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Author for correspondence:
Emilian I. Părău
e-mail: E.Parau@uea.ac.uk

Travelling hydroelastic waves in constant vorticity flows with stagnation points

Călin-Iulian Martin[1] and Emilian I. Părău[2]

[1]Faculty of Mathematics and Computer Science, Babes-Bolyai University, Mihail Kogalniceanu Street 1, Cluj-Napoca, Romania
[2]School of Engineering, Mathematics and Physics, University of East Anglia, Norwich NR4 7TJ, UK

EIP, 0000-0001-5134-2068

We present a bifurcation approach which delivers two-dimensional travelling hydroelastic water waves propagating at the free surface of a rotational ideal fluid of constant vorticity and finite depth, covered by a thin ice sheet which is modelled by using the special Cosserat theory of hyperelastic shells satisfying Kirchhoff's hypothesis. The approach is based on a reformulation of the water wave problem as a pseudodifferential equation for a function of one variable, giving the elevation of the free surface (allowed to have overhanging profiles). Moreover, the involved method permits the existence of stagnation points whose existence in the resulting solution flows is then proved rigorously.

1. Introduction

During the last decades, hydroelasticity—the study of elastic body deformations induced by hydrodynamic forces and the reciprocal influence of these deformations on the flow—has seen significant advances (see [1] for a review). In particular, the propagation of hydroelastic waves beneath ice covers has received considerable mathematical attention. As noted in [2], these waves are especially relevant in cold regions (e.g. Canada, Russia, Alaska and Antarctica), where frozen surfaces serve as transport routes or runways, and where air-cushioned vehicles are used to break the ice.

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Most of the existing literature is concerned with hydroelastic waves on elastic plates on top of inviscid fluids and irrotational flows. However, there is substantial evidence of currents and vorticity under floating ice. For example, Menemenlis & Farmer [3] measured the velocity of currents and the vorticity under the eastern Arctic Ocean. A recent analytical investigation of near-surface ocean currents near Antarctica was performed by Abrashkin & Constantin [4].

In the same vein, several measurements of horizontal and vertical currents at various depths under the flat thin ice were performed in the shallow Lake Saroma (or Saroma-ko Lagoon), which is located on the Okhotsk Sea coast of Hokkaido, Japan [5,6]. In this lagoon hydroelastic waves generated by moving loads (snowmobiles) were observed in field experiments performed in 1981 by Takizawa [7,8].

Some studies have explored linear hydroelastic waves in flows with vorticity from various perspectives. Marchenko & Prokhorov [9] identified several types of travelling hydroelastic waves with the same frequency on the surface of a finite-depth fluid with constant vorticity, covered by a thin elastic plate. They derived the dispersion relation for this case and examined wave diffraction at the edge of a semi-infinite elastic plate over such a fluid. Also, Bhattacharjee & Sahoo [10] investigated the effects of shear currents in finite-depth fluids on linear hydroelastic waves within the shallow-water approximation.

More recently, asymptotic and nonlinear results have been obtained for fully nonlinear hydroelastic waves in the presence of a constant vorticity in the underlying fluid. Gao *et al*. [11] computed travelling solitary hydroelastic waves for different vorticities in infinite-depth fluids and examined their stabillity using fully nonlinear time-dependent computations based on conformal mappings. Gao *et al*. [12] derived nonlinear Schrödinger-type equations using multiple-scale analysis for weakly nonlinear hydroelastic waves on fluids of finite depth and constant vorticity, and investigated various resonances and the modulational instability phenomenon. They also calculated solitary waves and generalized solitary waves for the full nonlinear problem and performed time-dependent computations to study their dynamics and the Benjamin–Feir instability in this case. Wang *et al*. [13] used a third-order Stokes expansion and fully nonlinear computations to calculate travelling hydroelastic waves on a linear shear current in a finite-depth fluid and to study the global bifurcation of periodic waves of the fully nonlinear equations. Particle trajectories and the flow structures beneath the elastic plate were also computed.

The existence of exact solutions is based on (equivalent) reformulations of the problem over a fixed domain. Then, bifurcation theory in the spirit of Crandall–Rabinowitz [14] is a very effective tool to prove existence results pertaining to the water wave problem, cf. the pioneering approach by Constantin & Strauss [15], see also [16,17]. For useful reformulations used when constructing rotational water waves in a wide range of physical regimes, we point to other studies [18–23].

Regarding the utilization of the bifurcation method for proving exact solutions in the hydroelastic scenario, we refer the reader to other works [24–26]. The existence of solutions for the hydroelastic problem was also considered from a variational perspective, cf. [27,28] as well as by spatial dynamics techniques [29].

A common feature of the aforementioned approaches to hydroelastic waves is that they do not account for stagnation points—locations where the gradient of the stream function vanishes. Physically, these points correspond to fluid particles with zero vertical velocity and horizontal velocity equal to the wave speed. In a reference frame moving with the wave, such particles appear stationary, hence the term stagnation points. The approach we use here allows us to prove that constant vorticity yields the emergence of stagnation points (cf. condition equation (4.21) below), which is a phenomenon that gives rise to critical layers. The latter are closed streamlines—that is, closed level sets of the stream function—enclosing the stagnation points, which do not exist in the irrotational setting. The critical layers were anticipated by a formal argument of Kelvin [30] and proved rigorously by Wahlén [31], see also [32,33]. For insightful numerical studies concerning stagnation points and overhanging profiles we refer the reader to the recent paper by Doak *et al*. [34]. Other studies on flows displaying stagnation points can be found in other works [31,35–37].

We adapt to our setting a method used by Constantin & Vărvărucă [38] to produce local curves of solutions admitting overhanging profiles and stagnation points in the setting of gravity surface waves over flows of constant vorticity. The method from [38] was shown to bear relevance to the case of capilary–gravity water waves over flows with constant vorticity, cf. [23]. We also employ to our hydroelastic scenario the methods developed in [23] when applying the Crandall–Rabinowitz theorem in §4.

Note that the assumption of constant vorticity is not merely a mathematical convenience but is physically relevant in a range of scenarios. For example, constant vorticity is meaningful when the waves are long compared with the water depth since in this case the existence of a non-zero mean vorticity is more important than its specific distribution, cf. the discussions in [39,40]. Moreover, tidal flows which are the most regular and predictable currents [41], are considered to be two-dimensional flows of constant vorticity: they feature negative constant vorticity when the flow of the current is directed towards the shore (flood) and positive constant vorticity when the current is directed out to the sea (ebb) [40].

The global bifurcation picture in the constant vorticity setting was presented by Constantin *et al*. [39,42], while the case with general vorticity was treated in [36]. The approach by Constantin & Vărvărucă [38], utilized first for the problem of waves on flows of infinite depth [43–45], uses the Babenko [46] reformulation of the water wave problem as a pseudodifferential equation for a function of one variable, giving the elevation of the free surface when the fluid domain is the conformal image of a half-plane.

Central to the bifurcation approach employed in this paper are the periodic Dirichlet–Neumann operator and the periodic Hilbert transform associated with a strip. These tools play a pivotal role in our analysis. The significance of the Dirichlet–Neumann operators is underscored by the surge of interest in recent years in both the full nonlinear water wave problem and various approximate nonlinear water wave models. This progress is driven by a combination of methods focussing on Dirichlet–Neumann operators and Hamiltonian systems, cf. [47–60], as well as methods for numerical computation of quantities, expressed by Dirichlet–Neumann operators [61–63], following the pioneer work of [64].

Following the presentation of the physical problem in §2, we provide in §3 a suitable reformulation of governing equations as a quasilinear pseudodifferential equation for a periodic function of one variable representing the free surface. The analysis is concluded in §4 by proving the existence of exact solutions, describing steady periodic hydroelastic waves, of (conformal) mean depth h (for some constant $h > 0$), and possessing a smooth profile with one crest and one trough per period, monotonic between crests and troughs and symmetric about any crest line.

2. The presentation of the problem

The object of interest is the free-boundary problem of steady periodic travelling hydroelastic waves with constant vorticity in flows of finite depth over a flat bed. We denote with (X, Y) the independent variables, where X stands for the direction of propagation, and Y represents the height variable. The water flow domain Ω in the XY-plane is bounded below by the impermeable flat bed $B = \{(X, 0); X \in R\}$, and above by an *a priori* unknown curve $S(t) = \{(u(t, s), v(t, s)); s \in \mathbb{R}\}$, for all $t \geq 0$, with

$$(u_s(t,s))^2 + (v_s(t,s))^2 > 0 \quad \text{for all } s \in \mathbb{R}, \quad t \geq 0 \tag{2.1}$$

and

$$u(t, s+L) = u(t,s) + L, \quad v(t, s+L) = v(t,s) \quad \text{for all } s \in \mathbb{R}. \tag{2.2}$$

The curve S represents the *a priori* unknown free surface of the water domain, is L-periodic in the X direction and admits overhanging profiles. Moreover, the free surface is considered as a thin elastic plate satisfying Kirchhoff's hypothesis, cf. [65–67]. (see [68,69] for a related model).

In addition, we assume that the water flow domain $\Omega \subset \mathbb{R}^2$ to be an *L-periodic strip-like domain* which means that Ω is contained in the upper half (X, Y)-plane and that its boundary is built of the real axis B and of a parametric curve S satisfying equations (2.1) and (2.2). If for some $d > 0$, we denote with $\mathcal{R}_d = \{(x,y) \in \mathbb{R}^2 : -d < y < 0\}$ the horizontal strip of height d, then for the L-periodic strip-like domain Ω specified earlier, its *conformal mean-depth* is defined to be the unique positive number h such that there exists an onto conformal mapping $U_h + iV_h : \mathcal{R}_h \to \Omega$ which can be extended to the closures of the domains $\mathcal{R}_h$ and Ω, respectively, with onto mappings

$$\{(x,0) : x \in \mathbb{R}\} \to S \quad \text{and} \quad \{(x,-h) : x \in \mathbb{R}\} \to B,$$

and such that

$$U_h(x+L,y) = U_h(x,y) + L \quad \text{and} \quad V_h(x+L,y) = V_h(x,y) \quad \text{for all } (x,y) \in \mathcal{R}_h.$$

Since under consideration are steady periodic waves travelling with speed $c > 0$ we make (as customary) the substitution $x = X - ct$, $y = Y$ and so the free surface S can be described by an implicit equation of the form $\mathcal{S}(x,y) = 0$, for some diffferentiable function $\mathcal{S}$. Denoting with $(\mathcal{U},\mathcal{V})$ the velocity field, with $\mathcal{P}$ the pressure, with g the acceleration of gravity and considering a homogeneous flow of normalized density, the equations of motion are the momentum conservation equations,

$$\left.\begin{aligned} &(\mathcal{U}-c)\mathcal{U}_x + \mathcal{V}\mathcal{U}_y = -\mathcal{P}_x \\ \text{and} \qquad &(\mathcal{U}-c)\mathcal{V}_x + \mathcal{V}\mathcal{V}_y = -\mathcal{P}_y - g, \end{aligned}\right\} \tag{2.3}$$

and the equation of mass conservation

$$\mathcal{U}_x + \mathcal{V}_y = 0, \tag{2.4}$$

which are complemented by the kinematic boundary conditions,

$$\left.\begin{aligned} &\mathcal{S}_x(\mathcal{U}-c) + \mathcal{S}_y\mathcal{V} = 0 \quad \text{on } S \\ \text{and} \qquad &\mathcal{V} = 0 \quad \text{on } B, \end{aligned}\right\} \tag{2.5}$$

and—owing to the Cosserat's theory of hyperelastic shells—by the dynamic condition

$$\mathcal{P} = \mathcal{D}\left(\kappa_{\mathfrak{s}\mathfrak{s}} + \frac{1}{2}\kappa^3\right) \quad \text{on } S, \tag{2.6}$$

where $\mathcal{D} = Ed^3/12(1-\nu^2)$ is a constant representing the coefficient of the flexural rigidity of the ice sheet, d is the ice thickness, E denotes Young's modulus, ν is the Poisson's ratio for ice, κ is the curvature of the fluid-ice interface, $\mathfrak{s}$ represents the arclength along this interface and $\kappa_{\mathfrak{s}\mathfrak{s}}$ denoes the second derivative of κ with respect to the arclength, that is $\kappa_{\mathfrak{s}\mathfrak{s}} = (1/\sqrt{u_s^2 + v_s^2})\partial_s((1/\sqrt{u_s^2 + v_s^2})\partial_s\kappa)$. From the equation of mass conservation we infer the existence of the stream function ψ which recovers the velocity field as

$$\nabla\psi = (-\mathcal{V}, \mathcal{U} - c).$$

Furthermore, setting

$$m = \int_0^{v(t,s_0)} (\mathcal{U}(u(t,s_0) - ct, y) - c)\,\mathrm{d}y,$$

(with $u(t,s_0) - ct, v(t,s_0))$ being the location of the wave trough) we have that m is a constant, cf. [70] and

$$\left.\begin{aligned} &\psi = 0 \quad \text{on } S \\ \text{and} \qquad &\psi = -m \quad \text{on } B. \end{aligned}\right\} \tag{2.7}$$

Further, if $\gamma = \mathcal{V}_x - \mathcal{U}_y$ denotes the (constant) vorticity, then from the momentum conservation equations (2.3) we obtain Bernoulli's law asserting that the expression

$$E = \frac{|\nabla\psi|^2}{2} + \mathcal{P} + gy + \gamma\psi \tag{2.8}$$

is constant within the fluid domain.

Choosing now a parametrization of the free surface S such that u and v are independent of t in the moving frame, writing the Bernoulli relation equation (2.8) on the free surface and utilizing equation (2.7) we obtain that the stream function ψ satisfies the boundary value problem,

$$\left.\begin{aligned} &\Delta\psi = -\gamma, \\ &\psi = 0 \quad \text{on } S, \\ &\psi = -m \quad \text{on } B \\ \text{and} \quad &|\nabla\psi|^2 + 2\mathcal{D}\left(\kappa_{\mathfrak{s}\mathfrak{s}} + \frac{1}{2}\kappa^3\right) + 2gv = Q \quad \text{on } S, \end{aligned}\right\} \tag{2.9}$$

where $Q := 2E$ is the hydraulic head. Note that, in terms of u and v, it holds that

$$\begin{aligned} \kappa_{\mathfrak{s}\mathfrak{s}} = {} & \frac{u'v^{(4)} - u^{(4)}v' + u''v^{(3)} - u^{(3)}v''}{(u'^2 + v'^2)^{5/2}} - 7\frac{(u'v^{(3)} - u^{(3)}v')(u'u'' + v'v'')}{(u'^2 + v'^2)^{7/2}} \\ & - 3\frac{(u'v'' - u''v')((u'')^2 + (v'')^2 + u'u^{(3)} + v'v^{(3)})}{(u'^2 + v'^2)^{7/2}} + 18\frac{(u'v'' - u''v')(u'u'' + v'v'')^2}{(u'^2 + v'^2)^{9/2}}, \end{aligned} \tag{2.10}$$

where the upper index $^{(j)}$, for $j = 3, 4$, indicates j times differentiation with respect to $\mathfrak{s}$.

3. Reformulation of the free-boundary problem as a pseudodifferential equation

The goal of this section is to reformulate the free-boundary problem equation (2.9) as a quasilinear pseudodifferential equation for a periodic function of one variable representing the height of the free surface. The main tools to achieve the proposed task are the periodic Dirichlet–Neumann operator $\mathcal{G}_d$ and the Hilbert transform $\mathcal{C}_d$ associated to a strip of finite width. The previously mentioned transforms act on suitable Hölder spaces that we introduce now. Note that we only list here the properties of the Dirichlet–Neumann operator and of the Hilbert transform. For detailed proofs, the reader is referred to [38].

Definition 3.1. For an integer $p \geq 0$ and $\alpha \in (0, 1)$ we let $C^{p,\alpha}$ denote the standard Hölder space of order p with exponent α over their domain of definition. By $C^{p,\alpha}_{\text{loc}}$ we refer to the space of functions of class $C^{p,\alpha}$ over any compact subset of their definition domain. To refer to 2π periodic functions and of class $C^{p,\alpha}_{\text{loc}}$ in $\mathbb{R}$ we use the notation $C^{p,\alpha}_{2\pi}$. Finally, by $C^{p,\alpha}_{2\pi,o}$ we denote the functions that are in $C^{p,\alpha}_{2\pi}$ and have zero mean over one period.

(a) The Dirichlet–Neumann operator

For any $d > 0$ let $\mathcal{R}_d := \{(x, y) \in \mathbb{R}^2 : -d < y < 0\}$.

Definition 3.2. For $w \in C^{p,\alpha}_{2\pi}$ let W be the unique solution of

$$\left.\begin{aligned} &\Delta W = 0 \quad \text{in } \mathcal{R}_d, \\ &W(x, -d) = 0, \quad x \in \mathbb{R} \\ \text{and} \quad &W(x, 0) = w(x), \quad x \in \mathbb{R}. \end{aligned}\right\} \tag{3.1}$$

It holds that the function $(x, y) \to W(x, y)$ is 2π-periodic in x throughout $\mathcal{R}_d$. Then, for $p \in \mathbb{Z}, p \geq 1$ and $\alpha \in (0, 1)$ the *periodic Dirichlet–Neumann operator* $\mathcal{G}_d$, associated to the strip $\mathcal{R}_d$ is defined through the formula

$$\mathcal{G}_d(w)(x) := W_y(x, 0), \quad x \in \mathbb{R}. \tag{3.2}$$

Remark 3.3. The Dirichlet–Neumann operator $\mathcal{G}_d$ satisfies the following properties.

(i) For all $p \geq 1$, $\mathcal{G}_d : C^{p,\alpha}_{2\pi} \to C^{p-1,\alpha}_{2\pi}$ is a bounded, linear operator.
(ii) If c is a real constant, then $\mathcal{G}_d(c) = c/d$.

(b) The Hilbert transform

For $w \in C^{p,\alpha}_{2\pi,o}$ we define the *periodic Hilbert transform* $\mathcal{C}_d(w)$ of w (relative to the strip $\mathcal{R}_d$) by the formula

$$\mathcal{C}_d(w)(x) := Z(x,0),$$

where $Z(x,y)$ is the unique (up to a constant) harmonic conjugate of W from equation (3.1). It follows that $(x,y) \to Z(x,y)$ is 2π-periodic in x. By specifying the constant in the definition of Z by the requirement that $x \to Z(x,0)$ has zero mean over one period we obtain that $\mathcal{C}_d$ has the following properties.

Lemma 3.4. *Let $d > 0$. Then for the Hilbert transform corresponding to the strip $\mathcal{R}_d$ it holds:*

(*i*) *The mapping $\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ is a bounded linear operator for all integer $p \geq 0$. Furthermore, $\mathcal{C}_d^{-1} = -\mathcal{C}_d : C^{p,\alpha}_{2\pi,o} \to C^{p,\alpha}_{2\pi,o}$ is also a bounded linear operator.*
(*ii*) *If $w \in C^{p,\alpha}_{2\pi,o}$ for $p \geq 1$, then $\mathcal{G}_d(w) = (\mathcal{C}_d(w))' = \mathcal{C}_d(w')$.*
(*iii*) *If $w \in C^{p,\alpha}_{2\pi,o}$ for $p \geq 1$, and $[w]$ denotes the average of w over one period, then $\mathcal{G}_d(w) = [w]/d + \mathcal{C}_d(w')$.*

An explicit formula for calculating the Hilbert transform holds for those w which belong $L^2_{2\pi,o}$, that is, the space of 2π-periodic locally square integrable functions (of one real variable) having zero mean over one period.

Lemma 3.5. *Let $w = \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ be the Fourier series expansion of $w \in L^2_{2\pi,o}$. Then,*

$$\mathcal{C}_d(w) = \sum_{n=1}^{\infty} a_n \coth(nd)\sin(nx) - \sum_{n=1}^{\infty} b_n \coth(nd)\cos(nx). \tag{3.3}$$

We are now ready to establish the equivalence of the free-boundary value problem equation (2.9) with a problem on the fixed strip $\mathcal{R}_{kh}$, where k is the wave number and h is the conformal mean-depth, introduced earlier in §2.

Theorem 3.6. *Let $k = 2\pi/L$. Then, if (Ω, ψ) of class $C^{4,\alpha}$ is a solution of equation (2.9), then there exists a positive number h, a positive function $v \in C^{4,\alpha}_{2\pi}$ with $[v] = h$ and a constant $\mathfrak{a} \in \mathbb{R}$ such that*

$$\left[\frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}\left(\frac{v^2}{2}\right) - v\mathcal{G}_{kh}(v)\right)\right]^2 = (Q - 2\mathcal{D}(P(v) + R(v)) - 2gv)(\mathcal{G}_{kh}(v)^2 + v'^2), \tag{3.4}$$

$$\text{the mapping } x \mapsto \left(\frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) \quad \text{is injective on } \mathbb{R}, \tag{3.5}$$

$$v'(x)^2 + \mathcal{G}_{kh}(v)(x)^2 \neq 0 \quad \text{for all } x \in \mathbb{R}, \tag{3.6}$$

$$S = \left\{\left(\mathfrak{a} + \frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x)\right) : x \in \mathbb{R}\right\}, \tag{3.7}$$

where

$$\begin{aligned}
P(v) = {} & \frac{\mathcal{G}_{kh}(v)v^{(4)} - \mathcal{G}_{kh}(v^{(3)})v' + \mathcal{G}_{kh}(v')v^{(3)} - \mathcal{G}_{kh}(v'')v^{(2)}}{(\mathcal{G}_{kh}(v)^2 + v'^2)^{5/2}} \\
& - 7\frac{(\mathcal{G}_{kh}(v)v^{(3)} - \mathcal{G}_{kh}(v'')v')(\mathcal{G}_{kh}(v)\mathcal{G}_{kh}(v') + v'v'')}{(\mathcal{G}_{kh}(v)^2 + v'^2)^{7/2}} \\
& - 3\frac{(\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v')(\mathcal{G}_{kh}(v')^2 + (v'')^2 + \mathcal{G}_{kh}(v)u^{(3)} + v'v^{(3)})}{(\mathcal{G}_{kh}(v)^2 + v'^2)^{7/2}} \\
& + 18\frac{(\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v')(\mathcal{G}_{kh}(v)\mathcal{G}_{kh}(v') + v'v'')^2}{(\mathcal{G}_{kh}(v)^2 + v'^2)^{9/2}}
\end{aligned} \tag{3.8}$$

and

$$R(v) = \frac{1}{2}\left(\frac{\mathcal{G}_{kh}(v)v'' - \mathcal{G}_{kh}(v')v'}{(\mathcal{G}_{kh}(v)^2 + v'^2)^{3/2}}\right)^3. \tag{3.9}$$

Vice versa, let $h > 0$ and let $v \in C^{4,\alpha}_{2\pi}$ be a positive function which satisfies equation (3.4) with equations (3.5) and (3.6), and Ω be the domain whose boundary consists of the bottom B and of the surface S defined by equation (3.7). Then there exists a function ψ in Ω such that (Ω, ψ) is a solution of class $C^{4,\alpha}$ of equation (2.9).

Proof. If (Ω, ψ) is a solution of class $C^{4,\alpha}$ of equation (2.9) and h is the conformal mean depth of Ω, then let $U_h + iV_h$ be the conformal mapping associated to Ω. Then the mapping $U + iV : \mathcal{R}_{kh} \to \Omega$ (for $k = \frac{2\pi}{L}$) defined through

$$U(x,y) = U_h\left(\frac{x}{k}, \frac{y}{k}\right), \quad V(x,y) = V_h\left(\frac{x}{k}, \frac{y}{k}\right), \quad (x,y) \in \mathcal{R}_{kh}, \tag{3.10}$$

has the property that $U, V \in C^{4,\alpha}(\overline{\mathcal{R}_{kh}})$ and $U + iV$ is conformal mapping from $\mathcal{R}_{kh}$ onto Ω which extends homeomorphically to the closures of $\mathcal{R}_{kh}$ and Ω, respectively, and, moreover, the mappings $\{(x, 0) : x \in \mathbb{R}\} \mapsto S$ and $\{(x, -kh) : x \in \mathbb{R}\} \mapsto B$ are onto. Setting

$$u(x) = U(x, 0), \quad v(x) = V(x, 0), \quad x \in \mathbb{R}, \tag{3.11}$$

we have from lemma 3.4,

$$u = \mathcal{C}_{kh}(v), \quad u' = \mathcal{G}_{kh}(v), \quad u'' = \mathcal{G}_{kh}(v'), \quad u^{(3)} = \mathcal{G}_{kh}(v''), \quad u^{(4)} = \mathcal{G}_{kh}(v^{(3)}). \tag{3.12}$$

Following the method of proof from [38] it follows that v is a strictly positive function with mean h and which belongs to $C^{4,\alpha}_{2\pi}$. Furthermore, the mapping $x \mapsto (x/k + \mathcal{C}_{kh}(v-h)(x), v(x))$ is injective and the free surface is given as $S = \{(t + x/k + \mathcal{C}_{kh}(v-h)(x), v(x)) : x \in \mathbb{R}\}$ where t is a constant that arises from the invariance of the free-boundary problem equation (2.9) under horizontal translations. Setting now

$$\Xi(x,y) := \psi(U(x,y), V(x,y)), \quad (x,y) \in \mathcal{R}_{kh}, \tag{3.13}$$

we obtain a function $\Xi : \mathcal{R}_{kh} \to \mathbb{R}$ which has the property that $\Xi + \gamma V^2/2$ is a harmonic function in $\mathcal{R}_{kh}$, the last assertion being true by the harmonicity of the function $(x,y) \to \psi(x,y) + \gamma y^2/2$ and the invariance of harmonic functions under conformal mappings. By the chain rule applied to the function Ξ we obtain (via the Cauchy–Riemann equations) from the last formula in equation (2.9) and equation (3.12) that

$$\Xi_x^2 + \Xi_y^2 = (Q - 2\mathcal{D}(P(v) + R(v)) - 2gv)(\mathcal{G}_{kh}(v)^2 + v'^2), \tag{3.14}$$

with P and R being the polynomials defined in equations (3.8) and (3.9), respectively. Defining now $\Theta : \mathcal{R}_{kh} \to \mathbb{R}$ by $\Theta := \Xi + m + \gamma V^2/2$ it follows that Θ satisfies the system

$$\left.\begin{aligned}
&\Delta\Theta = 0 \quad \text{in } \mathcal{R}_{kh},\\
&\Theta(x, -kh) = 0 \quad \text{for all } x \in \mathbb{R},\\
&\Theta(x, 0) = m + \frac{\gamma v^2(x)}{2} \quad \text{all } x \in \mathbb{R}\\
\text{and}\quad &\Theta_y - \gamma V V_y)^2 = (Q - 2\mathcal{D}(P(v) + R(v)) - 2gv)(\mathcal{G}_{kh}(v)^2 + v'^2) \quad \text{at } (x,0) \quad \text{all } x \in \mathbb{R}
\end{aligned}\right\} \tag{3.15}$$

which can be reformulated by means of the Dirichlet–Neumann operator and its properties (cf. lemma 3.4) as

$$\left[\frac{m}{kh} + \gamma\left(\mathcal{G}_{kh}\left(\frac{v^2}{2}\right) - v\mathcal{G}_{kh}(v)\right)\right]^2 = (Q - 2\mathcal{D}(P(v) + R(v)) - 2gv)(\mathcal{G}_{kh}(v)^2 + v'^2). \tag{3.16}$$

To prove the sufficiency, let $h>0$ and let $v\in C^{4,\alpha}_{2\pi}$ satisfy equations (3.4)–(3.7). We denote with $V:\mathcal{R}_{kh}\to\mathbb{R}$ the harmonic function for which it holds that

$$V(x,-kh)=0 \quad \text{and} \quad V(x,0)=v(x), \tag{3.17}$$

for all $x\in\mathbb{R}$. Now let $U:\mathcal{R}_{kh}\to\mathbb{R}$ such that $U+iV$ is holomorphic. Then $U+iV\in C^{4,\alpha}(\overline{\mathcal{R}_{kh}})$. Since $[v]=h$ we apply the reasoning from [38] and obtain

$$U(x+2\pi,y)=U(x,y)+\frac{2\pi}{k},\quad V(x+2\pi,y)=V(x,y)\quad \text{for all } (x,y)\in\mathcal{R}_{kh}. \tag{3.18}$$

From the injectivity of the map equation (3.5) we deduce that the curve given through equation (3.7) is non-self-intersecting and from the positivity of v it follows that equation (3.7) is contained in the upper-half plane. Denoting with Ω the domain which has S as its upper boundary and B as its lower boundary, theorem 3.4 from [71] yields that $U+iV:\mathcal{R}_{kh}\to\Omega$ is an onto conformal mapping, which extends to a homeomorphism between the closures of these domains, and the mappings $\{(x,0): x\in\mathbb{R}\}\mapsto S$ and $\{(x,-kh): x\in\mathbb{R}\}\mapsto B$ are onto. The previous arguments and equation (3.18) show that Ω is an L-periodic strip-like domain, for $L=2\pi/k$. Setting now $U_h(x,y):=U(kx,ky)$, $V_h(x,y):=V(kx,ky)$ we notice that the properties of the conformal mapping $U_h+iV_h:\mathcal{R}_h\to\Omega$ yield that h is the conformal mean depth of Ω. We now let Θ be the unique solution of

$$\Delta\Theta=0 \quad \text{in } \mathcal{R}_{kh},\quad \Theta|_{y=-kh}=0,\quad \Theta|_{y=0}=m+\frac{\gamma v^2(x)}{2}. \tag{3.19}$$

Then $\Theta\in C^{4,\alpha}(\overline{\mathcal{R}_{kh}})\cap C^\infty(\mathcal{R}_{kh})$. Now let $\Xi:=\Theta-m-\gamma V^2/2$ and let ψ be defined by $\psi(U(x,y),V(x,y)):=\Xi(x,y)$. We readily see that ψ satisfies

$$\Delta\psi=-\gamma,\quad \psi|_S=0,\quad \psi|_B=-m. \tag{3.20}$$

Invoking now equation (3.4) and utilizing the properties of the Dirichlet–Neumann map $\mathcal{G}_{kh}$ (cf. remark 3.3 and lemma 3.4) we see that ψ also satisfies the fourth formula in equation (2.9). ■

4. Exact solutions and their properties

(a) Local bifurcation

To prove the existence of solutions to equation (3.4) we use the reformulation equations (3.4)–(3.7) in conjunction with the Crandall–Rabinowitz theorem on bifurcation from simple eigenvalues for whose proof we refer the reader to [14]. The applicability of the Crandall–Rabinowitz theorem to our hydroelastic setting hinges upon some technical lemmas whose proofs rely partly on methods developed in [23].

Theorem 4.1 (Crandall–Rabinowitz). *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces, I be an open interval in $\mathbb{R}$ containing an element λ^* and $\mathscr{F}:I\times\mathbb{X}\to\mathbb{Y}$ be a continuous map with the properties*

(*i*) *$\mathscr{F}(\lambda,0)=0$ for all $\lambda\in I$.*
(*ii*) *The partial derivatives $\mathscr{F}_\lambda$, $\mathscr{F}_u$, $\mathscr{F}_{\lambda u}$ exist and are continuous.*
(*iii*) *The spaces $Y/Im(\mathscr{F}_u(\lambda^*,0))$ and $\ker(\mathscr{F}_u(\lambda^*,0))$ are one-dimensional, the latter being generated by an element u^*.*
(*iv*) *The transversality condition holds, that is, $\mathscr{F}_{\lambda u}(\lambda^*,0)(u^*)\notin Im(\mathscr{F}_u(\lambda^*,0))$.*

Then there exists a continuous local bifurcation curve $\{(\lambda(s),u(s)):|s|<\epsilon\}$ for some $\epsilon>0$ sufficiently small such that $(\lambda(0),u(0))=(\lambda^*,0)$ and there exists a neighbourhood $\mathcal{O}$ of $(\lambda^*,0)$ such that

$$\{(\lambda,u)\in\mathcal{O}:u\neq 0,\mathscr{F}(\lambda,u)=0\}=\{(\lambda(s),u(s)):0<|s|<\epsilon\}.$$

Moreover, for all s with $|s|<\epsilon$ we have

$$u(s)=su^*+o(s)\quad \text{in } \mathbb{X}.$$

In order to apply the Crandall–Rabinowitz theorem in our setting, we need first a few preparations. Since $[v]=h$ it follows that $w:=v-h$ satisfies $[w]=0$. Utilizing remark 3.3 and lemma 3.4 we have

$$\left.\begin{aligned}
u' &= \mathcal{G}_{kh}(v)=\mathcal{G}_{kh}(w+h)=\mathcal{C}_{kh}(w')+\frac{[w]}{kh}=\mathcal{C}_{kh}(w')+\frac{1}{k},\\
u'' &= \mathcal{G}_{kh}(v')=\mathcal{C}_{kh}(w''),\\
u^{(3)} &= \mathcal{G}_{kh}(v'')=\mathcal{C}_{kh}(w^{(3)})\\
\text{and}\qquad u^{(4)} &= \mathcal{G}_{kh}(v^{(3)})=\mathcal{C}_{kh}(w^{(4)}).
\end{aligned}\right\}\tag{4.1}$$

Therefore, we can write equation (3.4) as

$$\left.\begin{aligned}
&\left[\frac{m}{kh}+\gamma\left(\frac{[w^2]}{2kh}-\frac{w}{k}-\frac{h}{2k}+\mathcal{C}_{kh}(ww')-w\mathcal{C}_{kh}(w')\right)\right]^2\\
&\quad=(Q-2gh-2\mathcal{D}(P(w+h)+R(w+h))-2gw)\left[w'^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)^2\right],\\
&[w]=0,\\
&w(x)>-h\quad\text{for all } x\in\mathbb{R},\\
&\text{the mapping }\left(\frac{x}{k}+\mathcal{C}_{kh}(w)(x),w(x)+h\right)\quad\text{is injective on }\mathbb{R}\\
\text{and}\qquad &w'(x)^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(w')(x)\right)^2\neq 0\quad\text{for all } x\in\mathbb{R}.
\end{aligned}\right\}\tag{4.2}$$

To prove the existence of solutions $w\in C^{4,\alpha}_{2\pi}$ to equation (4.2) for fixed $\gamma\in\mathbb{R}, k>0, h>0$, we regard m and Q as parameters. We note first that $w=0\in C^{4,\alpha}_{2\pi,o}$ is a solution of equation (4.2) if and only if $Q-2gh=(m/h-\gamma h/2)^2$. The latter equality suggests to introduce parameters λ, μ defined as

$$\lambda:=\frac{m}{h}-\frac{\gamma h}{2}\quad\text{and}\quad\mu:=Q-2gh-\left(\frac{m}{h}-\frac{\gamma h}{2}\right)^2.\tag{4.3}$$

Note that $(\lambda,\mu):\mathbb{R}^2\mapsto\mathbb{R}^2$, is a bijection as a function of (m,Q). With the transformation equation (4.3), we can rewrite equation (4.2) as

$$\begin{aligned}
&\left[\frac{\lambda}{k}+\gamma\left(\frac{[w^2]}{2kh}-\frac{w}{k}+\mathcal{C}_{kh}(ww')-w\mathcal{C}_{kh}(w')\right)\right]^2\\
&\quad=(\lambda^2+\mu-2\mathcal{D}(P(w+h)+R(w+h))-2gw)\left[w'^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)^2\right].
\end{aligned}\tag{4.4}$$

To put the previous equation into a functional analytic setting we define the Banach spaces

$$\mathbb{X}=\mathbb{R}\times C^{p+4,\alpha}_{2\pi,0,e}\quad\text{and}\quad\mathbb{Y}=C^{p,\alpha}_{2\pi,e},$$

where

$$C^{p,\alpha}_{2\pi,0,e}:=\{f\in C^{p,\alpha}_{2\pi,o}:f(x)=f(-x)\quad\text{for all } x\in\mathbb{R}\},$$

and

$$C^{p,\alpha}_{2\pi,e}:=\{f\in C^{p,\alpha}_{2\pi}:f(x)=f(-x)\quad\text{for all } x\in\mathbb{R}\}.$$

We notice that we can further rewrite equation (4.4) as $\mathscr{F}(\lambda,(\mu,w))=0$ where $\mathscr{F}:\mathbb{R}\times\mathbb{X}\to\mathbb{Y}$ is given by

$$\begin{aligned}\mathscr{F}(\lambda,(\mu,w)) &= \gamma^2\left(\frac{[w^2]}{2kh}-\frac{w}{k}+\mathcal{C}_{kh}(ww')-w\mathcal{C}_{kh}(w')\right)^2\\ &\quad+\frac{2\gamma\lambda}{k}\left(\frac{[w^2]}{2kh}-\frac{w}{k}+\mathcal{C}_{kh}(ww')-w\mathcal{C}_{kh}(w')\right)\\ &\quad-(\mu-2\mathcal{D}(P(w+h)+R(w+h))-2gw)\left(w'^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)^2\right)\\ &\quad-\lambda^2\left(w'^2+\frac{2}{k}\mathcal{C}_{kh}(w')+(\mathcal{C}_{kh}(w'))^2\right).\end{aligned} \tag{4.5}$$

We first note that $\mathscr{F}(\lambda,(0,0))=0$, which means that the first condition from the local bifurcation theorem checks out. Let us compute

$$\mathscr{F}_{(\mu,w)}(\lambda,(0,0))(\nu,f)=\lim_{t\to0}\frac{\mathscr{F}(\lambda,t(\nu,f))-\mathscr{F}(\lambda,(0,0))}{t}.$$

Using the representation $f=\sum_{n=1}^{\infty}a_n\cos(nx)$, it follows from lemma 3.5 that

$$\begin{aligned}\mathscr{F}_{(\mu,w)}(\lambda,(0,0))(\nu,f) &= -\frac{2}{k^2}\left(\lambda\gamma f-gf+\lambda^2k\mathcal{C}_{kh}(f')-\mathcal{D}k^4f^{(4)}\right)-\frac{\nu}{k^2}\\ &= -\frac{2}{k^2}\sum_{n=1}^{\infty}(kn\coth(knh)\lambda^2+\gamma\lambda-g-\mathcal{D}k^4n^4)a_n\cos(nx)-\frac{\nu}{k^2}.\end{aligned} \tag{4.6}$$

Thus, we infer from lemma 3.4 that the bounded linear operator $\mathscr{F}_{(\mu,w)}(\lambda,(0,0)):\mathbb{X}\to\mathbb{Y}$ is invertible if and only if $kn\coth(knh)\lambda^2+\gamma\lambda-g-\mathcal{D}k^4n^4\neq0$ for all integers $n\geq1$. Thus, the bifurcation points are to be found among the solutions of the equation

$$kn\coth(knh)\lambda^2+\gamma\lambda-g-\mathcal{D}k^4n^4=0, \tag{4.7}$$

for some integer $n\geq1$. Given that our interest lies in solutions of equation (4.4) of minimal period 2π we take $n=1$ in equation (4.7). We denote by

$$\lambda_{\pm}^{*n}=-\frac{\gamma\tanh(knh)}{2kn}\pm\sqrt{\frac{\gamma^2\tanh^2(knh)}{4k^2n^2}+\frac{\tanh(knh)}{kn}(g+\mathcal{D}k^4n^4)},$$

the two solutions of equation (4.7).

Remark 4.2. The purpose of the next lemma is to show that the kernel of $\mathscr{F}_{(\mu,w)}(\lambda^*,(0,0))$ is at most two-dimensional if λ^* is among the solutions of equation (4.7). A criterion that ensures the one-dimensionality of the kernel of $\mathscr{F}_{(\mu,w)}(\lambda^*,(0,0))$ also emerge. In carrying out the following considerations it will be useful to use that $\lambda_{\pm}^{*1}(kn)=\lambda_{\pm}^{*n}(k)$ for all integers $n\geq1$ and all $k>0$.

Lemma 4.3. *If $\lambda(k):=\lambda_{+}^{*1}(k)$. Then λ has a maximum at $k=0$ and a unique local extremum which is a local minimum at $k=k_0>0$. Furthermore, there exists a strictly decreasing sequence $(k_n)_{n\geq2}$ such that $\lambda(k)=\lambda(nk)$ for $k>0$ if and only if $k=k_n$.*

Proof. To begin with we note that a computation shows that $\lambda'(0)=0$, and for all γ and all h it holds that

$$\lambda''(0)=\frac{h^3}{3}(\gamma\sqrt{\gamma^2h^2+4gh}-\gamma^2h-2g)<0,$$

showing that λ has at $k=0$ a local maximum. Denoting now $q(k)=k\coth(hk)$ we have $\lambda=-\gamma/2q+\sqrt{\gamma^2/4q^2+(g+k^4\mathcal{D})/q}$ which yields the relation

$$\lambda^2q=k^4\mathcal{D}+g-\lambda\gamma. \tag{4.8}$$

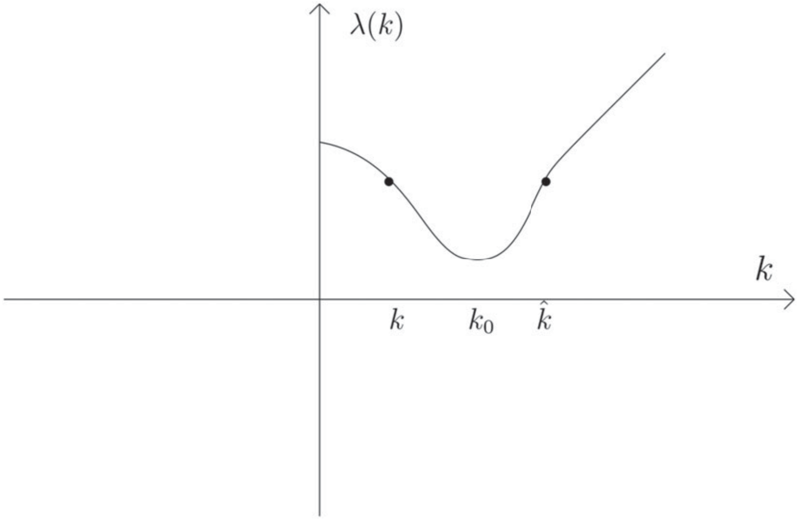

Figure 1. Graph of λ.

Differentiation of equation (4.8) gives

$$(\gamma + 2\lambda q)\lambda' = 4k^3\mathcal{D} - \lambda^2 q', \tag{4.9}$$

and

$$(\gamma + 2\lambda q)\lambda'' = 12k^2\mathcal{D} - \lambda^2 q'' - 4\lambda\lambda' q' - 2(\lambda')^2 q. \tag{4.10}$$

If k_0 is a potential critical point of λ, then equation (4.9) yields the relation $\lambda^2(k_0) = 4k_0^3\mathcal{D}/q'(k_0)$ and so it follows from equation (4.10) that

$$(\gamma + 2\lambda(k_0)q(k_0))\lambda''(k_0) = 4k_0^2\mathcal{D}\left(\frac{3q'(k_0) - k_0 q''(k_0)}{q'(k_0)}\right). \tag{4.11}$$

We claim now that

$$3q'(k_0) - k_0 q''(k_0) > 0. \tag{4.12}$$

To this end note that

$$3q'(k) - kq''(k) = \frac{3\sinh^2(kh)\cosh(kh) - kh\sinh(kh) - 2(kh)^2\cosh(kh)}{\sinh^3(kh)},$$

an expression which suggests investigation of the sign of the function

$$x \mapsto \tau(x) := 3\sinh^2(x)\cosh(x) - x\sinh(x) - 2x^2\cosh(x).$$

A computation shows that $\tau'(x) > 0$ for all $x > 0$, and thus $\tau(x) > 0$ for all $x > 0$. Thus, the inequality equation (4.12) is proved and since $q'(k_0) > 0$ and $\gamma + 2\lambda(k_0)q(k_0)) > 0$ we conclude from equation (4.10) that $\lambda''(k_0) > 0$. Since λ has a local maximum at $k = 0$ and $\lim_{k\to\infty}\lambda(k) = \infty$ we infer that λ has a unique local minimum at some $k_0 > 0$. To prove the existence of the sequence k_n with the claimed property, for a $k \in (0, k_0)$ let $\hat{k} > k_0$ be the unique element such that $\lambda(k) = \lambda(\hat{k})$, cf. figure 1. We define $F : (0, k_0) \mapsto (1, \infty)$ by $F(k) = \hat{k}/k$. We readily see that F is strictly decreasing and $\lim_{k\to 0} F(k) = \infty$ and $\lim_{k\to k_0} F(k) = 1$. Since F^{-1} is decreasing, it follows that the sequence k_n is well-defined and $\lim_{n\to\infty} k_n = 0$. If k is such that $\lambda(k) = \lambda(nk)$ then $F(k) = nk/k = n$ which shows that $k \in F^{-1}(\{n\}) = \{k_n\}$.

∎

Applying a similar reasoning as in lemma 4.3 we can prove the following result.

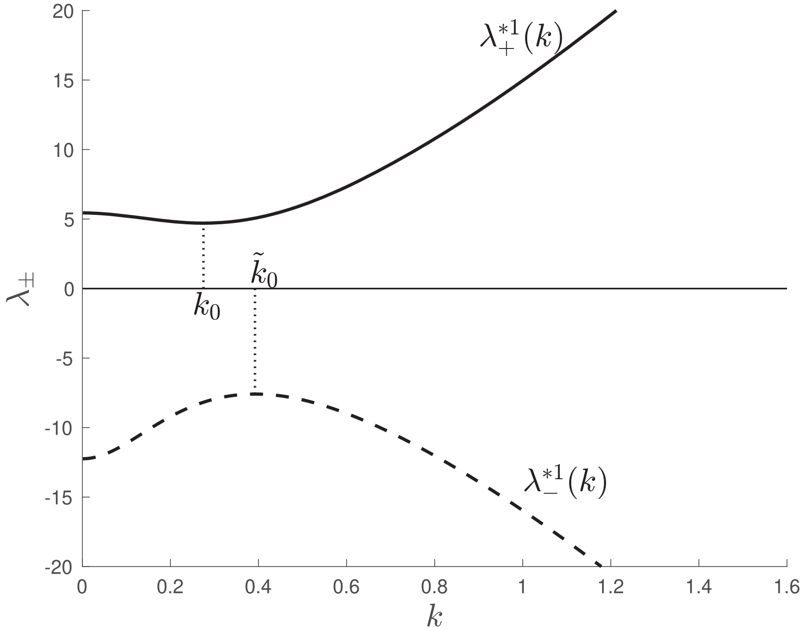

Figure 2. Graph of $\lambda_{\pm}^{*1}(k)$ versus k for $D = 2.35 \times 10^5$ N m, $h = 6.8$ m, $\rho = 1026$ kg m^{-3}, $g = 9.8$ m s^{-2} (note that $\mathcal{D} = D/\rho$), corresponding to Takizawa's experiments [7,8] at Lake Saroma in 1981. We choose here $\gamma = 1$ s^{-1} for illustration. In this case $k_0 = 0.275$ m^{-1}, $\tilde{k}_0 = 0.392$ m^{-1}.

Lemma 4.4. *Let $\tilde{\lambda}(k) := \lambda_-^{*1}(k)$. Then $\tilde{\lambda}$ has minimum at $k = 0$ and a unique local extremum (which is a local maximum) at $k = \tilde{k}_0 > 0$. Furthermore, there exists a strictly decreasing sequence $(\tilde{k}_n)_{n\geq 2}$ such that $\tilde{\lambda}(k) = \tilde{\lambda}(nk)$ for $k > 0$ if and only if $k = \tilde{k}_n$.*

We plot $\lambda_{\pm}^{*1}(k)$ from lemma 4.3–4.4 in figure 2 using the physical values of parameters corresponding to Takizawa's experiments at Lake Saroma [7,8].

Lemmas 4.3 and 4.4 supply us with a necessary and sufficient condition for the one-dimensionality of the kernel of $\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))$. We state this condition in the following proposition.

Proposition 4.5. *Let λ^{*1} be a solution of equation (4.7), i.e.*

$$\lambda_{\pm}^{*1} = -\frac{\gamma \tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + \frac{\tanh(kh)}{k}(g + \mathcal{D}k^4)}.$$

Then $\ker \mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))$ is one-dimensional if and only if $k \neq k_n$ and $k \neq \tilde{k}_n$ for all $n \geq 2$. In this situations, $\ker \mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))$ is generated by $(0, w^*) \in X$, where $w^*(x) = \cos(x)$ for all $x \in \mathbb{R}$.

To ensure the applicability of the Crandall–Rabinowitz theorem and prove thus the existence of exact solutions to equation (4.2) it remains to ensure the Fredholmness of $\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))$ as well as to show that the transversality condition holds. These aspects are treated in the next Proposition.

Proposition 4.6. *Let $\mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0)))$ denote the range of $\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))$. Then the following are true.*

(*i*) $\mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0))) = \{\mathfrak{f} \in Y : \int_{-\pi}^{\pi} \mathfrak{f}(x)\cos(x)\, dx = 0\}$.
(*ii*) *$\mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0)))$ is a closed subspace of Y.*
(*iii*) *The quotient $Y/\mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0)))$ is the one-dimensional subspace of Y generated by the function $w^*(x) = \cos(x)$.*
(*iv*) *The transversality condition $\mathscr{F}_{\lambda,(\mu,w)}(\lambda^*, (0,0))(0, w^*) \notin \mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*, (0,0)))$ holds.*

Proof. We see that (i), (ii) and (iii) are true by glancing at equation (4.6). To prove (iv), note that by equation (4.6) we have

$$\begin{aligned}\mathscr{F}_{\lambda,(\mu,w)}(\lambda^*,(0,0))(0,w^*) &= -\frac{2\gamma w^*}{k^2} - \frac{4\lambda^* k}{k^2}\mathcal{C}_{kh}((w^*)') \\ &= -\frac{2}{k^2}(\gamma + 2\lambda^* k\coth(kh))w^* \notin \mathcal{R}(\mathscr{F}_{(\mu,w)}(\lambda^*,(0,0)))\end{aligned} \tag{4.13}$$

since $\gamma + 2\lambda^* k\coth(kh) = 1/\lambda^*(g + k^4\mathcal{D} + (\lambda^*)^2 k\coth(kh)) \neq 0$. ∎

Theorem 4.7. *Let k_n (respectively, $\tilde{k}_n$) be the sequence from lemma 4.3 (respectively, lemma 4.4). Then, for all $h>0, \mathcal{D}>0$, $m, \gamma \in \mathbb{R}$ and all $k>0$ with $k \neq k_n (n \geq 2)$ and $k \neq \tilde{k}_n (n \geq 2)$ there exist steady periodic waves of small amplitude, with period $2\pi/k$ and conformal mean depth h, which have a smooth profile with one crest and one trough per period, monotonic between crests and troughs and symmetric about any crest line. These non-trivial solutions are triggered by the laminar flows with horizontal speeds at the flat free surface given by*

$$\lambda_\pm = -\frac{\gamma\tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{\tanh(kh)}{k}(g + \mathcal{D}k^4)}, \tag{4.14}$$

and whose mass flux equals

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma h\tanh(kh)}{2k} \pm h\sqrt{\frac{\gamma^2\tanh^2(kh)}{4k^2} + \frac{\tanh(kh)}{k}(g + \mathcal{D}k^4)}. \tag{4.15}$$

Proof. From lemmas 4.3, 4.4, propositions 4.5 and 4.6 it follows that $(\lambda^*,(0,0))$, (with $\lambda^* := \lambda^n_\pm(k)$ solution of equation (4.7) for some integer $n \geq 1$ and $k \neq k_n, k \neq \tilde{k}_n$) are bifurcation points in the spirit of the Crandall–Rabinowitz theorem 4.1. Since we are looking for solutions of equation (4.4) of minimal period 2π we set $n=1$ in equation (4.7) and obtain the bifurcation values given in equation (4.14). These solutions describe trivial flows in the fluid domain bounded below by the rigid bed $Y=0$ and above by the free surface $Y=h$, having as stream function

$$\psi(X,Y) = -\frac{\gamma Y^2}{2} + \left(\frac{m}{h} + \frac{\gamma h}{2}\right)Y - m, \quad X \in \mathbb{R},\ Y \in [0,h], \tag{4.16}$$

and velocity field

$$(\psi_Y, -\psi_X) = \left(-\gamma Y + \frac{m}{h} + \frac{\gamma h}{2}, 0\right), \quad X \in \mathbb{R},\ Y \in [0,h]. \tag{4.17}$$

Appealing now to the Crandall–Rabinowitz theorem 4.1 we derive the existence of a local curve,

$$\{(\lambda(s),(0+o(s), s\cos(x)+o(s))) : |s| < \epsilon\} \subset \mathbb{R}\times\mathbb{X},$$

consisting of solutions of equation (4.4) emerging from the bifurcation points $(\lambda_\pm,(0,0))$ where $\lambda_\pm$ are given by equation (4.14). A schematic of the previous situation is presented in figure 3. Since

$$w(x;s) = s\cos(x) + o(s) \quad \text{in} \quad C^{p+4,\alpha}_{2\pi}, \tag{4.18}$$

we have from lemma 3.4 (choosing also $\epsilon > 0$ small enough) that

$$w(x) > -h \quad \text{and} \quad \frac{1}{k} + \mathcal{C}_{kh}(w')(x) > 0 \quad \text{for all } x \in \mathbb{R}.$$

The latter implies that the corresponding non-flat surface S given by equation (3.7) where $v = w + h$ is the graph of a smooth function, symmetric with respect to the points corresponding to $x = n\pi, n \in \mathbb{Z}$. To complete the proof of the statement of the theorem we note that $s\frac{\mathrm{d}w}{\mathrm{d}x}(x;s) < 0$ for all $x \in (0,\pi)$ and all s with $0 < |s| < \epsilon$ for $\epsilon > 0$ small enough and $p \geq 1$. Together with the eveness of the function $x \to w(x;s)$ we infer that S has one crest and one trough per minimal period and is monotonic between consecutive crests and troughs.

∎

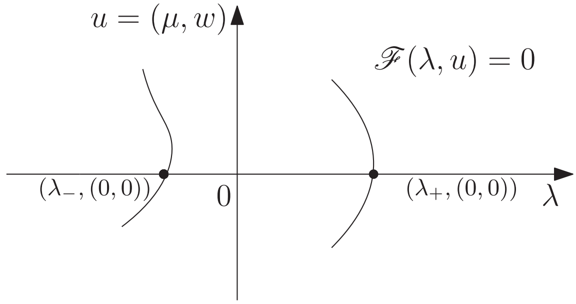

Figure 3. A schematic of the local bifurcation curves ensuing from $\lambda_\pm$ given by equation (4.14).

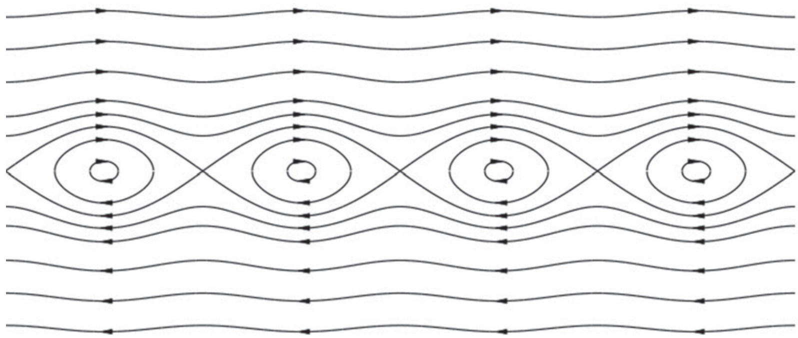

Figure 4. Sketch of a typical flow pattern (streamlines) with stagnation points: Kelvin's 'cat's eye' flow pattern.

(b) Existence of stagnation points

The purpose of this section is to show that the trivial flows equation (4.16) contain stagnation points, that is, points where the horizontal velocity equals the wave speed. Since the flows are laminar, the stagnation points form, in fact, horizontal lines. Then an analysis like in [31] shows that the non-trivial flows bifurcating from the trivial solutions have critical layers which are closed level sets of the stream function enclosing stagnation points, a scenario depicted in figure 4 showing the typical 'cat's eye' flow pattern of Thomson (Lord Kelvin) [30]. To prove that the flows equation (4.16) contain stagnation points we note first that equations (4.14) and (4.17) yield

$$(\psi_Y, -\psi_X) = (\gamma(h - Y) + \lambda_\pm, 0), \qquad X \in \mathbb{R},\ 0 \le Y \le h. \tag{4.19}$$

The previous relation shows that the necessary and sufficient condition for the occurrence of stagnation points is that the equation

$$\lambda_\pm + \gamma(h - Y) = 0 \tag{4.20}$$

admits a solution Y in $[0, h]$. Since $\lambda_+ > 0$ and $\lambda_- < 0$ (cf. equation (4.14)), we notice immediately that irrotational flows ($\gamma = 0$) do not feature stagnation points. For $\gamma \neq 0$ we argue by continuity and see that, in order for equation (4.20) to admit a solution in $[0, h]$, it is sufficient that the inequality

$$\lambda_\pm(\lambda_\pm + \gamma h) \le 0 \tag{4.21}$$

holds. To find scenarios in which equation (4.21) holds we split the discussion concerning the fulfilment of equation (4.21) into two cases according to the sign of γ.

Case (i): $\gamma > 0$.

We immediately notice that, if $\gamma > 0$, then equation (4.21) does not hold for the flow corresponding to λ_+. Since $\lambda_- < 0$, in the case of the flow corresponding to λ_-, condition equation (4.21) is equivalent to $\lambda_- + \gamma h \ge 0$, which can be reformulated as

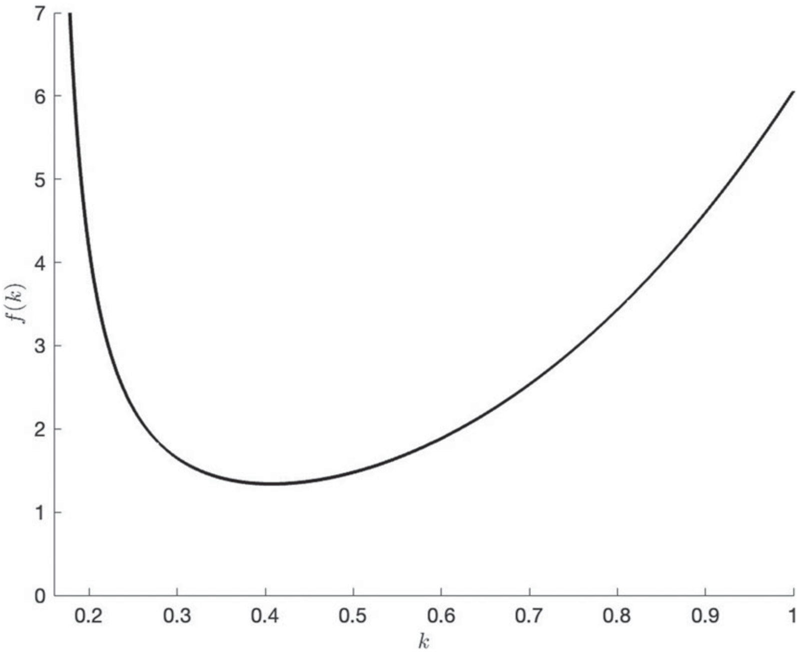

Figure 5. Graph of the function $k \longrightarrow (g + \mathcal{D}k^4)/6.8(6.8k - 1)$ from the right hand side of equation (4.24) from $k = 0.16$ to $k = 1$.

$$\tanh(kh) \le \frac{k\gamma^2 h^2}{g + \mathcal{D}k^4 + \gamma^2 h}. \tag{4.22}$$

We notice that inequality equation (4.22) is satisfied if, for instance,

$$\gamma^2 h(kh - 1) \ge g + \mathcal{D}k^4. \tag{4.23}$$

Note that, given $h > 0$, the inequality equation (4.23) is satisfied if and only if

$$k > \frac{1}{h} \quad \text{and} \quad \gamma^2 > \frac{g + \mathcal{D}k^4}{h(kh - 1)} =: f(k). \tag{4.24}$$

A few remarks are in order.

Remark 4.8. If $Y = h_0$ is a solution of equation (4.20), then the flow has a line of stagnation points given as $Y = h_0$, with $h - h_0 = -\frac{\lambda_\pm}{\gamma}$. Therefore, whenever inequality equation (4.24) is satisfied there is (in fact) a line of stagnation points $Y = h_0$ with

$$h - h_0 = \frac{\tanh(kh)}{2k} + \sqrt{\frac{\tanh^2(kh)}{4k^2} + \frac{g + \mathcal{D}k^4}{\gamma^2}\frac{\tanh(kh)}{k}}, \tag{4.25}$$

representing the distance from the line of stagnation points to the surface. We see from equation (4.25) that the distance from the stagnation line to the free surface decreases with γ^2. Furthermore, $|h - h_0|$ approaches 0 if and only if $\gamma > k^2$ and $k \to \infty$.

The following remark addresses the plausibility of conditions equations (4.24) and (4.25) from the physical point of view.

Remark 4.9.

(1) Taking $h = 6.8\,\text{m}$, which is the depth of Lake Saroma, we see from figure 5 that condition equation (4.24) is ensured by values of the wave number k and of the vorticity γ expected at Lake Saroma.

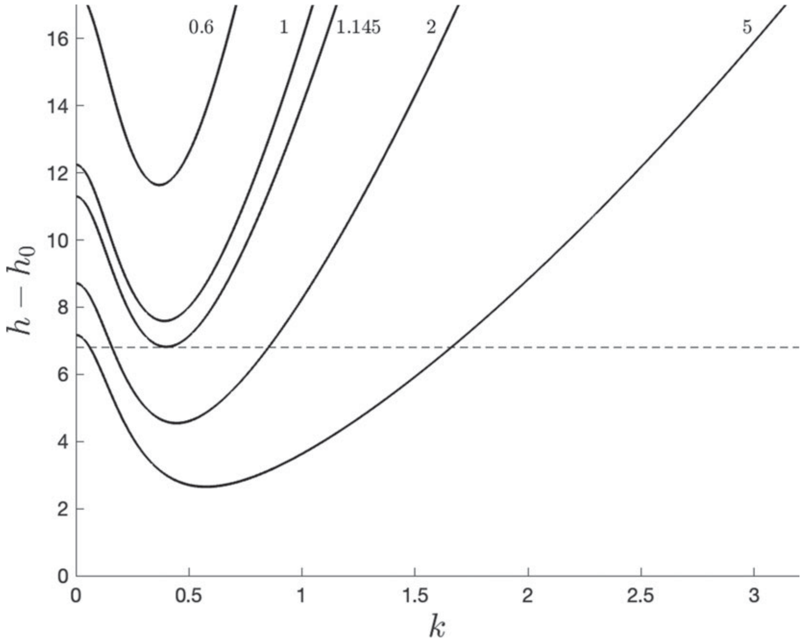

Figure 6. Graph of $h - h_0$ given by equation (4.25) versus k for $D = 2.35 \times 10^5$ N m, $h = 6.8$ m, $\rho = 1026$ kg m^{-3}, $g = 9.8$ m s^{-2} for different values of γ. From top to bottom $\gamma = 0.6; 1; 1.145; 2; 5$ s^{-1}. The dashed line corresponds to $h = 6.8$ m, the Lake Saroma's depth in the area of Takizawa's experiments.

(2) While constant vorticity is not explicitly measured, the slow stratification, the wind-sheltered conditions under the flat homogeneous ice cover together with the absence of wind forcing found by the Slope 2019 field campaign [5] suggest the idea that large regions of Lake Saroma might exhibit nearly uniform vorticity.

(3) Using again the physical values corresponding to Takizawa's experiments [7,8], we plot in figure 6 the distance $h - h_0$ (from equation (4.25)) versus k for different values of γ. Setting $h_S = 6.8$ m to be the Lake Saroma's depth, we observe that for small γ it holds that $h - h_0 > h_S = 6.8$ m, which means that there are no stagnation points in the fluid. For γ greater than a critical value γ_c, which in this case is $\gamma_c = 1.145$ s^{-1}, there is an interval $k_1 < k < k_2$ where $h - h_0 < h_S$, so the stagnation line $Y = h_0$ is in the fluid.

Case (ii): $\gamma < 0$

This discussion in this case is similar to the precedent one, in the sense that the flow corresponding to λ_- does not present stagnation points and the flow whose speed at the free surface is λ_+ has stagnation points if equation (4.21) holds. In this case the stagnation line is $Y = h_0$ with h_0 satisfying equation (4.25).

5. Conclusion

We have considered the two-dimensional hydroelastic water wave problem in a rotational ideal fluid of constant vorticity and finite depth, covered by a thin ice sheet. The model is based on the special Cosserat theory of hyperelastic shells satisfying Kirchhoff's hypothesis [65–69]. By conformally mapping the fluid domain to a horizontal strip we were able to reformulate the hydroelastic boundary value problem as a quasilinear pseudodifferential equation for a periodic function of one variable representing the height of the free surface. Employing the Crandall–Rabinowitz theorem of bifurcation from simple eigenvalues [14] we prove existence of free surface travelling hydroelastic water waves in constant vorticity water flows allowing for stagnation points and overhanging profiles. Following the existence proof we show that for certain values of the wave number and of the vorticity (cf. condition equation (4.24)) the solutions contain stagnation points and critical layers. To the best of our knowledge, the existence of stagnation points in the hydroelastic context has not been proved elsewhere.

Data accessibility. This article has no additional data.
Declaration of AI use. We have not used AI-assisted technologies in creating this article.
Authors' contributions. C.I.M.: conceptualization, formal analysis, funding acquisition, investigation, methodology, writing—original draft; E.I.P.: conceptualization, funding acquisition, investigation, writing—original draft.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.
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References

1. Korobkin A, Părău EI, Vanden-Broeck J-M. 2011 The mathematical challenges and modelling of the hydroelasticity. *Phil. Trans. R. Soc. A* **369**, 2803–2812. (doi:10.1098/rsta.2011.0116)
2. Ashton GD. 1986 *River and lake ice engineering*. Littleton, Co: Water Resources Publication.
3. Menemenlis D, Farmer DM. 1992 Acoustical measurement of current and vorticity beneath ice. *J. Atmos. Ocean. Technol.* **9**, 827–849. (doi:10.1175/1520-0426(1992)009<0827:AMOCAV>2.0.CO;2)
4. Abrashkin AA, Constantin A. 2023 A steady azimuthal stratified flow modelling the Antarctic Circumpolar Current. *J. Differ. Equ.* **374**, 632–641. (doi:10.1016/j.jde.2023.07.044)
5. Nomura D *et al.* 2020 Saroma-ko Lagoon Observations for sea ice Physico-chemistry and Ecosystems 2019 (SLOPE2019). *Bull. Glaciol. Res.* **38**, 1–12. (doi:10.5331/bgr.19R02)
6. Nomura D *et al.* 2024 Multidisciplinary research for sea ice in Saroma-ko Lagoon, Hokkaido, Japan 2023. *Bull. Glaciol. Res.* **42**, 19–37. (doi:10.5331/bgr.24R01)
7. Takizawa T. 1985 Deflection of a floating sea ice sheet induced by a moving load. *Cold Regions Sci. Tech.* **11**, 171–180. (doi:10.1016/0165-232X(85)90015-1)
8. Takizawa T. 1988 Response of a floating sea ice sheet to a steadily moving load. *J. Geophys. Res.* **93**, 5100–5112. (doi:10.1029/JC093iC05p05100)
9. Marchenko AV, Prokhorov IV. 1991 Linear waves in a fluid flow with constant vorticity located under an ice blanket. *J. Appl. Math. Mech.* **55**, 193–200. (doi:10.1016/0021-8928(91)90085-9)
10. Bhattacharjee J, Sahoo T. 2009 Interaction of flexural gravity waves with shear current in shallow water. *Ocean Eng.* **36**, 831–841. (doi:10.1016/j.oceaneng.2009.05.008)
11. Gao T, Milewski P, Vanden-Broeck J-M. 2019 Hydroelastic solitary waves with constant vorticity. *Wave Motion* **85**, 84–97. (doi:10.1016/j.wavemoti.2018.11.005)
12. Gao T, Wang Z, Milewski PA. 2019 Nonlinear hydroelastic waves on a linear shear current at finite depth. *J. Fluid Mech.* **876**, 55–86. (doi:10.1017/jfm.2019.528)
13. Wang Z, Guan X, Vanden-Broeck J-M. 2020 Progressive flexural–gravity waves with constant vorticity. *J. Fluid Mech.* **905**, A12. (doi:10.1017/jfm.2020.750)
14. Crandall MG, Rabinowitz PH. 1971 Bifurcation from simple eigenvalues. *J. Funct. Anal.* **8**, 321–340. (doi:10.1016/0022-1236(71)90015-2)
15. Constantin A, Strauss W. 2004 Exact steady periodic water waves with vorticity. *Comm. Pure Appl. Math.* **57**, 481–527. (doi:10.1002/cpa.3046)
16. Constantin A, Strauss W. 2011 Periodic traveling gravity water waves with discontinuous vorticity. *Arch. Ration. Mech. Anal.* **202**, 133–175. (doi:10.1007/s00205-011-0412-4)
17. Constantin A. 2011 *Nonlinear water waves with applications to wave-current interactions and tsunamis*, vol. 81. CBMS-NSF Conference Series in Applied Mathematics. Philadelphia: SIAM.

18. Escher J, Knopf P, Lienstromberg C, Matioc B-V. 2020 Stratified periodic water waves with singular density gradients. *Ann. Mat. Pura Appl.* **199**, 1923–1959. (doi:10.1007/s10231-020-00950-1)
19. Haziot SV, Hur VM, Strauss WA, Toland JF, Wahlén E, Walsh S, Wheeler MH. 2022 Traveling water waves–the ebb and flow of two centuries. *Quart. Appl. Math.* **80**, 317–401. (doi:10.1090/qam/1614)
20. Henry D. 2013 Large Amplitude steady periodic waves for fixed-depth rotational flows. *Comm. Part. Differ. Equ.* **38**, 1015–1037. (doi:10.1080/03605302.2012.734889)
21. Henry D, Matioc B-V. 2013 On the existence of steady periodic capillary-gravity stratified water waves. *Ann. della Scuola Normale Superiore di Pisa - Classe di Sci.* **12**, 955–974. (doi:10.2422/2036-2145.201108_001)
22. Henry D, Matioc A-V. 2014 Global bifurcation of capillary-gravity stratified water waves. *Proc. R. Soc. Edinb. Sect. A* **144**, 775–786. (doi:10.1017/S0308210512001990)
23. Martin CI. 2013 Local bifurcation and regularity for steady periodic capillary–gravity water waves with constant vorticity. *Nonlinear Anal.: Real World Appl.* **14**, 131–149. (doi:10.1016/j.nonrwa.2012.05.007)
24. Akers BF, Ambrose DM, Sulon DW. 2017 Periodic traveling interfacial hydroelastic waves with or without mass. *Z. Angew. Math. Phys.* **68**, 141. (doi:10.1007/s00033-017-0884-7)
25. Akers BF, Ambrose DM, Sulon DW. 2019 Periodic travelling interfacial hydroelastic waves with or without mass II: multiple bifurcations and ripples. *Eur. J. Appl. Math.* **30**, 756–790. (doi:10.1017/S0956792518000396)
26. Matioc B-V, Părău EI. 2024 Steady periodic hydroelastic waves in polar regions. *Water Waves.* **7**, 363–387. (doi:10.1007/s42286-024-00095-1)
27. Burton GR, Toland JF. 2011 Surface waves on steady perfect-fluid flows with vorticity. *Comm. Pure Appl. Math.* **64**, 975–1007. (doi:10.1002/cpa.20365)
28. Groves MD, Hewer B, Wahlén E. 2016 Variational existence theory for hydroelastic solitary waves. *C. R. Math. Acad. Sci. Paris* **354**, 1078–01086. (doi:10.1016/j.crma.2016.10.004)
29. Ahmad R, Groves M. 2024 Spatial dynamics and solitary hydroelastic surface waves. *Water Waves* **6**, 5–47. (doi:10.1007/s42286-023-00077-9)
30. Thomson(Lord Kelvin) W. 1880 On a disturbing infinity in Lord Rayleigh's solution for waves in a plane vortex stratum. *Nature* **23**, 45–46. (doi:10.1038/023045a0)
31. Wahlén E. 2009 Steady water waves with a critical layer. *J. Differ. Equ.* **246**, 2468–2483. (doi:10.1016/j.jde.2008.10.005)
32. Ehrnström M, Escher J, Wahlén E. 2011 Steady water waves with multiple critical layers. *SIAM J. Math. Anal.* **43**, 1436–1456. (doi:10.1137/100792330)
33. Ehrnström M, Escher J, Villari G. 2012 Steady water waves with multiple critical layers: interior dynamics. *J. Math. Fluid Mech.* **14**, 407–419. (doi:10.1007/s00021-011-0068-8)
34. Doak AG, Mikyoung Hur V, Vanden-Broeck J-M. 2025 Stokes waves in rotational flows: internal stagnation and overhanging profiles. *J. Fluid Mech.* **1015**, A30. (doi:10.1017/jfm.2025.10373)
35. Martin CI, Matioc B-V. 2013 Existence of Wilton ripples for water waves with constant vorticity and capillary effects. *SIAM J. Appl. Math.* **73**, 1582–1595. (doi:10.1137/120900290)
36. Wahlén E, Weber J. 2024 Large-amplitude steady periodic gravity water waves with general vorticity and critical layers. *Duke Math. J.* **173**, 2197–2258. (doi:10.1215/00127094-2023-0054)
37. Wang J, Xu F, Zhang Y. 2025 The existence of stratified linearly steady two-mode water waves with stagnation points. *J. Math. Fluid Mech.* **27**, 13. (doi:10.1007/s00021-024-00916-w)
38. Constantin A, Vărvărucă E. 2011 Steady periodic water waves with constant vorticity: regularity and local bifurcation. *Arch. Ration. Mech. Anal.* **199**, 33–67. (doi:10.1007/s00205-010-0314-x)
39. Constantin A, Strauss W, Vărvărucă E. 2021 Large-amplitude steady downstream water waves. *Comm. Math. Phys.* **387**, 237–266. (doi:10.1007/s00220-021-04178-9)
40. Teles da Silva AF, Peregrine DH. 1988 Steep, steady surface waves on water of finite depth with constant vorticity. *J. Fluid Mech.* **195**, 281–302. (doi:10.1017/S0022112088002423)
41. Jonsson IG. 1990 Wave-current interactions. In *The Sea: Ocean Engineering Science*, vol. 9, pp. 65–119. New York: Wiley.
42. Constantin A, Strauss W, Vărvărucă E. 2016 Global bifurcation of steady gravity water waves with critical layers. *Acta Math.* **217**, 195–162. (doi:10.1007/s11511-017-0144-x)
43. Buffoni B, Dancer EN, Toland JF. 2000 The regularity and local bifurcation of steady periodic water waves. *Arch. Rational Mech. Anal.* **152**, 207–240. (doi:10.1007/s002050000086)

44. Buffoni B, Dancer EN, Toland JF. 2000 The sub-harmonic bifurcation of Stokes waves. *Arch. Ration. Mech. Anal.* **152**, 241–271. (doi:10.1007/s002050000087)
45. Shargorodsky E, Toland JF. 2008 Bernoulli free-boundary problems. *Mem. Am. Math. Soc.* **196**, 914, 1–70. (doi:10.1090/memo/0914)
46. Babenko KI. 1987 Some remarks on the theory of surface waves of finite amplitude. *Dokl. Akad. Nauk* **294**, 1033–1037.
47. Alvarez-Samaniego B, Lannes D. 2008 Large time existence for 3D water-waves and asymptotics. *Invent. Math.* **171**, 485–541. (doi:10.1007/s00222-007-0088-4)
48. Baldi P, Berti M, Haus E, Montalto R. 2018 Time quasi-periodic gravity water waves in finite depth. *Invent. Math.* **214**, 739–911. (doi:10.1007/s00222-018-0812-2)
49. Berti M, Franzoi L, Maspero A. 2021 Traveling quasi-periodic water waves with constant vorticity. *Arch. Ration. Mech. Anal.* **240**, 99–202. (doi:10.1007/s00205-021-01607-w)
50. Bona JL, Lannes D, Saut J-C. 2008 Asymptotic models for internal waves. *J. Math. Pures Appl.* **89**, 538–566. (doi:10.1016/j.matpur.2008.02.003)
51. Clamond D. 2022 Explicit Dirichlet-Neumann operator for water waves. *J. Fluid Mech.* **950**, A33. (doi:10.1017/jfm.2022.830)
52. Constantin A, Ivanov R, Prodanov E. 2008 Nearly-Hamiltonian structure for water waves with constant vorticity. *J. Math. Fluid Mech.* **10**, 224–237. (doi:10.1007/s00021-006-0230-x)
53. Constantin A, Ivanov R, Martin CI. 2016 Hamiltonian formulation for wave-current interactions in stratified rotational flows. *Arch. Ration. Mech. Anal.* **221**, 1417–1447. (doi:10.1007/s00205-016-0990-2)
54. Constantin A, Ivanov RI. 2019 Equatorial wave-current interactions. *Comm. Math. Phys.* **370**, 1–48. (doi:10.1007/s00220-019-03483-8)
55. Craig W, Guyenne P, Kalisch H. 2005 Hamiltonian long wave expansions for free surfaces and interfaces. *Comm. Pure Appl. Math.* **58**, 1587–1641. (doi:10.1002/cpa.20098)
56. Ivanov RI, Martin CI, Todorov MD. 2022 Hamiltonian approach to modelling interfacial internal waves over variable bottom. *Physica D* **433**, 133190. (doi:10.1016/j.physd.2022.133190)
57. Lannes D. 2005 Well-posedness of the water-waves equations. *J. Am. Math. Soc.* **18**, 605–654. (doi:10.1090/S0894-0347-05-00484-4)
58. Nachbin A. 2018 A three-dimensional Dirichlet-to-Neumann operator for water waves over topography. *J. Fluid Mech.* **845**, 321–345. (doi:10.1017/jfm.2018.241)
59. Compelli A, Ivanov R, Todorov M. 2018 Hamiltonian models for the propagation of irrotational surface gravity waves over a variable bottom. *Phil. Trans. R. Soc. A* **376**, 20170091. (doi:10.1098/rsta.2017.0091)
60. Craig W, Guyenne P, Nicholls DP, Sulem C. 2005 Hamiltonian long-wave expansions for water waves over a rough bottom. *Proc. R. Soc. A* **461**, 839–873. (doi:10.1098/rspa.2004.1367)
61. Canning Gregory R, Nicholls DP. 2012 Numerical simulation of a weakly nonlinear model for internal waves. *Commun. Comput. Phys.* **12**, 1461–1481. (doi:10.4208/cicp.140811.060112a)
62. Guyenne P, Părău EI. 2016 An operator expansion method for computing nonlinear surface waves on a ferrofluid jet. *J. Comput. Phys.* **321**, 414–434. (doi:10.1016/j.jcp.2016.05.055)
63. Guyenne P. 2017 A high-order spectral method for nonlinear water waves in the presence of a linear shear current. *Comput. Fluids* **154**, 224–235. (doi:10.1016/j.compfluid.2017.06.004)
64. Craig W, Sulem C. 1993 Numerical simulation of gravity waves. *J. Comput. Phys.* **108**, 73–83. (doi:10.1006/jcph.1993.1164)
65. Guyenne P, Părău EI. 2012 Computations of fully nonlinear hydroelastic solitary waves on deep water. *J. Fluid Mech.* **713**, 307–0329. (doi:10.1017/jfm.2012.458)
66. Guyenne P, Părău EI. 2014 Finite depth effects on solitary waves in a floating ice sheet. *J. Fluids Struct.* **40**, 242–262. (doi:10.1016/j.jfluidstructs.2014.04.015)
67. Plotnikov PI, Toland JF. 2011 Modelling nonlinear hydroelastic waves. *Proc. R. Soc. A* **369**, 2942–2956. (doi:10.1098/rsta.2011.0104)
68. Milewski PA, Vanden-Broeck J-M, Wang Z. 2011 Hydroelastic solitary waves in deep water. *J. Fluid Mech.* **679**, 628–640. (doi:10.1017/jfm.2011.163)
69. Părău E, Dias F. 2002 Nonlinear effects in the response of a floating ice plate to a moving load. *J. Fluid Mech.* **460**, 281–305. (doi:10.1017/S0022112002008236)
70. Martin CI. 2013 Local bifurcation for steady periodic capillary water waves with constant vorticity. *J. Math. Fluid Mech.* **15**, 155–170. (doi:10.1007/s00021-012-0096-z)
71. Vărvărucă E. 2008 Bernoulli free-boundary problems in strip-like domains and a property of permanent waves on water of finite depth. *Proc. R. Soc. A* **138**, 1345–1362. (doi:10.1017/S0308210507000522)