variations are then computed in order to assess the stability of equilibria [37–39]. We introduce a generalized energy functional

$$\hat{E}[a, b] = \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{|k|} |b|^2 + (k^4 + 1) |a|^2 + Fk^2 |a|^2 \right] dk, \quad (3.22)$$

where $b = \partial a / \partial t$ and the final term can be interpreted as imposing the length constraint or quantifying work done on the beam ends. For a perturbation $[\delta a, \delta b]$, the first variation of the energy is

$$\delta \hat{E} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{|k|} (b \delta \bar{b} + \bar{b} \delta b) + (k^4 + Fk^2 + 1) (a \delta \bar{a} + \bar{a} \delta a) \right] dk, \quad (3.23)$$

and the second variation is

$$\delta^2 \hat{E} = \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{1}{|k|} |\delta b|^2 + (k^4 + Fk^2 + 1) |\delta a|^2 \right] dk. \quad (3.24)$$

For any permitted steady state, the first variation is zero, and the sign of the second variation gives the stability. This means that a steady state can be achieved with a single, static peak in a at the (real) value of k that satisfies $k^4 + Fk^2 + 1 = 0$, and so can be achieved for any $F \leq -2$. A *stable* steady state is then one for which $\delta^2 \hat{E} \geq 0$ for any perturbation, and is only satisfied if $k^4 + Fk^2 + 1 \geq 0$ for all k . This means that there is only one stable steady state, the one at $F = -2$.

Finally, we note that in the absence of dissipation ($\mu = 0$), the total energy is conserved and the $k^2 = 1$, $F = -2$ equilibrium outlined above cannot typically be reached. Despite this, we find in §5 that spectrally integrated (or ensemble-averaged) energies do appear to equilibrate at late time.

4. Numerical analysis

(a) Numerical method

Reintroducing $b(k, t) = \partial a / \partial t$, the problem equations (3.16) and (3.17) can be restated as finding a solution set a, b, F to the system

$$\frac{\partial a}{\partial t} = b, \quad (4.1)$$

$$\frac{\partial b}{\partial t} = -2\mu k^2 b - |k| (k^4 + Fk^2 + 1) a, \quad (4.2)$$

$$0 = 1 - \int_{-\infty}^{\infty} \frac{1}{2} k^2 |a|^2 dk, \quad (4.3)$$

given a set of initial conditions $a(k, 0)$, $b(k, 0)$ and $F(0)$. Owing to the integral constraint equation (4.3), which will become an algebraic constraint upon numerical discretization, equations (4.1)–(4.3) form what is canonically referred to as a system of differential-algebraic equations (DAEs) [40–42]. DAEs are typically classified by a so-called ‘differentiation index’, defined to be the minimum number of times the algebraic constraint equation (4.3) is to be differentiated in order to recast the problem in terms of an explicit first-order system [43].

Systems of DAEs with a single algebraic constraint with a differentiation index of $n > 1$ must, for consistency, satisfy $n - 1$ additional constraints obtained by differentiating the algebraic constraint $n - 1$ times [42]. The index of the system equations (4.1)–(4.3) is three, and hence, we obtain two additional constraints upon computing the first and second derivatives of equation (4.3),

namely,

$$\int_{-\infty}^{\infty} k^2 \left(a\bar{b} + \bar{a}b \right) dk = 0 \quad (4.4)$$

and

$$\int_{-\infty}^{\infty} k^2 \left[|b|^2 - |k| (k^4 + Fk^2 + 1) |a|^2 - \mu k^2 \left(a\bar{b} + \bar{a}b \right) \right] dk = 0. \quad (4.5)$$

The final constraint, [equation \(4.5\)](#), can be rearranged to yield an expression for F , given as

$$F = \frac{\int_{-\infty}^{\infty} k^2 \left[|b|^2 - |k| (k^4 + 1) |a|^2 - \mu k^2 \left(a\bar{b} + \bar{a}b \right) \right] dk}{\int_{-\infty}^{\infty} k^4 |k| |a|^2 dk}. \quad (4.6)$$

Differentiating [equation \(4.6\)](#) once in time and combining the result with [equations \(4.1\)](#) and [\(4.2\)](#) yields an explicit first-order system for (a, b, F) ; this verifies that the differentiation index of [equations \(4.1\)–\(4.3\)](#) is three.

By the inverse Fourier transform of [equation \(3.13\)](#), one can show that $a(k, t) = \bar{a}(-k, t)$, and hence, by symmetry, we restrict the solution domain to $k \geq 0$. By defining the weighted inner product

$$\langle f, g \rangle_{k^2} = \int_0^{\infty} k^2 f \bar{g} dk, \quad (4.7)$$

the length constraint and its first derivative are restated as

$$\langle a, a \rangle_{k^2} = 1 \quad (4.8)$$

and

$$\Re \left[\langle a, b \rangle_{k^2} \right] = 0, \quad (4.9)$$

respectively, where $\Re$ denotes the real part. [Equation \(4.6\)](#) for F obtained from the second derivative of the length constraint is then

$$F = \frac{\langle b - 2\mu k^2 a, b \rangle_{k^2} - \langle (k^5 + k)a, a \rangle_{k^2}}{\langle k^3 a, a \rangle_{k^2}}. \quad (4.10)$$

[Equations \(4.8\)](#) and [\(4.9\)](#) provide a geometric interpretation of the solution space. Solutions for a must lie on the surface of the unit sphere in the space of functions satisfying the inner product [equation \(4.7\)](#), while the corresponding velocity b must be orthogonal to a (i.e. tangent to the unit sphere).

To proceed numerically, the restricted wavenumber domain is truncated beyond some $k = k_{\text{end}}$, so we define the subdomain $k \in [0, k_{\text{end}}]$, which is discretized using a mesh of N ordered collocation points k_i for $i = 0, 1, \dots, N-1$, such that

$$0 = k_0 < k_1 < \dots < k_{N-1} = k_{\text{end}}. \quad (4.11)$$

Writing $a_i(t) = a(k_i, t)$ and $b_i(t) = b(k_i, t)$, the discrete form of the evolution [equations \(4.1\)](#) and [\(4.2\)](#) are given by

$$\frac{da_i}{dt} = b_i, \quad (4.12)$$

$$\frac{db_i}{dt} = -2\mu k_i^2 b_i - |k_i| (k_i^4 + Fk_i^2 + 1) a_i, \quad (4.13)$$

where F is calculated by applying the trapezoidal rule to [equation \(4.10\)](#). These equations are used to integrate the system forward in time using an eighth-order Runge–Kutta method via SciPy’s implementation of the DOP853 algorithm. After each integration step, the solution for a is renormalized to enforce [equation \(4.8\)](#), before the Gram–Schmidt process is used to subtract any component of b tangential to a in order to ensure [equation \(4.9\)](#), where the inner products are once again evaluated by the trapezoidal rule.

(b) Initial conditions

When equipping [equations \(4.1\)–\(4.3\)](#) with a set of initial conditions, care must be taken to ensure that any choice is consistent not only with the length constraint [equation \(4.3\)](#), but also the additional constraints [equations \(4.4\)](#) and [\(4.5\)](#) arising from its first and second derivatives.

To obtain a consistent set of initial conditions, an initial profile for $a(k, 0)$ is chosen, with an amplitude that has been normalized so that the constraint [equation \(4.3\)](#) is satisfied. In general, the initial velocity $b(k, t)$ must then be prescribed so that [equation \(4.4\)](#) is satisfied subject to the choice of $a(k, 0)$. In our simulations, we found that taking $b(k, 0) \neq 0$ did not yield qualitatively different results to the case $b(k, 0) = 0$. In the latter case, it is noted that the constraint [equation \(4.4\)](#) is satisfied for any $a(k, 0)$, which simplifies the numerical method. Therefore, we proceed by presenting results with $b(k, 0) \equiv 0$ used in all of our numerical simulations. A discussion of alternative initial conditions—including those with $b(k, 0) \neq 0$ —may be found in [appendix C](#). Having specified $a(k, 0)$ and $b(k, 0)$, a corresponding initial value $F(0)$ for the axial tension must then be calculated using [equation \(4.6\)](#), which in turn ensures that the final constraint, [equation \(4.5\)](#), will also be satisfied.

In the numerical results presented below, we prescribe a static initial wrinkle amplitude, a , in the form of a rectangle that spans a finite range of wavenumbers. Specifically,

$$a(k, 0) = A \left[\text{rect} \left(\frac{k - \alpha}{2\beta} \right) + \text{rect} \left(\frac{k + \alpha}{2\beta} \right) \right] \quad \text{and} \quad b(k, 0) = 0, \quad (4.14)$$

where $A^2 = 3/[2\beta(\beta^2 + 3\alpha^2)]$ and $0 < \beta < \alpha$ are parameters that characterize the width and centre of the rectangle. This choice corresponds to the physical (dimensionless) initial profile

$$w(x, 0) = A\beta \sqrt{\frac{L}{\pi}} \cos(\alpha x) \text{sinc} \left(\frac{\beta x}{\pi} \right). \quad (4.15)$$

5. Numerical observations

(a) Spectral profiles

We solve the governing equations presented in [§3c](#) via the numerical procedure outlined in [§4a](#) and present solutions as a function of wavenumber and time for the initial conditions [equation \(4.14\)](#) (see [figure 2](#)). The quantity $|ka(k, t)|^2$ measures the length accommodated by each wavenumber k as a function of time, and therefore, we made the decision to scale amplitudes a with a factor of k before plotting. The wave packet initially spreads in k -space, transferring length to lower values of k (longer wavelengths). In the absence of dissipation ($\mu = 0$), a broad spectrum of modes remains excited at late times ([figure 2a](#)), while when dissipation is present ($\mu \neq 0$) ([figure 2c,e](#)), the wave packet narrows at a rate that increases with increasing μ and the peak moves towards $k = 1$. Eventually, the dissipative solutions converge to the late-time steady-state equilibrium $k = 1$, as predicted from the energetic arguments made in [§3d](#). In tandem, the compressive force decays to its steady-state value $F = -2$ (see [figure 3a](#)). Interestingly, we observe decay to $F = -2$ even when $\mu = 0$, despite the fact that the steady-state wrinkle profile associated with this force cannot actually be achieved in the absence of dissipation (see [figure 3a](#)).

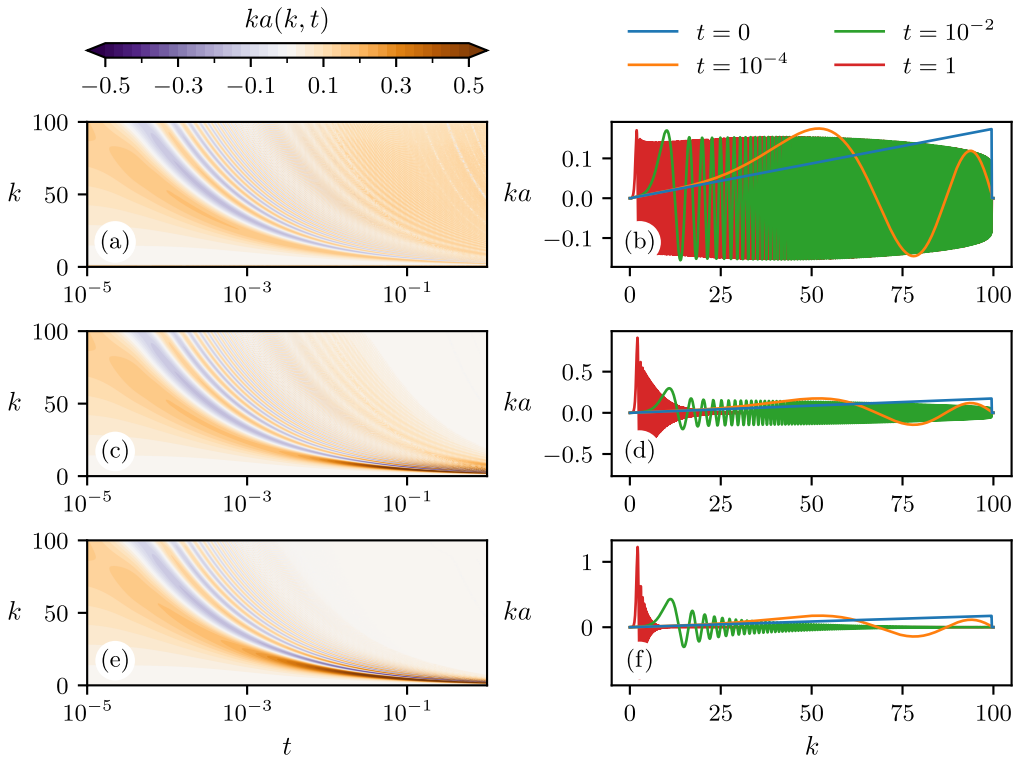


Figure 2. Numerical solutions to the mode amplitude equations (4.1)–(4.3), for rectangular initial data in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$. (a),(c),(e): contour plot of $ka(k, t)$, the length accommodated across wavenumbers k as a function of time t . (b),(d),(f): ka as a function of k at times $t = 0, 10^{-4}, 10^{-2}$ and 1 . The dissipation coefficient μ is set to $\{0, 0.005, 0.05\}$ ordered top to bottom.

For the solutions presented in figure 2, the domain was truncated at $k_{\text{end}} = 100$ with $N = 120\,000$ points. However, the profiles for ka become increasingly oscillatory in k as time increases (a physical explanation for this is given in §5b), and so continuing to later time in figure 3 required higher resolution. Motivated by the narrowing wave packet in figure 2d,f and the procedure used in [29], integration was paused at $t = 10$ and the solution was interpolated onto a shorter k -domain with a finer mesh before restarting integration (with new values $k_{\text{end}} = 6$ for $\mu = 0.05$, $k_{\text{end}} = 20$ for $\mu = 0.005$ and again $N = 120\,000$). For $\mu = 0$, truncating the k -domain was problematic owing to the persistence of amplitudes at large k -values (see figure 2b), and the numerics diverged beyond $t = 1$.

Further examining figure 2, we observe that, although the spectral profiles exhibit significant variability across different values of μ , the peak in ka appears to follow a similar trajectory. We investigate this further by defining a dominant wavenumber

$$k_{\text{dom}}(t) = \operatorname{argmax}_{k>0} (|a(k, t)|), \quad (5.1)$$

which is plotted in figure 3b as a function of time: this dominant wavenumber appears to be approximately independent of μ . The early-time dynamics align loosely with the $t^{-2/5}$ trend suggested by O’Kiely *et al.* [29] for a regime dominated by inertia and bending, but deviate slightly owing to the nonlinear inextensibility constraint, equation (4.3), as expected. Beyond early time, buoyancy becomes significant, wrinkle growth and coarsening are slowed and $k_{\text{dom}}(t)$ then decays towards its equilibrium value $k_{\text{dom}} = 1$, which emerges from a balance between bending and buoyancy.

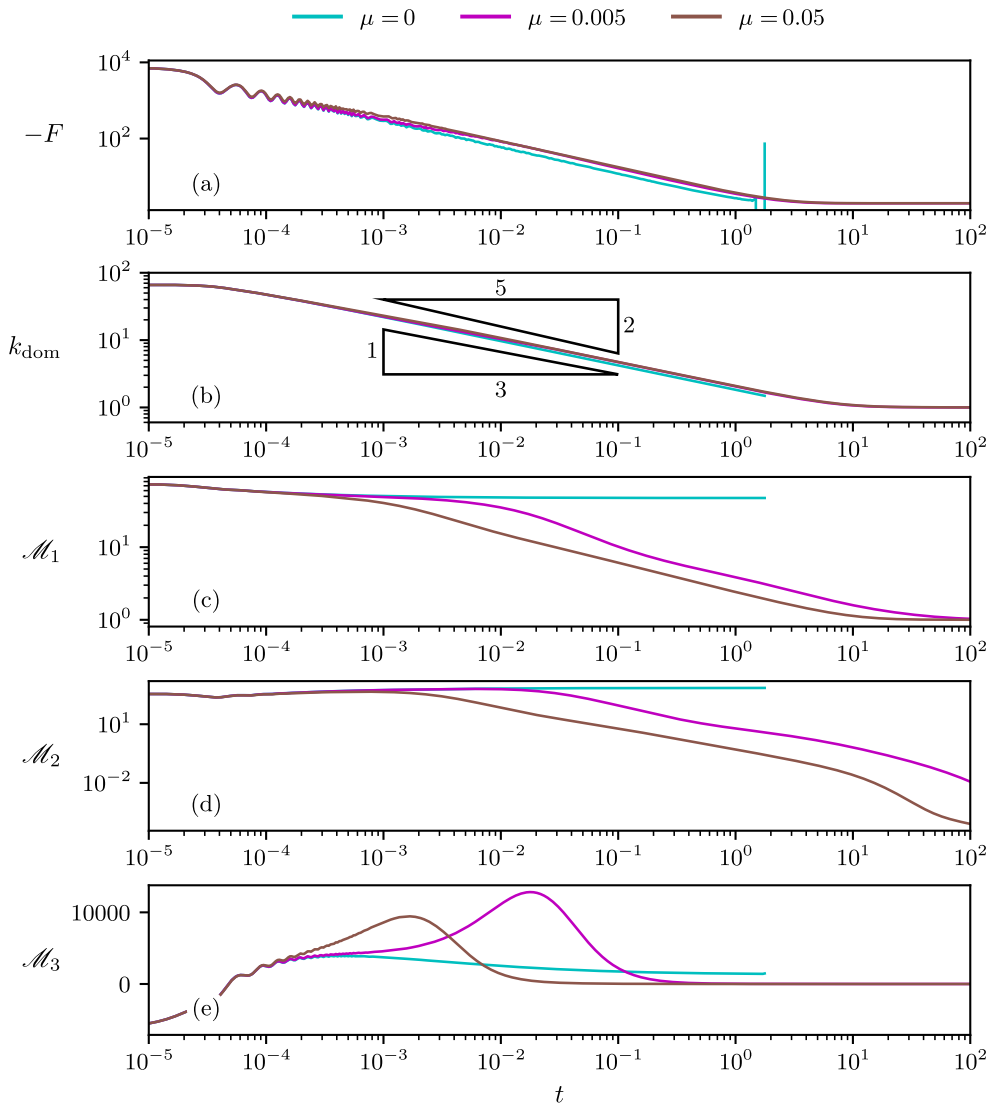


Figure 3. (a) The magnitude of the compressive force, $-F(t)$, (b) the dominant wavenumber k_{dom} and (c)–(e) the moments $\mathcal{M}_i$ for $i = 1, 2, 3$, for rectangular initial data in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$. In (b), line segments of gradient $-2/5$ and $-1/3$ are included for a comparison with the $t^{-2/5}$ and $t^{-1/3}$ power-law predictions suggested by Draux *et al.* [22] and O’Kiely *et al.* [29], respectively. For the case $\mu = 0$, numerical convergence could not be maintained beyond $t = 1$, as evidenced by the emergence of noise in the plot for F and subsequent truncation of the integration procedure.

(b) Wavenumber distribution metrics

While the dominant wavenumber equation (5.1) provides a useful metric for analysing wrinkle dynamics, it fails to take into account the non-negligible contributions from the remaining Fourier modes. For this reason, in this section, we introduce alternative (statistical) metrics that may be used to further analyse the numerical results presented above by considering the central moments (namely, the mean, variance and skewness) that arise through treating our solution spectra as density functions associated with the length allocated to each wrinkling mode. That is, we define

$$\langle k^n \rangle = \int_0^\infty k^n P(k, t) dk, \quad (5.2)$$

where $P(k, t) = k^2 |a(k, t)|^2$ denotes the length accommodated by a wavenumber k with amplitude a , and $\int_0^\infty P(k, t) dk = 1$ by the length constraint [equation \(3.9\)](#), so that P takes the form of a standard probability density distribution. The mean, variance and (non-normalized) skewness can then be defined as

$$\mathcal{M}_1(t) = \langle k \rangle, \quad \mathcal{M}_2(t) = \langle (k - \langle k \rangle)^2 \rangle \quad \text{and} \quad \mathcal{M}_3(t) = \langle (k - \langle k \rangle)^3 \rangle. \quad (5.3)$$

The energy-minimizing equilibrium profile in which only $k = 1$ is excited corresponds to $\mathcal{M}_1 = 1$, $\mathcal{M}_2 = \mathcal{M}_3 = 0$.

Unlike the dominant wavenumber, the mean value $\mathcal{M}_1$ evolves differently for different values of the dissipation coefficient μ (see [figure 3c](#)). For $\mu = 0$, the mean does not converge to $\mathcal{M}_1 = 1$ even at later times. This is consistent with the late-time spread observed in [figure 2a,b](#). For $\mu = 0.05$, the largest dissipation considered here, once the early-time transients have passed, the mean $\mathcal{M}_1$ appears to track the peak k_{dom} closely; this is consistent with the narrow focusing observed in [figure 2e,f](#). Solutions corresponding to the intermediate value $\mu = 0.005$ (plotted in [figure 2c,d](#)) show that the value of $\mathcal{M}_1$ does evolve towards $\mathcal{M}_1 = 1$, but does so more slowly than $k_{\text{dom}}(t)$.

Similar to the mean, the variance, $\mathcal{M}_2(t)$, also exhibits significant variability with respect to changes in the dissipation parameter, μ (see [figure 3d](#)). Specifically, when $\mu = 0$, the variance plateaus to a constant (non-zero) value, whereas when $\mu \neq 0$, the variance approaches zero at late times, at a rate that increases with an increase in the dissipation coefficient. For a uniform initial condition, there is a bias in potential energy towards higher wavenumbers (as is confirmed by [equation \(B 4\)](#)) owing to the fact that high-frequency wrinkles store more bending energy, which scales with the square of the wrinkle curvature. Minimizing the potential energy therefore relies upon transferring potential energy from higher-to-lower wavenumbers, which by the length constraint modifies the amplitude of each mode over time, and hence populates the modes with kinetic energy. When populated with kinetic energy, inertial effects then provide a mechanism for the broadening of the wave packet, as observed in [figure 3d](#), before dissipative effects overcome inertia leading to narrowing again.

To gain further insight, we consider the evolution [equation \(3.16\)](#) through the lens of a damped harmonic oscillator and treat the compressive force $F < -2$ as though it were fixed, with $\mu \ll 1$, which yields approximate analytical solutions:

$$a(k, t) = A(k)e^{-\mu|k|^2 t} \cos \left(t \sqrt{|k|(k^4 + Fk^2 + 1)} \right), \quad (5.4)$$

from which we can deduce that each wrinkle mode oscillates in time at a frequency that increases with both $|k|$ and the distance from the roots k^2 of $k^4 + Fk^2 + 1 = 0$, which we interpret as quasi-steady ‘preferred’ wavenumbers. The oscillations in time transform to vertical velocities in physical space and are penalized by the viscous dissipation term $2\mu\partial a/\partial t$ in [equation \(3.16\)](#), which again increases with $|k|$ and the distance from the quasi-steady wavenumbers and is also proportional to μ . In agreement with our numerical results, [equation \(5.4\)](#) suggests that the peak k_{dom} evolves in tandem with the compressive force $F(t)$, which is plotted in [figure 3a](#) and decays towards $F = -2$ as $k_{\text{dom}} \rightarrow 1$, as expected. Viscous damping focuses the wrinkle spectrum around this peak; this explains the trends for $\mathcal{M}_1$ for $\mu \neq 0$ in [figure 3c](#) and also suggests that—following initial transients—the variance $\mathcal{M}_2$ should decay in time at a rate that depends on μ , as observed in [figure 3d](#). [Equation \(5.4\)](#) also helps to inform the dynamics of the skewness (plotted in [figure 3e](#)), where we see that the focusing of the wave packet should be asymmetric about k_{dom} , with large- k

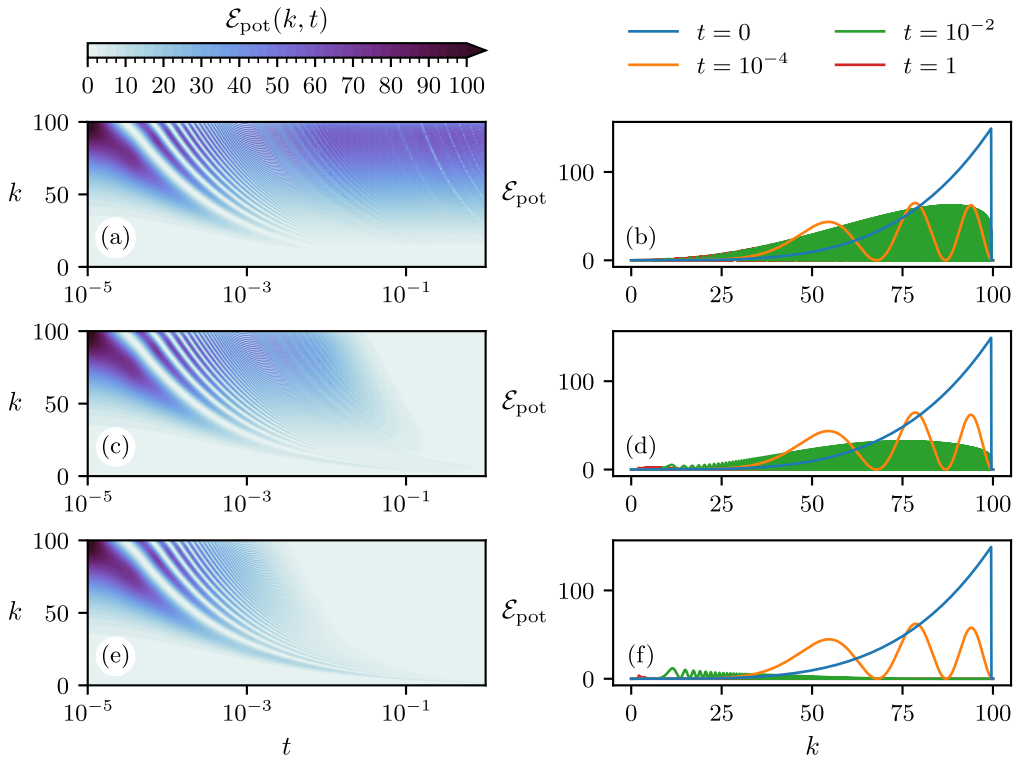


Figure 4. (a),(c),(e): potential energy density contours. (b),(d),(f): energy density at chosen times. The dissipation values μ are $\{0, 0.005, 0.05\}$ top to bottom. Rectangular initial data are in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$.

modes dissipating more strongly, so that a positive skew grows over a time scale that is faster for smaller μ and subsequently decays as the spectrum converges on the equilibrium configuration.

(c) Energy

We plot the potential and kinetic energy distributions in figures 4 and 5, respectively; these correspond to the spectra illustrated in figure 2. The initial potential energy distribution has a monotonic $k^4 + 1$ shape in the excited k -range, but it develops rapidly into a wave-train (a wave packet with approximately uniform-in- k envelope height) for all dissipation values, while the corresponding kinetic energy, which is zero initially, rapidly develops a local peak. As time evolves, the potential energy profile broadens as lower modes are populated with potential energy, and the kinetic energy increases from zero everywhere; if the system has dissipation, it later decreases back towards zero everywhere. Energy is continuously transferred between kinetic and potential until at late time when the potential energy is concentrated around the equilibrium solution $k = 1$. As expected from figure 2, both the potential and kinetic energy profiles become increasingly oscillatory in k as time increases, and the oscillations are wider at smaller k .

The position of the peak in potential energy moves gradually and is independent of dissipation, consistent with the amplitude peak k_{dom} in figure 3b. Nonetheless, the profile is influenced somewhat by dissipation: in the absence of dissipation, some potential energy remains around the position of the initial peak, while increasing the dissipation parameter from zero narrows the peak of the profile, consistent with the trend in $\mathcal{M}_2$ illustrated in figure 3d. In the case of kinetic energy, the position of the peak as a function of time depends strongly on the dissipation parameter. When $\mu = 0.05$, the peak in kinetic energy closely tracks the peak in the potential

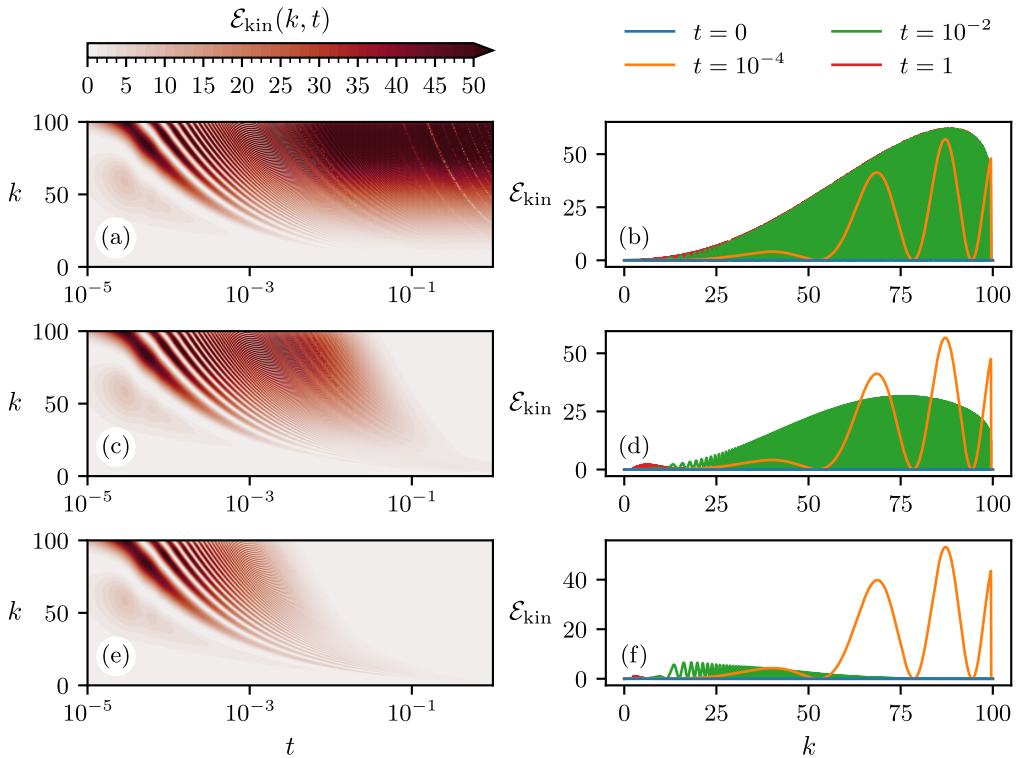


Figure 5. (a),(c),(e): kinetic energy density contours. (b),(d),(f): energy density at chosen times. The dissipation values μ are $\{0, 0.005, 0.05\}$ top to bottom. Rectangular initial data are in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$.

energy, because modes away from this peak are dissipated strongly. However, as μ decreases, the movement of the peak in kinetic energy is delayed, and when $\mu = 0$, it remains close to the initial peak. We seek some additional insight by plotting total energies in figure 6; that is, by integrating the energy distributions from figures 4 and 5 in k . For the purpose of these plots, we measure potential as an excess above the value of 2 that corresponds to the energy-minimizing steady state. From figure 6, it is clear that the systems considered in our numerical simulations all start with no kinetic energy (as imposed by the initial conditions) and some potential energy. We observe that excess potential is transferred into kinetic energy until equipartition is reached, with or without dissipation. With dissipation, this corresponds to complete removal of both kinetic and excess potential energy, while in the absence of dissipation, this corresponds to equal sharing of the initial (spectrally averaged) excess between potential and kinetic forms. The energy transfer occurs concurrently with the drop in compressive force F and the transfer of energy and length from high-to-low wavenumbers. For a non-static initial condition, the initial kinetic energy would of course be non-zero, but we anticipate that the broad dynamics would be analogous: global transfer of kinetic and potential energy towards an equipartition, and dissipative loss towards the energy minimum. A preliminary study is presented in appendix C.

For the case $\mu = 0$, the dynamics remain fully conservative, and individual Fourier modes exhibit oscillatory exchange between kinetic and potential energy, analogous to harmonic motion (see equation (5.4) and figures 4 and 5). When considering spectrally integrated (or ensemble-averaged) quantities, an increase in kinetic energy from its initial value is accompanied by a corresponding decrease in potential energy (figure 6). This should not be interpreted as a monotonic transfer at the level of individual modes, but rather as a redistribution across wavenumbers. In particular, different k -modes oscillate with distinct frequencies and phases, and the resulting phase mixing suppresses coherent oscillations in integrated quantities. In this

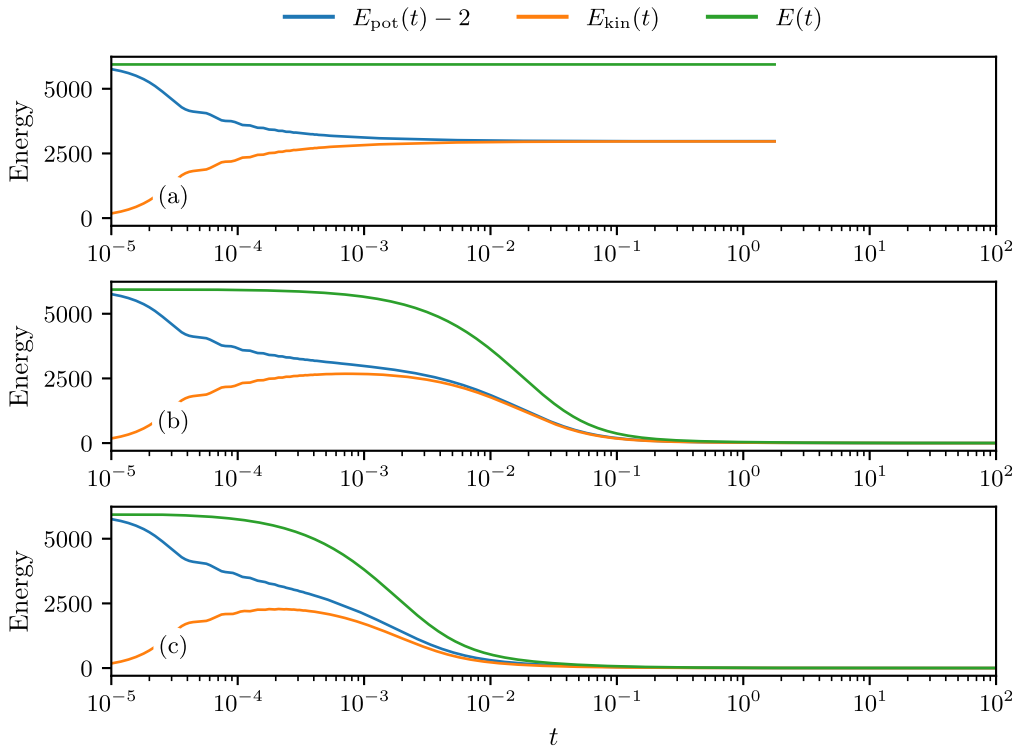


Figure 6. Excess potential, kinetic and total energies as a function of time, where the excess potential is measured relative to the minimum potential of 2 in steady state. The dissipation values used for μ are $\{0, 0.005, 0.05\}$ top to bottom (a)–(c). Rectangular initial data are in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$.

sense, the observed increase in kinetic energy corresponds to a net transfer of spectral weight towards modes closer to $k = 1$, even though the underlying dynamics remains oscillatory and non-dissipative at the level of each mode.

6. Conclusions

When the ends of a thin elastic sheet are pushed together, the elastic sheet typically buckles out of plane, accommodating its end-to-end displacement while preserving its arclength. It is well-known that for small strains, the resulting buckling pattern takes a sinusoidal form, with a wavelength determined by a competition between bending stiffness, which resists curvature, and restoring stiffnesses, which resist displacement. Examples of resistive stiffnesses include the buoyancy or elastic stiffness of a liquid or soft solid substrate, inertia and elastic contributions from the sheet itself associated with transverse tension or curvature. In particular, when the resistive stiffness is due to buoyancy, wrinkles have an equilibrium wavelength controlled by density and gravity [16–18], while when the resistive stiffness is due to inertia, wrinkle wavelengths increase inexorably in time [19,29]. In reality, these correspond to early- and late-time limits, and inertia will eventually give way to buoyancy; this is the research gap addressed by this work. Here, we have presented a study of wrinkles in a uniaxially compressed floating sheet, evolving from an initial out-of-plane perturbation, through early-time inertia-dominated growth, to late-time gravity-controlled equilibrium. During model development, we find that dissipation is required for the system to reach the expected late-time gravity-dominated equilibrium, and we also find that the governing equations can only be satisfied by a spectrum of wrinkle modes—a single mode with an evolving wavelength is not permitted mathematically. With these findings

to hand, we consider an infinitely long sheet on an infinite low-viscosity bath and decompose the wrinkle profile in the thin elastic sheet into Fourier modes, whose amplitudes evolve in time under the combined action of compression, inertia and gravity, and are coupled by a geometrically non-linear length constraint. Upon numerical solution of the amplitude equations, our model captures simultaneous coarsening of the wrinkles and collapse of the compressive stress in the system: the peak of the wrinkle spectrum shifts as wrinkles of different wavelengths grow and decay. At late time, the system reaches an energetically favourable steady state at a single buoyancy-selected wavenumber, provided that there is a mechanism for excess energy to be dissipated from the system. Our model therefore provides a bridge between the early-time and steady-state solutions presented in [16–19] and [29].

While the use of a spectrum is consistent with previous early-time analysis [29], we go beyond this previous analysis in a number of ways: (i) extension beyond early time to include late-time equilibration, (ii) inclusion of dissipation to facilitate this equilibration, (iii) tracking of energy in the system, and (iv) consideration of more than one way of quantifying an effective wavenumber for the system. For point (iv), we changed from tracking the peak of $a(k, t)$ (cf. [29]) to tracking the peak of $ka(k, t)$; the former choice is natural for thinking about visual appearance, while the latter allows for consideration of the length allocated to each mode. The observation of broad wrinkling spectra then motivated the introduction of an alternative metric $k_{\text{dom}} = \langle k \rangle$, the average value of k in the spectrum a . We view this not as a replacement for tracking the peak but rather as a context-appropriate supplement.

Dissipation was included in the model in the form of a small-but-finite viscous contribution from the bath of liquid that the sheet floats on. Here, we adapted an existing analysis of surface waves in a fluid, invoking a Helmholtz decomposition to resolve the flow in the underlying fluid bath and hence derive a viscous drag on the sheet which acts in opposition to its vertical velocity. We found that viscous dissipation narrows the spectrum of excited wrinkling modes: each mode oscillates with a frequency that increases with distance from the spectrum's peak, and so modes in the tail of the amplitude distribution are affected disproportionately by drag. It is therefore surprising that our results also suggest that both the position of the peak of the spectrum and the compressive force in the sheet evolve along a path that is independent of the strength of dissipation. We hope this intriguing finding will stimulate further study of dynamic wrinkle coarsening.

Focusing on energy, we observed that evolution of the wrinkle spectrum is closely tied to the transfer of energy, as expected. This transfer occurs both between potential and kinetic energy and between modes. Any initial amplitude that is not a narrow peak at $k = 1$ has an excess potential energy, which induces motion and transfer to kinetic energy. With dissipation, this kinetic energy is continually removed and therefore so is excess potential until the steady state is reached. Without dissipation, the initial excess energy is shared between kinetic and potential and so a steady state is never reached.

The modelling in this paper was motivated by a thought experiment in which a floating elastic sheet is compressed uniaxially. We now consider restrictions on the applicability of our model to a real-world experimental realization. We initiate the buckling as an out-of-plane disturbance that obeys an inextensibility constraint, [equation \(2.13\)](#). In reality, this inextensibility constraint is only valid after a very-early-time window when the compressive force is sufficiently large to initiate centreline shortening that is comparable to the length absorbed by out-of-plane wrinkling. For a 100 nm thick sheet of polystyrene, floating on water and compressed with 1% strain, this would occur when the residual stress F computed here is of order 10^6 . A second concern might be that, at very early time, viscous effects could dominate at very short length scales. However, the shrinking of length scales is counterbalanced by the fact that velocities grow very large at early times, and so if the [inequality \(2.3\)](#) holds, then viscous effects will remain small even at early time, although the model will break down once time is so short that the application of the end-to-end displacement can no longer be considered instantaneous. Finally, if the imposed end-to-end

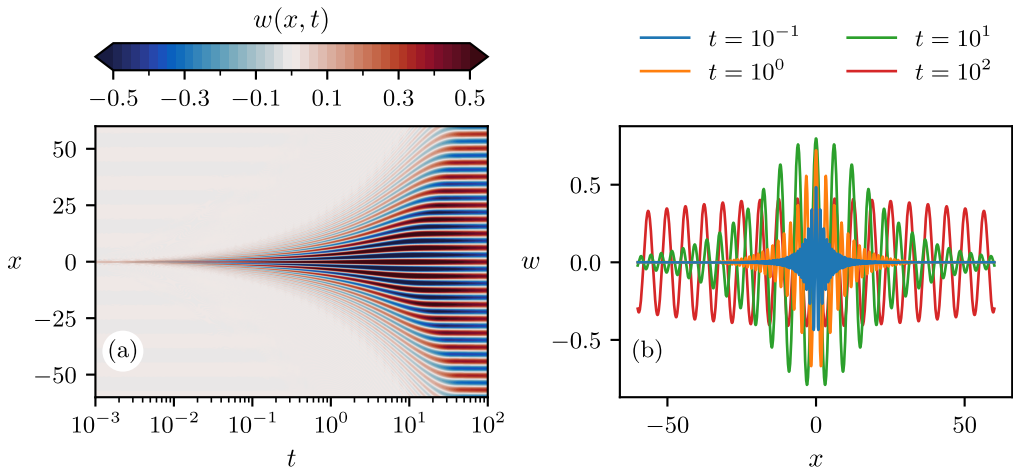


Figure 7. (a): heat map of wrinkle profiles in space (vertical axis), evolving in time (horizontal axis). (b): wrinkle profiles in space at chosen times. The dissipation value used for μ is 0.05. Rectangular initial data are in the form of equation (4.14) with $\alpha = 50$ and $\beta = 49.5$.

displacement is too large, then the system will transform from regular sinusoidal wrinkles to folds via a secondary instability. For quasi-static wrinkling, this can certainly be avoided by ensuring that the fractional end-shortening Δ^*/L^* is below $(\lambda^*/L^*)^2$ [44]. For dynamic wrinkling, it remains an open question whether a sheet might initially attempt to fold at very early times, before forming short-wavelength regular wrinkles that coarsen in line with the theory presented here, although inertia would certainly oppose such early-time folding.

Finally, we can reconstruct the wrinkle profile in physical space from the amplitude $a(k, t)$. We do this in figure 7, which shows the spatial wrinkles and their evolution through time. The results nicely demonstrate the effects discussed in terms of energy and modes, above. In particular, initially fine-scale, localized wrinkles coarsen throughout the dynamics to the equilibrium configuration with dimensionless wavenumber 1 (wavelength 2π), and this coarsening is supplemented by an amplitude increase to maintain the arclength of the sheet centreline, as expected.

Data accessibility. This article has no additional data.

Declaration of AI use. We have not used AI-assisted technologies in creating this article.

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Appendix A. Derivation of the fluid-mechanical equations

A.1. Governing equations

The fluid flow is governed by the Navier–Stokes equations, which in the absence of nonlinear fluid inertia are given by

$$\nabla^* \cdot \mathbf{v}^* = 0, \quad (\text{A } 1)$$

$$\rho^* \frac{\partial \mathbf{v}^*}{\partial t^*} = -\nabla^* p^* + \mu^* \nabla^{*2} \mathbf{v}^* - \rho^* g^* \mathbf{k}, \quad (\text{A } 2)$$

where $\mathbf{v}^* = v_x^* \mathbf{i} + v_z^* \mathbf{k}$ is the fluid velocity field, t^* is the time, p^* is the hydrodynamic pressure, ρ^* is the fluid density, μ^* is the dynamic viscosity, g^* is the force owing to gravity and ∇^* is the spatially two-dimensional gradient operator.

There are two boundary conditions to be imposed at the interface between the beam and the fluid. The first is the usual kinematic condition

$$v_z^* = \frac{\partial w^*}{\partial t^*} + v_x^* \frac{\partial w^*}{\partial x^*}, \quad \text{at} \quad z^* = w^*. \quad (\text{A } 3)$$

The second condition enforces stress continuity at the fluid–beam interface, and is given by

$$(-p^* I_2 + 2\mu^* e^*) \cdot \mathbf{n} = \sigma^* \cdot \mathbf{n}, \quad \text{at} \quad z^* = w^*, \quad (\text{A } 4)$$

where I_2 is the 2×2 identity matrix, e^* is the rate-of-strain tensor, σ^* is the Cauchy stress tensor of the sheet and $\mathbf{n}$ is the unit normal, which by convention is positive when oriented outwards relative to the fluid.

The small-amplitude displacement regime considered here means that the unit normal points predominantly in the $\mathbf{k}$ direction and hence the normal and tangential components of the stress balance (A 4) are given by

$$-p^* + 2\mu^* \frac{\partial v_z^*}{\partial z^*} = \sigma_{22}^*, \quad \text{at} \quad z^* = w^*, \quad (\text{A } 5)$$

and

$$\mu^* \left(\frac{\partial v_x^*}{\partial z^*} + \frac{\partial v_z^*}{\partial x^*} \right) = \sigma_{12}^*, \quad \text{at} \quad z^* = w^*. \quad (\text{A } 6)$$

We discuss far-field boundary and initial conditions on the fluid velocity in the main body of the paper.

A.2. Potential flow

We now resolve the fluid velocity field into irrotational and solenoidal parts using the Helmholtz decomposition: [34]

$$\mathbf{v}^* = \nabla^* \phi^* + \nabla^* \times \boldsymbol{\psi}^* = \left(\frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial z^*} \right) \mathbf{i} + \left(\frac{\partial \phi^*}{\partial z^*} + \frac{\partial \psi^*}{\partial x^*} \right) \mathbf{k}, \quad (\text{A } 7)$$

where $\boldsymbol{\psi} = \psi^* \mathbf{j}$. Substituting equation (A 7) into equation (A 1), it follows that ϕ^* satisfies Laplace's equation

$$\nabla^{*2} \phi^* = 0. \quad (\text{A } 8)$$

Similarly, on substituting [equation \(A7\)](#) into [equation \(A2\)](#) and making use of [equation \(A8\)](#), we obtain the following pair of equations:

$$\rho^* \frac{\partial}{\partial t^*} \left(\frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial z^*} \right) = -\frac{\partial p^*}{\partial x^*} - \mu^* \frac{\partial}{\partial z^*} \left(\nabla^{*2} \psi^* \right) \quad (\text{A9})$$

$$\rho^* \frac{\partial}{\partial t^*} \left(\frac{\partial \phi^*}{\partial z^*} + \frac{\partial \psi^*}{\partial x^*} \right) = -\frac{\partial p^*}{\partial z^*} + \mu^* \frac{\partial}{\partial x^*} \left(\nabla^{*2} \psi^* \right) - \rho^* g^*. \quad (\text{A10})$$

By differentiating [equation \(A9\)](#) in z^* and [equation \(A10\)](#) in x^* , subtracting and then integrating, one can deduce that ψ^* satisfies

$$\mu^* \nabla^{*2} \psi^* = \rho^* \frac{\partial \psi^*}{\partial t^*}, \quad (\text{A11})$$

where it is noted that by virtue of having integrated in two spatial dimensions, one has the freedom to apply a time and/or spatially linear gauge transformation to ψ^* , with a corresponding transformation also required to be imposed on the velocity potential ϕ^* for consistency with [equation \(A7\)](#). Since the resultant fluid velocity v^* is invariant to any gauge transformation of this form, we proceed by taking [equation \(A11\)](#) in isolation. We also note that it is unsurprising that ψ^* satisfies the diffusion equation, with the diffusion coefficient equal to the kinematic viscosity.

By considering [equation \(A10\)](#), making use of [equation \(A11\)](#) and then integrating, it is found that

$$\rho^* \frac{\partial \phi^*}{\partial t^*} = -p^* - \rho^* g^* z^*, \quad (\text{A12})$$

where the arbitrary constant has been absorbed into the potential ϕ^* . In terms of the potentials ϕ^* and ψ^* , the kinematic equation is given by

$$\frac{\partial \phi^*}{\partial z^*} + \frac{\partial \psi^*}{\partial x^*} = \frac{\partial w^*}{\partial t^*} + \left(\frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial z^*} \right) \frac{\partial w^*}{\partial x^*}, \quad \text{at} \quad z^* = w^*. \quad (\text{A13})$$

Using [equation \(A11\)](#), the tangential component of the stress condition, [equation \(A6\)](#), is given by

$$2\mu^* \left(\frac{\partial^2 \phi^*}{\partial z^* \partial x^*} - \frac{\partial^2 \psi^*}{\partial z^{*2}} \right) + \rho^* \frac{\partial \psi^*}{\partial t^*} = \sigma_{12}^*. \quad (\text{A14})$$

Finally, eliminating the hydrodynamic pressure between [equation \(A12\)](#) and the normal-stress balance, [equation \(A5\)](#), gives to leading order

$$\rho^* \frac{\partial \phi^*}{\partial t^*} + \rho^* g^* w^* + 2\mu^* \left(\frac{\partial^2 \phi^*}{\partial z^{*2}} + \frac{\partial^2 \psi^*}{\partial z^* \partial x^*} \right) = \sigma_{22}^*, \quad \text{at} \quad z^* = w^*. \quad (\text{A15})$$

When combined with appropriate far-field fluid-velocity conditions, [equations \(A8\)](#), [\(A11\)](#) and [\(A13\)–\(A15\)](#) constitute the fluid-mechanical governing system under consideration.

Appendix B. Energetics

In this appendix, we derive an expression for the energy budget of the system [equations \(3.16\)](#) and [\(3.17\)](#). We begin by first taking the sum of the product between $\partial \bar{a}/\partial t$ and [equation \(3.16\)](#), and the product between $\partial a/\partial t$ and the conjugate of [equation \(3.16\)](#), to obtain

$$\frac{1}{|k|} \left(\frac{\partial \bar{a}}{\partial t} \frac{\partial^2 a}{\partial t^2} + \frac{\partial a}{\partial t} \frac{\partial^2 \bar{a}}{\partial t^2} \right) + 4\mu |k| \frac{\partial a}{\partial t} \frac{\partial \bar{a}}{\partial t} + (k^4 + Fk^2 + 1) \left(\frac{\partial \bar{a}}{\partial t} a + \frac{\partial a}{\partial t} \bar{a} \right) = 0. \quad (\text{B1})$$

Using the product rule for differentiation, [equation \(B 1\)](#) may be written as

$$\frac{1}{2|k|} \frac{\partial}{\partial t} \left| \frac{\partial a}{\partial t} \right|^2 + 2\mu |k| \left| \frac{\partial a}{\partial t} \right|^2 + \frac{1}{2} (k^4 + Fk^2 + 1) \frac{\partial}{\partial t} |a|^2 = 0. \quad (\text{B } 2)$$

Integrating [equation \(B 2\)](#) over all wavenumbers $k \in \mathbb{R}$, exchanging the order of time derivatives with the integrals and rearranging, it follows that

$$\frac{d}{dt} \left[\int_{-\infty}^{\infty} \frac{1}{2|k|} \left| \frac{\partial a}{\partial t} \right|^2 + \frac{1}{2} (k^4 + 1) |a|^2 dk \right] = -2\mu \int_{-\infty}^{\infty} |k| \left| \frac{\partial a}{\partial t} \right|^2 dk, \quad (\text{B } 3)$$

where the term involving F has vanished by virtue of the integral constraint [equation \(3.17\)](#).

[Equation \(B 3\)](#) is an energy budget¹ for the system [equations \(3.16\)](#) and [\(3.17\)](#). The left-hand side of [equation \(B 3\)](#) represents the change in total energy within the system. From left to right, the terms in the integrand represent the modal contributions to: the kinetic energy (density times the square of velocity), the bending cost of the beam in response to the fluid and the substrate cost (cost of moving the fluid surface against pressure).

The absence of the compressive force in [equation \(B 3\)](#) is a reflection that—by inextensibility—there is no energetic penalty associated with a change in beam length. The right-hand side of [equation \(B 3\)](#) represents losses in energy owing to viscous dissipation. We refer to the terms within the integrand of [equation \(B 3\)](#), which are given by

$$\mathcal{E}_{\text{kin}}(k, t) = \frac{1}{2|k|} \left| \frac{\partial a}{\partial t} \right|^2 \quad \text{and} \quad \mathcal{E}_{\text{pot}}(k, t) = \frac{1}{2} (k^4 + 1) |a|^2, \quad (\text{B } 4)$$

as the kinetic and potential energy densities (energy per unit wavenumber). The sum of these energy components defines the total energy density

$$\mathcal{E}(k, t) = \mathcal{E}_{\text{kin}}(k, t) + \mathcal{E}_{\text{pot}}(k, t). \quad (\text{B } 5)$$

The system's total kinetic and potential energy may then be obtained by integrating over all wavenumbers:

$$E_{\text{kin}}(t) = \int_{-\infty}^{\infty} \mathcal{E}_{\text{kin}} dk, \quad E_{\text{pot}}(t) = \int_{-\infty}^{\infty} \mathcal{E}_{\text{pot}} dk, \quad E(t) = \int_{-\infty}^{\infty} \mathcal{E} dk. \quad (\text{B } 6)$$

The energy budget, [equation \(B 3\)](#), can now be expressed more succinctly as

$$\frac{dE}{dt} = -2\mu \int_{-\infty}^{\infty} |k| \left| \frac{\partial a}{\partial t} \right|^2 dk. \quad (\text{B } 7)$$

Hence, when $\mu = 0$, the energy of the beam is conserved, as expected; otherwise, it is non-increasing with non-zero velocities leading to energy losses through viscous dissipation.

Appendix C. Initial conditions and starting velocities

C.1. Velocity initial conditions and projection procedure

When prescribing initial conditions with non-zero velocity, additional care is required to ensure consistency with the constraint [equation \(4.4\)](#), which enforces orthogonality between $a(k, 0)$ and

¹It is noted that it is possible to arrive at the energy budget [equation \(B 3\)](#) by means of an alternative *physical approach*, in which one can derive an energy equation for w before substituting in the Fourier decompositions, [equations \(3.11\)–\(3.13\)](#), from the main body.

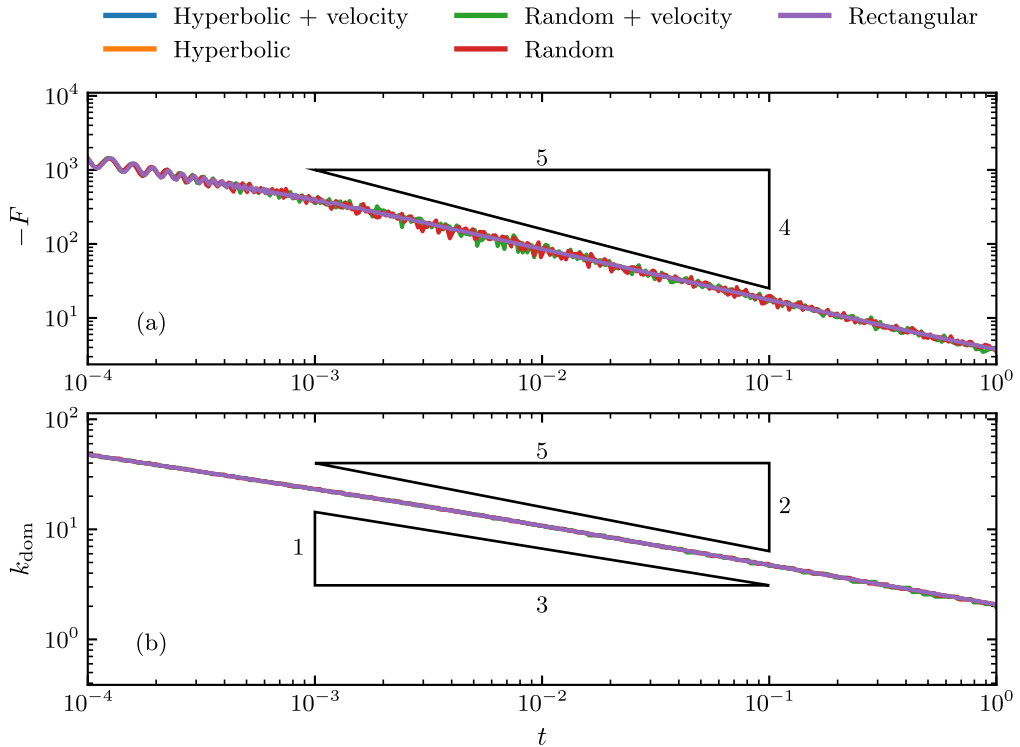


Figure 8. (a) The magnitude of the compressive force, $-F(t)$, (b) the dominant wavenumber k_{dom} , for the smoothed rectangle, random noise, rectangle, smoothed rectangle with appropriate velocity and noise with appropriate velocity where $\alpha = 50$, $\beta = 49.5$ and $\mu = 0.05$. The chosen starting velocities correspond to approximately 0.1% of the total kinetic energy.

$b(k, 0)$ under the weighted inner product

$$\langle f, g \rangle_{k^2} = \int_0^\infty k^2 f \bar{g} \, dk. \quad (\text{C } 1)$$

In general, a naïve choice of $b(k, 0)$ will not satisfy this condition and must therefore be corrected.

We apply an analytic Gram–Schmidt projection. Given an initial profile $a(k, 0)$ and a raw velocity $b_{\text{raw}}(k, 0)$ (i.e. some function that is linearly independent of $a(k, 0)$), we define

$$b(k, 0) = b_{\text{raw}}(k, 0) - \frac{\langle a, b_{\text{raw}} \rangle_{k^2}}{\langle a, a \rangle_{k^2}} a(k, 0), \quad (\text{C } 2)$$

which ensures that $\langle a, b \rangle_{k^2} = 0$, and hence that equation (4.4) is satisfied. In practice, the inner products are evaluated numerically using a trapezium rule on the k -grid. Following projection, the velocity field in equation (C 2) can be normalized to achieve an initial kinetic energy of appropriate amplitude while still satisfying equation (C 1).

C.2. Comparing numerical results for different initial conditions

We consider the following classes of initial conditions.

1. **Hyperbolic tangent (smoothed rectangle) profiles.** A smoothed rectangular spectrum is constructed using a pair of hyperbolic tangent functions,

$$a(k, 0) \propto \tanh(s(k - \alpha + \beta)) - \tanh(s(k - \alpha - \beta)), \quad (\text{C } 3)$$

where s controls the edge steepness. This provides a smooth approximation to the rectangular profile [equation \(4.14\)](#). For the corresponding velocity field, we take a derivative with respect to the centre parameter α , yielding a pair of localized edge contributions,

$$b_{\text{raw}}(k, 0) \propto -s \operatorname{sech}^2(s(k - \alpha + \beta)) + s \operatorname{sech}^2(s(k - \alpha - \beta)), \quad (\text{C } 4)$$

which is then projected using [equation \(C 2\)](#) and rescaled so that kinetic energy is approximately 0.1% of the total energy.

2. **Random initial conditions.** Random spectra are generated by drawing $a(k, 0)$ independently and uniformly from the interval $[-1/2, 1/2]$, with $a(0) = 0$ enforced. The resulting profile is subsequently normalized to satisfy the length constraint [equation \(4.3\)](#). We also draw $b_{\text{raw}}(k, 0)$ independently and uniformly from $[-1/2, 1/2]$, before applying the projection step [equation \(C 2\)](#) and renormalizing.

For the initial conditions 1 and 2 above, plus the rectangular case presented in the main body of the paper, we simulate dynamics up to $t = 1.0$ with $\alpha = 50$, $\beta = 49.5$ and $\mu = 0.05$, in a broad k domain, $[0, 100]$. The results for k_{dom} and $-F(t)$ are presented in [figure 8](#). The strong agreement between all sets of initial conditions in [figure 8](#) highlights that the qualitative global dynamics is insensitive to the shape of initial spectral distribution, and insensitive to changes from rest to small starting velocities. For larger initial velocities, there is a technical challenge associated with finding non-zero initial velocities that are compatible with the constraints but also physically meaningful—we do not address that challenge in this paper.

References

1. Greiner A, Kaessmair S, Budday S. 2021 Physical aspects of cortical folding. *Soft Matter* **17**, 1210–1222. ([doi:10.1039/d0sm02209h](#))
2. Balbi V, Destrade M, Goriely A. 2020 Mechanics of human brain organoids. *Phys. Rev. E* **101**, 022403. ([doi:10.1103/PhysRevE.101.022403](#))
3. Karzbrun E, Kshirsagar A, Cohen SR, Hanna JH, Reiner O. 2018 Human brain organoids on a chip reveal the physics of folding. *Nat. Phys.* **14**, 515–522. ([doi:10.1038/s41567-018-0046-7](#))
4. Youn J, Kim D, Kwak H, Lee A, Kim DS. 2024 Tissue-scale in vitro epithelial wrinkling and wrinkle-to-fold transition. *Nat. Commun.* **15**, 7118. ([doi:10.1038/s41467-024-51437-z](#))
5. Edwards CM, Chapman SJ. 2007 Biomechanical modelling of colorectal crypt budding and fission. *Bull. Math. Biol.* **69**, 1927–1942. ([doi:10.1007/s11538-007-9199-8](#))
6. Rozman J, Krajnc M, Zihlerl P. 2021 Morphologies of compressed active epithelial monolayers. *Eur. Phys. J. E* **44**, 99. ([doi:10.1140/epje/s10189-021-00094-x](#))
7. Geisel S, Secchi E, Vermant J. 2022 The role of surface adhesion on the macroscopic wrinkling of biofilms. *Elife* **11**, e76027. ([doi:10.7554/eLife.76027](#))
8. Trejo M, Douarche C, Bailleux V, Poulard C, Mariot S, Regeard C, Raspaud E. 2013 Elasticity and wrinkled morphology of *Bacillus subtilis* pellicles. *Proc. Natl Acad. Sci. USA* **110**, 2011–2016. ([doi:10.1073/pnas.1217178110](#))
9. Fei C, Mao S, Yan J, Alert R, Stone HA, Bassler BL, Wingreen NS, Košmrlj A. 2020 Nonuniform growth and surface friction determine bacterial biofilm morphology on soft substrates. *Proc. Natl Acad. Sci. USA* **117**, 7622–7632. ([doi:10.1073/pnas.1919607117](#))
10. Harrison C, Stafford CM, Zhang W, Karim A. 2004 Sinusoidal phase grating created by a tunably buckled surface. *Appl. Phys. Lett.* **85**, 4016–4018. ([doi:10.1063/1.1809281](#))
11. Lee E, Zhang M, Cho Y, Cui Y, Van der Spiegel J, Engheta N, Yang S. 2014 Tilted pillars on wrinkled elastomers as a reversibly tunable optical window. *Adv. Mater.* **26**, 4127–4133. ([doi:10.1002/adma.201400711](#))
12. Chung JY, Youngblood JP, Stafford CM. 2007 Anisotropic wetting on tunable micro-wrinkled surfaces. *Soft Matter* **3**, 1163–1169. ([doi:10.1039/b705112c](#))
13. Rogers JA, Huang Y. 2009 A curvy, stretchy future for electronics. *Proc. Natl Acad. Sci. USA* **106**, 10875–10876. ([doi:10.1073/pnas.0905723106](#))

14. Vandeparre H, Gabriele S, Brau F, Gay C, Parker KK, Damman P. 2010 Hierarchical wrinkling patterns. *Soft Matter* **6**, 5751–5756. (doi:10.1039/C0SM00394H)
15. Tobasco I, Timounay Y, Todorova D, Leggat GC, Paulsen JD, Katifori E. 2022 Exact solutions for the wrinkle patterns of confined elastic shells. *Nat. Phys.* **18**, 1099–1104. (doi:10.1038/s41567-022-01672-2)
16. Landau LD, Lifshitz EM. 2012 *Theory of elasticity: volume 7*, 3rd edn. Amsterdam, The Netherlands: Elsevier.
17. Cerda E, Mahadevan L. 2003 Geometry and physics of wrinkling. *Phys. Rev. Lett.* **90**, 074302. (doi:10.1103/PhysRevLett.90.074302)
18. Paulsen JD, Hohlfeld E, King H, Huang J, Qiu Z, Russell TP, Menon N, Vella D, Davidovitch B. 2016 Curvature-induced stiffness and the spatial variation of wavelength in wrinkled sheets. *Proc. Natl Acad. Sci. USA* **113**, 1144–1149. (doi:10.1073/pnas.1521520113)
19. Box F, O’Kiely D, Kodio O, Inizan M, Castrejón-Pita AA, Vella D. 2019 Dynamics of wrinkling in ultrathin elastic sheets. *Proc. Natl Acad. Sci. USA* **116**, 20875–20880. (doi:10.1073/pnas.1905755116)
20. Box F, Kodio O, O’Kiely D, Cantelli V, Goriely A, Vella D. 2020 Dynamic buckling of an elastic ring in a soap film. *Phys. Rev. Lett.* **124**, 198003. (doi:10.1103/PhysRevLett.124.198003)
21. Kodio O, Griffiths I, Vella D. 2017 Lubricated wrinkles: imposed constraints affect the dynamics of wrinkle coarsening. *Phys. Rev. Fluids* **2**, 014202. (doi:10.1103/PhysRevFluids.2.014202)
22. Draux A, Rizzo M, Vella D, Démery V, Brau F, Damman P. 2026 Universal wrinkling dynamics of a sheet on viscous liquid. <https://doi.org/10.48550/arXiv.2601.11682>.
23. Chatterjee S, McDonald C, Niu J, Velankar SS, Wang P, Huang R. 2015 Wrinkling and folding of thin films by viscous stress. *Soft Matter* **11**, 1814–1827. (doi:10.1039/c4sm02501f)
24. Guan X, Nguyen N, Cerda E, Pocivavsek L, Velankar SS. 2023 Ridge localization driven by wrinkle packets. *Soft Matter* **19**, 9206–9214. (doi:10.1039/d3sm01287e)
25. Guan X, Nguyen N, Pocivavsek L, Cerda E, Velankar SS. 2023 Flat, wrinkled, or ridged: relaxation of an elastic film on a viscous substrate undergoing continuous compression. *Int. J. Solids Struct.* **275**, 112242. (doi:10.1016/j.ijsolstr.2023.112242)
26. Vella D, Huang J, Menon N, Russell TP, Davidovitch B. 2015 Indentation of ultrathin elastic films and the emergence of asymptotic isometry. *Phys. Rev. Lett.* **114**, 014301. (doi:10.1103/PhysRevLett.114.014301)
27. Vella D, Davidovitch B. 2018 Regimes of wrinkling in an indented floating elastic sheet. *Phys. Rev. E* **98**, 013003. (doi:10.1103/PhysRevE.98.013003)
28. Ripp MM, Démery V, Zhang T, Paulsen JD. 2020 Geometry underlies the mechanical stiffening and softening of an indented floating film. *Soft Matter* **16**, 4121–4130. (doi:10.1039/d0sm00250j)
29. O’Kiely D, Box F, Kodio O, Whiteley J, Vella D. 2020 Impact on floating thin elastic sheets: a mathematical model. *Phys. Rev. Fluids* **5**, 014003. (doi:10.1103/PhysRevFluids.5.014003)
30. Vermorel R, Vandenberghe N, Villiermaux E. 2007 Rubber band recoil. *Proc. R. Soc. A* **463**, 641–658. (doi:10.1098/rspa.2006.1781)
31. Vermorel R, Vandenberghe N, Villiermaux E. 2009 Impacts on thin elastic sheets. *Proc. R. Soc. A* **465**, 823–842. (doi:10.1098/rspa.2008.0297)
32. Duchemin L, Vandenberghe N. 2014 Impact dynamics for a floating elastic membrane. *J. Fluid Mech.* **756**, 544–554. (doi:10.1017/jfm.2014.471)
33. Kodio O, Goriely A, Vella D. 2020 Dynamic buckling of an inextensible elastic ring: linear and nonlinear analyses. *Phys. Rev. E* **101**, 053002. (doi:10.1103/PhysRevE.101.053002)
34. Dias F, Dyachenko AI, Zakharov VE. 2008 Theory of weakly damped free-surface flows: a new formulation based on potential flow solutions. *Phys. Lett. A* **372**, 1297–1302. (doi:10.1016/j.physleta.2007.09.027)
35. O’Kiely D, Xin M, Lecointre P, Vella D. 2025 Wrinkling of a bilayer with spatially varying stiffness: from wrinkle branching to cascades. *Proc. R. Soc. A* **481**, 20240842. (doi:10.1098/rspa.2024.0842)
36. Taffetani M, Vella C. 2017 Regimes of wrinkling in pressurized elastic shells. *Phil. Trans. R. Soc. A* **375**, 20160330. (doi:10.1098/rsta.2016.0330)
37. Gelfand IM, Fomin SV. 1963 *Calculus of variations*. Englewood Cliffs, NJ: Prentice-Hall.
38. Lanczos C. 1970 *The variational principles of mechanics*, 4th edn. Toronto, Canada: University of Toronto Press.

39. Pignataro M, Rizzi N, Luongo A. 1991 *Stability, bifurcation and postcritical behaviour of elastic structures*. Amsterdam, The Netherlands: Elsevier.
40. Gear C. 1971 Simultaneous numerical solution of differential-algebraic equations. *IEEE Trans. Circuit Theory* **18**, 89–95. (doi:10.1109/TCT.1971.1083221)
41. Campbell SL, Linh VH, Petzold LR. 2008 Differential-algebraic equations. *Scholarpedia* **3**, 2849. (doi:10.4249/scholarpedia.2849).
42. Ascher UM, Petzold LR. 1998 *Computer methods for ordinary differential equations and differential-algebraic equations*. Philadelphia, PA: SIAM.
43. Campbell SL, Gear CW. 1995 The index of general nonlinear DAEs. *Numer. Math.* **72**, 173–196. (doi:10.1007/s002110050165)
44. Oshri O, Brau F, Diamant H. 2015 Wrinkles and folds in a fluid-supported sheet of finite size. *Phys. Rev. E* **91**, 052408. (doi:10.1103/PhysRevE.91.052408)