
Equilibrium Configurations of Ferrofluid Droplets and Toroids

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Abstract

The presented work provides a novel study into the equilibrium shapes of toroidal droplets of ferrofluid. We present a mathematical model describing the equilibrium shapes of ferrofluid toroids surrounding a circular loop of current-carrying wire. The obtained governing equations are then discretised and solved numerically using a boundary-integral technique. Validation of the numerical results is achieved via an asymptotic analysis of the special case of a toroidal droplet of small fluid volume.

The results obtained from the numerical solution of the mathematical model provides evidence to confirm the existence of toroidal droplets of ferrofluid that are in equilibrium. The droplets in question are parameterised by their fluid volume and a magnetic Bond number, which is the ratio of magnetic effects to surface tension effects. An analysis of the equilibrium shape obtained as these parameters are varied is presented, determining regions in parameter space in which no equilibrium shapes can be found, as well as regions in parameter space where two equilibrium shapes can be found.

Finally, we consider how the model that was developed for toroidal droplets could be extended to include simply-connected ferrofluid droplets. Some preliminary results for the equilibrium shape of such droplets is presented and some possible links to toroidal droplets are identified.

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Contents

Abstract	2
List of Figures	9
Acknowledgements	17
1 Introduction	18
1.1 Toroidal Droplets	18
1.1.1 Background	18
1.1.2 Static Toroidal Droplets	21
1.2 Ferrofluids	25
1.2.1 An Introduction to Ferrofluids	25
1.2.2 Ferrofluids and Instability	28
1.3 Outline	30
2 Development of the Mathematical Model	32
2.1 Maxwell's Equations	32

2.1.1	Maxwell's Equations in a Vacuum	32
2.1.2	Polarisation and Magnetisation	33
2.1.3	Constitutive Laws	36
2.2	Equation of Motion for a Ferrofluid	37
2.2.1	The Stress Tensor for a Ferrofluid	37
2.2.2	The Navier–Stokes Equations with Magnetisation	40
2.2.3	The FHD Bernoulli Equation	42
2.3	Boundary and Interface Conditions	43
2.3.1	Electromagnetic Fields Across an Interface	43
2.3.2	Dynamic Boundary Condition	44
2.4	Linear Ferrofluid Magnetisation	46
2.4.1	Equilibrium Magnetisation of a Ferrofluid	46
2.4.2	The Linear Limit	47
2.4.3	FHD Bernoulli Equation with Linear Magnetisation	49
2.4.4	Linear Magnetisation in the Dynamic Boundary Condition	50
2.5	Integral Representation for the Magnetic Induction	50
2.5.1	Helmholtz Decomposition for Piecewise Continuous Vector Fields	50
2.5.2	Integral Representation for Piecewise Uniform μ	51
2.6	Limit of a Thin Circular Current-Carrying Wire	53
2.7	Boundary Integral Equation for a Ferrofluid Toroid	57

2.7.1	Coordinates and Unit Vectors	57
2.7.2	Azimuthal Integrals	61
2.7.3	Wire Contribution $\mathbf{B}_w$	61
2.7.4	Free Surface Contribution $\mathbf{B}_s$	62
2.7.5	Final Expression	63
2.7.6	Dynamic Boundary Condition	63
2.8	Non-Dimensionalisation	64
2.8.1	Dimensionless Dynamic Boundary Condition	65
2.8.2	Dimensionless Boundary-Integral Equation	66
3	Numerical Method	69
3.1	Full Problem Statement	69
3.1.1	Problem Set-up	69
3.1.2	Comment on the Azimuthal Integrals $\tilde{\mathbf{P}}$ and $\mathbf{f}$	71
3.1.3	Computation of the Fluid Volume	72
3.2	Analytic Treatment of the Singularity	73
3.3	Set-up for the Boundary-Integral Method	78
3.3.1	Variables	78
3.3.2	Surface geometry	79
3.3.3	Magnetic Field Expansion	82
3.3.4	Governing Equations	83
3.4	Implementing the Boundary-Integral Method	87

3.4.1	Solution Method	87
3.4.2	Preliminary Results	91
3.4.3	Convergence of Numerical Solutions as $N \rightarrow \infty$	93
4	Asymptotic Analysis for a Toroid with a Small Aspect Ratio	99
4.1	Problem Statement	99
4.1.1	Problem Set-up	99
4.1.2	Non-Dimensionalisation	101
4.1.3	Problem in Polar Coordinates	105
4.2	Analysis as $\varepsilon \rightarrow 0$	108
4.2.1	Premise	108
4.2.2	Leading-Order Solution	109
4.2.3	Series Expansions for $ \varepsilon \ll 1$	111
4.2.4	Symmetries	111
4.3	Asymptotic Solution for $ \varepsilon \ll 1$	113
4.3.1	Fourier Series Representation and Volume Constraint . . .	113
4.3.2	Dynamic Boundary Condition	114
4.3.3	Tangential Field Equation	115
4.3.4	Values of the Fourier Coefficients	117
4.3.5	Normal Field Corrections	119
4.3.6	Statement of Asymptotic Solution	120
4.4	Comparison with Numerical Method	121

5	Equilibrium Configurations of Ferrofluid Toroids	128
5.1	Problem Statement and Parameters	128
5.1.1	Reminder of the Dimensionless Problem	128
5.1.2	Method of Numerical Solution	130
5.1.3	The Value of $\tilde{m}$	131
5.2	Varying V_f with Fixed Bo	132
5.3	Extreme Values of Bo	140
5.3.1	Limit of Zero Magnetic Bond Number	140
5.3.2	Limit of Infinite Magnetic Bond Number	148
5.4	Existence of Equilibrium Boundaries	150
5.4.1	Type I Boundaries	152
5.4.2	Type II Boundaries	154
6	Extension to Ferrofluid Droplets	158
6.1	Extension of the Mathematical Model	158
6.1.1	Motivation and Premise	158
6.1.2	Dimensionless Problem Statement	161
6.1.3	Analytic Treatment of the Singularity	162
6.2	Numerical Method for the Droplet Problem	165
6.2.1	Variables	166
6.2.2	Surface Geometry	167
6.2.3	Governing Equations	170

6.2.4	Method of Solution	175
6.3	Results	175
6.3.1	The Case of $Bo = 0$	176
6.3.2	Droplets Surrounding the Wire-Loop	177
6.3.3	Case Where the Wire-Loop Lies Outside of the Droplet . .	180
6.4	Concluding Remarks	183
7	Conclusion	184
7.1	Summary	184
7.2	Future Work	185
	Appendices	187
A	Surface Geometry and Curvatures	187
B	The Magnetic Stress Tensor	192
C	Evaluation of the Boundary Integral Equation for a Point on the Ferrofluid Boundary	196
D	Analytic Evaluation of the Ring Integrals	199
E	Integrals Needed for the Asymptotic Analysis	204
	References	210

List of Figures

1.1.1	Diagram showing the set-up and coordinates for the cross-section of a static fluid toroid.	22
1.1.2	Plots of the numerically computed z , and analytically known z^* , for the system of ODEs describing a smooth axisymmetric toroidal droplet.	24
1.2.1	The characteristic peaks and valleys found on the surface of a ferrofluid subject to a sufficiently strong magnetic field (Arrighi, 2021).	28
2.7.1	Diagram showing the setup and coordinates for a toroidal volume of ferrofluid surrounding a circular wire-loop carrying a current I , for the case where $D > 0$	59
2.7.2	Diagram showing the setup and coordinates for a toroidal volume of ferrofluid surrounding a circular wire-loop carrying a current I , for the case where $D < 0$	59
3.3.1	The setup for the boundary-integral method for the dimensionless problem of an axisymmetric toroidal volume of ferrofluid.	79

3.4.1	log-log plot of $\ \mathbf{F}(\mathbf{w}_s)\ $ against XTOL, using $N = 100$ points on the boundary and parameter values $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.1$.	91
3.4.2	log-log plot of $\ \mathbf{w}_s - \mathbf{w}^{**}\ $ against XTOL, using $N = 100$ points on the boundary and parameter values $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.1$.	91
3.4.3	Numerically computed equilibrium boundaries for some example values of the problem parameters: $\tilde{m}$, Bo , and V_f . The number of boundary points was taken to be $N = 100$.	93
3.4.4	Numerically computed H_t and B_n for some example values of the problem parameters: $\tilde{m}$, Bo , and V_f . The number of boundary points was taken to be $N = 100$.	94
3.4.5	Numerically computed equilibrium boundary for increasing values of N , with $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.	96
3.4.6	Graph of $ \rho_{\max} - \rho_0 $ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.	96
3.4.7	Graph of $ P_{\max} - P $ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.	97
3.4.8	log-log plot of $ \rho_{\max} - \rho_0 $ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.	97
3.4.9	log-log plot of $ P_{\max} - P $ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.	98
4.1.1	Diagrams of the setup for the problem of solving for the equilibrium boundary of a ferrofluid toroid in the two cases $\varepsilon > 0$ and $\varepsilon < 0$.	105

4.2.1	Sketch of the field lines due to an infinitely long straight wire, shown in the plane with the wire located at the origin and the current directed into the plane. The direction of the field is determined using a right hand screw rule.	109
4.2.2	Reflection of the $\varepsilon > 0$ problem, as illustrated in figure 4.1.1a, in the z -axis.	112
4.4.1	Comparison of asymptotic and numerical solutions for ρ , where $Bo = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$	123
4.4.2	Comparison of asymptotic and numerical solutions for H_t , where $Bo = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$	124
4.4.3	Comparison of asymptotic and numerical solutions for ρ , where $Bo = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$	124
4.4.4	log-log plot of E_P (the error between the numerical and asymptotic solutions for the over-pressure P) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.	125
4.4.5	log-log plot of E_ρ (the error between the numerical and asymptotic solutions for the ferrofluid boundary ρ) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.	125
4.4.6	log-log plot of E_H (the error between the numerical and asymptotic solutions for the tangential component H_t) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.	126

4.4.7	log-log plot of E_B (the error between the numerical and asymptotic solutions for the normal component B_n) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.	126
4.4.8	Semi-log plot of the asymptotic and numerical solutions for the rightmost point of the ferrofluid boundary x_r against ε	127
4.4.9	log-log plot of E_x (the relative error between the asymptotic and numerical solutions for the rightmost point of the ferrofluid boundary x_r) against ε	127
5.2.1	Equilibrium boundaries for $Bo = 0.5$, each with a different fluid volume V_f	132
5.2.2	Equilibrium boundaries for $Bo = 2$, each with a different fluid volume V_f	133
5.2.3	Equilibrium boundaries for $Bo = 10$, each with a different fluid volume V_f	133
5.2.4	Equilibrium boundaries for $Bo = 0.5$, each with a different fixed value of x_l	134
5.2.5	Equilibrium boundaries for $Bo = 2$, each with a different fixed value of x_l	135
5.2.6	Equilibrium boundaries for $Bo = 10$, each with a different fixed value of x_l	135
5.2.7	Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 0.5$	136
5.2.8	Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 2$	136

5.2.9	Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 10$	137
5.2.10	Type I boundary for $Bo = 2$ and $V_f = 1$	138
5.2.11	Type II boundary for $Bo = 2$ and $V_f = 1$	138
5.2.12	Numerically computed H_t for the Type I and Type II boundaries obtained when $Bo = 2$ and $V_f = 1$	139
5.2.13	Numerically computed B_n for the Type I and Type II boundaries obtained when $Bo = 2$ and $V_f = 1$	139
5.3.1	Diagram showing the set-up and coordinates for a toroid of fluid pinned to the wire-loop.	141
5.3.2	Equilibrium boundary obtained in the $Bo = 0$ case, with $\tilde{\kappa}_0 = 0.003$	144
5.3.3	Equilibrium boundary obtained in the $Bo = 0$ case, with $\tilde{\kappa}_0 = 1$	144
5.3.4	Equilibrium boundary obtained in the $Bo = 0$ case, with $\tilde{\kappa}_0 = 10$	145
5.3.5	Plot of fluid volume $\hat{V}_0$ against $\hat{x}_0$ for equilibrium boundaries obtained when $Bo = 0$	145
5.3.6	Comparison of equilibrium boundaries for small values of Bo with the $Bo = 0$ boundary, for $V_f = 0.48$	146
5.3.7	Comparison of equilibrium boundaries for $Bo = 0.003$ and $Bo = 0$, where $V_f = 0.2$	147
5.3.8	Comparison of equilibrium boundaries for $Bo = 0.003$ and $Bo = 0$, where $V_f = 0.2$, close to the wire at the origin.	148
5.3.9	Equilibrium boundaries in the infinite Bond number case, each with a different fixed value of x_l	149

5.3.10	Plot of the fluid volume V_f against x_l for equilibrium boundaries obtained when the Bond number is infinite.	150
5.3.11	Type I equilibrium boundaries obtained for increasibly large values of the Bond number.	151
5.4.1	Plot of the maximum fluid volume V_{max} , for which a Type I equilibrium boundary can be shown to exist, against Bo.	152
5.4.2	Plot of the maximum fluid volume V_{max} against Bo for just small values of Bo.	153
5.4.3	Plot of the maximum fluid volume V_{max} against Bo for just large values of Bo.	153
5.4.4	Plots of V_{least} and V_{max} against Bo for just small values of Bo.	155
5.4.5	Plots of V_{least} and V_{max} against Bo for just large values of Bo.	156
5.4.6	Type II equilibrium boundary with the smallest observed fluid volume for Bo = 0.5.	156
5.4.7	Type II equilibrium boundary with the smallest observed fluid volume for Bo = 5.	157
5.4.8	Type II equilibrium boundary with the smallest observed fluid volume for Bo = 100.	157
6.1.1	The setup for the dimensionless problem of an axisymmetric ferrofluid droplet	163
6.2.1	The setup for the discretised problem of an axisymmetric ferrofluid droplet	167
6.3.1	Equilibrium boundaries for Bo = 0, each with a different fluid volume V_f	176

6.3.2	Equilibrium boundaries for $Bo = 5$, each with a different fluid volume V_f . In this instance, the fluid volume is becoming increasingly large.	177
6.3.3	Equilibrium boundaries for $Bo = 5$, each with a different fluid volume V_f . In this instance, the fluid volume is being decreased.	178
6.3.4	Equilibrium boundaries for $V_f = 3$, each with a different value of the Bond number Bo . Here the Bond number is becoming increasingly large.	179
6.3.5	Equilibrium boundaries for $V_f = 3$, each with a different value of the Bond number Bo . Here, the Bond number is becoming increasingly small. The $Bo = 0$ solution is also plotted.	179
6.3.6	Equilibrium boundaries for $Bo = 0.1$ for varying values of the rightmost point x_r	180
6.3.7	Equilibrium boundaries for $Bo = 5$ for varying values of the rightmost point x_r	181
6.3.8	Equilibrium boundaries for $Bo = 10$ for varying values of the rightmost point x_r	181
6.3.9	Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the rightmost point x_r for $Bo = 0.1$	182
6.3.10	Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the rightmost point x_r for $Bo = 5$	182
6.3.11	Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the rightmost point x_r for $Bo = 10$	183
A.0.1	Coordinate system for the closed cross-section of a surface of revolution, parameterised using plane polar coordinates.	189

-
- D.0.1 Plots of the complete elliptic integrals of the first and second
kind, denoted as E_k and E_e respectively. 200
- D.0.2 Semi-log plots of the functions I_1 and I_2 against m , taking $A = 1.203$

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Introduction

1.1 Toroidal Droplets

1.1.1 Background

It is well-known that, in the absence of any external forces, the equilibrium shape of a liquid droplet is spherical as a result of surface tension. Surface tension seeks to minimise the surface area for any volume of fluid, and the minimal surface that achieves this is that of a perfect sphere. Toroidal droplets are thus inherently unstable, and will seek to transform into the energetically favourable spherical droplet should they occur transiently as part of the fluid motion.

To gain an understanding of how surface tension acts as a driving mechanism for instability, we can consider a static droplet of fluid forming a perfect torus. The Laplace pressure difference ΔP across the fluid interface, is given by

$$\Delta P = \gamma \kappa, \tag{1.1.1}$$

where γ is the coefficient of surface tension and κ is the total curvature of the torus.

It can be shown that κ , and consequentially ΔP , must be larger on the outside region of the fluid interface than on the inside (Yao and Bowick, 2011). This pressure difference drives the torus to shrink until it collapses in on itself to form

a spherical droplet (Fragopolous et al., 2017).

Although unstable, toroidal droplets can be induced via the influence of external forces. This was first observed by Plateau (1857), who studied experimentally the nature of freely suspended droplets rotated at high speed. In doing so, he observed a variety of droplet shapes that were not spherical, including that of a torus. Plateau’s work was built upon by Brown and Scriven (1980), who studied computationally the equilibrium shapes and stability of the rotating drops first observed by Plateau. This study greatly furthered the understanding of nonspherical drops, with axisymmetric droplets, as well as two-lobed, three-lobed, and four-lobed droplets, being analysed and their equilibria and stability understood. However, whilst observed by Plateau, the analysis of Brown and Scriven (1980) did not extend to the toroidal droplet.

It is not just in a rotating fluid that toroidal droplets can be obtained. Droplets falling in an immiscible fluid can become toroidal over the course of their motion. For example, transient toroidal droplets are observed in the behaviour of falling raindrops (Villermaux and Bossa, 2009). Additionally, large viscoelastic drops falling through a viscous fluid can attain a toroidal shape as a steady state (Sostarecz and Belmonte, 2003). Alternatively, toroidal droplets can be formed after a spherical droplet of water impacts with a superhydrophobic surface (Renardy et al., 2003).

Whilst it can be seen that toroidal droplets exist, we know that they need necessarily collapse into spherical droplets once the influences of any external forces are removed. As such, a question can be asked about exactly how this collapse takes place. Pairam and Fernández-Nieves (2009) developed a simple method for generating toroidal droplets and analysing how they transformed into spherical droplets. This method involved the precise injection of liquid into a rotating bath containing a viscous continuous phase. The viscous drag acting on the injected fluid induces a curved jet at the exit of the injection needle. Should the rotational velocity be sufficiently large, then this curved jet forms a

toroidal droplet, which can then be studied as it evolves in time.

Pairam and Fernández-Nieves (2009) observed two mechanisms by which the generated toroidal droplets could collapse into spherical droplets. Firstly, the toroid could shrink until such time that it coalesced onto itself. Additionally, the well-known hydrodynamic Rayleigh–Plateau instability (Lord Rayleigh, 1892; Plateau, 1873), which is responsible for the breakup of cylindrical fluid columns and jets, could cause the breakup of the toroidal droplet into multiple spheres. Moreover, the aspect ratio of the toroids (ratio of the major radius to the minor radius) greatly impacted the route by which the toroids transformed. Whilst the shrinking instability was found to be present regardless of the aspect ratio of the toroid, the Rayleigh–Plateau instability was only observed for sufficiently thin toroids. Therefore, it is possible for thin toroidal droplets to collapse into multiple spheres, whilst sufficiently fat toroidal droplets always collapse into a single sphere.

Following on from the work of Pairam and Fernández-Nieves (2009), a number of theoretical studies have been conducted in order to further understand the two mechanisms of toroidal droplet collapse. For example, Mehrabian and Feng (2013) produced a numerical investigation into the nature of the capillary breakup of a liquid torus, including an analysis of the growth of the capillary waves. Furthermore, work by Yao and Bowick (2011), Zabarankin et al. (2015), and Fragopolous et al. (2017) has culminated in a comprehensive description of the shrinking instability in the Stokes flow regime.

Of a particular interest here is the work conducted by Zabarankin et al. (2015). Here, an investigation was carried out into the deformation of a toroidal drop embedded in axisymmetric compressional Stokes flow. The viscosity of the toroidal droplet was assumed to be the same as that of the surrounding fluid. It was found that, as long as the intensity of the flow was weak in comparison to the surface tension, a stationary toroidal shape could be achieved as part of the droplet’s evolution. However, if no flow is present whatsoever, then the toroidal

droplet collapsed in accordance with the experimental observations made by Pairam and Fernández-Nieves (2009).

Crucially, it follows that toroidal droplets can be seen to exist in equilibrium, at least in the case of viscous toroidal droplets surrounded by viscous fluid. We have so far considered examples of viscoelastic toroidal droplets being in equilibrium when falling through a viscous fluid (Sostarecz and Belmonte, 2003), and viscous toroidal droplets being in equilibrium when embedded in a Stokes flow of relatively weak intensity (Zabaranin et al., 2015).

Additionally, Gulliver (1984) provides a rigorous mathematical proof of the existence of toroidal solutions to the equilibrium equations governing the rotating droplets observed by Plateau (1857). We notice that a commonality between all scenarios in which toroidal droplets have been shown to be in equilibrium is the presence of fluid flow. We know that a static toroidal droplet, in the absence of any fluid flow, must collapse into a sphere. However, we can ask the question of whether said toroidal droplet is ever in equilibrium, albeit an equilibrium that is unstable.

1.1.2 Static Toroidal Droplets

Consider a static axisymmetric toroidal volume of fluid under surface tension and in the absence of gravity. We consider the two-dimensional problem for the shape of the cross-section of the toroid in a meridional plane, such that the toroid is in equilibrium. A schematic of the problem is shown in figure 1.1.1. We work in Cartesian (x, z) coordinates and define s to be the arc-length around the cross-section of the toroid, with $s = 0$ corresponding to the point $x = x_0, z = 0$, which is taken to be the inner-most point of the cross-section in the plane. We denote the angle the tangent vector at a point on the boundary makes with the vertical by ϑ . Finally, we let V be the incremental fluid volume between $x = x_0$ and $x = x$.

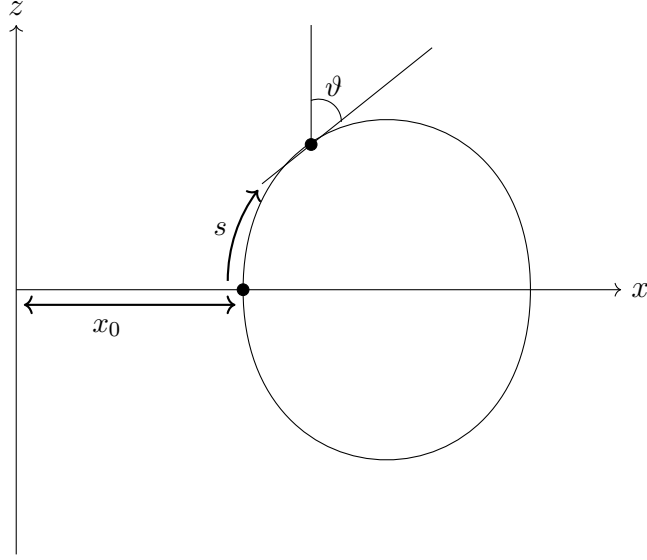


Figure 1.1.1: Diagram showing the set-up and coordinates for the cross-section of a static fluid toroid.

The governing equations can be obtained as follows. Firstly, as shown in appendix A, the total curvature κ is given by

$$\kappa = \frac{d\vartheta}{ds} - \frac{\cos \vartheta}{x}. \quad (1.1.2)$$

Secondly, by considering standard trigonometric arguments on infinitesimal changes in x , z , and s , we obtain

$$\frac{dx}{ds} = \sin \vartheta, \quad (1.1.3)$$

$$\frac{dz}{ds} = \cos \vartheta. \quad (1.1.4)$$

Finally, a consideration of an infinitesimal volume element of the fluid, yields

$$\frac{dV}{dx} = 4\pi xz. \quad (1.1.5)$$

For the droplet to be in equilibrium, the pressure difference across the surface of the fluid must be balanced by the surface tension, as in equation (1.1.1). Moreover, for a static solution, the pressure difference across the fluid surface must be constant. As such, since the surface tension γ is also a constant, the

curvature κ must be constant.

Next, we use equation (1.1.2) to eliminate s in (1.1.4), yielding

$$\frac{dz}{d\vartheta} = \frac{\cos \vartheta}{\left(\kappa + \frac{\cos \vartheta}{x}\right)}. \quad (1.1.6)$$

Should a smooth solution exist for the shape of the fluid cross-section, we would need to be able to solve equation (1.1.6), subject to the boundary conditions $z(0) = z(\pi) = 0$.

It can be shown that (1.1.6) can never be satisfied subject to $z(0) = z(\pi) = 0$. To see this, we consider the modified equation

$$\frac{dz^*}{d\vartheta} = \frac{\cos \vartheta}{\kappa}, \quad (1.1.7)$$

subject to $z^*(0) = z^*(\pi) = 0$. Since κ is a constant, we have that

$$z^* = \frac{\sin \vartheta}{\kappa}, \quad (1.1.8)$$

is the solution to this problem. Now, for $0 < \vartheta < \pi/2$, we have that

$$0 < \frac{dz}{d\vartheta} < \frac{dz^*}{d\vartheta}, \quad (1.1.9)$$

and $z(0) = z^*(0) = 0$. Therefore, it must be the case that $z(\pi/2) < z^*(\pi/2)$.

However, for $\pi/2 < \vartheta < \pi$, we have that

$$\frac{dz}{d\vartheta} < \frac{dz^*}{d\vartheta} < 0, \quad (1.1.10)$$

and $z(\pi) = z^*(\pi) = 0$. Therefore, it must be the case that $z(\pi/2) > z^*(\pi/2)$.

We thus have a contradiction, proving that no solution to (1.1.6) with $z(0) = z(\pi) = 0$ exists.

To see an explicit example of this, we note that we can use (1.1.2) to eliminate s

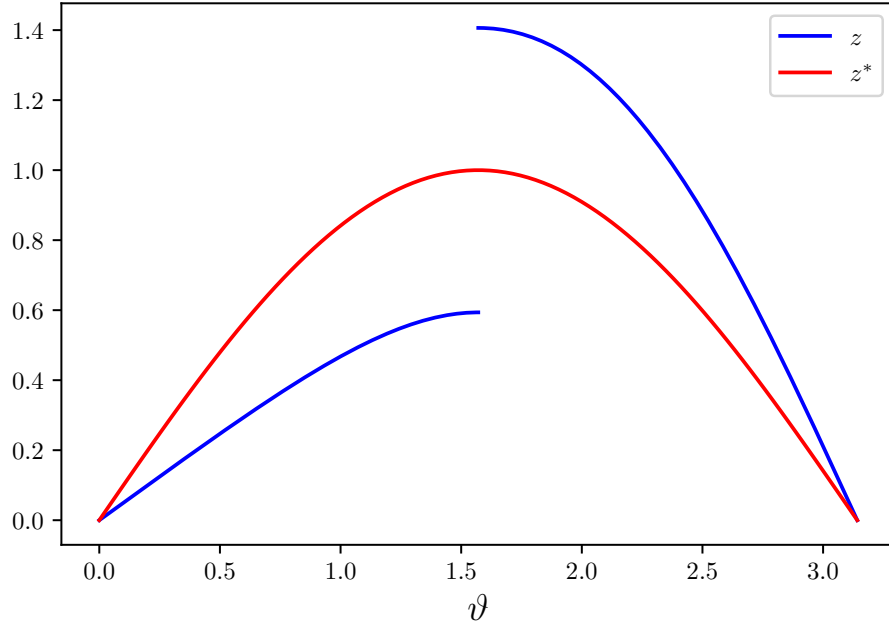


Figure 1.1.2: Plots of the numerically computed z , and analytically known z^* , for the system of ODEs describing a smooth axisymmetric toroidal droplet.

in (1.1.3), yielding

$$\frac{dx}{d\vartheta} = \frac{\sin \vartheta}{\left(\kappa + \frac{\cos \vartheta}{x}\right)}. \quad (1.1.11)$$

We then seek to solve (1.1.6) and (1.1.11) as a system of coupled ODEs, subject to $z(0) = z(\pi) = 0$ and $x(\pi) > x(0) = x_0$. We can then pick values of κ , x_0 , and $x(\pi)$, and integrate the ODEs numerically. The numerical integration required for solving the ODEs present in this problem is carried out in Python, using the *solve_ivp* algorithm from the SciPy package. *solve_ivp* solves initial value problems for systems of ODEs using a form of Runge–Kutta integration.

We choose to take $\kappa = 1$, $x_0 = 1$, and $x(\pi) = 3$. First, we integrate (1.1.6) and (1.1.11) from $\vartheta = 0$ to $\vartheta = \pi/2$. Then, we integrate the same equations from $\vartheta = \pi$ to $\vartheta = \pi/2$. Figure 1.1.2 shows a plot of the numerically computed z for this choice of parameters, in addition to the corresponding z^* , as given by (1.1.8). As predicted, z is discontinuous at $\vartheta = \pi/2$ and so a static smooth axisymmetric toroidal droplet cannot exist in equilibrium here.

We have thus shown that static toroidal droplets cannot exist in equilibrium, at

least not in the absence of external forces. Moreover, toroidal droplets that exist in equilibrium as a result of rotation or viscosity, as discussed in section 1.1.1, are not static.

Static liquid toroids have been observed, for example, when liquid oxygen in the Leidenfrost state is moulded into a ring through the use of an annular magnet (Texier et al., 2013), toroidal films on a solid substrate are generated using dielectrowetting (Edwards et al., 2021), and, when liquid is placed into an axisymmetric angled groove machined into a circular substrate and coated with a superhydrophobic spray (Falcon et al., 2021). In each case, only part of the surface of the toroid is open to the air. As such, static toroidal droplets with an entirely free surface have not been observed.

We are thus motivated to investigate a possible mechanism of external forcing by which static toroidal droplets, with a surface completely open to the surrounding air, can exist in equilibrium. In particular, we investigate the possible equilibria of static toroidal droplets of ferrofluid in the presence of an applied magnetic field.

1.2 Ferrofluids

1.2.1 An Introduction to Ferrofluids

Ferrofluids are colloidal fluids, comprised of nanoscale particles of magnetic solid suspended in a carrier liquid (Cowley and Rosensweig, 1967). The exact magnetic solid, or indeed carrier liquid, which is used can vary between different types of ferrofluid. Typical examples of the magnetic solid include magnetite, hematite, and cobalt, whilst the carrier fluid is typically water, kerosene, or some kind of oil (Rosenweig, 1985). Agglomeration of the magnetic nanoparticles is typically inhibited through the coating of these nanoparticles with a steric layer of surfactant, such as oleic acid (a naturally occurring fatty

acid) (Shilomis, 1974; Rosensweig, 1985).

Ferrofluids are viscous Newtonian fluids, which, in the absence of any external magnetic fields, behave according to the equations governing Newtonian fluid flow. However, when a magnetic field is applied ferrofluids become magnetised, which in turn can cause a magnetic body force to act upon the ferrofluid. This magnetisation is a result of the alignment of the nanoparticles, each of which is considered as a magnetic dipole with a net magnetic moment, with the applied magnetic field (Rosenweig, 1985). This induced alignment in the direction of the field is a property known as paramagnetism. Paramagnetism typically occurs in nature when atoms have unpaired electrons (see Griffiths, 1981, for example). The magnetisation observed in ferrofluids hence occurs at much lower field strengths than that required for ordinary paramagnets. As such, ferrofluids are classified as exhibiting superparamagnetism. Notably, as they are paramagnetic, ferrofluids do not retain their magnetisation when the applied magnetic field is removed.

Ferrofluids are synthetic liquids, with interesting origins in the field of space travel. Arrighi (2021) gives an account of the first recorded synthesis of ferrofluids, back in the 1960s. At that time, rocket designers working for NASA were growing increasingly interested in methods through which it could be ensured that liquid rocket fuel could be pumped into the combustion chamber. As the engines may need to be restarted during missions, it was crucial that any method of pumping would work in the absence of gravity. Steven Papell, a World War II veteran turned mechanical engineer, was working for NASA at the time, at what is now known as the ‘Glenn Research Center’. As the interest in space travel increased, Papell switched his focus to low-gravity fluid mechanics. Papell’s research ultimately led to him suggesting the use of a stable suspension of finely ground iron oxide particles in the standard rocket fuel. Such fuel could then be drawn into the combustion chamber by the use of an electromagnet.

Papell’s magnetic rocket fuel represented the first synthetically produced

ferrofluid and, most significantly, it was found to be fit for purpose. However, around the same time, alternative methods of pumping the rocket fuel into the combustion chamber were developed, which proved to be simpler and cheaper. As such, ferrofluids were, unfortunately, never actually used as rocket fuel. However, although originally developed to be used as rocket fuel, ferrofluids have been found to have multiple applications much closer to home. A key appeal of ferrofluids is the ability to hold them in place, or move them to a specific location, by means of a non-invasive magnetic field.

Ferrofluids are commonly used as a sealant in spinning disc drives, such as the hard-drive of a computer (Raj et al., 2004). In such drives, ferrofluids form a microfluidic seal between the spinning disc and the surrounding casing, protecting the inside of the drive without restricting the rotation of the disc. Ferrofluids are also often found in loudspeakers, wherein they serve more than one purpose (Scherer and Figueiredo Neto, 2005). Firstly, ferrofluids can be seen to dissipate the heat produced by the vibrating voice coil producing the sound in the speaker, thus facilitating cooling of the speaker. Additionally, ferrofluids act to dampen unwanted resonances produced by the coil, thus improving the sound quality of the speaker.

Ferrofluids also have a variety of interesting biomedical applications. Contrast in MRI can be enhanced through the use of biocompatible ferrofluids, whereby different types of tissue uptake different amounts of the magnetic nanoparticles suspended in the ferrofluid and, as such, can become increasingly distinguishable on an MRI scan (Pankhurst et al., 2003). Additionally, the use of ferrofluids in magnetic drug targeting has been investigated (Asfer et al., 2017), whereby a drug can bind with the nanoparticles in the ferrofluid and then be appropriately directed to the target site via a non-invasive magnetic field. The final application considered here, is that of Magnetic Hyperthermia (MH) as a treatment for cancer. In MH, ferrofluid is injected into the site of the tumour and a rapidly oscillating current field, generating a magnetic field, is applied. As a result, the ferrofluid heats up and releases heat into the

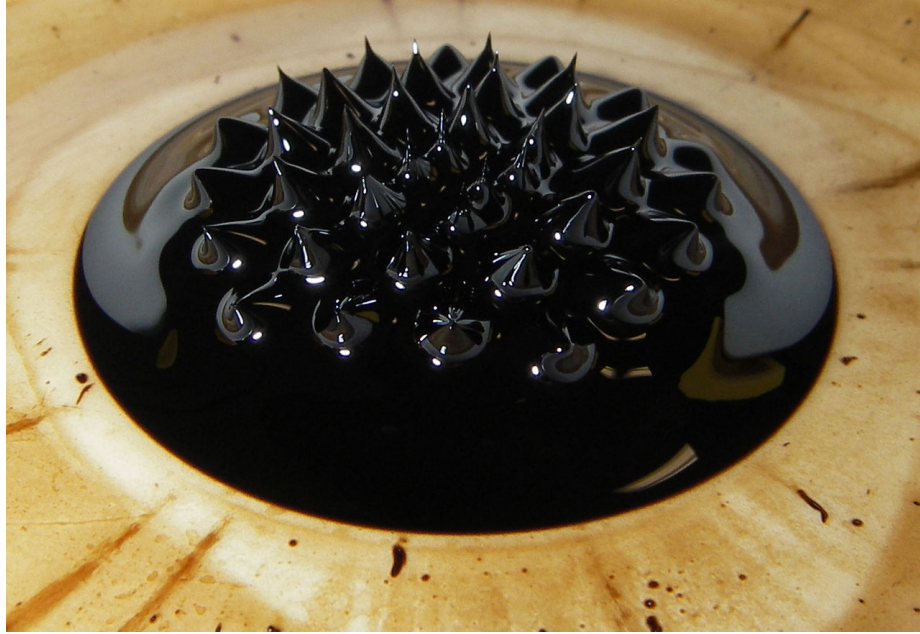


Figure 1.2.1: The characteristic peaks and valleys found on the surface of a ferrofluid subject to a sufficiently strong magnetic field (Arrighi, 2021).

surrounding cancerous tissue, which can kill the cancerous cells (Liu, 2020; Scherer and Figueiredo Neto, 2005).

1.2.2 Ferrofluids and Instability

The nature of the magnetic body force experienced by a magnetised ferrofluid is discussed further in Chapter 2. However, of specific interest is the potential for this force to either drive or inhibit fluid instabilities. The first, and most famous, instability discovered in ferrofluids was that of the Normal-Field (or Rosensweig) instability (Cowley and Rosensweig, 1967). As its name suggests, this instability can arise when an applied magnetic field has a component normal to the interface between a ferrofluid and a surrounding non-magnetic fluid. In this instance, the magnetic force is destabilising, opposing surface-tension and gravity. For a sufficiently strong field, the surface is deformed into a characteristic configuration of peaks and valleys, which is the minimum energy configuration for the system. An image showing the characteristic ‘spikey’ surface as a result of this instability is given in figure 1.2.1, taken from Arrighi (2021).

On the other hand, there are circumstances under which the magnetic body force experienced by a ferrofluid can stabilise an otherwise unstable configuration. For example, consider the case of a long cylindrical column or jet of fluid. It is well-known that such columns and jets are unstable and break up into spherical droplets, a phenomenon known as the Rayleigh–Plateau instability (Lord Rayleigh, 1892; Plateau, 1873).

Arkhipenko et al. (1980) considered the case of a stationary cylindrical column of ferrofluid surrounding a straight length of current-carrying wire, which produces a magnetic field. Ordinarily, such columns would be expected to break up into droplets as a result of the Rayleigh–Plateau instability, however, there is another force at play here. Arkhipenko et al. (1980) showed, both theoretically and experimentally, that the magnetic body force could stabilise the cylindrical column for sufficiently large magnetic field strengths.

In the theory, the magnetic field was taken to be that produced by an infinitely long current-carrying wire, which is purely tangential to a cylindrical fluid column surrounding the wire. In such case, the cylindrical column was shown to be an equilibrium solution to the governing equations. Gravity was neglected and a magnetic Bond number Bo_m introduced, quantifying the relative strength of magnetic effects to surface tension effects. It was shown, via a linear stability analysis of wave-like perturbations introduced to the cylindrical column, that for $Bo_m > 1$ the column is stable to all wave-like perturbations considered. However, for $Bo_m < 1$, there is a wave number for which the perturbations become unstable, causing an instability of the column and yielding the decay into droplets.

Experimentally, a thin conductor surrounded by ferrofluid was placed into an aqueous solution of glycerine with a density equal to that of the ferrofluid. This neutral bouyancy allowed for the effect of gravity to be ignored. A current was established in the conductor corresponding to a value of $Bo_m > 1$. As predicted, the ferrofluid took the shape of a cylindrical fluid column. As Bo_m was reduced below unity, the cylindrical column became unstable and the break-down to

droplets observed.

The stabilising effect of the magnetic body force observed by Arkhipenko et al. (1980), provides encouragement that other fluid configurations that are known to be subject to instability, such as that of a toroidal droplet, may also be stabilised by virtue of the magnetic body force. We return now to our consideration of an axisymmetric static toroidal droplet of fluid. Moreover, we note that such toroidal droplets are not only unstable, but do not exist as an equilibrium configuration at all. The primary aim of this thesis is to investigate the possibility that the magnetic body force allows for static toroidal droplets of ferrofluid to exist in equilibrium. The study of the equilibria of static toroidal droplets of ferrofluid is entirely novel.

1.3 Outline

The primary setup of consideration is that of an axisymmetric toroidal volume of ferrofluid surrounding a circular loop of current-carrying wire. In Chapter 2, we develop a mathematical model that governs the behaviour of the described ferrofluid toroid. Specifically, we derive a boundary-integral representation for the present magnetic field, and a dynamic boundary condition pertaining to the balance of normal stresses on the free surface of the ferrofluid.

In Chapter 3, we discuss how the obtained mathematical model can be discretised and solved numerically using a boundary-integral technique. Moreover, we present some preliminary results suggesting the existence of ferrofluid toroids in equilibrium, and verify the sensible convergence of these solutions as the number of discretisation points is increased.

In Chapter 4, we validate the numerical method discussed in Chapter 3 by considering the special case of a toroidal droplet with a small aspect ratio (that is, a characteristic minor radius much smaller than the major radius). In this limit, an analytic solution in the form of an asymptotic expansion is obtained,

up to the second-order correction terms, and these are compared with the corresponding numerical solution. We show a strong agreement between the asymptotic solutions and the numerical solutions.

The results of the numerical analysis of toroidal droplets of ferrofluid in equilibrium are presented, and discussed, in Chapter 5. We find that there is a global maximum fluid volume beyond which ferrofluid toroids cannot exist in equilibrium, regardless of the strength of the applied magnetic field. Moreover, the nature of toroidal droplets in equilibrium is characterised by a pair of parameters, the fluid volume and the magnetic Bond number. In particular, for any given Bond number, we find that there is a local maximum fluid volume beyond which toroidal droplets do not exist in equilibrium. However, for any fluid volume below this maximum, a toroidal droplet can always be shown to exist in equilibrium and its shape can be determined numerically. Additionally, for a fixed pair of values for the Bond number and the fluid volume for which a solution exists, we find that there may exist a second, qualitatively different, toroidal droplet that is also in equilibrium. The existence and shape of these secondary solutions is considered.

In Chapter 6, we consider how the mathematical model derived in Chapter 2 can be extended to account for axisymmetric simply-connected ferrofluid droplets. A brief analysis of the equilibria of these droplets, obtained numerically via a method analogous to that developed in Chapter 3, is considered.

In Chapter 7, we conclude the thesis by summarising the key results from the analysis and providing suggestions for future work and possible applications.

Development of the Mathematical Model

In this chapter, we focus on deriving a mathematical model that describes a static toroidal volume of ferrofluid surrounding a circular loop of current-carrying wire. The aim is to derive an equation governing the magnetic field, accounting for the specific geometry of the problem of interest. Additionally, we will seek to derive a dynamic boundary condition capturing the balance of forces on the free surface of the ferrofluid.

2.1 Maxwell's Equations

2.1.1 Maxwell's Equations in a Vacuum

In order to develop the mathematics that underpins ferrohydrodynamics (FHD), we need to familiarise ourselves with some fundamental ideas from electrodynamics.

Electric and magnetic fields are generated by charges and currents in accordance with Maxwell's equations. For an electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, charge density ρ , and current density $\mathbf{J}$, Maxwell's equations are given by (see Griffiths,

1981, for example)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad (2.1.1)$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.1.2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.1.3)$$

$$\nabla \wedge \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad (2.1.4)$$

where ε_0 and μ_0 are constants known as the permittivity and permeability of free space, respectively.

The above form of Maxwell's equations are most useful when studying electrodynamics in a vacuum. If we wish to consider the nature of electrodynamics in matter, then a more useful form of Maxwell's equations can be presented. In order to obtain this preferable form of Maxwell's equations, we must first briefly discuss the notions of polarisation and magnetisation.

2.1.2 Polarisation and Magnetisation

Electrically conductive materials contain charges that are free to move around the material. These charges are in the form of electrons that are not bound to any specific atom or molecule comprising the material. Dielectric materials are electrically insulative, with all charges confined to the atoms or molecules comprising the material. However, this does not mean that no response is elicited when a dielectric material is placed in an electric field.

Neutral atoms are made up of a positively charged nucleus and a negatively charged cloud of electrons. When placed in an electric field, the nucleus is shifted in the direction of the electric field, whilst the electron cloud is shifted in the opposite direction. This creates an electric dipole moment aligned with the electric field. Additionally, polar molecules (such as water), which act as permanent dipoles, will tend to align with the direction of any applied electric

field.

Hence, when an electric field is applied to a material, even a dielectric material, electric dipole moments that point in the direction of the field are induced in the atoms or molecules which make up the material. When this occurs the material is said to have become polarised. We define the polarisation $\mathbf{P}$, to be the density of these dipole moments. The alignment of dipole moments creates a distribution of charge within the material, which is known as the bound charge (Griffiths, 1981).

Defining ϱ_b to be the bound charge density, we have that

$$\varrho_b = -\nabla \cdot \mathbf{P}. \quad (2.1.5)$$

The total charge density for a material can be written as

$$\varrho = \varrho_b + \varrho_f, \quad (2.1.6)$$

where ϱ_f is called the free charge density and is made up of all charge which is not a result of polarisation, such as the unbound electrons found in electrically conductive materials. If the polarisation field is changing in time, then this corresponds to a motion of the bound charge. The rate of change of the bound charge density corresponds to a current density $\mathbf{J}_p$, known as the polarisation current density, which is given by

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}. \quad (2.1.7)$$

Magnetisation is a very similar idea to polarisation. When a material is placed in a magnetic field, an alignment of magnetic dipole moments is induced within the material. Magnetic dipole moments can be thought of as tiny current loops (Griffiths, 1981). We define the magnetisation $\mathbf{M}$ of a material, to be the density of the magnetic dipole moments. The alignment of these magnetic dipole moments leads to a distribution of current known as the bound current,

as it is fixed by the atomic structure of the material. Defining $\mathbf{J}_b$ to be the bound current density, we have that

$$\mathbf{J}_b = \nabla \wedge \mathbf{M}. \quad (2.1.8)$$

Magnetisation is more complicated than polarisation, as the alignment of the dipole moments is not necessarily in the same direction as the magnetic field. The total current density of a material can be written as

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_p + \mathbf{J}_b, \quad (2.1.9)$$

where $\mathbf{J}_f$ is called the free current density and consists of all current that is not a result of magnetisation or polarisation.

Using (2.1.5)–(2.1.9), we can rewrite (2.1.1) and (2.1.4) as

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \varrho_f, \quad (2.1.10)$$

and

$$\nabla \wedge \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}), \quad (2.1.11)$$

respectively. We now define the displacement field $\mathbf{D}$, and the auxiliary field $\mathbf{H}$, via the relations

$$\mathbf{E} = \frac{1}{\varepsilon_0} (\mathbf{D} - \mathbf{P}), \quad (2.1.12)$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (2.1.13)$$

so that (2.1.10) and (2.1.11) become

$$\nabla \cdot \mathbf{D} = \varrho_f, \quad (2.1.14)$$

and

$$\nabla \wedge \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \quad (2.1.15)$$

respectively. We are now able to present an equivalent form of Maxwell's equations that are more convenient for the study of electrodynamics in matter. In particular,

$$\nabla \cdot \mathbf{D} = \varrho_f, \quad (2.1.16)$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.1.17)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.1.18)$$

$$\nabla \wedge \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \quad (2.1.19)$$

At this stage, a comment needs to be made around the somewhat awkward nomenclature regarding the $\mathbf{B}$ and $\mathbf{H}$ fields. It is very common in the literature to see $\mathbf{H}$ referred to as the magnetic field, whilst $\mathbf{B}$ is given some other name such as the magnetic flux density or magnetic induction. In keeping with this, $\mathbf{H}$ will hereafter be referred to as the magnetic field, whilst $\mathbf{B}$ will be referred to as the magnetic induction.

2.1.3 Constitutive Laws

For our purposes, we require a constitutive law that relates the magnetisation to the magnetic field. For nonlinear isotropic magnetisation, which is typical of colloidal ferrofluids (Rosensweig, 1985), the constitutive law is given by

$$\mathbf{M} = \chi_m(\mathbf{H})\mathbf{H}. \quad (2.1.20)$$

The quantity χ_m is called the magnetic susceptibility, and is a measure of how strongly a material will become magnetised in the presence of an applied magnetic field. An analogous quantity called the electrical susceptibility χ_e , measures how strongly a material will become polarised in the presence of an applied electric field. As such, nonlinear isotropic polarisation is described by the constitutive

law

$$\mathbf{P} = \varepsilon_0 \chi_e(\mathbf{E}) \mathbf{E}. \quad (2.1.21)$$

Defining the permeability of a material as $\mu = \mu_0 (1 + \chi_m)$, we notice that using (2.1.13) and (2.1.20), we can write

$$\mathbf{B} = \mu(\mathbf{H}) \mathbf{H}. \quad (2.1.22)$$

Similarly, we could define the permittivity of a material as $\varepsilon = \varepsilon_0 (1 + \chi_e)$. Then, using (2.1.12) and (2.1.21), we have that

$$\mathbf{D} = \varepsilon(\mathbf{E}) \mathbf{E}. \quad (2.1.23)$$

2.2 Equation of Motion for a Ferrofluid

2.2.1 The Stress Tensor for a Ferrofluid

The magnetic stress tensor as presented by Cowley and Rosensweig (1967) provides a quantitative description of the magnetic force and its distribution. An abridged form of this derivation is given in appendix B. Here, we will simply state the form of the magnetic stress tensor from Cowley and Rosensweig (1967). For a ferrofluid with magnetisation $\mathbf{M}$ and density ρ , in the presence of a magnetic field $\mathbf{H}$, the magnetic stress tensor $\mathbf{T}_m$, is given by

$$\mathbf{T}_m = - \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B} \mathbf{H}, \quad (2.2.1)$$

where $\nu = \rho^{-1}$ is the specific volume of the ferrofluid, $M = |\mathbf{M}|$, $H = |\mathbf{H}|$, T is the absolute temperature of the ferrofluid, $\mathbf{I}$ is the identity tensor, and $\mathbf{B} = \mu(\mathbf{H}) \mathbf{H}$, as in (2.1.22), with μ being the permeability of the ferrofluid. The subscript H, T in the integral term in (2.2.1) indicates that the differentiation is being carried out with H and T held constant.

Note that if $\mathbf{M} = \mathbf{0}$, then the integral in (2.2.1) is zero, and so (2.2.1) reduces to the standard Maxwell stress tensor in the absence of electric fields (see Jackson, 1962, for example).

The magnetic force density $\mathbf{f}_m$, is obtained from the magnetic stress tensor via the relation

$$\mathbf{f}_m = \nabla \cdot \mathbf{T}_m. \quad (2.2.2)$$

Using (2.2.1), we obtain

$$\mathbf{f}_m = -\nabla \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} + \mathbf{H} (\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{H}. \quad (2.2.3)$$

We note that, from Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$. Additionally, since the vectors $\mathbf{B}$ and $\mathbf{H}$ are collinear, as a consequence of (2.1.22), we have that

$$(\mathbf{B} \cdot \nabla) \mathbf{H} = \frac{B}{H} (\mathbf{H} \cdot \nabla) \mathbf{H}, \quad (2.2.4)$$

where $B = |\mathbf{B}|$. We can now make use of the vector identity

$$(\mathbf{F} \cdot \nabla) \mathbf{F} = \nabla \left(\frac{1}{2} |\mathbf{F}|^2 \right) - \mathbf{F} \wedge (\nabla \wedge \mathbf{F}), \quad (2.2.5)$$

which holds for any arbitrary vector $\mathbf{F}$ (see Acheson, 1990, for example), to obtain

$$(\mathbf{H} \cdot \nabla) \mathbf{H} = \nabla \left(\frac{1}{2} |\mathbf{H}|^2 \right) - \mathbf{H} \wedge (\nabla \wedge \mathbf{H}). \quad (2.2.6)$$

From Maxwell's equations, we have that

$$\nabla \wedge \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \quad (2.2.7)$$

However, if we assume magnetostatics, that is that all currents are steady, then we have that

$$\nabla \wedge \mathbf{H} = \mathbf{J}_f, \quad (2.2.8)$$

where the free current $\mathbf{J}_f$ must be independent of time. Moreover, for a non-

conducting ferrofluid $\mathbf{J}_f = \mathbf{0}$ as there is no free current in a non-conducting material. We thus have that $\mathbf{H}$ is curl-free, and so

$$(\mathbf{H} \cdot \nabla) \mathbf{H} = \nabla \left(\frac{1}{2} |\mathbf{H}|^2 \right) = \frac{1}{2} \nabla (H^2). \quad (2.2.9)$$

It follows that

$$(\mathbf{B} \cdot \nabla) \mathbf{H} = \frac{B}{H} \frac{1}{2} \nabla (H^2) = B \nabla H. \quad (2.2.10)$$

The magnetic force density (2.2.3) hence becomes

$$\mathbf{f}_m = -\nabla \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} + B \nabla H, \quad (2.2.11)$$

which can be rewritten as

$$\mathbf{f}_m = -\nabla \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH \right\} + \mu_0 M \nabla H, \quad (2.2.12)$$

using the relationship $B = \mu_0(H + M)$, which follows from (2.1.13) when $\mathbf{H}$, $\mathbf{B}$ and $\mathbf{M}$ are collinear. Note that the collinearity of $\mathbf{M}$ with $\mathbf{H}$ and $\mathbf{B}$ follows from the isotropic constitutive laws (2.1.20) and (2.1.22).

We now define the magnetostrictive pressure

$$p_s = \mu_0 \int_0^H \nu \left(\frac{\partial M}{\partial \nu} \right)_{H,T} dH, \quad (2.2.13)$$

and the fluid-magnetic pressure

$$p_m = \mu_0 \int_0^H M dH, \quad (2.2.14)$$

such that (2.2.12) can be written as

$$\mathbf{f}_m = -\nabla (p_s + p_m) + \mu_0 M \nabla H. \quad (2.2.15)$$

The magnetic stress tensor is not the complete stress tensor for a ferrofluid, as

it accounts only for magnetic effects and not the other effects one would expect to see acting on a fluid, such as pressure and viscosity. For an incompressible Newtonian fluid, the stress tensor is given by (see Acheson, 1990, for example)

$$\mathbf{T}_f = -p\mathbf{I} + \eta \left\{ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right\}, \quad (2.2.16)$$

where p is the fluid pressure, η is a viscosity coefficient, and $\mathbf{u}$ is the fluid velocity. The stress tensor for an incompressible Newtonian ferrofluid is therefore given by

$$\mathbf{T} = \mathbf{T}_m + \mathbf{T}_f, \quad (2.2.17)$$

where $\mathbf{T}_m$ is as in (2.2.1).

Moreover, since $\mathbf{T}_m$ reduces to the Maxwell stress tensor when $\mathbf{M} = \mathbf{0}$, (2.2.17) remains valid for non-magnetic fluids, such as air, where we can take $\mathbf{M} = \mathbf{0}$ due to the extremely weak density of induced magnetic dipole moments in such fluids.

2.2.2 The Navier–Stokes Equations with Magnetisation

The Navier–Stokes equations for an incompressible Newtonian fluid, are given by (see Acheson, 1990, for example)

$$\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{g}, \quad (2.2.18)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.2.19)$$

where ρ is the fluid density, p is the fluid pressure, η is a viscosity coefficient, $\mathbf{T}$ is the stress tensor for the fluid, $\mathbf{u}$ is the fluid velocity, $\mathbf{g}$ is the gravitational vector, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla), \quad (2.2.20)$$

is the material derivative.

For an incompressible Newtonian ferrofluid, we have that the stress tensor $\mathbf{T}$, is given by (2.2.17). Hence, we have that

$$\nabla \cdot \mathbf{T} = \nabla \cdot \mathbf{T}_m + \nabla \cdot \mathbf{T}_f. \quad (2.2.21)$$

From (2.2.2) and (2.2.15), we have that

$$\nabla \cdot \mathbf{T}_m = -\nabla (p_s + p_m) + \mu_0 M \nabla H. \quad (2.2.22)$$

Moreover, it is easily shown that for an incompressible fluid

$$\nabla \cdot \mathbf{T}_f = \nabla \cdot \left(-p\mathbf{I} + \eta \left\{ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right\} \right) = -\nabla p + \eta \nabla^2 \mathbf{u}. \quad (2.2.23)$$

It follows that

$$\nabla \cdot \mathbf{T} = -\nabla (p_s + p_m) + \mu_0 M \nabla H - \nabla p + \eta \nabla^2 \mathbf{u}. \quad (2.2.24)$$

We now define

$$p^* = p + p_s + p_m, \quad (2.2.25)$$

to be a composite pressure, which yields

$$\nabla \cdot \mathbf{T} = -\nabla p^* + \eta \nabla^2 \mathbf{u} + \mu_0 M \nabla H. \quad (2.2.26)$$

Finally, upon substituting (2.2.26) into (2.2.18), we obtain

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p^* + \eta \nabla^2 \mathbf{u} + \rho \mathbf{g} + \mu_0 M \nabla H. \quad (2.2.27)$$

Equation (2.2.27) is the equation of motion for a ferrofluid in the presence of a magnetic field.

2.2.3 The FHD Bernoulli Equation

If the ferrofluid is assumed to be inviscid and have uniform density, then we can obtain the so-called FHD Bernoulli equation as an integral of the equation of motion (2.2.27).

We begin by expanding the material derivative and dropping the viscous term $\eta \nabla^2 \mathbf{u}$ in (2.2.27), to obtain

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p^* + \rho \mathbf{g} + \mu_0 M \nabla H. \quad (2.2.28)$$

We now make use of the vector identity (2.2.5), to note that

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) - \mathbf{u} \wedge (\nabla \wedge \mathbf{u}). \quad (2.2.29)$$

Additionally, by an extension of the Leibniz formula, we have that

$$\nabla \int_0^H M \, dH = M \nabla H + \int_0^H \nabla M \, dH, \quad (2.2.30)$$

where ∇M is evaluated at constant H . If M depends only on the field H and the absolute temperature T , as is the case for non-isothermal but sensibly constant-density materials (Rosensweig, 1985), then we can write (2.2.30) as

$$\nabla \int_0^H M \, dH = M \nabla H + \int_0^H \frac{\partial M}{\partial T} \nabla T \, dH. \quad (2.2.31)$$

Moreover, if we assume that the flow-field is isothermal then $\nabla T = 0$, and so

$$\nabla \int_0^H M \, dH = M \nabla H. \quad (2.2.32)$$

We now substitute (2.2.29) and (2.2.32) into (2.2.28), to obtain

$$\rho \frac{\partial \mathbf{u}}{\partial t} - \rho \mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = -\nabla \left(p^* + \rho \frac{|\mathbf{u}|^2}{2} - \rho \mathbf{g} \cdot \mathbf{x} - \mu_0 \int_0^H M \, dH \right), \quad (2.2.33)$$

where $\mathbf{x}$ is the position vector and the gravitational force and fluid density are

assumed to be constant. If we suppose that the fluid is irrotational, then $\nabla \wedge \mathbf{u} = \mathbf{0}$, and we can introduce a velocity potential ϕ , such that $\mathbf{u} = \nabla \phi$. Hence, we can write (2.2.33) as

$$\nabla \left(\rho \frac{\partial \phi}{\partial t} + p^* + \rho \frac{1}{2} |\nabla \phi|^2 - \rho \mathbf{g} \cdot \mathbf{x} - \mu_0 \bar{M} H \right) = \mathbf{0}, \quad (2.2.34)$$

where $\bar{M}$ is the field-averaged magnetisation, given by

$$\bar{M} = \frac{1}{H} \int_0^H M \, dH. \quad (2.2.35)$$

Spatial integration of (2.2.34) yields the FHD Bernoulli equation for an inviscid irrotational ferrofluid of uniform density

$$\rho \frac{\partial \phi}{\partial t} + p^* + \rho \frac{1}{2} |\nabla \phi|^2 - \rho \mathbf{g} \cdot \mathbf{x} - \mu_0 \bar{M} H = C(t), \quad (2.2.36)$$

where C is an arbitrary function of time.

2.3 Boundary and Interface Conditions

2.3.1 Electromagnetic Fields Across an Interface

Maxwell's equations are accompanied by the following conditions at an interface between different materials (see Jackson, 1962, for example)

$$[\hat{\mathbf{n}} \cdot \mathbf{D}] = \rho_s, \quad (2.3.1)$$

$$[\hat{\mathbf{n}} \wedge \mathbf{E}] = \mathbf{0}, \quad (2.3.2)$$

$$[\hat{\mathbf{n}} \cdot \mathbf{B}] = 0, \quad (2.3.3)$$

$$[\hat{\mathbf{n}} \wedge \mathbf{H}] = \mathbf{J}_s, \quad (2.3.4)$$

where ρ_s is the surface charge density, $\mathbf{J}_s$ is the surface current density, and $\hat{\mathbf{n}}$ is the unit normal to the interface. The square brackets denote the change across

the interface, defined as the value of the quantity on the side of the interface $\hat{\mathbf{n}}$ points into minus the value of the quantity on the other side.

We can see directly from (2.3.1) and (2.3.2) that in the absence of surface charges, both the normal component of $\mathbf{D}$ and the tangential component of $\mathbf{E}$ are continuous across an interface. Similarly, from (2.3.3) and (2.3.4), in the absence of surface currents we have that both the normal component of $\mathbf{B}$ and the tangential component of $\mathbf{H}$ are continuous across an interface.

Additionally, since $\mathbf{D} = \varepsilon \mathbf{E}$, we expect discontinuities in the normal component of $\mathbf{E}$ and the tangential component of $\mathbf{D}$ at an interface where there is a discontinuity in the permittivity ε , even in the absence of surface charges. Similarly, since $\mathbf{B} = \mu \mathbf{H}$, we expect discontinuities in the normal component of $\mathbf{H}$ and the tangential component of $\mathbf{B}$ at an interface where there is a discontinuity in the permeability μ , even in the absence of surface currents.

2.3.2 Dynamic Boundary Condition

The normal stress condition at the interface between two fluids is given by (see Batchelor, 1967, for example)

$$[\hat{\mathbf{n}} \cdot \mathbf{T}] = \kappa \gamma \hat{\mathbf{n}}, \quad (2.3.5)$$

where $\hat{\mathbf{n}}$ is the unit normal to the interface, $\mathbf{T}$ is the stress tensor for a given fluid, κ is the total curvature of the interface, and γ is the coefficient of interfacial tension between the fluids.

At a fluid interface in which we have a discontinuity in the magnetic permeability μ , the corresponding discontinuities in $\mathbf{H}$ and $\mathbf{B}$ will lead to a discontinuity in the magnetic pressure experienced by the ferrofluid. In order for the ferrofluid to be in equilibrium, there must be a discontinuity in the hydrodynamic pressure to balance this. This is captured by the normal stress condition (2.3.5).

Here, we consider the normal stress condition at the interface of two immiscible fluids with differing permeabilities. First, recall that the stress tensor is given by (2.2.17), which can be written as

$$\mathbf{T} = - \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B}\mathbf{H} - p\mathbf{I} + \eta \left\{ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right\}. \quad (2.3.6)$$

If we assume that all fluids are inviscid then $\eta = 0$, and we have that

$$\mathbf{T} = - \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B}\mathbf{H} - p\mathbf{I}. \quad (2.3.7)$$

Using our definitions of the magnetostrictive pressure p_s (2.2.13), the fluid-magnetic pressure p_m (2.2.14), and the composite pressure p^* (2.2.25), (2.3.7) becomes

$$\mathbf{T} = - \left\{ p^* + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B}\mathbf{H}. \quad (2.3.8)$$

Therefore, substituting (2.3.8) into (2.3.5), we obtain

$$\left[-p^* \hat{\mathbf{n}} + (\mathbf{B} \cdot \hat{\mathbf{n}}) \mathbf{H} - \frac{1}{2} \mu_0 H^2 \hat{\mathbf{n}} \right] = \kappa \gamma \hat{\mathbf{n}}. \quad (2.3.9)$$

We can decompose $\mathbf{H}$ into components tangential and normal to the interface. Namely, we have that

$$\mathbf{H} = \mathbf{H}_t + H_n \hat{\mathbf{n}}, \quad (2.3.10)$$

where $\mathbf{H}_t = \mathbf{H} \cdot (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}})$ is the tangential component, and $H_n = \mathbf{H} \cdot \hat{\mathbf{n}}$ is the normal component. Using (2.1.13), we can write (2.3.10) as

$$\mathbf{H} = \mathbf{H}_t + (\mu_0^{-1} B_n - M_n) \hat{\mathbf{n}}, \quad (2.3.11)$$

with $B_n = \mathbf{B} \cdot \hat{\mathbf{n}}$ and $M_n = \mathbf{M} \cdot \hat{\mathbf{n}}$.

Using (2.3.11), we can write (2.3.9) as

$$\left[-p^* \hat{\mathbf{n}} + B_n (\mathbf{H}_t + (\mu_0^{-1} B_n - M_n) \hat{\mathbf{n}}) - \frac{1}{2} \mu_0 (|\mathbf{H}_t|^2 + \mu_0^{-2} B_n^2 - 2\mu_0^{-1} B_n M_n + M_n^2) \hat{\mathbf{n}} \right] = \kappa \gamma \hat{\mathbf{n}}. \quad (2.3.12)$$

From (2.3.3) and (2.3.4), we have that $[B_n] = 0$ and $[\mathbf{H}_t] = \mathbf{0}$, in the absence of surface currents. Therefore, (2.3.12) becomes

$$\left[-p^* \hat{\mathbf{n}} - \frac{1}{2} \mu_0 M_n^2 \hat{\mathbf{n}} \right] = \kappa \gamma \hat{\mathbf{n}}, \quad (2.3.13)$$

where we have also noted the cancellation of terms proportional to $B_n M_n \hat{\mathbf{n}}$. Finally, taking the normal component of (2.3.13), we arrive at our dynamic boundary condition, namely

$$\left[-p^* - \frac{1}{2} \mu_0 M_n^2 \right] = \kappa \gamma. \quad (2.3.14)$$

2.4 Linear Ferrofluid Magnetisation

We previously stated that the magnetisation of a ferrofluid is typically taken to be nonlinear. All the analysis conducted up to this point has been valid for a ferrofluid with a nonlinear constitutive law of magnetisation of the form (2.1.20). In this section, we will motivate the assumption of linear magnetisation and see how it affects the FHD Bernoulli equation (2.2.36) and the dynamic boundary condition (2.3.14).

2.4.1 Equilibrium Magnetisation of a Ferrofluid

If the magnetisation of a ferrofluid is assumed to be in equilibrium, then the relationship between $\mathbf{M}$ and $\mathbf{H}$ is well-modelled using a constitutive law of the

form (Rosensweig, 1985; Yecko, 2010)

$$\mathbf{M} = M_s L(\alpha) \frac{\mathbf{H}}{H}, \quad (2.4.1)$$

where

$$L(\alpha) = \coth \alpha - \frac{1}{\alpha}, \quad (2.4.2)$$

is the Langevin function, and M_s is the saturation value of the magnetisation. The saturation value of the magnetisation of a material is the value of the magnetisation beyond which an increase in the applied magnetic field will no longer increase the magnetisation.

The parameter α is given by

$$\alpha = \frac{\pi D^3}{6} \frac{\mu_0 M_d}{kT} H, \quad (2.4.3)$$

where k is Boltzmann's constant, T is the temperature in the ferrofluid, and M_d is the saturation magnetisation of magnetisable nanoparticles of diameter D present in the ferrofluid, related to the saturation magnetisation of the ferrofluid by $M_s = M_d \phi_V$, where ϕ_V is the bulk volume fraction of nanoparticles in the ferrofluid. The parameters D , M_d , ϕ_V and M_s are all constants whose values depend on the physical properties of the ferrofluid being considered.

2.4.2 The Linear Limit

Standard series expansions show that for small α , we have

$$L(\alpha) \sim \frac{\alpha}{3}. \quad (2.4.4)$$

Hence, for small α , we have the constitutive law

$$\mathbf{M} \sim \frac{M_s \alpha}{3H} \mathbf{H}. \quad (2.4.5)$$

Since α depends linearly on H , the quantity

$$\frac{M_s \alpha}{3H}, \quad (2.4.6)$$

is independent of H , giving us a linear relationship between $\mathbf{M}$ and $\mathbf{H}$. Moreover, if we take T to be a constant, then (2.4.6) is also a constant.

This linear relationship is typically written as

$$\mathbf{M} = \chi_m \mathbf{H}, \quad (2.4.7)$$

with χ_m being the magnetic susceptibility of the ferrofluid, which is a constant, given by

$$\chi_m = \frac{M_s \alpha}{3H} = \frac{\pi D^3}{18} \frac{\mu_0 M_d^2 \phi_V}{kT}. \quad (2.4.8)$$

Recalling that

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (2.4.9)$$

we also obtain a linear relationship between $\mathbf{B}$ and $\mathbf{H}$, namely

$$\mathbf{B} = \mu \mathbf{H}, \quad (2.4.10)$$

where $\mu = \mu_0(1 + \chi_m)$ is the permeability of the ferrofluid.

In the context of the magnetisation of a ferrofluid, since the only variable on which α depends is H , linear magnetisation is based on the assumption that the applied magnetic field is relatively weak, and the temperature is constant. In Weidner (2017), the use of a linear magnetisation law was seen to be accurate to a factor of less than 5% as long as $H < 1.5 \times 10^4 \text{ A/m}$.

Additionally, Kaiser and Rosensweig (1969) provide tabulated values of D , ϕ_V , and M_d , for a variety of different types of ferrofluid. As such, taking room temperature to be $T = 300\text{K}$, we can calculate some typical values for χ_m , as in (2.4.8), when the magnetisation is assumed to be linear. We find a smallest value

of $\chi_m \approx 0.08$ and a largest value of $\chi_m \approx 1.3$. Additionally, ferrofluids studied experimentally by Weidner (2017) had susceptibilities in the range $1 < \chi_m < 3.4$. As such, values of χ_m in the range $0 < \chi_m < 4$ do not seem to be atypical for ferrofluids. Although this is by no means a comprehensive study of the value of the magnetic susceptibility for ferrofluids.

For the remainder of this work, we will assume linear isotropic magnetisation in all materials.

2.4.3 FHD Bernoulli Equation with Linear Magnetisation

The magnetic term in the FHD Bernoulli equation (2.2.36) involves the field-averaged magnetisation $\bar{M}$, given by

$$\bar{M} = \frac{1}{H} \int_0^H M \, dH. \quad (2.4.11)$$

If we assume linear magnetisation, then $M = \chi_m H$, and so

$$\bar{M} = \frac{1}{2} \chi_m H. \quad (2.4.12)$$

Therefore, the FHD Bernoulli equation (2.2.36) can be written as

$$\rho \frac{\partial \phi}{\partial t} + p^* + \rho \frac{1}{2} |\nabla \phi|^2 - \rho \mathbf{g} \cdot \mathbf{x} - \frac{1}{2} \mu_0 \chi_m H^2 = F(t). \quad (2.4.13)$$

It is useful for us to note that if the fluid is static, (2.4.13) can be written as

$$p^* = p_0^* + \frac{1}{2} \mu_0 \chi_m H^2 + \rho \mathbf{g} \cdot \mathbf{x}, \quad (2.4.14)$$

where p_0^* is a constant composite pressure.

2.4.4 Linear Magnetisation in the Dynamic Boundary Condition

When the magnetisation is linear, it follows that

$$M_n = \frac{\chi_m}{\mu} B_n. \quad (2.4.15)$$

We now substitute (2.4.14) and (2.4.15) into the dynamic boundary condition (2.3.14), making use of the decomposition (2.3.11), to obtain

$$\left[-p_0^* - \frac{1}{2} \mu_0 \chi_m \{ |\mathbf{H}_t|^2 + \mu^{-2} B_n^2 \} - \rho \mathbf{g} \cdot \mathbf{x} - \frac{1}{2} \mu_0 \chi_m^2 \mu^{-2} B_n^2 \right] = \kappa \gamma. \quad (2.4.16)$$

By using the fact $\mu = \mu_0(1 + \chi_m)$ in (2.4.16), we arrive finally at

$$\left[-p_0^* - \frac{\mu_0 \chi_m}{2} |\mathbf{H}_t|^2 - \frac{\chi_m}{2\mu_0(1 + \chi_m)} B_n^2 - \rho \mathbf{g} \cdot \mathbf{x} \right] = \kappa \gamma, \quad (2.4.17)$$

which is the dynamic boundary condition in the case of linear magnetisation.

2.5 Integral Representation for the Magnetic Induction

In this section, we will seek an integral representation for the magnetic induction $\mathbf{B}$ at a general point in space, where the free current distribution $\mathbf{J}_f$ is prescribed and steady, and the magnetic permeability μ is prescribed and piecewise uniform. In this instance $\mathbf{B}$ is piecewise continuous, with discontinuities only between materials of differing permeability.

2.5.1 Helmholtz Decomposition for Piecewise Continuous Vector Fields

It is well-known that a suitably well-behaved vector field can be expressed as the sum of a solenoidal field and an irrotational field. This idea is known as the

Helmholtz decomposition. We are unable to readily use the Helmholtz decomposition to obtain an expression for $\mathbf{B}$ here, due to the fact that it is discontinuous across interfaces at which there is a discontinuity in the permeability.

Kustepeli (2016) employs distribution theory to extend the notion of the Helmholtz decomposition to include piecewise continuous vector fields. In particular, for a vector field $\mathbf{V}$, which decays at infinity and is well-behaved everywhere except across a surface of discontinuity, the Helmholtz decomposition is given by

$$\begin{aligned} \mathbf{V}(\mathbf{x}) = & -\nabla \left(\iiint_{\mathbb{R}^3 \setminus S} \frac{\nabla_{\xi} \cdot \mathbf{V}(\xi)}{4\pi|\mathbf{x} - \xi|} dV_{\xi} + \iint_S \frac{\hat{\mathbf{n}}_{\xi} \cdot [\mathbf{V}(\xi)]}{4\pi|\mathbf{x} - \xi|} dS_{\xi} \right) \\ & + \nabla \wedge \left(\iiint_{\mathbb{R}^3 \setminus S} \frac{\nabla_{\xi} \wedge \mathbf{V}(\xi)}{4\pi|\mathbf{x} - \xi|} dV_{\xi} + \iint_S \frac{\hat{\mathbf{n}}_{\xi} \wedge [\mathbf{V}(\xi)]}{4\pi|\mathbf{x} - \xi|} dS_{\xi} \right), \end{aligned} \quad (2.5.1)$$

where S is the overall surface of discontinuity (which may be the union of finitely many individual surfaces of discontinuity), ∇_{ξ} denotes spatial differentiation with respect to the integration variable ξ , $\hat{\mathbf{n}}_{\xi}$ is the unit normal to the surface S at the point ξ , and square brackets denote the change in a quantity across an interface. Note that in (2.5.1) the quantities $\mathbf{V}$, $\nabla_{\xi} \cdot \mathbf{V}$, and $\nabla_{\xi} \wedge \mathbf{V}$ are evaluated only where they are well-defined. Hence, S is excluded from the domain of integration in the volume integrals.

2.5.2 Integral Representation for Piecewise Uniform μ

Before expressing $\mathbf{B}$ in the form of (2.5.1), we recall the magnetostatic form of Maxwell's equations, namely

$$\nabla \cdot \mathbf{B} = 0, \quad (2.5.2)$$

$$\nabla \wedge \mathbf{H} = \mathbf{J}_f, \quad (2.5.3)$$

where $\mathbf{J}_f$ is the free current density. By employing equations (2.5.2) and (2.5.3) we are assuming that all currents are steady. We also have the constitutive law

$$\mathbf{B} = \mu \mathbf{H}, \quad (2.5.4)$$

where μ is piecewise constant. Additionally, in the absence of surface currents, the conditions

$$[\hat{\mathbf{n}} \cdot \mathbf{B}] = 0, \quad (2.5.5)$$

$$[\hat{\mathbf{n}} \wedge \mathbf{H}] = \mathbf{0}, \quad (2.5.6)$$

must be satisfied at an interface between materials with a discontinuity in μ .

From equations (2.5.2) and (2.5.5), we can immediately see that the first volume and surface integrals in the Helmholtz decomposition for $\mathbf{B}$, as in (2.5.1), will vanish. Hence, we have

$$\mathbf{B}(\mathbf{x}) = \nabla \wedge \left(\iiint_{\mathbb{R}^3 \setminus S} \frac{\nabla_{\xi} \wedge \mathbf{B}(\xi)}{4\pi|\mathbf{x} - \xi|} dV_{\xi} + \iint_S \frac{\hat{\mathbf{n}}_{\xi} \wedge [\mathbf{B}(\xi)]}{4\pi|\mathbf{x} - \xi|} dS_{\xi} \right). \quad (2.5.7)$$

By using (2.5.4), we can write (2.5.7) as

$$\mathbf{B}(\mathbf{x}) = \nabla \wedge \left(\iiint_{\mathbb{R}^3 \setminus S} \frac{\mu \nabla_{\xi} \wedge \mathbf{H}(\xi)}{4\pi|\mathbf{x} - \xi|} dV_{\xi} + \iint_S \frac{\hat{\mathbf{n}}_{\xi} \wedge [\mu \mathbf{H}(\xi)]}{4\pi|\mathbf{x} - \xi|} dS_{\xi} \right), \quad (2.5.8)$$

where we have taken μ outside of the curl in the volume integral as we are not integrating over any surfaces across which μ is discontinuous.

Moreover, we have that

$$\hat{\mathbf{n}}_{\xi} \wedge [\mu \mathbf{H}(\xi)] = [\mu] (\hat{\mathbf{n}}_{\xi} \wedge \mathbf{H}(\xi)), \quad (2.5.9)$$

since $\hat{\mathbf{n}}_{\xi} \wedge \mathbf{H}(\xi)$ is continuous across S by (2.5.6). By also making use of (2.5.3), (2.5.8) becomes

$$\mathbf{B}(\mathbf{x}) = \nabla \wedge \left(\iiint_{\mathbb{R}^3 \setminus S} \frac{\mu \mathbf{J}_f(\xi)}{4\pi|\mathbf{x} - \xi|} dV_{\xi} + \iint_S \frac{[\mu] (\hat{\mathbf{n}}_{\xi} \wedge \mathbf{H}(\xi))}{4\pi|\mathbf{x} - \xi|} dS_{\xi} \right). \quad (2.5.10)$$

Evaluating the curl of (2.5.10) and taking the $\boldsymbol{\xi}$ dependence of $\mathbf{J}_f$, $\mathbf{H}$, and $\hat{\mathbf{n}}$ to be understood, we arrive finally at

$$\mathbf{B}(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3 \setminus S} \frac{\mu \mathbf{J}_f \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dV_{\boldsymbol{\xi}} + \frac{1}{4\pi} \iint_S \frac{[\mu](\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_{\boldsymbol{\xi}}. \quad (2.5.11)$$

We have hence obtained an expression for $\mathbf{B}$ in terms of an integral over space and an integral over the interfaces between materials.

Whilst $\mathbf{B}$ itself is not well-defined on S , we can consider neighbouring points on either side of S to determine how $\mathbf{B}(\mathbf{x})$ computed by (2.5.11) relates to the values of $\mathbf{B}$ either side of S if $\mathbf{x} \in S$. Additionally, we can verify that the interface conditions (2.5.5) and (2.5.6) are satisfied. This analysis is conducted in appendix C, and yields the fact that for $\mathbf{x} \in S$, (2.5.11) gives

$$\mathbf{B}(\mathbf{x}) = \bar{\mu} \mathbf{H}_t + B_n \hat{\mathbf{n}}, \quad (2.5.12)$$

where $\bar{\mu}$ is the arithmetic mean of the two values of μ on either side of the interface.

2.6 Limit of a Thin Circular Current-Carrying Wire

For the problem we are considering, we suppose that the only current is being carried along a circular loop of thin wire of constant circular cross-section. The circular cross-section of the wire is of radius a and the total flux carried by the wire is denoted as I . We denote the permeability of the wire as μ_w and the permeability of the material surrounding the wire as μ_s . This surrounding material could either be ferrofluid or air.

As we are assuming that the only current is being carried along the wire, the volume integral in (2.5.11) is given by

$$\mathbf{I}_V = \frac{1}{4\pi} \iiint_{V_w} \frac{\mu_w \mathbf{J}_f \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dV_{\boldsymbol{\xi}}, \quad (2.6.1)$$

where V_w is the region of space containing the wire.

We also have the component of the surface integral that comes from the interface between the wire and the material surrounding the wire. That is

$$\mathbf{I}_S = \frac{1}{4\pi} \iint_{S_{w,s}} (\mu_s - \mu_w) (\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_{\boldsymbol{\xi}}, \quad (2.6.2)$$

where $S_{w,s}$ is the interface between the wire and its surroundings.

The total contribution to (2.5.11) due to the wire, which we will call $\mathbf{B}_w$, is hence given by

$$\mathbf{B}_w = \mathbf{I}_V + \mathbf{I}_S. \quad (2.6.3)$$

By taking the wire to be thin, specifically that we have $a \ll \kappa_w^{-1}$, where κ_w is the curvature of the wire, and that we consider points with $|\mathbf{x} - \boldsymbol{\xi}| \gg a$ for all $\boldsymbol{\xi}$ in V_w , we assume that we can treat the $(\mathbf{x} - \boldsymbol{\xi})/|\mathbf{x} - \boldsymbol{\xi}|^3$ part of the integrand of the volume integral as a constant with respect to an integration over the cross-sectional area of the wire. The integral of $\mathbf{J}_f$ over the cross-section of the wire is precisely $I\hat{\mathbf{t}}$, where $\hat{\mathbf{t}}$ is a unit tangent vector in the direction of the current. Hence, we are left with an integral over the centre-line of the wire, with length element $I d\mathbf{l}_{\boldsymbol{\xi}}$. This can be written as

$$\mathbf{I}_V \sim -\frac{1}{4\pi} \int_w \mu_w I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_{\boldsymbol{\xi}}. \quad (2.6.4)$$

Note that (2.6.4) is the standard form of the Biot–Savart law for a thin wire (see Griffiths, 1981, for example).

In order to make progress in evaluating $\mathbf{I}_S$, we need to know the value of $\hat{\mathbf{n}} \wedge \mathbf{H}$ on the surface of the wire. For a thin circular loop of current-carrying wire with circular cross-section, the magnetic field in the immediate neighbourhood of the wire is approximately that of an infinitely long straight wire of circular cross-section. The field due to an infinitely long straight wire is given by (see Griffiths,

1981, for example)

$$\mathbf{H} = \frac{I}{2\pi a} f(r) \hat{\boldsymbol{\theta}}, \quad (2.6.5)$$

where $\hat{\boldsymbol{\theta}}$ is the azimuthal tangent vector to the surface of the wire, and

$$f(r) = \begin{cases} \frac{r}{a}, & \text{if } r \leq a \\ \frac{a}{r}, & \text{if } r > a \end{cases}, \quad (2.6.6)$$

where the circular cross-section of the wire is being described using polar coordinates (r, θ) such that $r = a$ is the boundary of the wire.

Since the field is purely azimuthal and $\hat{\mathbf{n}} \wedge \mathbf{H}$ is continuous across the interface, as in (2.5.6), the above result would be unaffected by the presence of a ferrofluid surrounding the wire. We hence observe that, for a circular wire loop,

$$\hat{\mathbf{n}} \wedge \mathbf{H} \sim \frac{I}{2\pi a} \hat{\mathbf{t}}. \quad (2.6.7)$$

We write the surface element in (2.6.2) as $dS_\xi = a d\theta dl_\xi$. For points with $|\mathbf{x} - \boldsymbol{\xi}| \gg a$, which are far from the wire, the leading-order contribution to the surface integral comes from a uniform azimuthal integral over θ , with $\boldsymbol{\xi}$ replaced by a point on the centre-line of the wire. To see this, first consider the surface integral in question, which is given by

$$\mathbf{I}_S \sim \frac{1}{4\pi} \int_w \int_0^{2\pi} (\mu_s - \mu_w) \frac{I}{2\pi a} \hat{\mathbf{t}} \wedge \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} a d\theta dl_\xi. \quad (2.6.8)$$

The integration variable $\boldsymbol{\xi}$, which is on the surface, can be written as $\boldsymbol{\xi} = \boldsymbol{\xi}' + a\hat{\mathbf{r}}(\theta)$, where $\boldsymbol{\xi}'$ is a point on the centre-line of the wire and $\hat{\mathbf{r}}$ is the radial polar coordinate unit vector. By computing a series expansion of (2.6.2) about $a = 0$, we find that, at leading-order, we have

$$\mathbf{I}_S \sim \frac{1}{4\pi} \int_w \int_0^{2\pi} (\mu_s - \mu_w) \frac{I}{2\pi} \hat{\mathbf{t}} \wedge \frac{(\mathbf{x} - \boldsymbol{\xi}')}{|\mathbf{x} - \boldsymbol{\xi}'|^3} d\theta dl_\xi. \quad (2.6.9)$$

We now note that $d\mathbf{l}_\xi = \hat{\mathbf{t}} d\ell_\xi$. Hence, we obtain

$$\mathbf{I}_S \sim -\frac{1}{4\pi} \int_w \int_0^{2\pi} (\mu_s - \mu_w) \frac{I}{2\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}')}{|\mathbf{x} - \boldsymbol{\xi}'|^3} \wedge d\theta d\mathbf{l}_\xi, \quad (2.6.10)$$

and therefore, upon carrying out the uniform θ integral,

$$\mathbf{I}_S \sim -\frac{1}{4\pi} \int_w (\mu_s - \mu_w) I \frac{(\mathbf{x} - \boldsymbol{\xi}')}{|\mathbf{x} - \boldsymbol{\xi}'|^3} \wedge d\mathbf{l}_\xi. \quad (2.6.11)$$

From (2.6.3), (2.6.4), and (2.6.11), we have that

$$\mathbf{B}_w \sim -\frac{1}{4\pi} \int_w \mu_w I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi - \frac{1}{4\pi} \int_w (\mu_s - \mu_w) I \frac{(\mathbf{x} - \boldsymbol{\xi}')}{|\mathbf{x} - \boldsymbol{\xi}'|^3} \wedge d\mathbf{l}_\xi. \quad (2.6.12)$$

We then arrive finally at

$$\mathbf{B}_w \sim -\frac{1}{4\pi} \int_w \mu_s I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi. \quad (2.6.13)$$

We have shown that, in the case of a thin circular loop of wire with circular cross-section, the contribution to $\mathbf{B}$ as given in (2.5.11) is given by a single line integral over the centre-line of the wire which depends on the permeability of the material surrounding the wire, but not on the permeability of the wire itself. Moreover, since the only current in our problem is in the wire, then from (2.5.11) we can see that adding on a surface integral over any remaining interfaces between materials to $\mathbf{B}_w$ yields the full integral representation for $\mathbf{B}$. In particular,

$$\mathbf{B}(\mathbf{x}) = -\frac{1}{4\pi} \int_w \mu_s I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi + \frac{1}{4\pi} \iint_S \frac{[\mu](\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_\xi, \quad (2.6.14)$$

where S is the surface comprising any remaining interfaces, not including that between the wire and its surroundings.

2.7 Boundary Integral Equation for a Ferrofluid Toroid

We now consider the integral representation (2.6.14) for a static toroid of ferrofluid surrounding a thin wire forming a circle of radius R and carrying a current I . The permeability of the ferrofluid is given by μ_f and the permeability of the surrounding air is given by μ_0 . The equation (2.6.14) gives an expression for the magnetic induction $\mathbf{B}$ at a general point $\mathbf{x}$ in terms of an integral over the current-carrying wire and the ferrofluid free surface. In (2.6.14), we separate the contribution due to the wire $\mathbf{B}_w$ from the contribution induced by the ferrofluid free surface $\mathbf{B}_s$. That is

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}_w(\mathbf{x}) + \mathbf{B}_s(\mathbf{x}), \quad (2.7.1)$$

with

$$\mathbf{B}_w = -\frac{1}{4\pi} \int_w \mu_f I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi, \quad (2.7.2)$$

where w is the centre-line of the wire, and

$$\mathbf{B}_s = \frac{1}{4\pi} (\mu_0 - \mu_f) \iint_S \frac{(\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_\xi, \quad (2.7.3)$$

where S is the free surface of the ferrofluid.

2.7.1 Coordinates and Unit Vectors

With the foresight of exploiting some symmetries in the asymptotic analysis conducted in Chapter 4, we consider the case where the circular wire-loop, and hence the ferrofluid toroid, can bend either clockwise or anticlockwise. The centre of curvature of the wire-loop is chosen to be located on the x -axis at $x = -D$, where $R = |D|$.

We choose Cartesian coordinates aligned with the plane of the wire and the

meridional cross-section containing the point $\mathbf{x}$. The meridional cross-section is defined to be the semi-infinite plane extending from $x = -D$ that contains the point $\mathbf{x}$. The point $\mathbf{x}$ is described using its Cartesian coordinates in the meridional plane, such that

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}. \quad (2.7.4)$$

We now introduce some additional coordinates. First, we let ϕ be the azimuthal angle of the point $\boldsymbol{\xi}$ relative to the meridional plane containing $\mathbf{x}$. We then define s to be an arc-length coordinate around the intersection of the ferrofluid free surface with the meridional plane. The origin of the (s, ϕ) coordinates is taken such that $s = \phi = 0$ corresponds to the point $\mathbf{x}$. The angle $\vartheta(s)$ is then defined to be the angle at each point that the free surface makes with the vertical in the meridional plane.

The azimuthal angle ϕ is defined differently depending on the centre of curvature of the wire-loop. If $D > 0$ then the centre of curvature of the wire is located at $x < 0$ and we take ϕ as going anticlockwise. If $D < 0$, then the centre of curvature of the wire is located at $x > 0$ and ϕ is taken as going clockwise. Schematics showing the setup and coordinates for this problem, for both signs of D , are given in figures 2.7.1 and 2.7.2.

The free surface of the ferrofluid toroid is hence described by

$$\boldsymbol{\xi}(s, \phi) = \begin{pmatrix} -D + (\xi(s) + D) \cos \phi \\ \pm(\xi(s) + D) \sin \phi \\ \zeta(s) \end{pmatrix}, \quad (2.7.5)$$

where ξ and ζ are, respectively, the x and z coordinates of the free surface in the meridional plane containing the point $\boldsymbol{\xi}$. In (2.7.5), the upper signs are taken if $D > 0$ and the lower signs are taken if $D < 0$. This sign convention will be assumed from hereon in. Additionally, the intersection of the ferrofluid free

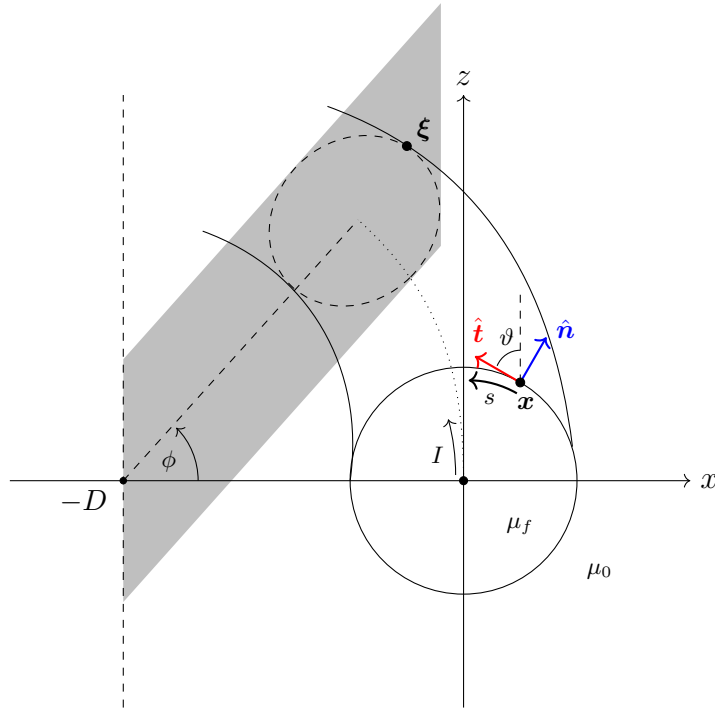


Figure 2.7.1: Diagram showing the setup and coordinates for a toroidal volume of ferrofluid surrounding a circular wire-loop carrying a current I , for the case where $D > 0$.

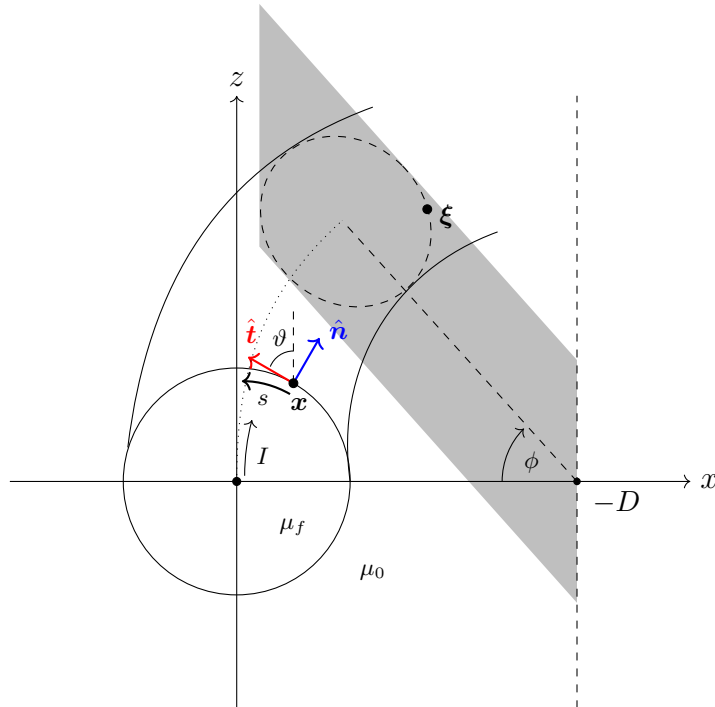


Figure 2.7.2: Diagram showing the setup and coordinates for a toroidal volume of ferrofluid surrounding a circular wire-loop carrying a current I , for the case where $D < 0$.

surface with the plane $\phi = 0$ is described by $\boldsymbol{\xi}_0(s)$, where

$$\boldsymbol{\xi}_0(s) = \begin{pmatrix} \xi(s) \\ 0 \\ \zeta(s) \end{pmatrix}. \quad (2.7.6)$$

We now define a triad of orthogonal unit vectors relative to the surface at any point. These are: a normal vector $\hat{\mathbf{n}}$, a meridional tangent $\hat{\mathbf{t}}$, and an azimuthal tangent $\hat{\boldsymbol{\phi}}$. These are obtained by deriving the unit vectors in the cross-sectional plane $\phi = 0$, and then rotating anticlockwise if $D > 0$ or clockwise if $D < 0$.

In the plane $\phi = 0$, we have that

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos \vartheta \\ 0 \\ \sin \vartheta \end{pmatrix}, \quad \hat{\mathbf{t}} = \begin{pmatrix} -\sin \vartheta \\ 0 \\ \cos \vartheta \end{pmatrix}, \quad \hat{\boldsymbol{\phi}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad (2.7.7)$$

The rotation matrix $\mathbf{R}(\phi)$ describing an anticlockwise rotation by an angle ϕ about the z -axis, is given by

$$\mathbf{R}(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.7.8)$$

The clockwise rotation is then obtained by taking $\mathbf{R}(-\phi) = \mathbf{R}^T(\phi)$.

Hence, the unit vectors take the general form

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos \vartheta \cos \phi \\ \pm \cos \vartheta \sin \phi \\ \sin \vartheta \end{pmatrix}, \quad \hat{\mathbf{t}} = \begin{pmatrix} -\sin \vartheta \cos \phi \\ \mp \sin \vartheta \sin \phi \\ \cos \vartheta \end{pmatrix}, \quad \hat{\boldsymbol{\phi}} = \begin{pmatrix} \mp \sin \phi \\ \cos \phi \\ 0 \end{pmatrix}. \quad (2.7.9)$$

2.7.2 Azimuthal Integrals

In order to proceed, we will need to deal with integrals of the form

$$\tilde{\mathbf{P}}(\mathbf{x}, \xi_0(s)) = \int_0^{2\pi} \frac{(\mathbf{x} - \xi(s, \phi)) \wedge \hat{\phi}}{|\mathbf{x} - \xi(s, \phi)|^3} d\phi. \quad (2.7.10)$$

Analysis of integrals of the form (2.7.10) for the coordinate system described in section 2.7.1 is given in appendix D. We find that we can express $\tilde{\mathbf{P}}$ as

$$\tilde{\mathbf{P}}(\mathbf{x}, \xi_0(s)) = (z - \zeta(s)) (I_1 - 2I_2) \hat{\mathbf{x}} + (-(x + \xi(s) + 2D) I_1 + 2(x + D) I_2) \hat{\mathbf{z}}, \quad (2.7.11)$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x and z directions respectively, and

$$I_1 = \frac{4}{(1-m)A^{\frac{3}{2}}} \text{Ee}(m), \quad I_2 = \frac{4}{mA^{\frac{3}{2}}} \left(\frac{\text{Ee}(m)}{(1-m)} - \text{Ek}(m) \right). \quad (2.7.12)$$

Here, Ek and Ee are the complete elliptic integrals of the first and second kind respectively, $m = B/A$, and

$$A = (x + \xi(s) + 2D)^2 + (z - \zeta(s))^2, \quad B = 4(x + D)(\xi(s) + D). \quad (2.7.13)$$

Hence, the integral $\tilde{\mathbf{P}}$ can be computed analytically.

2.7.3 Wire Contribution $\mathbf{B}_w$

For a circular wire-loop with the centre of curvature located at $x = -D$, we have that

$$d\mathbf{l} = |D| \hat{\phi} d\phi. \quad (2.7.14)$$

Hence, (2.7.2) becomes

$$\mathbf{B}_w = -\frac{\mu_f I}{4\pi} \int_0^{2\pi} |D| \frac{(\mathbf{x} - \xi) \wedge \hat{\phi}}{|\mathbf{x} - \xi|^3} d\phi. \quad (2.7.15)$$

Noting that for the wire, $\xi = \zeta = 0$, we have

$$\mathbf{B}_w = -\frac{\mu_f I}{4\pi} \mathbf{P}(\mathbf{x}, \mathbf{0}), \quad (2.7.16)$$

where

$$\mathbf{P}(\mathbf{x}, \mathbf{0}) = |D| \tilde{\mathbf{P}}(\mathbf{x}, \mathbf{0}). \quad (2.7.17)$$

2.7.4 Free Surface Contribution $\mathbf{B}_s$

We begin by noting that, by symmetry, $\mathbf{H}$ cannot have a component in the $\hat{\phi}$ direction. It is then easy to show that regardless of the sign of D , we have that $\hat{\mathbf{n}} \wedge \mathbf{H} = -H_t \hat{\phi}$, where $H_t = \mathbf{H} \cdot \hat{\mathbf{t}}$ is the component of $\mathbf{H}$ tangent to the ferrofluid free surface.

Hence, (2.7.3) can be written as

$$\mathbf{B}_s = \frac{1}{4\pi} (\mu_0 - \mu_f) \iint_S H_t(\boldsymbol{\xi}_0(s)) \frac{(\mathbf{x} - \boldsymbol{\xi}) \wedge \hat{\phi}}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS. \quad (2.7.18)$$

We can compute the integral around the ferrofluid free surface by first computing an azimuthal integral around the toroid and then integrating around the intersection of the free surface and the meridional plane containing $\mathbf{x}$ (henceforth called the ferrofluid boundary).

Therefore, we have that

$$dS = |\xi + D| d\phi ds, \quad (2.7.19)$$

and our integral becomes

$$\mathbf{B}_s = \frac{1}{4\pi} (\mu_0 - \mu_f) \int_{-C}^C \int_0^{2\pi} H_t(\boldsymbol{\xi}_0(s)) |\xi + D| \frac{(\mathbf{x} - \boldsymbol{\xi}) \wedge \hat{\phi}}{|\mathbf{x} - \boldsymbol{\xi}|^3} d\phi ds, \quad (2.7.20)$$

where $2C$ is the total arc-length of the ferrofluid boundary. We then have that

$$\mathbf{B}_s = \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s); D) ds, \quad (2.7.21)$$

where

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = |\boldsymbol{\xi} + D| \tilde{\mathbf{P}}(\mathbf{x}, \boldsymbol{\xi}_0(s)). \quad (2.7.22)$$

2.7.5 Final Expression

Using (2.7.1), (2.7.16), and (2.7.20), we arrive at the full expression for the integral representation, namely

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_f I}{4\pi} \mathbf{P}(\mathbf{x}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) \, ds. \quad (2.7.23)$$

We now note that since H_t depends on $\mathbf{B}$, (2.7.23) is an implicit equation for $\mathbf{B}$, requiring knowledge of the value of H_t on the ferrofluid boundary. Additionally, since $\tilde{\mathbf{P}}$ can be computed analytically, the first term in (2.7.23) is known analytically. For a given ferrofluid boundary, (2.7.23) could be solved numerically for $\mathbf{B}$ with only the ferrofluid boundary needing to be discretised and not the entire domain. We have hence derived an implicit boundary-integral equation for the magnetic induction for a static toroid of ferrofluid with a prescribed interface surrounding a circular loop of current-carrying wire.

2.7.6 Dynamic Boundary Condition

Alongside the boundary-integral equation, we also have the dynamic boundary condition, which is needed to determine the equilibrium shape of the toroid.

To arrive at a form of the dynamic boundary condition for the problem at hand, we begin by dropping the gravity term in (2.4.17), to obtain the dynamic boundary condition in the absence of gravity, namely

$$\left[-p_0^* - \frac{\mu_0 \chi_m}{2} |\mathbf{H}_t|^2 - \frac{\chi_m}{2\mu_0(1 + \chi_m)} B_n^2 \right] = \kappa \gamma. \quad (2.7.24)$$

We then recall that the magnetic susceptibility χ_m of a material, is given by

$$\chi_m = \frac{\mu}{\mu_0} - 1, \quad (2.7.25)$$

where μ is the permeability of the material. From this, it is clear that outside of the ferrofluid, $\chi_m = 0$. Hence, the dynamic boundary condition (2.7.24) becomes

$$(p_{\text{in}} - p_{\text{out}}) + \frac{\mu_0 \chi_m}{2} H_t^2 + \frac{\chi_m}{2\mu_0 (1 + \chi_m)} B_n^2 = \kappa\gamma, \quad (2.7.26)$$

where p_{in} is the composite pressure inside the ferrofluid, p_{out} is the composite pressure outside the ferrofluid and χ_m is the susceptibility of the ferrofluid. Letting $P = p_{\text{in}} - p_{\text{out}}$, we have that

$$P + \frac{\mu_0 \chi_m}{2} H_t^2 + \frac{\chi_m}{2\mu_0 (1 + \chi_m)} B_n^2 = \kappa\gamma. \quad (2.7.27)$$

2.8 Non-Dimensionalisation

To conclude this chapter, we derive dimensionless forms of the dynamic boundary condition (2.4.17) and the boundary-integral representation (2.7.23).

We choose to non-dimensionalise lengths on the radius R of the wire-loop, magnetic inductions on $\mu_0 I/R$, magnetic fields on I/R , and pressures on the surface tension scale γ/R . We now define two dimensionless parameters that will appear in the problem of interest. The first is the permeability ratio

$$\tilde{m} = \frac{\mu_f}{\mu_0}, \quad (2.8.1)$$

which gives the relative permeability of the ferrofluid to the surrounding air. The second is the magnetic Bond number

$$\text{Bo} = \frac{\mu_0 I^2}{R\gamma}, \quad (2.8.2)$$

which quantifies the relative strength of magnetic effects to surface tension effects on the free surface.

In accordance with our choice of scalings, we introduce dimensionless variables (denoted by quantities with hats) as follows

$$\mathbf{B} = \frac{\mu_0 I}{R} \hat{\mathbf{B}}, \quad B_n = \frac{\mu_0 I}{R} \hat{B}_n, \quad H_t = \frac{I}{R} \hat{H}_t, \quad \mathbf{x} = R \hat{\mathbf{x}}, \quad \boldsymbol{\xi} = R \hat{\boldsymbol{\xi}}, \quad s = R \hat{s}. \quad (2.8.3)$$

We also introduce the non-dimensional over-pressure $\hat{P}$, such that

$$P = \frac{\gamma}{R} \hat{P}, \quad (2.8.4)$$

and the non-dimensional curvature $\hat{\kappa}$, where

$$\kappa = \frac{1}{R} \hat{\kappa}. \quad (2.8.5)$$

Finally, we note the fact that $\chi_m = \tilde{m} - 1$.

2.8.1 Dimensionless Dynamic Boundary Condition

Introducing dimensionless variables as described by (2.8.3), (2.8.4), and (2.8.5) into (2.7.27), we obtain

$$\frac{\gamma}{R} \hat{P} + \frac{1}{2} (\tilde{m} - 1) \left\{ \mu_0 \frac{I^2}{R^2} \hat{H}_t^2 + \frac{1}{\mu_0 \tilde{m}} \frac{I^2}{R^2} \mu_0^2 \hat{B}_n^2 \right\} = \gamma \frac{1}{R} \hat{\kappa}, \quad (2.8.6)$$

or, using our definition of the Bond number (2.8.2) and dropping the hats,

$$P + \frac{1}{2} (\tilde{m} - 1) \text{Bo} \{ H_t^2 + \tilde{m}^{-1} B_n^2 \} = \kappa, \quad (2.8.7)$$

which is the dimensionless dynamic boundary condition. The over-pressure P is unknown, and needs to be determined as part of the problem.

2.8.2 Dimensionless Boundary-Integral Equation

We now consider the boundary-integral equation (2.7.23). We begin by considering the function $\tilde{\mathbf{P}}$ as in (2.7.10).

Introducing dimensionless variables as described by (2.8.3), we have that

$$\tilde{\mathbf{P}}(\mathbf{x}, \boldsymbol{\xi}_0) = \frac{1}{R^2} \int_0^{2\pi} \frac{(\hat{\mathbf{x}} - \hat{\boldsymbol{\xi}}) \wedge \hat{\boldsymbol{\phi}}}{|\hat{\mathbf{x}} - \hat{\boldsymbol{\xi}}|^3} d\phi = \frac{1}{R^2} \hat{\tilde{\mathbf{P}}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}), \quad (2.8.8)$$

where $\hat{\tilde{\mathbf{P}}}$ is the dimensionless form of the function $\tilde{\mathbf{P}}$.

We note that, in dimensionless variables, the centre of curvature of the wire-loop is located at $\hat{x} = -D/R = \mp 1$, since $R = |D|$. Throughout, we will take the upper sign if $D > 0$ and the lower sign if $D < 0$. The analytic expression for $\hat{\tilde{\mathbf{P}}}$, analogous to (2.7.11), is then given by

$$\hat{\tilde{\mathbf{P}}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0) = (\hat{z} - \hat{\zeta}) (I_1 - 2I_2) \hat{\mathbf{x}} + \left(-(\hat{x} + \hat{\xi} \pm 2) I_1 + 2(\hat{x} \pm 1) I_2 \right) \hat{\mathbf{z}}, \quad (2.8.9)$$

where

$$\hat{\boldsymbol{\xi}}_0 = \begin{pmatrix} \hat{\xi} \\ 0 \\ \hat{\zeta} \end{pmatrix}, \quad (2.8.10)$$

$\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are, as before, the unit vectors in the x and z directions respectively, and

$$I_1 = \frac{4}{(1-m)A^{\frac{3}{2}}} \text{Ee}(m), \quad I_2 = \frac{4}{mA^{\frac{3}{2}}} \left(\frac{\text{Ee}(m)}{(1-m)} - \text{Ek}(m) \right), \quad (2.8.11)$$

with Ek and Ee being the complete elliptic integrals of the first and second kind respectively, $m = B/A$, and

$$A = (\hat{x} + \hat{\xi} \pm 2)^2 + (\hat{z} - \hat{\zeta})^2, \quad B = 4(\hat{x} \pm 1)(\hat{\xi} \pm 1). \quad (2.8.12)$$

We can now obtain the dimensionless forms of the functions $\mathbf{P}$ and $\mathbf{f}$, given by (2.7.17) and (2.7.22) respectively. Starting with $\mathbf{P}$, we have that

$$\mathbf{P}(\mathbf{x}, \mathbf{0}) = |D| \frac{1}{R^2} \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \mathbf{0}). \quad (2.8.13)$$

Since $R = |D|$, we obtain

$$\mathbf{P}(\mathbf{x}, \mathbf{0}) = \frac{1}{R} \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \mathbf{0}). \quad (2.8.14)$$

Now, considering $\mathbf{f}$, we have that

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = |R\hat{\xi} + D| \frac{1}{R^2} \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})). \quad (2.8.15)$$

We know that $D/R = \pm 1$, and so

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = \frac{1}{R} |\hat{\xi} \pm 1| \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})). \quad (2.8.16)$$

We now define

$$\hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})) = |\hat{\xi} \pm 1| \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})), \quad (2.8.17)$$

to be the dimensionless form of the function $\mathbf{f}$, yielding

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = \frac{1}{R} \hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})). \quad (2.8.18)$$

We are now in a position to obtain a dimensionless form of the boundary-integral equation. Substituting (2.8.3), (2.8.14), and (2.8.18) into (2.7.23), we have that

$$\frac{\mu_0 I}{R} \hat{\mathbf{B}}(\hat{\mathbf{x}}) = -\frac{\mu_f I}{4\pi} \frac{1}{R} \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-\hat{C}}^{\hat{C}} \frac{I}{R} \hat{H}_t(\hat{\boldsymbol{\xi}}_0(\hat{s})) \frac{1}{R} \hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})) R d\hat{s}, \quad (2.8.19)$$

where $C = \lambda \hat{C}$. After some simplification, and using the definition of the permeability ratio $\tilde{m}$ (2.8.1), we arrive at

$$\hat{\mathbf{B}}(\hat{\mathbf{x}}) = -\frac{\tilde{m}}{4\pi} \hat{\hat{\mathbf{P}}}(\hat{\mathbf{x}}, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_{-\hat{C}}^{\hat{C}} \hat{H}_t(\hat{\boldsymbol{\xi}}_0(\hat{s})) \hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\boldsymbol{\xi}}_0(\hat{s})) d\hat{s}. \quad (2.8.20)$$

Finally, dropping hats for clarity, we arrive at

$$\mathbf{B}(\mathbf{x}) = -\frac{\tilde{m}}{4\pi} \tilde{\mathbf{P}}(\mathbf{x}, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) \, ds, \quad (2.8.21)$$

which is the dimensionless boundary-integral equation for the magnetic induction $\mathbf{B}$.

3

Numerical Method

In the previous chapter, we derived the equations governing a static ferrofluid toroid surrounding a circular loop of current carrying wire. In this section, we will discretise these equations and seek to solve them numerically to determine equilibrium configurations of such ferrofluid toroids. Some preliminary results will be presented, showing the cross-sections of some ferrofluid toroids in equilibrium. Finally, we will discuss the convergence of the numerical solutions as the number of discretised points on the boundary is increased.

3.1 Full Problem Statement

3.1.1 Problem Set-up

We consider a static axisymmetric toroidal volume V of ferrofluid surrounding a thin wire forming a circular loop of radius R and carrying a current I . We denote the permeability of the ferrofluid by μ_f and note that the surrounding air will have permeability μ_0 . The surface tension on the fluid interface is denoted by γ . Our aim is to find the position of the ferrofluid free surface, such that the toroid is in equilibrium. We work in Cartesian coordinates and use dimensionless variables, with lengths scaled on R , magnetic inductions scaled on $\mu_0 I/R$, magnetic fields on I/R , and pressures on γ/R . We also define $\tilde{m} = \mu_f/\mu_0$ to be the dimensionless

permeability ratio and

$$\text{Bo} = \frac{\mu_0 I^2}{R\gamma}, \quad (3.1.1)$$

to be the magnetic Bond number.

The first equation we need to satisfy is the boundary-integral equation (2.8.21), which can be used to determine the magnetic field given terms of the position of the free surface of the ferrofluid. The boundary-integral equation for the magnetic induction $\mathbf{B}$ at a point in space $\mathbf{x}$, is given by

$$\mathbf{B}(\mathbf{x}) = -\frac{\tilde{m}}{4\pi} \tilde{\mathbf{P}}(\mathbf{x}, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) \, ds, \quad (3.1.2)$$

where H_t is the component of the magnetic field $\mathbf{H}$ tangential to the ferrofluid boundary, $\tilde{\mathbf{P}}$ and $\mathbf{f}$ are as defined in (2.7.10)–(2.7.13) and (2.7.22), $\tilde{m} = \mu_f/\mu_0$ is the dimensionless permeability ratio, s is arc-length around the ferrofluid boundary, $2C$ is the total arc-length of the ferrofluid boundary, and $\boldsymbol{\xi}_0(s)$ describes the ferrofluid boundary in a meridional plane.

As we have seen, there is no azimuthal dependence in (3.1.2), as this is all contained within the functions $\tilde{\mathbf{P}}$ and $\mathbf{f}$, which can be evaluated analytically, independently of the azimuthal angle. Hence, we have a two-dimensional problem to be solved for the position of the ferrofluid boundary. It follows that we can restrict our analysis to two-dimensional vectors in the meridional x, z plane. We write

$$\mathbf{x} = \begin{pmatrix} x \\ z \end{pmatrix}, \quad \boldsymbol{\xi}_0(s) = \begin{pmatrix} \xi(s) \\ \zeta(s) \end{pmatrix}. \quad (3.1.3)$$

In accordance with (2.7.7), the unit normal vector to the ferrofluid boundary, $\hat{\mathbf{n}}$, is given by

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix}, \quad (3.1.4)$$

where ϑ is the angle which the ferrofluid boundary makes with the vertical.

Similarly, the unit tangent vector to the ferrofluid boundary, $\hat{\mathbf{t}}$, is given by

$$\hat{\mathbf{t}} = \begin{pmatrix} -\sin \vartheta \\ \cos \vartheta \end{pmatrix}. \quad (3.1.5)$$

Additionally, we need to satisfy the dynamic boundary condition (2.8.7) on the ferrofluid boundary. The dynamic boundary condition is given by

$$P + \frac{1}{2}(\tilde{m} - 1)\text{Bo} \{H_t^2 + \tilde{m}^{-1}B_n^2\} = \kappa, \quad (3.1.6)$$

where B_n is the component of $\mathbf{B}$ normal to the ferrofluid boundary, κ is the total curvature of the ferrofluid toroid and P is the dimensionless over-pressure.

Without loss of generality we choose the centre of curvature of the wire-loop to be located at $x = -1$. We therefore work in the $(x > -1, z)$ half-plane, where the center of the cross-section of the wire-loop is located at the point $x = 0, z = 0$, and the line $x = -1$ serves as the axis of rotational symmetry for the ferrofluid toroid. We describe the ferrofluid boundary using plane polar coordinates (r, θ) , which are centred on the wire at $x = 0, z = 0$. If the radial coordinates of the ferrofluid boundary are given by $r = \rho(\theta)$, then the Cartesian coordinates of a point $\mathbf{X}$ on the ferrofluid boundary, are given by

$$\mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos \theta \\ \rho \sin \theta \end{pmatrix}. \quad (3.1.7)$$

3.1.2 Comment on the Azimuthal Integrals $\tilde{\mathbf{P}}$ and $\mathbf{f}$

The choice of coordinates described in section 3.1.1 corresponds to taking $D > 0$ in the expressions for the integrals $\tilde{\mathbf{P}}$ and $\mathbf{f}$ given in Chapter 2 (2.7.10)–(2.7.13), (2.7.22). Therefore, we recall that

$$\tilde{\mathbf{P}}(\mathbf{x}, \boldsymbol{\xi}_0) = (z - \zeta)(I_1 - 2I_2)\hat{\mathbf{x}} + (-(x + \xi + 2)I_1 + 2(x + 1)I_2)\hat{\mathbf{z}}, \quad (3.1.8)$$

and

$$\mathbf{f}(\mathbf{x}, \xi_0(s)) = (\xi + 1) \tilde{\mathbf{P}}(\mathbf{x}, \xi_0), \quad (3.1.9)$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x and z directions respectively,

$$I_1 = \frac{4}{(1-m)A^{\frac{3}{2}}} \text{Ee}(m), \quad I_2 = \frac{4}{mA^{\frac{3}{2}}} \left(\frac{\text{Ee}(m)}{(1-m)} - \text{Ek}(m) \right), \quad (3.1.10)$$

Ek and Ee are the complete elliptic integrals of the first and second kind respectively, $m = B/A$, and

$$A = (x + \xi + 2)^2 + (z - \zeta)^2, \quad B = 4(x + 1)(\xi + 1). \quad (3.1.11)$$

3.1.3 Computation of the Fluid Volume

Finally, we note the additional constraint that the volume of the ferrofluid toroid needs to be prescribed. Since a toroidal volume of ferrofluid is a volume of revolution, the volume V of the ferrofluid toroid can be calculated using Pappus's centroid theorem, which states that

$$V = 2\pi \bar{R} A', \quad (3.1.12)$$

where A' is the cross sectional area of the toroid and the distance $\bar{R}$ of the centroid from the axis, is given by

$$\bar{R} = \frac{\int_0^{2\pi} \int_0^\rho (1 + r \cos \theta) \, dA}{A'}. \quad (3.1.13)$$

It follows that the volume is given by

$$V = 2\pi \bar{R} A' = 2\pi \left(\frac{\int_0^{2\pi} \int_0^\rho (1 + r \cos \theta) \, dA}{A'} \right) A', \quad (3.1.14)$$

or more simply

$$V = 2\pi \int_0^{2\pi} \int_0^\rho (1 + r \cos \theta) \, dA. \quad (3.1.15)$$

We note that the area element is given by $dA = r dr d\theta$, therefore

$$V = 2\pi \int_0^{2\pi} \int_0^\rho (r + r^2 \cos \theta) dr d\theta. \quad (3.1.16)$$

Completing the integration with respect to r , we have that

$$V = 2\pi \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3} \rho \cos \theta \right) \rho^2 d\theta. \quad (3.1.17)$$

The formula (3.1.17) can be used to calculate the fluid volume if the position of the ferrofluid boundary is known.

3.2 Analytic Treatment of the Singularity

We recall that, using the original integral definition of $\tilde{\mathbf{P}}$ (2.7.10), the function $\mathbf{f}$ is given by

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = (\xi + 1) \int_0^{2\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}) \wedge \hat{\boldsymbol{\phi}}}{|\mathbf{x} - \boldsymbol{\xi}|^3} d\phi, \quad (3.2.1)$$

where ϕ is the azimuthal angle about the z axis. If we take $\mathbf{x} = \mathbf{X}$ to be a point on the ferrofluid boundary, and choose $s = 0$ to be the arc-length coordinate of the point $\mathbf{X}$, then the integrand of (3.2.1) has a singularity at $s = \phi = 0$, which behaves like $(s^2 + x^2 \phi^2)^{-1}$. Once the ϕ -integral has been completed in (3.2.1), we are left with an s^{-1} singularity in $\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s))$ appearing in the s -integral in (3.1.2). Whilst this singularity is not strictly integrable, it can be interpreted by the Cauchy principal value, and so the integral is well-defined.

However, we wish to deal with this integral numerically, which is an issue if the integrand contains a singularity. To deal with the singularity in $H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$, we add and subtract a simple function $\mathbf{f}^*$ that shares the same singularity at $s = 0$. The added copy can be replaced by its analytic integral, whilst the copy which is subtracted yields a well-behaved integrand $H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) - \mathbf{f}^*$.

To determine the function $\mathbf{f}^*$, we first need to study the behaviour of $\mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$ near the singularity at $s = 0$. We begin by considering a Taylor series expansion of $\boldsymbol{\xi}_0(s)$ about $s = 0$. We have

$$\boldsymbol{\xi}_0(s) = \boldsymbol{\xi}_0(0) + s\boldsymbol{\xi}'_0(0) + \frac{1}{2}s^2\boldsymbol{\xi}''_0(0) + O(s^3), \quad (3.2.2)$$

where primes denote derivatives with respect to s . Since $s = 0$ is chosen to be the arc-length coordinate for the point $\mathbf{X}$, we have that $\boldsymbol{\xi}_0(0) = \mathbf{X}$. Additionally, by definition, $\boldsymbol{\xi}'_0(s)$ is the unit tangent vector to the surface at $\boldsymbol{\xi}_0(s)$. We have an expression for this already, namely $\hat{\mathbf{t}}$ as in (3.1.5). We then have that

$$\boldsymbol{\xi}''_0(s) = \frac{d\hat{\mathbf{t}}}{ds}. \quad (3.2.3)$$

In appendix A, we analyse both the Frenet-Serret formulae and the curvatures for the set-up we are considering. In accordance with appendix A, we have that

$$\frac{d\hat{\mathbf{t}}}{ds} = -\kappa_m \hat{\mathbf{n}}, \quad (3.2.4)$$

where κ_m is the plane curvature of the ferrofluid boundary and $\hat{\mathbf{n}}$ is as in (3.1.4).

We hence have that

$$\boldsymbol{\xi}_0(s) = \mathbf{X} + s\hat{\mathbf{t}}_0 - \frac{1}{2}s^2\kappa_{m,0}\hat{\mathbf{n}}_0 + O(s^3), \quad (3.2.5)$$

where $\hat{\mathbf{n}}_0$ and $\hat{\mathbf{t}}_0$ are the unit normal and tangent vectors to the ferrofluid boundary at the point $\mathbf{X}$, where $s = 0$, and $\kappa_{m,0}$ is the plane curvature of the ferrofluid boundary at the point $\mathbf{X}$.

Using (3.1.4) and (3.1.5), we can obtain expressions for $\xi(s)$ and $\zeta(s)$, namely

$$\xi(s) = x - s \sin \vartheta_0 - \frac{1}{2}s^2\kappa_{m,0} \cos \vartheta_0 + O(s^3), \quad (3.2.6)$$

and

$$\zeta(s) = z + s \cos \vartheta_0 - \frac{1}{2}s^2\kappa_{m,0} \sin \vartheta_0 + O(s^3), \quad (3.2.7)$$

where ϑ_0 is the angle which the ferrofluid boundary makes with the vertical at the point $\mathbf{X}$, where $s = 0$.

We can now study the behaviour of $\mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$, as described by (3.1.8)–(3.1.11), close to the singularity at $s = 0$, by substituting in the expressions (3.2.6) and (3.2.7) and expanding in the limit $s \ll 1$. The expansion is computed using Maple, and yields

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = \frac{\mathbf{a}_{-1}}{s} + (\mathbf{b}_0 \log |s| + \mathbf{a}_0) + s(\mathbf{b}_1 \log |s| + \mathbf{a}_1) + O(s^2 \log |s|), \quad (3.2.8)$$

where

$$\mathbf{a}_{-1} = 2(\cos \vartheta_0 \hat{\mathbf{x}} + \sin \vartheta_0 \hat{\mathbf{z}}), \quad (3.2.9)$$

$$\mathbf{b}_0 = \frac{1}{x+1} \hat{\mathbf{z}}, \quad (3.2.10)$$

$$\mathbf{a}_0 = \left(\kappa_{m,0} + \frac{\cos \vartheta_0}{x+1} \right) (-\sin \vartheta_0 \hat{\mathbf{x}} + \cos \vartheta_0 \hat{\mathbf{z}}) - \frac{\log(8(x+1))}{x+1} \hat{\mathbf{z}}, \quad (3.2.11)$$

$$\mathbf{b}_1 = \frac{1}{4(x+1)^2} (3 \cos \vartheta_0 \hat{\mathbf{x}} + \sin \vartheta_0 \hat{\mathbf{z}}). \quad (3.2.12)$$

Note that $\mathbf{a}_1$ is not given above. This is because $\mathbf{a}_1$ requires knowledge of the $O(s^3)$ terms in the expansions of ξ and ζ , which we have not included.

For use in the boundary-integral, we need to resolve the Cartesian unit vectors into their components normal and tangent to the ferrofluid boundary. Using (3.1.4) and (3.1.5), it is easy to see that

$$\hat{\mathbf{x}} = \hat{\mathbf{n}} \cos \vartheta - \hat{\mathbf{t}} \sin \vartheta, \quad \hat{\mathbf{z}} = \hat{\mathbf{n}} \sin \vartheta + \hat{\mathbf{t}} \cos \vartheta. \quad (3.2.13)$$

We can hence express the coefficients (3.2.9)–(3.2.12), as

$$\mathbf{a}_{-1} = 2\hat{\mathbf{n}}_0, \quad (3.2.14)$$

$$\mathbf{b}_0 = \frac{1}{x+1} (\hat{\mathbf{n}}_0 \sin \vartheta_0 + \hat{\mathbf{t}}_0 \cos \vartheta_0), \quad (3.2.15)$$

$$\mathbf{a}_0 = \left(\kappa_{m,0} + \frac{\cos \vartheta_0}{x+1} \right) \hat{\mathbf{t}}_0 - \frac{\log(8(x+1))}{x+1} (\hat{\mathbf{n}}_0 \sin \vartheta_0 + \hat{\mathbf{t}}_0 \cos \vartheta_0), \quad (3.2.16)$$

$$\mathbf{b}_1 = \frac{1}{4(x+1)^2} (\hat{\mathbf{n}}_0 (1 + 2 \cos^2 \vartheta_0) - 2\hat{\mathbf{t}}_0 \sin \vartheta_0 \cos \vartheta_0). \quad (3.2.17)$$

The tangential field H_t is expected to have a regular series expansion about $s = 0$. That is,

$$H_t(\boldsymbol{\xi}_0(s)) = h_0 + h_1 s + h_2 s^2 + \cdots. \quad (3.2.18)$$

The values of the coefficients h_i are derived from H_t , where $h_0 = H_t(\mathbf{X})$, $h_1 = H'_t(\mathbf{X})$, and so on. These are estimated using finite differences from the values of H_t at the point $\mathbf{X}$ and the surrounding points in the numerical method.

Using the expansions (3.2.8) and (3.2.18), we see that, close to $s = 0$, we have

$$\begin{aligned} H_t(\boldsymbol{\xi}_0) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) &= \frac{h_0 \mathbf{a}_{-1}}{s} + h_1 \mathbf{a}_{-1} + h_2 \mathbf{a}_{-1} s + h_0 \mathbf{b}_0 \log |s| + h_1 \mathbf{b}_0 s \log |s| \\ &\quad + h_0 \mathbf{a}_0 + h_1 \mathbf{a}_0 s + h_0 \mathbf{b}_1 s \log |s| + h_0 \mathbf{a}_1 s + O(s^2 \log |s|). \end{aligned} \quad (3.2.19)$$

We seek a function $\mathbf{f}^*$ that when subtracted from (3.2.19) will remove terms that are singular, or have a singular first derivative, at $s = 0$. This prompts a first guess for $\mathbf{f}^*$ of the form

$$\mathbf{f}^*(\mathbf{X}, s) = \frac{h_0 \mathbf{a}_{-1}}{s} + h_0 \mathbf{b}_0 \log |s| + (h_1 \mathbf{b}_0 + h_0 \mathbf{b}_1) s \log |s|. \quad (3.2.20)$$

However, this function is not continuous at $s = \pm C$, neither is its first derivative.

It will aid the numerical integration if we can rectify this, so we instead choose

$$\begin{aligned} \mathbf{f}^*(\mathbf{X}, s) = h_0 \mathbf{a}_{-1} \left(\frac{1}{s} - \frac{s}{C^2} \right) + h_0 \mathbf{b}_0 \left(\log |s| - \frac{s^2}{2C^2} \right) \\ + (h_1 \mathbf{b}_0 + h_0 \mathbf{b}_1) (s \log |s| - s \log C), \end{aligned} \quad (3.2.21)$$

which is continuous, and has continuous first derivative, at $s = \pm C$.

We now note that we can decompose $\mathbf{f}^*$ as

$$\mathbf{f}^*(\mathbf{X}, s) = h_0 \mathbf{f}_0^*(\mathbf{X}, s) + h_1 \mathbf{f}_1^*(\mathbf{X}, s), \quad (3.2.22)$$

where we have

$$\mathbf{f}_0^*(\mathbf{X}, s) = \mathbf{a}_{-1} \left(\frac{1}{s} - \frac{s}{C^2} \right) + \mathbf{b}_0 \left(\log |s| - \frac{s^2}{2C^2} \right) + \mathbf{b}_1 (s \log |s| - s \log C), \quad (3.2.23)$$

and

$$\mathbf{f}_1^*(\mathbf{X}, s) = \mathbf{b}_0 (s \log |s| - s \log C), \quad (3.2.24)$$

which will be useful when implementing the numerical method later on. The analytic integral of $\mathbf{f}^*$ is easy to compute from (3.2.21). Noting that the first and third groups in (3.2.21) are odd in s , and hence integrate to zero, it follows that

$$\int_{-C}^C \mathbf{f}^*(\mathbf{X}, s) \, ds = h_0 \mathbf{b}_0 \int_{-C}^C \left(\log |s| - \frac{s^2}{2C^2} \right) \, ds, \quad (3.2.25)$$

and so

$$\int_{-C}^C \mathbf{f}^*(\mathbf{X}, s) \, ds = h_0 \mathbf{b}_0 \left[s \log |s| - s - \frac{s^3}{6C^2} \right]_{-C}^C. \quad (3.2.26)$$

Therefore, we have an analytic expression for the integral of $\mathbf{f}^*$, that is

$$\int_{-C}^C \mathbf{f}^*(\mathbf{X}, s) \, ds = h_0 \mathbf{b}_0 \left(2 \log C - \frac{7}{3} \right) C. \quad (3.2.27)$$

It remains to determine the correct limiting value of the integrand $H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)$, as $s \rightarrow 0$. Accounting for the constructed

cancellation of the $O(s^{-1})$, $O(\log |s|)$, and $O(s \log |s|)$ terms, we have that

$$\begin{aligned} H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s) &= (h_0 \mathbf{a}_0 + h_1 \mathbf{a}_{-1}) \\ &+ \left((h_0 \mathbf{a}_1 + h_1 \mathbf{a}_0 + h_2 \mathbf{a}_{-1}) + \frac{h_0 \mathbf{a}_{-1}}{C^2} + (h_0 \mathbf{b}_1 + h_1 \mathbf{b}_0) \log C \right) s \\ &+ O(s^2 \log |s|). \end{aligned}$$

The limiting value $\mathbf{L}(\mathbf{X})$, to be used in the numerical method at $s = 0$, is hence taken to be

$$\mathbf{L}(\mathbf{X}) = h_0 \mathbf{a}_0 + h_1 \mathbf{a}_{-1}. \quad (3.2.28)$$

3.3 Set-up for the Boundary-Integral Method

3.3.1 Variables

We discretise the problem by representing the ferrofluid boundary by a set of N points $\mathbf{X}_i$, where $i = 0, 1, 2, \dots, N-1$. If we impose the polar angle θ_i of each point $\mathbf{X}_i$, then the points $\mathbf{X}_i$ are characterised by their radial coordinate ρ_i . In particular, we have that

$$\mathbf{X}_i = \begin{pmatrix} x_i \\ z_i \end{pmatrix} = \begin{pmatrix} \rho_i \cos \theta_i \\ \rho_i \sin \theta_i \end{pmatrix}, \quad (3.3.1)$$

where the θ_i are prescribed and the ρ_i need to be determined.

The magnetic field at each point $\mathbf{X}_i$ is described by the normal component $B_{n,i}$ of $\mathbf{B}$ and the tangential component $H_{t,i}$ of $\mathbf{H}$. As the over-pressure P is also unknown, we hence have $3N+1$ unknowns for which we need to solve. A schematic for this problem is shown in figure 3.3.1.

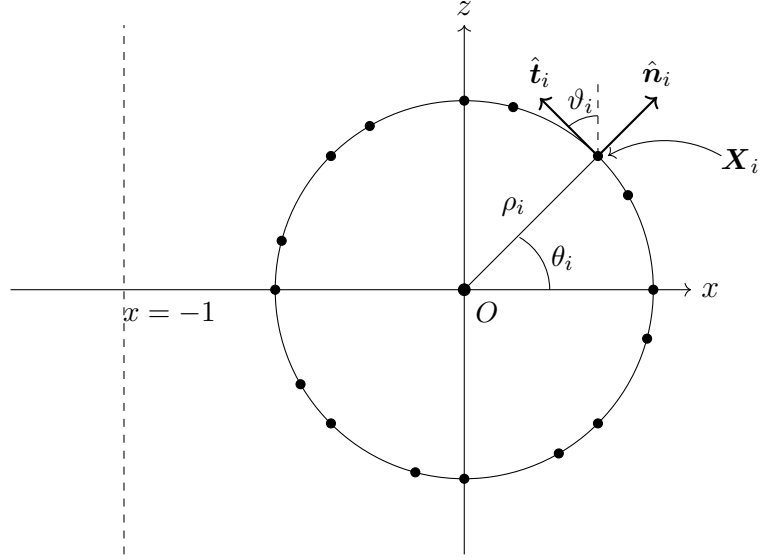


Figure 3.3.1: The setup for the boundary-integral method for the dimensionless problem of an axisymmetric toroidal volume of ferrofluid.

3.3.2 Surface geometry

In order to solve the problem numerically, we need to compute some quantities pertaining to the geometry of the free surface. We begin by defining the finite-difference angular derivative approximations

$$\rho_i'^+ = \frac{\rho_{i+1} - \rho_i}{\theta_{i+1} - \theta_i}, \quad \rho_i'^- = \frac{\rho_i - \rho_{i-1}}{\theta_i - \theta_{i-1}}, \quad \rho_i' = \frac{\rho_{i+1} - \rho_{i-1}}{\theta_{i+1} - \theta_{i-1}}, \quad (3.3.2)$$

and

$$\rho_i'' = \frac{2(\rho_i'^+ - \rho_i'^-)}{\theta_{i+1} - \theta_{i-1}}. \quad (3.3.3)$$

The distance along the ferrofluid boundary between neighbouring points either side of $\mathbf{X}_i$, denoted d_i^+ for the distance to the point $\mathbf{X}_{i+1}$ or d_i^- for the distance to the point $\mathbf{X}_{i-1}$, can be approximated using the polar coordinate arc-length formula

$$(ds)^2 = (d\rho)^2 + \rho^2 (d\theta)^2. \quad (3.3.4)$$

We can rewrite (3.3.4) as

$$(ds)^2 = \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right) (d\theta)^2. \quad (3.3.5)$$

For our problem, we have that either

$$\frac{d\rho}{d\theta} \approx \rho_i^+, \quad \rho \approx \frac{\rho_{i+1} + \rho_i}{2}, \quad d\theta \approx \theta_{i+1} - \theta_i, \quad (3.3.6)$$

for d_i^+ , or

$$\frac{d\rho}{d\theta} \approx \rho_i^-, \quad \rho \approx \frac{\rho_i + \rho_{i-1}}{2}, \quad d\theta \approx \theta_i - \theta_{i-1}, \quad (3.3.7)$$

for d_i^- . Hence, we can write

$$d_i^+ = (\theta_{i+1} - \theta_i) \left((\rho_i^+)^2 + \frac{1}{4} (\rho_{i+1} + \rho_i)^2 \right)^{\frac{1}{2}}, \quad (3.3.8)$$

and

$$d_i^- = (\theta_i - \theta_{i-1}) \left((\rho_i^-)^2 + \frac{1}{4} (\rho_i + \rho_{i-1})^2 \right)^{\frac{1}{2}}. \quad (3.3.9)$$

We then define

$$d_i = \frac{1}{2} (d_i^+ + d_i^-), \quad (3.3.10)$$

which will be used as the cell width in the trapezium rule, and

$$s_i = \sum_{j=0}^{i-1} d_j^+, \quad (3.3.11)$$

as the distance along the ferrofluid boundary from $\mathbf{X}_0$ to $\mathbf{X}_i$. The parameter C can hence be determined by recalling that the total arc-length along the ferrofluid boundary is given by $2C$. Hence, we have that

$$C = \frac{1}{2} \sum_{i=0}^{N-1} d_i^+. \quad (3.3.12)$$

We also need to compute the signed free surface arc-length between two points $\mathbf{X}_i$ and $\mathbf{X}_j$, requiring that it lies within the interval $(-C, C]$. We thus define

$$s_{ij} = \begin{cases} s_i - s_j + 2C & : s_i - s_j \leq -C \\ s_i - s_j - 2C & : s_i - s_j > C \\ s_i - s_j & : \text{otherwise} \end{cases} \quad (3.3.13)$$

The unit normal and tangential vectors to the free surface at the point $\mathbf{X}_i$, are given by

$$\hat{\mathbf{n}}_i = \begin{pmatrix} \cos \vartheta_i \\ \sin \vartheta_i \end{pmatrix}, \quad \hat{\mathbf{t}}_i = \begin{pmatrix} -\sin \vartheta_i \\ \cos \vartheta_i \end{pmatrix}, \quad (3.3.14)$$

where ϑ_i is the angle the free surface makes with the vertical at the point $\mathbf{X}_i$.

We require expressions for $\sin \vartheta_i$ and $\cos \vartheta_i$ in terms of ρ_i and θ_i . As the unit tangent to any curve is given by the derivative of the position vector with respect to arc-length, we have that

$$\frac{d\mathbf{X}}{ds} = \hat{\mathbf{t}}, \quad (3.3.15)$$

where $\mathbf{X}$ is as in (3.1.7) and $\hat{\mathbf{t}}$ is as in (3.1.5). It follows that

$$\sin \vartheta = -\frac{dx}{ds}, \quad \cos \vartheta = \frac{dz}{ds}. \quad (3.3.16)$$

Since

$$x = \rho \cos \theta, \quad z = \rho \sin \theta, \quad (3.3.17)$$

and noting that, via the chain rule, we can write

$$\sin \vartheta = -\frac{dx}{d\theta} \frac{d\theta}{ds}, \quad \cos \vartheta = \frac{dz}{d\theta} \frac{d\theta}{ds}, \quad (3.3.18)$$

we obtain

$$\sin \vartheta = \left\{ -\frac{d\rho}{d\theta} \cos \theta + \rho \sin \theta \right\} \frac{d\theta}{ds}, \quad (3.3.19)$$

and

$$\cos \vartheta = \left\{ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta \right\} \frac{d\theta}{ds}. \quad (3.3.20)$$

Moreover, it follows from (3.3.5) that

$$\frac{d\theta}{ds} = \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{1}{2}}. \quad (3.3.21)$$

We now take equations (3.3.19) and (3.3.20), together with (3.3.21), and approximate the angular derivative $d\rho/d\theta$, at the point $\mathbf{X}_i$, by the centred

difference approximation ρ'_i , to obtain

$$\sin \vartheta_i = \frac{-\rho'_i \cos \theta_i + \rho_i \sin \theta_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{1}{2}}}, \quad (3.3.22)$$

and

$$\cos \vartheta_i = \frac{\rho'_i \sin \theta_i + \rho_i \cos \theta_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{1}{2}}}. \quad (3.3.23)$$

Finally, we require expressions for the plane curvature κ_m and the total curvature κ of the ferrofluid toroid at $\mathbf{X}_i$. From the work in appendix A, we have that, at the point $\mathbf{X}$, the curvatures are given by

$$\kappa_m = \left\{ \rho^2 + 2 \left(\frac{d\rho}{d\theta} \right)^2 - 2\rho \frac{d^2\rho}{d\theta^2} \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{3}{2}}, \quad (3.3.24)$$

and

$$\kappa = \kappa_m + \frac{\cos \vartheta}{(x+1)}. \quad (3.3.25)$$

We hence have that the curvatures at the point $\mathbf{X}_i$, are given by

$$\kappa_{m,i} = \frac{\rho_i^2 + 2(\rho'_i)^2 - \rho_i \rho''_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{3}{2}}}, \quad (3.3.26)$$

and

$$\kappa_i = \kappa_{m,i} + \frac{\cos \vartheta_i}{(x_i+1)}, \quad (3.3.27)$$

where we have used the approximations

$$\frac{d\rho}{d\theta} \approx \rho'_i, \quad \frac{d^2\rho}{d\theta^2} \approx \rho''_i. \quad (3.3.28)$$

3.3.3 Magnetic Field Expansion

We consider a local approximation of H_t near the point $\mathbf{X} = \mathbf{X}_i$, by writing

$$H_t = h_0 + h_1 s + h_2 s^2 + O(s^3), \quad (3.3.29)$$

where s is arc-length from $\mathbf{X}_i$. We choose the coefficients in order to make the approximation exact at the points $\mathbf{X} = \mathbf{X}_{i-1}, \mathbf{X}_i, \mathbf{X}_{i+1}$. The interpolating quadratic that fits the points $(s = -d_i^-, H_t = H_{t,i-1})$, $(s = 0, H_t = H_{t,i})$, and $(s = d_i^+, H_t = H_{t,i+1})$, is given by

$$H_t - H_{t,i} = \frac{s(s - d_i^+)}{d_i^-(d_i^- + d_i^+)}(H_{t,i-1} - H_{t,i}) + \frac{s(s + d_i^-)}{d_i^+(d_i^- + d_i^+)}(H_{t,i+1} - H_{t,i}). \quad (3.3.30)$$

Comparison with (3.3.29) yields

$$h_0 = H_{t,i}, \quad (3.3.31)$$

$$h_1 = \frac{-d_i^+ H_{t,i-1}}{d_i^-(d_i^+ + d_i^-)} + \frac{(d_i^+ - d_i^-)H_{t,i}}{d_i^+ d_i^-} + \frac{d_i^- H_{t,i+1}}{d_i^+(d_i^+ + d_i^-)}, \quad (3.3.32)$$

together with a similar expression for h_2 . However, this is never used and so we do not state it here.

3.3.4 Governing Equations

The equations to be satisfied on the ferrofluid boundary are the boundary-integral equation (3.1.2) and the dynamic boundary condition (3.1.6). Additionally, we have the constraint that the volume of the ferrofluid must be prescribed.

In appendix C, we considered the value for $\mathbf{B}(\mathbf{x})$ computed by the integral representation, when $\mathbf{x}$ is a point on the free surface of the ferrofluid. Using this result (C.0.12), for the two-dimensional problem considered here, we have that

$$\mathbf{B}(\mathbf{X}) = \frac{1}{2} (1 + \tilde{m}) H_t(\mathbf{X}) \hat{\mathbf{t}} + B_n(\mathbf{X}) \hat{\mathbf{n}}, \quad (3.3.33)$$

which has been suitably non-dimensionalised using the scalings described in section 3.1.1.

Moreover, following the work in section 3.2, we write the boundary-integral

equation (3.1.2) as

$$\begin{aligned} \mathbf{B}(\mathbf{X}) = & -\frac{\tilde{m}}{4\pi} \mathbf{P}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1-\tilde{m})}{4\pi} \int_{-C}^C (H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + H_t(\mathbf{X}) \frac{(1-\tilde{m})}{4\pi} \mathbf{b}_0 \left(2 \log C - \frac{7}{3} \right) C, \end{aligned} \quad (3.3.34)$$

where the arc-length s is measured from $\mathbf{X}$ so that $\boldsymbol{\xi}_0(0) = \mathbf{X}$.

Tangential Field Equation

For each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we take the component of the boundary-integral equation (3.3.34) in the direction of the tangent $\hat{\mathbf{t}}_i$ at $\mathbf{X}_i$.

Using (3.3.33) and (3.3.34), we have that

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_t(\mathbf{X}) = & -\frac{\tilde{m}}{4\pi} \hat{\mathbf{t}} \cdot \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1-\tilde{m})}{4\pi} \hat{\mathbf{t}} \cdot \int_{-C}^C (H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + H_t(\mathbf{X}) \frac{(1-\tilde{m})}{4\pi} \frac{\cos \vartheta}{x+1} \left(2 \log C - \frac{7}{3} \right) C, \end{aligned} \quad (3.3.35)$$

where we have used the fact that

$$\mathbf{b}_0 = \frac{1}{x+1} (\hat{\mathbf{n}} \sin \vartheta + \hat{\mathbf{t}} \cos \vartheta). \quad (3.3.36)$$

Discretising (3.3.35), and using the trapezium rule for the integral, we have that

$$\sum_{j=0}^{N-1} \left(\mathbb{A}_{ij}^t + \frac{(1-\tilde{m})}{4\pi} \{ \mathbb{B}_{ij}^t - \mathbb{C}_{ij}^t + \mathbb{D}_{ij}^t \} \right) H_{t,j} = b_i^t, \quad (3.3.37)$$

where

$$\mathbb{A}_{ij}^t = \begin{cases} -\frac{1}{2}(1 + \tilde{m}) + \frac{(1-\tilde{m})C \cos \vartheta_i}{4\pi(x_i+1)} (2 \log C - \frac{7}{3}) C & : i = j \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.38)$$

$$\mathbb{B}_{ij}^t = \begin{cases} 0 & : i = j \\ \hat{\mathbf{t}}_i \cdot \mathbf{f}(\mathbf{X}_i, \mathbf{X}_j) d_j & : \text{otherwise} \end{cases} \quad (3.3.39)$$

$$\mathbb{C}_{ij}^t = \begin{cases} \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, s_{ki}) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, s_{ki}) \right) d_k & : j = i \\ \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, s_{ki}) \right) d_k & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.40)$$

$$\mathbb{D}_{ij}^t = \begin{cases} \left(\kappa_{m,i} + \frac{\cos \vartheta_i}{x_i+1} - \frac{\log(8(x_i+1)) \cos \vartheta_i}{x_i+1} \right) d_i & : i = j \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.41)$$

and

$$b_i^t = \frac{\tilde{m}}{4\pi} \hat{\mathbf{t}}_i \cdot \tilde{\mathbf{P}}(\mathbf{X}_i, \hat{\mathbf{x}}). \quad (3.3.42)$$

In deriving these matrices, we have used the decomposition of $\mathbf{f}^*$ given in (3.2.21), and the expressions for h_0 and h_1 given by (3.3.31) and (3.3.32) respectively. Here, $\mathbb{A}_{ij}^t$ contains the direct contributions from $H_t(\mathbf{X}_i)$, $\mathbb{B}_{ij}^t$ and $\mathbb{C}_{ij}^t$ are the two terms in the integral excluding the singular point, $\mathbb{D}_{ij}^t$ is the contribution from the singular point, and b_i^t is the forcing from the wire loop. The contribution from the singular point is obtained by taking the component of $\mathbf{L}(\mathbf{X}_i)$ in the direction of $\hat{\mathbf{t}}_i$, where $\mathbf{L}(\mathbf{X})$ is as in (3.2.28).

This discretisation of the tangential field equation sets up a system of N implicit linear equations for the $H_{t,i}$ values in terms of the ρ_i values.

Normal Field Equation

For each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we take the component of the boundary-integral equation (3.3.34) in the direction of the normal $\hat{\mathbf{n}}_i$ at $\mathbf{X}_i$.

Using (3.3.33) and (3.3.34), we have that

$$\begin{aligned} B_n(\mathbf{X}) = & -\frac{\tilde{m}}{4\pi} \hat{\mathbf{n}} \cdot \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1-\tilde{m})}{4\pi} \hat{\mathbf{n}} \cdot \int_{-C}^C (H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + H_t(\mathbf{X}) \frac{(1-\tilde{m})}{4\pi} \frac{\sin \vartheta}{x+1} \left(2 \log C - \frac{7}{3} \right) C, \end{aligned} \quad (3.3.43)$$

where, again, we have used the fact that

$$\mathbf{b}_0 = \frac{1}{x+1} (\hat{\mathbf{n}} \sin \vartheta + \hat{\mathbf{t}} \cos \vartheta). \quad (3.3.44)$$

Discretising (3.3.43), and using the trapezium rule for the integral, we have that

$$B_{n,i} = \frac{(1-\tilde{m})}{4\pi} \sum_{j=0}^{N-1} (\mathbb{A}_{ij}^n + \mathbb{B}_{ij}^n - \mathbb{C}_{ij}^n + \mathbb{D}_{ij}^n) - b_i^n, \quad (3.3.45)$$

where

$$\mathbb{A}_{ij}^n = \begin{cases} \frac{C \sin \vartheta_i}{x_i+1} \left(2 \log C - \frac{7}{3} \right) C & : i = j \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.46)$$

$$\mathbb{B}_{ij}^n = \begin{cases} 0 & : i = j \\ \hat{\mathbf{n}}_i \cdot \mathbf{f}(\mathbf{X}_i, \mathbf{X}_j) d_j & : \text{otherwise} \end{cases} \quad (3.3.47)$$

$$\mathbb{C}_{ij}^n = \begin{cases} \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, s_{ki}) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, s_{ki}) \right) d_k & : j = i \\ \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\frac{\pm 2d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, s_{ki}) \right) d_k & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.48)$$

$$\mathbb{D}_{ij}^n = \begin{cases} \left(-\frac{\log(8(x_i+1)) \sin \vartheta_i}{x_i+1} + \frac{2(d_i^+ - d_i^-)}{d_i^- d_i^+} \right) d_i & : i = j \\ \frac{\pm 2d_i^\mp}{d_i^\pm (d_i^- + d_i^+)} d_i & : j = i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (3.3.49)$$

and

$$b_i^n = \frac{\tilde{m}}{4\pi} \hat{\mathbf{n}}_i \cdot \tilde{\mathbf{P}}(\mathbf{X}_i, \hat{\mathbf{x}}). \quad (3.3.50)$$

In deriving these matrices, we have used the decomposition of $\mathbf{f}^*$ given in (3.2.21), and the expressions for h_0 and h_1 given by (3.3.31) and (3.3.32) respectively. Here, $\mathbb{A}_{ij}^n$ contains the direct contributions from $H_t(\mathbf{X}_i)$, $\mathbb{B}_{ij}^n$ and $\mathbb{C}_{ij}^n$ are the two terms in the integral excluding the singular point, $\mathbb{D}_{ij}^n$ is the contribution from the singular point, and b_i^n is the forcing from the wire-loop. The contribution from the singular point is obtained by taking the component of $\mathbf{L}(\mathbf{X}_i)$ in the direction of $\hat{\mathbf{n}}_i$, where $\mathbf{L}(\mathbf{X})$ is as in (3.2.28).

This discretisation of the normal field equation sets up a system of N explicit linear equations for the $B_{n,i}$ values in terms of the ρ_i and $H_{t,i}$ values.

Dynamic Boundary Condition

At each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we apply the dynamic boundary condition (3.1.6). This yields a system of N nonlinear equations involving the ρ_i , $H_{t,i}$, and $B_{n,i}$ values. In particular, we have

$$P + \frac{1}{2}(\tilde{m} - 1)\text{Bo} \{H_{t,i}^2 + \tilde{m}^{-1}B_{n,i}^2\} = \kappa_i, \quad (3.3.51)$$

where κ_i is the total curvature of the ferrofluid toroid at the point $\mathbf{X}_i$.

3.4 Implementing the Boundary-Integral Method

3.4.1 Solution Method

We are seeking the position of a ferrofluid boundary such that the corresponding ferrofluid toroid is in equilibrium. Such ferrofluid boundaries are hereafter referred to as equilibrium boundaries. In order for this to be the case, the forces on the surface of the ferrofluid need to be balanced. Hence, the

dynamic boundary condition (3.1.6) needs to be satisfied.

To find an equilibrium boundary numerically, we begin by setting values for the parameters in the problem. These are: the permeability ratio $\tilde{m}$, the magnetic Bond number Bo , and the total volume of ferrofluid V_f .

Since the values of $H_{t,i}$ and $B_{n,i}$ can be computed directly from the ρ_i values, we need to only consider a solution vector, $\mathbf{w}$, of size $N + 1$. The $N + 1$ unknowns that make up $\mathbf{w}$ are then the N values of ρ_i , and the over-pressure P . The $N + 1$ equations to be solved are the N nonlinear equations (3.3.51) from the dynamic boundary condition, and the volume constraint, which states that

$$V = V_f, \quad (3.4.1)$$

where V is the volume of the ferrofluid toroid, which can be computed numerically from (3.1.17) using the trapezium rule for the integral.

We prescribe equally spaced angular coordinates θ_i , that is

$$\theta_i = i\alpha, \quad (3.4.2)$$

where $\alpha = 2\pi/N$, and make an initial guess $\mathbf{w}_0$ for the unknowns ρ_i and P . The values of $H_{t,i}$ can then be computed by solving the linear system given by the tangential field equation (3.3.37), after which we can obtain the values of $B_{n,i}$ directly from the normal field equation (3.3.45). We can also calculate V using the trapezium rule.

Once the values of $H_{t,i}$, $B_{n,i}$, and V are obtained, we substitute them, along with the ρ_i values and P , into (3.3.51) and (3.4.1). Should the equations not be satisfied, then the initial guess $\mathbf{w}_0$ needs to be adjusted and the process repeated, until the solution vector $\mathbf{w}$ converges to a solution that suitably satisfies the dynamic boundary condition and the volume constraint. Overall, this amounts

to solving a nonlinear system of the form

$$\mathbf{F}(\mathbf{w}) = \mathbf{0}, \quad (3.4.3)$$

where the vector function $\mathbf{F}$ returns the $N + 1$ residuals from equations (3.3.51) and (3.4.1) for the given set of values in $\mathbf{w}$.

We implement the numerical scheme in Python, using the nonlinear solver *fsolve* from the SciPy package, which algorithmically performs iterative adjustments to $\mathbf{w}$ until it converges to a solution that is deemed to be sufficiently close to the true solution to (3.3.51) and (3.4.1).

fsolve is based upon MINPACK-1, which is a package of FORTRAN subprogrammes that solve systems of nonlinear equations and nonlinear least-squares problems numerically (Moré, 1980). The subroutines in MINPACK-1 provide methods of solving nonlinear equations via a modified form of Powell's hybrid algorithm (Powell, 1970).

Let the value of $\mathbf{w}$ at step k be denoted by $\mathbf{w}_k$ and the value of the true solution to the system be denoted by $\mathbf{w}^*$. At each step, *fsolve* attempts to check if the convergence condition

$$\|\mathbf{D}(\mathbf{w}_k - \mathbf{w}^*)\| \leq \text{XTOL}\|\mathbf{D}\mathbf{w}^*\|, \quad (3.4.4)$$

is satisfied. Here, $\mathbf{D}$ is a diagonal scaling matrix computed as part of the algorithm, $\|\cdot\|$ is the Euclidean norm, and XTOL is a specified tolerance level typically taken to be 10^{-5} or smaller. Since $\mathbf{w}^*$ is unknown, the condition (3.4.4) cannot be applied directly. Instead, *fsolve* approximates (3.4.4) by

$$\Delta \leq \text{XTOL}\|\mathbf{D}\mathbf{w}_k\|, \quad (3.4.5)$$

where Δ is a bound on the step size between successive iterates $\mathbf{w}_k$, set internally as part of the algorithm. If (3.4.5) is satisfied, then *fsolve* is said to have converged

and the current $\mathbf{w}_k$ is taken to be the numerical solution to (3.3.51) and (3.4.1).

Therefore, if *fsolve* converges, we have obtained a numerical solution for the equilibrium boundary of a ferrofluid toroid with parameters $\tilde{m}$, Bo, and V_f .

Before moving on, we consider a brief analysis of the convergence behaviour of the numerical scheme as XTOL becomes increasingly small. To do this, we apply the numerical scheme to solve (3.3.51) and (3.4.1) for increasingly small values of XTOL, denoting the numerical solution obtained by $\mathbf{w}_s$. We can then substitute $\mathbf{w}_s$ back into equations (3.3.51) and (3.4.1), and study how the size of the residuals behaves as XTOL becomes increasingly small.

A plot of $\|\mathbf{F}(\mathbf{w}_s)\|$ against XTOL, using $N = 100$ points on the boundary and parameter values $\tilde{m} = 2$, Bo = 2, and $V_f = 0.1$, is given in figure 3.4.1. As we would expect, the value of $\|\mathbf{F}(\mathbf{w}_s)\|$ decreases monotonically as XTOL becomes increasingly small. It is also visible from the graph that there are two instances wherein the value of $\mathbf{w}_s$ that satisfies the convergence condition (3.4.5) for a given value of XTOL also satisfies (3.4.5) for the next smallest value of XTOL considered.

We then approximate the true solution to the system using the numerical solution obtained for the smallest value of XTOL considered, specifically $\text{XTOL} = 10^{-11}$. We denote the numerical solution obtained when $\text{XTOL} = 10^{-11}$ by $\mathbf{w}^{**}$. Figure 3.4.2 shows a plot of $\|\mathbf{w}_s - \mathbf{w}^{**}\|$ against XTOL for the same parameters as above. The monotonic decrease in $\|\mathbf{w}_s - \mathbf{w}^{**}\|$ as XTOL becomes increasingly small indicates that $\mathbf{w}_s$ is converging toward $\mathbf{w}^{**}$ as XTOL is decreased.

Therefore, the numerical solution obtained using the scheme described above appears to converge towards the true solution as the tolerance value XTOL becomes increasingly small. For the remainder of this work, *fsolve* will always take Python's default value of $\text{XTOL} = 1.49012 \times 10^{-8}$.

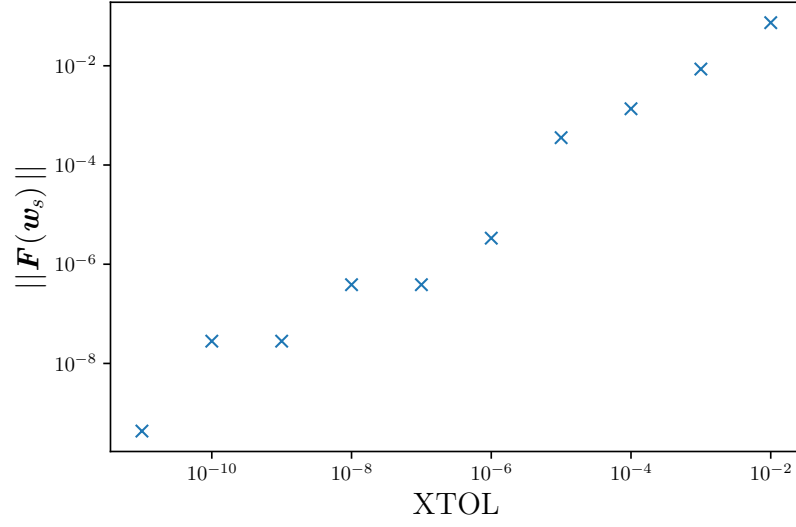


Figure 3.4.1: log-log plot of $\|F(w_s)\|$ against XTOL, using $N = 100$ points on the boundary and parameter values $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.1$.

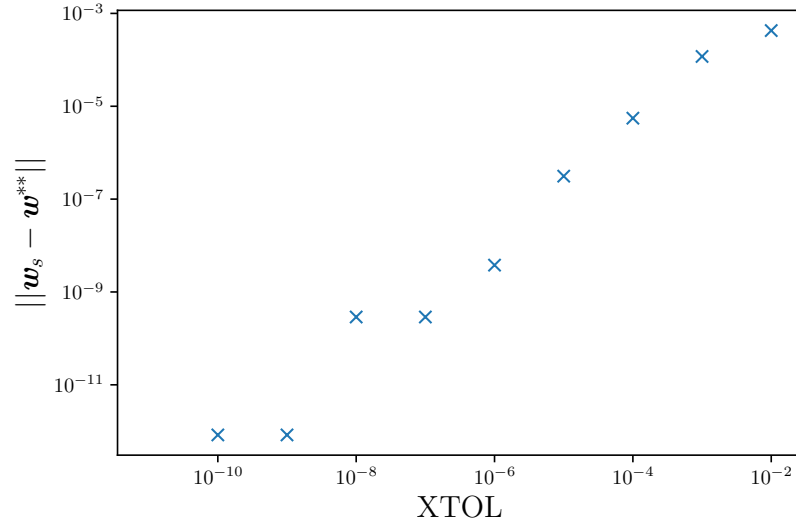


Figure 3.4.2: log-log plot of $\|w_s - w^{**}\|$ against XTOL, using $N = 100$ points on the boundary and parameter values $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.1$.

3.4.2 Preliminary Results

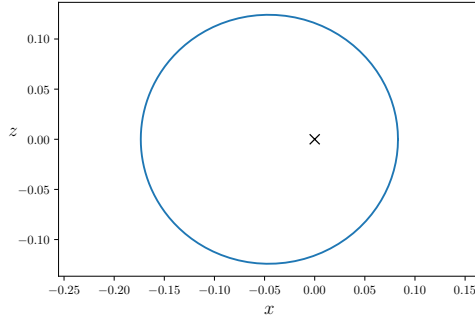
We now present some initial results obtained via the numerical method detailed above. Figure 3.4.3 shows numerically computed equilibrium boundaries for some representative values of the problem parameters $\tilde{m}$, Bo , and V_f . Crucially, the fact that we have been able to find these equilibrium boundaries confirms the existence of ferrofluid toroids in equilibrium.

Additionally, we can see that the position of the equilibrium boundary depends heavily on the problem parameters. For example, figure 3.4.3a (where $\tilde{m} = 2$, $Bo = 2$, and $V_f = 0.3$) shows an almost circular equilibrium boundary somewhat off-centred from the wire, whereas figure 3.4.3d (where $\tilde{m} = 2$, $Bo = 0.2$, and $V_f = 0.5$) shows a more ovalar equilibrium boundary with the wire located very close to the rightmost point of the boundary. We can also see that the equilibrium boundaries shown in figures 3.4.3a and 3.4.3c (where $\tilde{m} = 2$, $Bo = 10$, and $V_f = 1$) are similar in shape, however the almost circular boundary shown in figure 3.4.3c is larger than that in figure 3.4.3a.

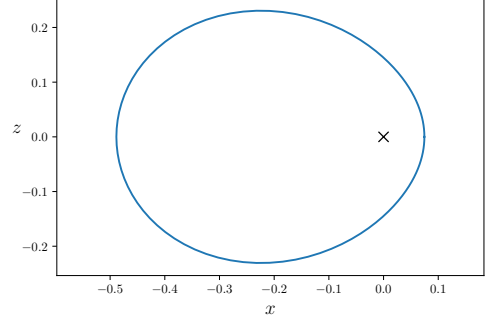
Figure 3.4.4 shows the numerically computed values of H_t and B_n for the equilibrium boundaries shown in figure 3.4.3. We will not consider a detailed analysis of the magnetic field associated with ferrofluid toroids in equilibrium as part of this work. However, there is something of interest to be noted about the behaviour of B_n here.

In Chapter 1, we discussed the Normal-Field instability, in which a non-zero component of the magnetic field normal to the ferrofluid free surface can introduce an instability that results in a characteristic ‘spikey’ free surface (Cowley and Rosensweig, 1967). It is clear from figure 3.4.4, that B_n is not typically zero all the way around the equilibrium boundary of a ferrofluid toroid, nor is it necessarily small in comparison to H_t . This opens up an interesting area of investigation, albeit one that is beyond the scope of this work, that the Normal-Field instability may manifest on the free surface of ferrofluid toroids.

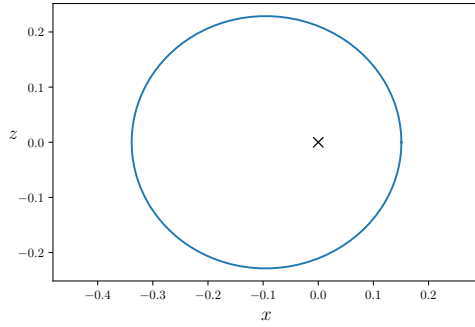
We have seen that the size and shape of equilibrium configurations of ferrofluid toroids can vary greatly depending on the parameters in the problem. A full discussion of the nature of the equilibrium boundaries of ferrofluid toroids is presented in Chapter 5. First, we need to investigate the validity of results obtained using the numerical method described in this chapter.



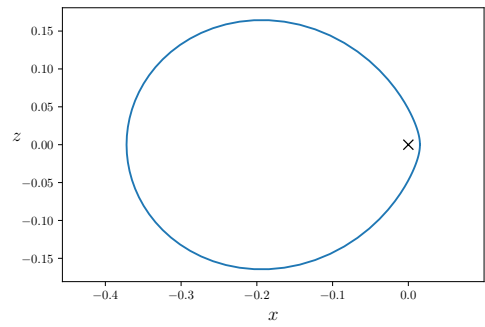
(a) Equilibrium boundary for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.



(b) Equilibrium boundary for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 1$.



(c) Equilibrium boundary for $\tilde{m} = 2$, $\text{Bo} = 10$, and $V_f = 1$.



(d) Equilibrium boundary for $\tilde{m} = 2$, $\text{Bo} = 0.2$, and $V_f = 0.5$.

Figure 3.4.3: Numerically computed equilibrium boundaries for some example values of the problem parameters: $\tilde{m}$, Bo , and V_f . The number of boundary points was taken to be $N = 100$. In each graph, the location of the wire is marked by a black cross.

3.4.3 Convergence of Numerical Solutions as $N \rightarrow \infty$

For the numerical results we obtain to be valid, the solution obtained from the numerical method must converge to the true solution as the number of points on the boundary tends to infinity. Since the true solution is unknown we are unable to verify this directly. However, we can still check that the numerical solution is converging to something sensible as N gets large.

To do this, we consider the case where $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$, and solve for the equilibrium boundary and the over-pressure P numerically, for increasingly large values of N .

Figure 3.4.5 shows the numerically computed equilibrium boundary for

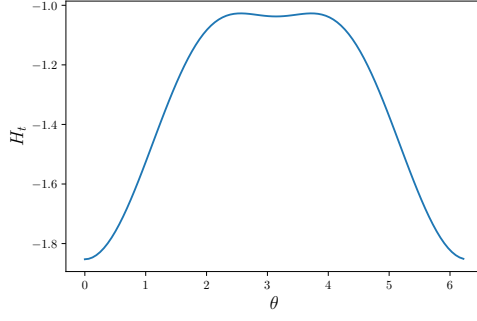
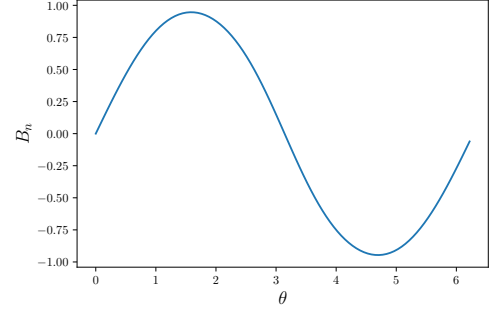
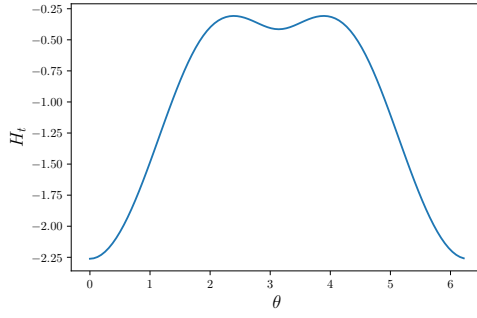
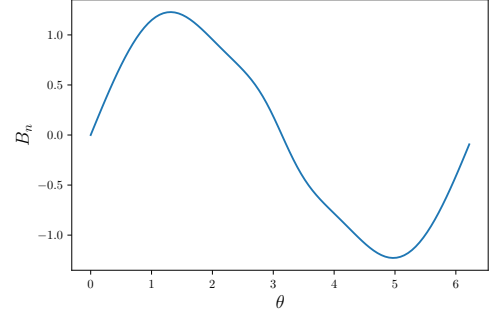
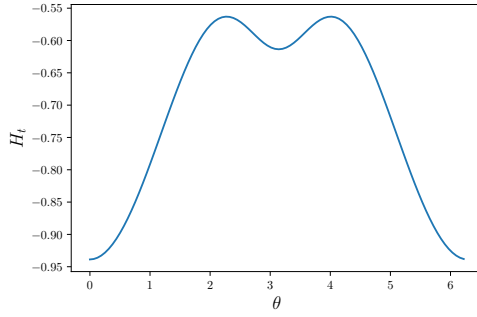
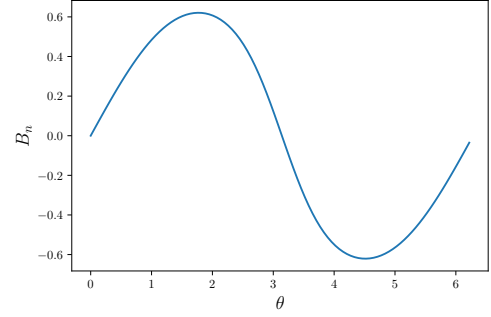
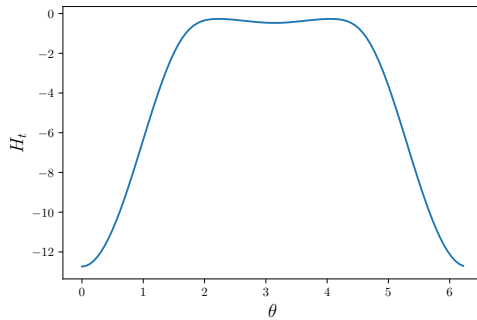
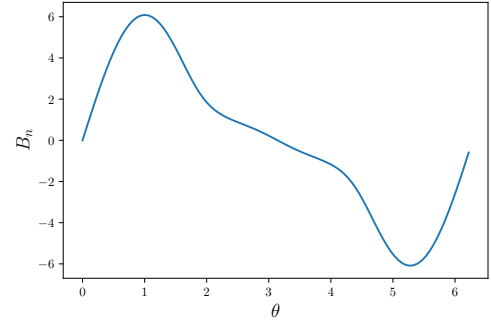
(a) H_t for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.(b) B_n for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.(c) H_t for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 1$.(d) B_n for $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 1$.(e) H_t for $\tilde{m} = 2$, $\text{Bo} = 10$, and $V_f = 1$.(f) B_n for $\tilde{m} = 2$, $\text{Bo} = 10$, and $V_f = 1$.(g) H_t for $\tilde{m} = 2$, $\text{Bo} = 0.2$, and $V_f = 0.5$.(h) B_n for $\tilde{m} = 2$, $\text{Bo} = 0.2$, and $V_f = 0.5$.

Figure 3.4.4: Numerically computed H_t and B_n for some example values of the problem parameters: $\tilde{m}$, Bo , and V_f . The number of boundary points was taken to be $N = 100$.

increasingly large values of N . We observe that these equilibrium boundaries are overlapping, with only the $N = 1000$ case actually visible on the graph. By treating the $N = 1000$ case as the ‘true’ solution, we can seek to verify that the numerical solution is converging to the $N = 1000$ solution as N is increased.

To do this, we note that the point ρ_0 is the ρ coordinate of the rightmost point of the equilibrium boundary, which lies on the x -axis, irrespective of the value of N . Therefore, we should find that ρ_0 converges toward the $N = 1000$ solution as N is increased up to 1000. We denote ρ_{max} to be the value of ρ_0 when $N = 1000$, then consider the behaviour of $|\rho_{max} - \rho_0|$ as N increases. Figure 3.4.6 shows the graph of $|\rho_{max} - \rho_0|$ against N , from which we can see that $|\rho_{max} - \rho_0|$ decays to the constructed value of zero when $N = 1000$.

Similarly, we expect the over-pressure P to converge toward the $N = 1000$ solution as N is increased up to 1000. Denoting P_{max} to be the value of the over-pressure when $N = 1000$, the behaviour of $|P_{max} - P|$ as N increases is shown in figure 3.4.7. Again, we can see the decay of $|P_{max} - P|$ to the constructed value of zero when $N = 1000$. Additionally, plots of $|\rho_{max} - \rho_0|$ and $|P_{max} - P|$ against N on logarithmically scaled axes are given in figures 3.4.8 and 3.4.9.

Therefore, we can see that both the equilibrium boundary and the over-pressure are converging to the $N = 1000$ solution as N is increased. Hence, we do appear to have sensible convergence of our numerical solution as N gets large.

To complete the validation of the numerical solutions obtained via this method, we need to consider a special case of the problem in which an analytical solution can be computed and compared with the numerical solution. In the next chapter, we consider such a case, whereby an asymptotic solution can be constructed when the aspect ratio of the toroidal droplet is small.

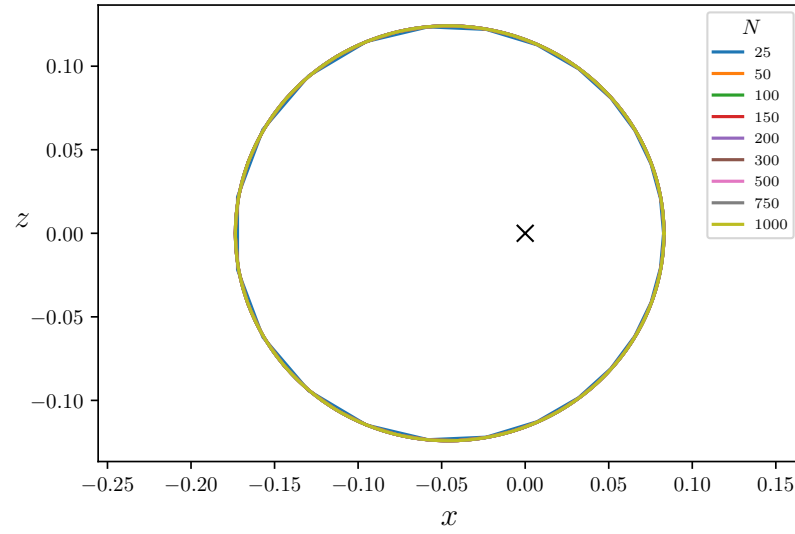


Figure 3.4.5: Numerically computed equilibrium boundary for increasing values of N , with $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.

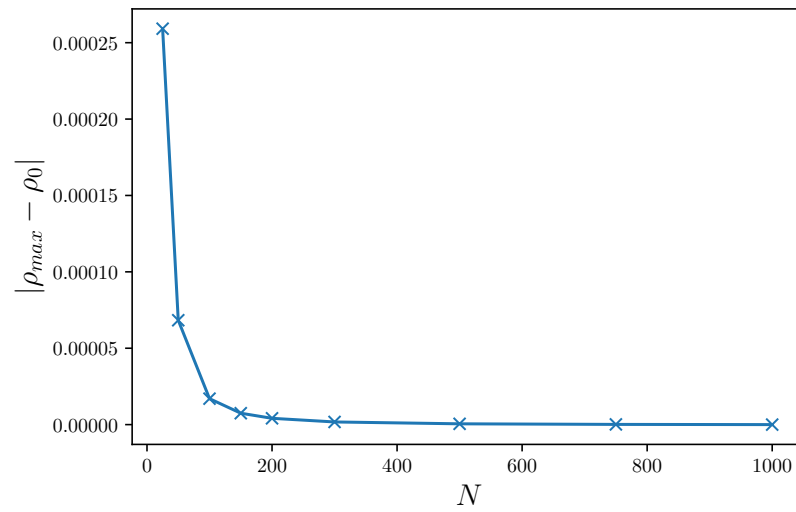


Figure 3.4.6: Graph of $|\rho_{max} - \rho_0|$ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.

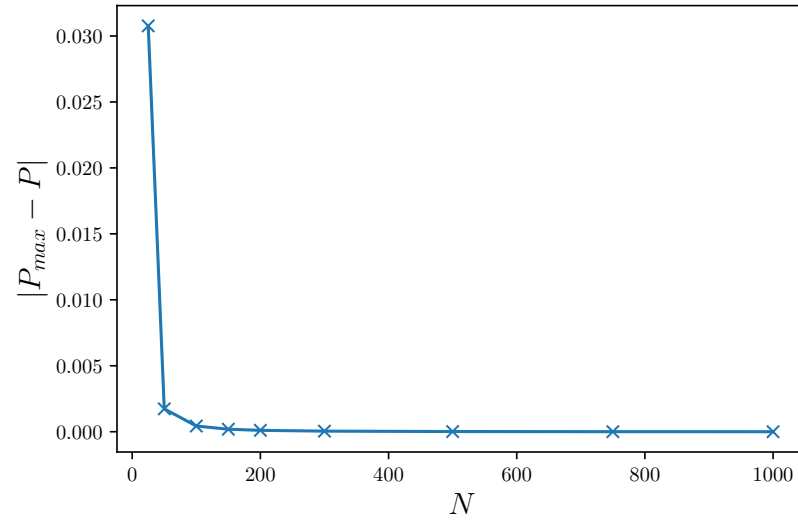


Figure 3.4.7: Graph of $|P_{max} - P|$ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.

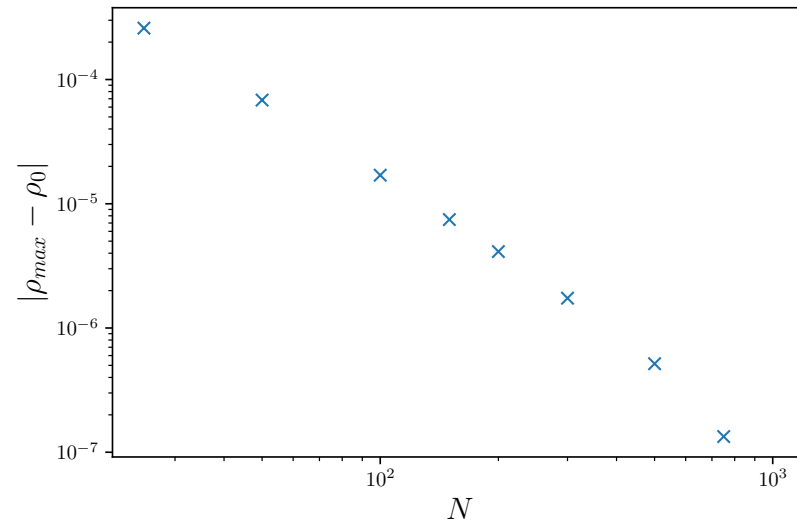


Figure 3.4.8: log-log plot of $|\rho_{max} - \rho_0|$ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.

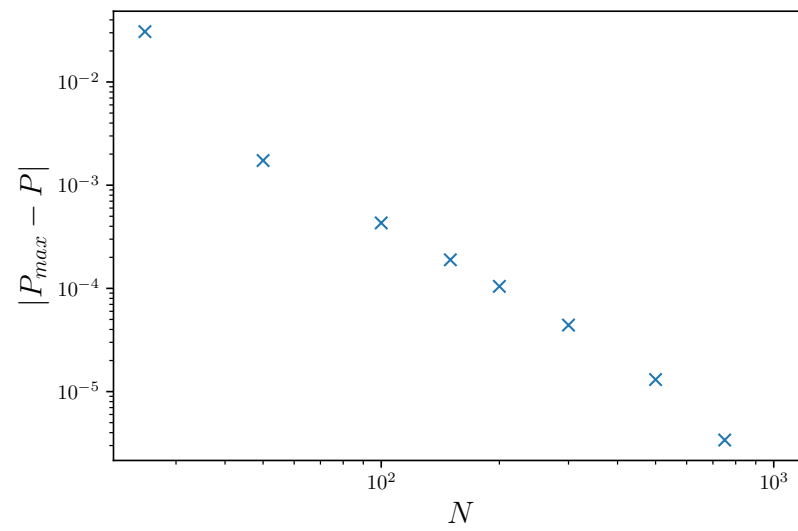


Figure 3.4.9: log-log plot of $|P_{max} - P|$ against N for the numerical solution obtained when $\tilde{m} = 2$, $\text{Bo} = 2$, and $V_f = 0.3$.

Asymptotic Analysis for a Toroid with a Small Aspect Ratio

In Chapter 3, we discussed the numerical method that we will be implementing in order to solve for ferrofluid toroids in equilibrium. In order to validate this method, we seek to consider a special case of the problem for which we can determine an analytical solution. In this chapter, we will consider the case where the toroid is thin in comparison to the radius of the circular wire-loop that it surrounds. An asymptotic solution for this special case, up to the second-order correction terms, is found and then compared with the results obtained by solving the same problem using the numerical method.

4.1 Problem Statement

4.1.1 Problem Set-up

We consider a static axisymmetric toroidal volume V_f of ferrofluid surrounding a thin wire forming a circular loop of radius R and carrying a current I . We denote the permeability of the ferrofluid by μ_f and note that the surrounding air will have permeability μ_0 . The aim is to determine the position of the ferrofluid free surface, such that the toroid is in equilibrium. From our previous work, we know that this is a two-dimensional problem, which needs to be solved for the

position of the ferrofluid boundary, the magnetic induction $\mathbf{B}$ on the ferrofluid boundary, and the constant over-pressure P . In this chapter, we focus on the case that V_f/R^3 is small.

We work in Cartesian coordinates, and choose the centre of curvature for the wire-loop to be located at $x = -D$, $z = 0$, where $R = |D|$. We work in either the $(x > -D, z)$ half-plane if $D > 0$, or the $(x < -D, z)$ half-plane if $D < 0$. The origin of the (x, z) coordinates thus corresponds to the position of the centre of the cross-section of the wire-loop. The axis of rotation for the toroid is the line $x = -D$. The direction of the current is always taken to be into the plane. Let $\mathbf{X}$ be a point on the ferrofluid boundary, described by its Cartesian coordinates as

$$\mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix}. \quad (4.1.1)$$

Moreover, the ferrofluid boundary is described by $\boldsymbol{\xi}_0(s)$, where

$$\boldsymbol{\xi}_0(s) = \begin{pmatrix} \xi(s) \\ \zeta(s) \end{pmatrix}, \quad (4.1.2)$$

and s is an arc-length coordinate around the ferrofluid boundary.

The unit normal vector to the ferrofluid boundary, $\hat{\mathbf{n}}$, at the point $\mathbf{X}$, is given by

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix}, \quad (4.1.3)$$

where ϑ is the angle which the ferrofluid boundary makes with the vertical at the point $\mathbf{X}$. Similarly, the unit tangent vector to the ferrofluid boundary, $\hat{\mathbf{t}}$, at the point $\mathbf{X}$, is given by

$$\hat{\mathbf{t}} = \begin{pmatrix} -\sin \vartheta \\ \cos \vartheta \end{pmatrix}. \quad (4.1.4)$$

The magnetic induction $\mathbf{B}$ at the point $\mathbf{X}$, is given by the boundary-integral

equation (2.7.23)

$$\mathbf{B}(\mathbf{X}) = -\frac{\mu_f I}{4\pi} \mathbf{P}(\mathbf{X}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) \, ds, \quad (4.1.5)$$

where where $H_t = \mathbf{H} \cdot \hat{\mathbf{t}}$ is the component of the magnetic field $\mathbf{H}$ tangential to the ferrofluid boundary at $\mathbf{X}$, $\mathbf{P}$ and $\mathbf{f}$ are as defined in (2.7.10)–(2.7.13), (2.7.17), (2.7.22), and $2C$ is the total arc-length of the ferrofluid boundary.

The dynamic boundary condition on the ferrofluid boundary, at the point $\mathbf{X}$, is given by (2.7.27)

$$P + \frac{\mu_0 \chi_m}{2} H_t^2 + \frac{\chi_m}{2\mu_0 (1 + \chi_m)} B_n^2 = \kappa \gamma, \quad (4.1.6)$$

where $B_n = \mathbf{B} \cdot \hat{\mathbf{n}}$ is the component of the magnetic induction normal to the ferrofluid boundary at $\mathbf{X}$, χ_m is the (constant) magnetic susceptibility of the ferrofluid, κ is the total curvature of the ferrofluid toroid at the point $\mathbf{X}$, and γ is the coefficient of surface tension.

Finally, we note that the total volume of ferrofluid, V_f , needs to be prescribed.

4.1.2 Non-Dimensionalisation

Choice of Scalings

We are considering the case where V_f is small compared with the major radius $R = |D|$. In this case, an appropriate length scale for the cross section is given by

$$\lambda = \sqrt{\frac{V_f}{2\pi^2 |D|}}, \quad (4.1.7)$$

which is the minor radius we would have if the ferrofluid formed a perfect torus of major radiues $|D|$. To encapsulate the smallness of V_f , we introduce the parameter

$$\varepsilon = \frac{\lambda}{D}. \quad (4.1.8)$$

Since $\lambda > 0$, the sign of ε corresponds to the sign of D . And we assume $|\varepsilon| \ll 1$.

The other dimensionless parameters in the problem are the permeability ratio $\tilde{m}$, given by

$$\tilde{m} = \frac{\mu_f}{\mu_0}. \quad (4.1.9)$$

and the magnetic Bond number Bo , defined as

$$\text{Bo} = \frac{1}{2} (\tilde{m} - 1) \frac{\mu_0 I^2}{4\pi^2 \lambda \gamma}. \quad (4.1.10)$$

Note that the definition of the Bond number given in (4.1.10) differs to the definition used in Chapters 2 and 3. The magnetic Bond number is chosen to quantify the relative strength of magnetic effects to surface tension effects on the ferrofluid free surface, and both forms of Bo defined in this work accomplish this. However, the form given in (4.1.10) happens to be particularly useful for the problem we are considering in this chapter, hence the change in definition.

We now introduce dimensionless variables, denoted by quantities with hats, as follows

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{2\pi\lambda} \hat{\mathbf{B}}, \quad B_n = \frac{\mu_0 I}{2\pi\lambda} \hat{B}_n, \quad H_t = \frac{I}{2\pi\lambda} \hat{H}_t, \quad \mathbf{X} = \lambda \hat{\mathbf{X}}, \\ \xi_0 &= \lambda \hat{\xi}_0, \quad s = \lambda \hat{s}, \quad P = \frac{1}{2} (\tilde{m} - 1) \frac{\mu_0 I^2}{4\pi^2 \lambda^2} \hat{P}, \quad \kappa = \frac{1}{\lambda} \hat{\kappa}. \end{aligned} \quad (4.1.11)$$

Additionally, we can introduce dimensionless forms of the functions $\mathbf{P}$ and $\mathbf{f}$, namely

$$\mathbf{P} = \frac{1}{\lambda} \hat{\mathbf{P}}, \quad \mathbf{f} = \frac{1}{\lambda} \hat{\mathbf{f}}, \quad (4.1.12)$$

where we have recalled the dimensions of $\mathbf{P}$ and $\mathbf{f}$ from Chapter 2.

Dynamic Boundary Condition

First, we recall the fact that $\chi_m = \tilde{m} - 1$. Substituting our scaled variables (4.1.11) into the dynamic boundary condition (4.1.6), yields

$$\frac{1}{2}(\tilde{m} - 1) \frac{\mu_0 I^2}{4\pi^2 \lambda^2} \hat{P} + \frac{1}{2}(\tilde{m} - 1) \left\{ \frac{\mu_0 I^2}{4\pi^2 \lambda^2} \hat{H}_t^2 + \frac{\mu_0 I^2}{4\pi^2 \lambda^2} \tilde{m}^{-1} \hat{B}_n^2 \right\} = \frac{1}{\lambda} \hat{\kappa} \gamma. \quad (4.1.13)$$

From here, we can tidy up, and use the definition of the Bond number (4.1.10), to obtain

$$\text{Bo} \left(\hat{P} + \hat{H}_t^2 + \tilde{m}^{-1} \hat{B}_n^2 \right) = \hat{\kappa}, \quad (4.1.14)$$

or finally, upon dropping hats,

$$P + H_t^2 + \tilde{m}^{-1} B_n^2 = \frac{\kappa}{\text{Bo}}. \quad (4.1.15)$$

Boundary-Integral Equation

First, we note that, in dimensionless coordinates, the centre of curvature of the wire-loop is located at $\hat{x} = -D/\lambda = -\varepsilon^{-1}$. In accordance with Chapter 2, we can write down expressions for $\hat{P}(\hat{\mathbf{X}}, \mathbf{0})$ and $\hat{f}(\hat{\mathbf{X}}, \hat{\xi}_0(\hat{s}))$. In particular, we have that

$$\hat{P}(\hat{\mathbf{X}}, \mathbf{0}) = |\varepsilon^{-1}| \hat{\hat{P}}(\hat{\mathbf{X}}, \mathbf{0}), \quad (4.1.16)$$

and

$$\hat{f}(\hat{\mathbf{X}}, \hat{\xi}_0(\hat{s})) = \left| \hat{\xi} + \varepsilon^{-1} \right| \hat{\hat{P}}(\hat{\mathbf{X}}, \hat{\xi}_0(\hat{s})), \quad (4.1.17)$$

where

$$\hat{\hat{P}}(\hat{\mathbf{X}}, \hat{\xi}_0) = (\hat{z} - \hat{\xi}) (I_1 - 2I_2) \hat{\mathbf{x}} + \left(-(\hat{x} + \hat{\xi} + 2\varepsilon^{-1}) I_1 + 2(\hat{x} + \varepsilon^{-1}) I_2 \right) \hat{\mathbf{z}}, \quad (4.1.18)$$

$\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x and z directions respectively,

$$I_1 = \frac{4}{(1-m)A^{\frac{3}{2}}} \text{Ee}(m), \quad I_2 = \frac{4}{mA^{\frac{3}{2}}} \left(\frac{\text{Ee}(m)}{(1-m)} - \text{Ek}(m) \right), \quad (4.1.19)$$

with E_k and E_e being the complete elliptic integrals of the first and second kind respectively, $m = B/A$, and

$$A = \left(\hat{x} + \hat{\xi} + 2\varepsilon^{-1}\right)^2 + \left(\hat{z} - \hat{\zeta}\right)^2, \quad B = 4\left(\hat{x} + \varepsilon^{-1}\right)\left(\hat{\xi} + \varepsilon^{-1}\right). \quad (4.1.20)$$

We now substitute the dimensionless quantities, given by (4.1.11), (4.1.12), (4.1.16), and (4.1.17), into the boundary-integral equation (4.1.5), to obtain

$$\frac{\mu_0 I}{2\pi\lambda} \hat{\mathbf{B}}(\hat{\mathbf{X}}) = -\frac{\mu_f I}{4\pi} \frac{1}{\lambda} \hat{\mathbf{P}}(\hat{\mathbf{X}}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-\hat{C}}^{\hat{C}} \frac{I}{2\pi\lambda} \hat{H}_t(\hat{\xi}_0) \frac{1}{\lambda} \hat{\mathbf{f}}(\hat{\mathbf{X}}, \hat{\xi}_0) \lambda d\hat{s}, \quad (4.1.21)$$

where $C = \lambda\hat{C}$. After a bit of algebra, and dropping the hats for clarity, we arrive at

$$\mathbf{B}(\mathbf{X}) = -\frac{\tilde{m}}{2} \mathbf{P}(\mathbf{X}, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_{-C}^C H_t(\xi_0) \mathbf{f}(\mathbf{X}, \xi_0) ds, \quad (4.1.22)$$

which is our dimensionless boundary-integral equation.

Finally, we recall from appendix C that the general boundary-integral equation evaluated at a point $\mathbf{X}$ on the ferrofluid boundary, yields

$$\mathbf{B}(\mathbf{X}) = \frac{1}{2} (1 + \tilde{m}) H_t \hat{\mathbf{t}} + B_n \hat{\mathbf{n}}, \quad (4.1.23)$$

which has been suitably non-dimensionalised using (4.1.11).

Fluid Volume

The dimensional volume of ferrofluid was chosen such that it corresponded to the volume of a perfect torus of minor radius λ and major radius $|D|$. Hence, with lengths scaled on λ , the dimensionless volume of ferrofluid (with the hat immediately dropped) is given by

$$V_f = 2\pi^2 |\varepsilon^{-1}|, \quad (4.1.24)$$

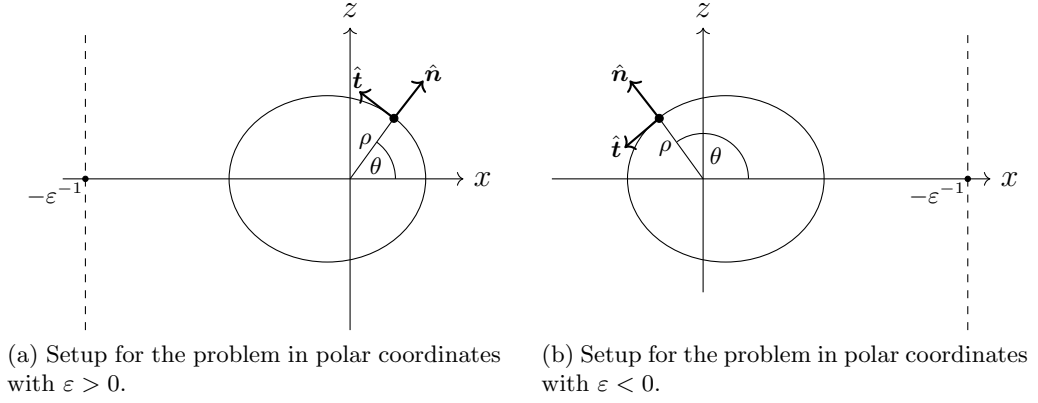


Figure 4.1.1: Diagrams of the setup for the problem of solving for the equilibrium boundary of a ferrofluid toroid in the two cases $\varepsilon > 0$ and $\varepsilon < 0$.

that is, the volume of a perfect torus of minor radius 1 and major radius $|\varepsilon^{-1}|$.

4.1.3 Problem in Polar Coordinates

Variables and Governing Equations

We now choose to describe the points on the ferrofluid boundary using plane polar coordinates (r, θ) . If the position of the ferrofluid boundary is given by $r = \rho(\theta)$, then we have that

$$\mathbf{X}(\theta) = \begin{pmatrix} x(\theta) \\ z(\theta) \end{pmatrix} = \begin{pmatrix} \rho(\theta) \cos \theta \\ \rho(\theta) \sin \theta \end{pmatrix}. \quad (4.1.25)$$

Consequently, both H_t and B_n can be expressed as functions of θ , as can the curvature κ . Diagrams showing the setup in polar coordinates for both the $\varepsilon > 0$ and $\varepsilon < 0$ cases are given in figure 4.1.1.

In particular, following the work in appendix A, we have that κ can be written,

independently of the sign of ε , as

$$\begin{aligned} \kappa(\theta) = & \frac{\rho(\theta)^2 + 2(\rho'(\theta))^2 - \rho(\theta)\rho''(\theta)}{\left((\rho'(\theta))^2 + \rho(\theta)^2\right)^{\frac{3}{2}}} \\ & + \varepsilon \frac{\rho'(\theta)\sin\theta + \rho(\theta)\cos\theta}{\left((\rho'(\theta))^2 + \rho(\theta)^2\right)^{1/2} (1 + \varepsilon\rho(\theta)\cos\theta)}. \end{aligned} \quad (4.1.26)$$

It is easy to see that the dynamic boundary condition (4.1.15) is unchanged when written in terms of θ . In order to express the boundary-integral equation (4.1.22) in terms of θ , we need to convert the arc-length integral in (4.1.22) into a θ integral. To do this, we note the polar coordinate arc-length relation, namely

$$ds = \left(r^2 + \left(\frac{dr}{d\theta} \right)^2 \right) d\theta. \quad (4.1.27)$$

Since the arc-length integral is carried out around the ferrofluid boundary $r = \rho(\theta)$, we have that

$$\mathbf{B}(\theta) = -\frac{\tilde{m}}{2} \mathbf{P}(\theta, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} H_t(\theta') \mathbf{f}(\theta, \theta') \left(\rho(\theta')^2 + \left(\frac{d\rho(\theta')}{d\theta'} \right)^2 \right) d\theta'. \quad (4.1.28)$$

Tangential and Normal Field Equations

By comparing equations (4.1.23) and (4.1.28), we obtain both the tangential field equation

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_t(\theta) = & \hat{\mathbf{t}} \cdot \left(-\frac{\tilde{m}}{2} \mathbf{P}(\theta, \mathbf{0}) \right) \\ & + \hat{\mathbf{t}} \cdot \left(\frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} H_t(\theta') \mathbf{f}(\theta, \theta') \left(\rho(\theta')^2 + \left(\frac{d\rho(\theta')}{d\theta'} \right)^2 \right)^{\frac{1}{2}} d\theta' \right), \end{aligned} \quad (4.1.29)$$

and the normal field equation

$$B_n(\theta) = \hat{\mathbf{n}} \cdot \left(-\frac{\tilde{m}}{2} \mathbf{P}(\theta, \mathbf{0}) \right) + \hat{\mathbf{n}} \cdot \left(\frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} H_t(\theta') \mathbf{f}(\theta, \theta') \left(\rho(\theta')^2 + \left(\frac{d\rho(\theta')}{d\theta'} \right)^2 \right)^{\frac{1}{2}} d\theta' \right). \quad (4.1.30)$$

Whilst the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$, (4.1.3) and (4.1.4), are defined in terms of the surface angle ϑ , they can be expressed in terms of the polar angle θ via the relations derived in Chapter 3 (3.3.5), (3.3.19), and (3.3.20). In particular, we have that

$$\sin \vartheta = \left\{ -\frac{d\rho(\theta)}{d\theta} \cos \theta + \rho(\theta) \sin \theta \right\} \left(\left(\frac{d\rho(\theta)}{d\theta} \right)^2 + \rho(\theta)^2 \right)^{-\frac{1}{2}}, \quad (4.1.31)$$

$$\cos \vartheta = \left\{ \frac{d\rho(\theta)}{d\theta} \sin \theta + \rho(\theta) \cos \theta \right\} \left(\left(\frac{d\rho(\theta)}{d\theta} \right)^2 + \rho(\theta)^2 \right)^{-\frac{1}{2}}. \quad (4.1.32)$$

Volume Constraint

The fluid volume V_f for the ferrofluid toroid can be calculated using Pappus's centroid theorem (see Chapter 3). Using Pappus's centroid theorem, we find that V_f is given by

$$V_f = 2\pi \int_0^{2\pi} \int_0^\rho \pm(\varepsilon^{-1} + r \cos \theta) r dr d\theta, \quad (4.1.33)$$

where the upper sign is taken if $\varepsilon > 0$ and the lower sign is taken if $\varepsilon < 0$. The volume constraint then states that

$$2\pi^2 |\varepsilon^{-1}| = 2\pi \int_0^{2\pi} \int_0^\rho \pm(\varepsilon^{-1} + r \cos \theta) r dr d\theta. \quad (4.1.34)$$

Moreover, since $|\varepsilon^{-1}| = \pm \varepsilon^{-1}$, it follows that the volume constraint is given by

$$2\pi^2 \varepsilon^{-1} = 2\pi \int_0^{2\pi} \int_0^\rho (\varepsilon^{-1} + r \cos \theta) r dr d\theta, \quad (4.1.35)$$

and is independent of the sign of ε . Computing the r integral in (4.1.35) and multiplying through by ε , yields the final expression for the dimensionless volume constraint,

$$\pi = \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3}\varepsilon\rho \cos\theta \right) \rho^2 d\theta. \quad (4.1.36)$$

The full problem has now been expressed entirely in terms of a single independent variable θ , and the parameters ε , $\tilde{m}$, and Bo .

4.2 Analysis as $\varepsilon \rightarrow 0$

4.2.1 Premise

The parameter ε (4.1.8) is an aspect ratio describing how thick the ferrofluid toroid is in comparison to the radius of the circular wire-loop. We wish to consider the case when $|\varepsilon|$ is small. A small value of $|\varepsilon|$ can be interpreted physically as either there being a small total volume of ferrofluid or the axis of rotation of the toroid being far away from its cross-section. The two interpretations are entirely analogous mathematically.

Before considering the case where $|\varepsilon| \ll 1$, we first choose to briefly study the case where $\varepsilon = 0$. Physically, this corresponds to either having no volume of ferrofluid whatsoever, or the axis of rotation being taken out to infinity. If the axis of rotation is taken out to infinity, then the circular wire-loop becomes an infinitely long straight wire. Thus, we can think of the $\varepsilon = 0$ case as considering an infinitely long axially symmetric column of ferrofluid surrounding an infinitely long straight current-carrying wire.

The magnetic field due to an infinitely long straight current-carrying wire, which we will denote $\mathbf{H}_W$, is well-known. Crucially, we note that the field is two-dimensional, where the field lines are concentric circles centred on the wire and the field strength depends only on the radial distance from the wire, decaying

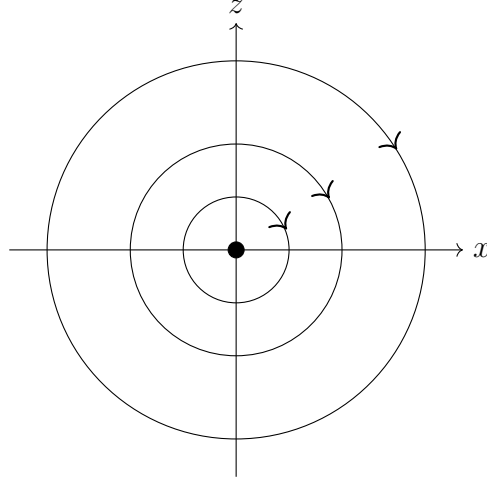


Figure 4.2.1: Sketch of the field lines due to an infinitely long straight wire, shown in the plane with the wire located at the origin and the current directed into the plane. The direction of the field is determined using a right hand screw rule.

like $1/r$ (see Griffiths, 1981, for example). A sketch of the magnetic field is given in figure 4.2.1.

We thus expect that for $\varepsilon = 0$, the ferrofluid boundary will take the shape of a field-line of $\mathbf{H}_W$, and so H_t and B_n will be constants. In fact, it is clear that we should expect B_n to be zero. Having determined the values of H_t and B_n , the value of P can then be computed from the dynamic boundary condition.

4.2.2 Leading-Order Solution

We suppose that the ferrofluid boundary is coincident with a field-line of $\mathbf{H}_W$, that is, a circle centred at the origin. Hence, we have that $\rho(\theta)$ is a constant, the value of which can be determined using the volume constraint (4.1.36). Setting $\varepsilon = 0$ in (4.1.36), it is easy to see that $\rho = 1$ is the appropriate solution.

We now consider an expansion of the tangential field equation (4.1.29) about $\varepsilon = 0$. This series expansion, along with all series expansions computed in this

chapter, is evaluated using Maple. At $O(1)$, we find that

$$\frac{1}{2}(1 + \tilde{m})H_t(\theta) = -\frac{\tilde{m}}{\rho(\theta)} + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} \frac{H_t(\theta')}{\rho(\theta)} \left(\rho(\theta')^2 + \left(\frac{d\rho(\theta')}{d\theta'} \right)^2 \right) d\theta'. \quad (4.2.1)$$

Since $\rho(\theta) = 1$, and we are expecting H_t to be a constant, we can write (4.2.1) as

$$\frac{1}{2}(1 + \tilde{m})H_t = -\tilde{m} + \frac{(1 - \tilde{m})}{4\pi} H_t \int_0^{2\pi} d\theta'. \quad (4.2.2)$$

Carrying out the trivial integration in (4.2.2), it is easy to show that $H_t = -1$.

We now consider a series expansion of the normal field equation (4.1.30) about $\varepsilon = 0$. At $O(1)$, we find that

$$B_n(\theta) = \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} \frac{H_t(\theta')}{\rho(\theta)} \left(\rho(\theta')^2 + \left(\frac{d\rho(\theta')}{d\theta'} \right)^2 \right) \left(\frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} \right) d\theta'. \quad (4.2.3)$$

Substituting $\rho(\theta) = 1$ and $H_t(\theta) = -1$ into (4.2.3), we have that

$$B_n(\theta) = -\frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} \left(\frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} \right) d\theta'. \quad (4.2.4)$$

Moreover, it can be shown (see appendix E), that the integral in (4.2.4) is zero for all θ . It follows that $B_n = 0$, as expected.

It remains to determine the value of P . Before substituting our obtained values of H_t and B_n into the dynamic boundary condition (4.1.15), we need to determine the value of the curvature. Setting $\rho(\theta) = 1$ and $\varepsilon = 0$ into (4.1.26), we find that $\kappa = 1$. Therefore, from (4.1.15), we have that

$$P + (-1)^2 + \tilde{m}^{-1}(0)^2 = \frac{1}{\text{Bo}}, \quad (4.2.5)$$

and so $P = -1 + 1/\text{Bo}$.

As all the equations are satisfied at leading-order, we have shown that our assumption about the form of the solution for $\varepsilon = 0$ was correct.

4.2.3 Series Expansions for $|\varepsilon| \ll 1$

Having obtained a form for the leading-order solution to the problem (when $\varepsilon = 0$), we now seek an asymptotic solution to the problem when $|\varepsilon| \ll 1$. We posit an ansatz of the form

$$\rho(\theta) = 1 + \varepsilon \log |\varepsilon| \rho_1(\theta) + \varepsilon \rho_2(\theta) + \cdots \quad (4.2.6)$$

$$B_n(\theta) = 0 + \varepsilon \log |\varepsilon| B_{n,1}(\theta) + \varepsilon B_{n,2}(\theta) + \cdots \quad (4.2.7)$$

$$H_t(\theta) = -1 + \varepsilon \log |\varepsilon| H_{t,1}(\theta) + \varepsilon H_{t,2}(\theta) + \cdots \quad (4.2.8)$$

$$P = -1 + \frac{1}{\text{Bo}} + \varepsilon \log |\varepsilon| P_1 + \varepsilon P_2 + \cdots, \quad (4.2.9)$$

where all functions of θ must be 2π -periodic. The leading-order terms in (4.2.6)–(4.2.9) are precisely the exact solution obtained in the case of $\varepsilon = 0$. Notice that the first-order correction terms are anticipated to be $O(\varepsilon \log |\varepsilon|)$. This is due to the presence of terms of this order when the tangential and normal field equations, (4.1.29) and (4.1.30), are expanded for small $|\varepsilon|$. These expansions are not explicitly presented here.

4.2.4 Symmetries

Before we attempt to find the correction terms in (4.2.6)–(4.2.9), we first consider the symmetrical nature of the problem. In particular, if we have a solution for the equilibrium boundary in the $\varepsilon > 0$ case, then we would expect the reflection of this equilibrium boundary, in the z -axis, to be a solution in the $\varepsilon < 0$ case, with the same over-pressure and volume.

We now take the problem for $\varepsilon < 0$ and reflect it in the z -axis. Based on the diagram in figure 4.1.1a, we can easily sketch the reflection of the problem for $\varepsilon > 0$. This is given in figure 4.2.2, in which the transformation matrix describing a reflection in the z -axis is denoted by $\mathbf{R}$.

We can now compare the reflected $\varepsilon > 0$ problem illustrated in figure 4.2.2, with

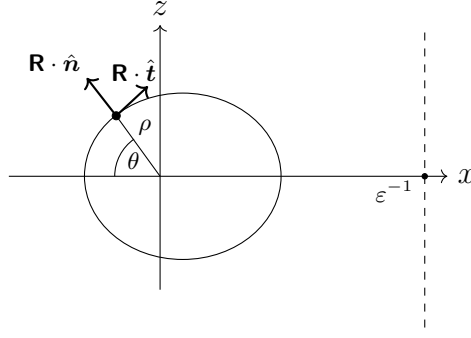


Figure 4.2.2: Reflection of the $\varepsilon > 0$ problem, as illustrated in figure 4.1.1a, in the z -axis.

the the $\varepsilon < 0$ problem illustrated in figure 4.1.1b. From this, we can see that the problem under $\varepsilon \mapsto -\varepsilon$ is equivalent to the reflected problem with the additional transformations $\theta \mapsto \pi - \theta$, $\hat{\mathbf{t}} \mapsto -\mathbf{R} \cdot \hat{\mathbf{t}}$. Under this reflection $\mathbf{B} \mapsto -\mathbf{R} \cdot \mathbf{B}$, since $\mathbf{B}$ is a pseudovector.

Hence, we have found that under $\varepsilon \mapsto -\varepsilon$, we have

$$\mathbf{B}(\theta) \mapsto -\mathbf{R} \cdot \mathbf{B}(\pi - \theta). \quad (4.2.10)$$

Moreover, since $\hat{\mathbf{n}} \mapsto \mathbf{R} \cdot \hat{\mathbf{n}}$ and $\hat{\mathbf{t}} \mapsto -\mathbf{R} \cdot \hat{\mathbf{t}}$, it follows from (4.2.10), that

$$B_n(\theta) \mapsto -B_n(\pi - \theta), \quad (4.2.11)$$

$$H_t(\theta) \mapsto H_t(\pi - \theta). \quad (4.2.12)$$

Finally, we note that $\rho(\theta) \mapsto \rho(\pi - \theta)$, and the over-pressure P remains unchanged. It can be verified that the governing equations for this problem (4.1.15), (4.1.28), and (4.1.36), are invariant under the transformations described above.

We now consider the implications this has for our asymptotic solutions (4.2.6)–(4.2.9). If $\rho(\theta)$ is a solution for ε , then $\rho(\pi - \theta)$ must be a solution for $-\varepsilon$. Similarly, if $H_t(\theta)$ is a solution for ε , then $H_t(\pi - \theta)$ must be a solution for $-\varepsilon$.

It follows that the correction terms ρ_1 , ρ_2 , $H_{t,1}$, and $H_{t,2}$, must all satisfy

$$f(\theta) = -f(\pi - \theta). \quad (4.2.13)$$

Additionally, we note that if $B_n(\theta)$ is a solution for ε , then $-B_n(\pi - \theta)$ must be a solution for $-\varepsilon$. Hence, $B_{n,1}$ and $B_{n,2}$ must satisfy the relation

$$g(\theta) = g(\pi - \theta). \quad (4.2.14)$$

Finally, we note that since P remains unchanged when $\varepsilon \mapsto -\varepsilon$, it must be that $P_1 = P_2 = 0$.

4.3 Asymptotic Solution for $|\varepsilon| \ll 1$

4.3.1 Fourier Series Representation and Volume Constraint

Since we know ρ must be 2π -periodic in θ , we seek Fourier series representations of the correction terms ρ_1 and ρ_2 . That is,

$$\rho_i(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta), \quad i \in \{1, 2\}, \quad (4.3.1)$$

where the a_n and b_n are to be determined. Note that we would expect different values of the Fourier coefficients depending on the value of i , but this dependence has been dropped for the simplicity of exposition moving forward.

We now substitute the asymptotic expansion for ρ , (4.2.6), into the volume constraint (4.1.36). This yields

$$\pi = \frac{1}{2} \int_0^{2\pi} d\theta + \varepsilon \log |\varepsilon| \int_0^{2\pi} \rho_1(\theta) d\theta + \varepsilon \int_0^{2\pi} \left(\rho_2(\theta) + \frac{1}{3} \cos \theta \right) d\theta + \dots \quad (4.3.2)$$

At leading-order we have $\pi = \pi$, which balances by construction. Equating terms

at $O(\varepsilon \log |\varepsilon|)$ and $O(\varepsilon)$, we see that

$$\int_0^{2\pi} \rho_1(\theta) d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \rho_2(\theta) d\theta = 0. \quad (4.3.3)$$

If we use the Fourier series representations (4.3.1), then it is clear from (4.3.3) that $a_0 = 0$ for both ρ_1 and ρ_2 . Hence,

$$\rho_i(\theta) = \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta), \quad i \in \{1, 2\}. \quad (4.3.4)$$

4.3.2 Dynamic Boundary Condition

Substituting the asymptotic expansions (4.2.6)–(4.2.9) into the dynamic boundary condition (4.1.15), we have that

$$\begin{aligned} & \left(-1 + \frac{1}{\text{Bo}} + \dots \right) + (-1 + \varepsilon \log |\varepsilon| H_{t,1}(\theta) + \varepsilon H_{t,2}(\theta) + \dots)^2 \\ & \quad + \tilde{m}^{-1} (\varepsilon \log |\varepsilon| B_{n,1}(\theta) + \varepsilon B_{n,2}(\theta) + \dots)^2 \\ & = \frac{1}{\text{Bo}} (1 - \{\rho_1(\theta) + \rho_1''(\theta)\} \varepsilon \log |\varepsilon| + \{\cos \theta - (\rho_2(\theta) + \rho_2''(\theta))\} \varepsilon + \dots), \end{aligned} \quad (4.3.5)$$

where we have noted $P_1 = P_2 = 0$. An expansion of κ (4.1.26) about $\varepsilon = 0$ has been used to determine the right-hand-side of (4.3.5).

At leading-order, everything balances by construction. At $O(\varepsilon \log |\varepsilon|)$, we have that

$$H_{t,1}(\theta) = \frac{1}{2\text{Bo}} \{\rho_1(\theta) + \rho_1''(\theta)\}, \quad (4.3.6)$$

and at $O(\varepsilon)$, we have

$$H_{t,2}(\theta) = \frac{1}{\text{Bo}} (\{\rho_2(\theta) + \rho_2''(\theta)\} - \cos \theta). \quad (4.3.7)$$

It follows from the Fourier series representations (4.3.4), that

$$H_{t,1}(\theta) = \frac{1}{2\text{Bo}} \left(\sum_{n=1}^{\infty} a_n (1 - n^2) \cos(n\theta) + b_n (1 - n^2) \sin(n\theta) \right), \quad (4.3.8)$$

and

$$H_{t,2}(\theta) = \frac{1}{2\text{Bo}} \left(\sum_{n=1}^{\infty} a_n (1 - n^2) \cos(n\theta) + b_n (1 - n^2) \sin(n\theta) \right) - \frac{1}{2\text{Bo}} \cos \theta. \quad (4.3.9)$$

4.3.3 Tangential Field Equation

We now substitute the asymptotic expansions (4.2.6)–(4.2.9) into the expansion of the tangential field equation (4.1.29) about $\varepsilon = 0$, computed using Maple. At leading-order, everything balances by construction. At $O(\varepsilon \log |\varepsilon|)$, we find that

$$\frac{1}{2} (1 + \tilde{m}) H_{t,1}(\theta) = -\frac{\tilde{m}}{2} (-2\rho_1(\theta) + \cos \theta) + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} (I(\rho_1) + J_1) d\theta', \quad (4.3.10)$$

where

$$J_1 = -\cos \theta + H_{t,1}(\theta'), \quad (4.3.11)$$

and

$$I(h) = \left[1 - \frac{1 - \cos(\theta + \theta')}{(\cos \theta - \cos \theta')^2} \right] (h(\theta) - h(\theta')) - \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} h'(\theta), \quad (4.3.12)$$

for a general function h . Using (4.3.8), it is easy to see that

$$\int_0^{2\pi} J_1 d\theta' = -\int_0^{2\pi} \cos \theta d\theta' = -2\pi \cos \theta. \quad (4.3.13)$$

Therefore, we have that

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_{t,1}(\theta) = & -\frac{\tilde{m}}{2} (-2\rho_1(\theta) + \cos \theta) \\ & + \frac{(\tilde{m} - 1)}{2} \cos \theta + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} I(\rho_1) d\theta'. \end{aligned} \quad (4.3.14)$$

Using the Fourier series representations (4.3.4) and (4.3.8), we arrive, after some manipulation, at the $O(\varepsilon \log |\varepsilon|)$ integral equation

$$\begin{aligned} \frac{\cos \theta}{2} = & \tilde{m} \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \\ & - \frac{(1 + \tilde{m})}{4\text{Bo}} \sum_{n=1}^{\infty} a_n (1 - n^2) \cos(n\theta) + b_n (1 - n^2) \sin(n\theta) \\ & + \frac{(1 - \tilde{m})}{4\pi} \sum_{n=1}^{\infty} a_n \int_0^{2\pi} I(\cos(n\theta)) d\theta' + b_n \int_0^{2\pi} I(\sin(n\theta)) d\theta'. \end{aligned} \quad (4.3.15)$$

At $O(\varepsilon)$, we find that

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_{t,2}(\theta) = & -\frac{\tilde{m}}{2} (-2\rho_2(\theta) - 3 \log(2) \cos \theta) \\ & + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} (I(\rho_2) + J_2) d\theta', \end{aligned} \quad (4.3.16)$$

where

$$J_2 = -\frac{\log(1 - \cos(\theta' - \theta))}{2} \cos \theta + \frac{5 \log(2) - 1}{2} \cos \theta - \frac{\cos \theta'}{2} + H_{t,2}(\theta'), \quad (4.3.17)$$

Using (4.3.9), the integral of J_2 in (4.3.16) is straightforward, with the exception of the first term in (4.3.17). In appendix E, we show that

$$\int_0^{2\pi} \log(1 - \cos(\theta' - \theta)) d\theta' = -2\pi \log 2, \quad (4.3.18)$$

which yields

$$\int_0^{2\pi} J_2 d\theta' = \pi (6 \log(2) - 1) \cos \theta. \quad (4.3.19)$$

We thus have

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_{t,2}(\theta) = & -\frac{\tilde{m}}{2} (-2\rho_2(\theta) - 3 \log(2) \cos \theta) \\ & + \frac{(1 - \tilde{m})}{4} (6 \log(2) - 1) \cos \theta + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} I(\rho_2) d\theta'. \end{aligned} \quad (4.3.20)$$

Using the Fourier series representations (4.3.4) and (4.3.9), we arrive, after some

manipulation, at the $O(\varepsilon)$ integral equation,

$$\begin{aligned} \frac{1}{4} \left(1 - 6 \log(2) - \tilde{m} - \frac{(1 + \tilde{m})}{\text{Bo}} \right) \cos \theta &= \tilde{m} \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \\ &\quad - \frac{(1 + \tilde{m})}{4\text{Bo}} \sum_{n=1}^{\infty} a_n (1 - n^2) \cos(n\theta) + b_n (1 - n^2) \sin(n\theta) \\ &\quad + \frac{(1 - \tilde{m})}{4\pi} \sum_{n=1}^{\infty} a_n \int_0^{2\pi} I(\cos(n\theta)) \, d\theta' + b_n \int_0^{2\pi} I(\sin(n\theta)) \, d\theta'. \end{aligned} \quad (4.3.21)$$

We now notice that both integral equations, (4.3.15) and (4.3.21), can be written in the form

$$\begin{aligned} K \cos \theta &= \tilde{m} \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta) \\ &\quad - \frac{(1 + \tilde{m})}{4\text{Bo}} \sum_{n=1}^{\infty} a_n (1 - n^2) \cos(n\theta) + b_n (1 - n^2) \sin(n\theta) \\ &\quad + \frac{(1 - \tilde{m})}{4\pi} \sum_{n=1}^{\infty} a_n \int_0^{2\pi} I(\cos(n\theta)) \, d\theta' + b_n \int_0^{2\pi} I(\sin(n\theta)) \, d\theta', \end{aligned} \quad (4.3.22)$$

where K is a constant. The coefficients in the Fourier series representations of the correction terms ρ_1 , ρ_2 , $H_{t,1}$, and $H_{t,2}$, are hence determined by solving (4.3.22).

4.3.4 Values of the Fourier Coefficients

To solve for the Fourier coefficients, we first note the following results derived in appendix E

$$\int_0^{2\pi} I(\cos(n\theta)) \, d\theta' = -2\pi (n - 1) \cos(n\theta), \quad (4.3.23)$$

and

$$\int_0^{2\pi} I(\sin(n\theta)) \, d\theta' = -2\pi (n - 1) \sin(n\theta). \quad (4.3.24)$$

Using these results in (4.3.22), we obtain

$$K \cos \theta = \sum_{n=1}^{\infty} C_n \cos(n\theta) + D_n \sin(n\theta), \quad (4.3.25)$$

where $C_n = a_n E_n$, $D_n = b_n E_n$, and

$$E_n = \left(\tilde{m} + \frac{(\tilde{m} - 1)}{2} (n - 1) + \frac{(1 + \tilde{m})}{4\text{Bo}} (n^2 - 1) \right). \quad (4.3.26)$$

It is then straightforward to read off the values of C_n and D_n from (4.3.25). In particular, we have that $D_n = 0$ for all n , $C_1 = K$, and $C_n = 0$ for all $n > 1$. Moreover, since $\tilde{m} = \mu_f/\mu_0$, it must be true that $\tilde{m} > 1$, as ferrofluids have a magnetic permeability greater than the permeability of free space. Since $\tilde{m} > 1$, it follows that $E_n > 0$. Since E_n is non-zero, we have that $b_n = 0$ for all n , and $a_n = 0$ for all $n > 1$. Finally, when $n = 1$, we have that $E_1 = \tilde{m}$. Hence, $a_1 = K/\tilde{m}$.

In the $O(\varepsilon \log |\varepsilon|)$ integral equation (4.3.15), we have $K = 1/2$. Hence, using (4.3.4) and (4.3.8), we have that

$$\rho_1(\theta) = \frac{1}{2\tilde{m}} \cos \theta, \quad (4.3.27)$$

and

$$H_{t,1}(\theta) = 0. \quad (4.3.28)$$

Similarly, in the $O(\varepsilon)$ integral equation (4.3.21), we have

$$K = \frac{1}{4} \left(1 - 6 \log(2) - \tilde{m} - \frac{(1 + \tilde{m})}{\text{Bo}} \right). \quad (4.3.29)$$

Hence, using (4.3.4) and (4.3.9), we have that

$$\rho_2(\theta) = \frac{1}{4\tilde{m}} \left(1 - 6 \log(2) - \tilde{m} - \frac{(1 + \tilde{m})}{\text{Bo}} \right) \cos \theta, \quad (4.3.30)$$

and

$$H_{t,2}(\theta) = -\frac{1}{2\text{Bo}} \cos \theta. \quad (4.3.31)$$

4.3.5 Normal Field Corrections

It remains to determine the values of $B_{n,1}$ and $B_{n,2}$. To do this, we substitute the asymptotic expansions (4.2.6)–(4.2.9), with the newly found values of ρ_1 , ρ_2 , $H_{t,1}$, and $H_{t,2}$, into the expansion of the normal field equation (4.1.30) about $\varepsilon = 0$, computed using Maple. Once again, everything balances at leading-order. At $O(\varepsilon \log |\varepsilon|)$, we find that

$$B_{n,1}(\theta) = -\frac{\tilde{m}}{2} \left(\frac{1}{\tilde{m}} \sin \theta + \sin \theta \right) + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} \left(\frac{1}{2\tilde{m}} \sin \theta' - \sin \theta \right) d\theta'. \quad (4.3.32)$$

Hence,

$$B_{n,1}(\theta) = -\frac{\tilde{m}}{2} \left(\frac{1}{\tilde{m}} \sin \theta + \sin \theta \right) + \frac{(\tilde{m} - 1)}{2} \sin \theta = -\sin \theta. \quad (4.3.33)$$

At $O(\varepsilon)$, we find

$$B_{n,2}(\theta) = -\frac{\tilde{m}}{2} \left\{ (-3 \log(2) + 1) \sin \theta + 2 \frac{K_2}{\tilde{m}} \sin \theta \right\} + \frac{(1 - \tilde{m})}{4\pi} \int_0^{2\pi} (S_1 + S_2) d\theta', \quad (4.3.34)$$

where

$$S_1 = -\frac{\log(1 - \cos(\theta' - \theta))}{2} \sin \theta + \frac{5 \log(2) - 1}{2} \sin \theta + \left(\frac{K_2}{\tilde{m}} + \frac{1}{2} \right) \sin \theta', \quad (4.3.35)$$

$$S_2 = -\frac{1}{2\text{Bo}} \left(\frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} \right) \cos \theta', \quad (4.3.36)$$

and

$$K_2 = \frac{1}{4} \left(1 - 6 \log(2) - \tilde{m} - \frac{(1 + \tilde{m})}{\text{Bo}} \right). \quad (4.3.37)$$

Using (4.3.18), it follows that

$$\int_0^{2\pi} S_1 d\theta' = \pi (6 \log(2) - 1) \sin \theta. \quad (4.3.38)$$

Additionally, using the results given in appendix E, it is easily shown that

$$\int_0^{2\pi} \left(\frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} \right) \cos \theta' = -2\pi \sin \theta. \quad (4.3.39)$$

Therefore, we have that

$$\int_0^{2\pi} S_2 d\theta' = \frac{\pi}{\text{Bo}} \sin \theta. \quad (4.3.40)$$

Hence, (4.3.34) can be written as

$$\begin{aligned} B_{n,2}(\theta) = -\frac{\tilde{m}}{2} \left\{ (-3 \log(2) + 1) \sin \theta + 2 \frac{K_2}{\tilde{m}} \sin \theta \right\} \\ + \frac{(1 - \tilde{m})}{4} \left\{ 6 \log(2) - 1 + \frac{1}{\text{Bo}} \right\} \sin \theta, \end{aligned} \quad (4.3.41)$$

which simplifies to

$$B_{n,2}(\theta) = \frac{1}{2} \left(6 \log(2) - 1 + \frac{1}{\text{Bo}} \right) \sin \theta. \quad (4.3.42)$$

4.3.6 Statement of Asymptotic Solution

We have hence shown that the asymptotic solution to the problem for $|\varepsilon| \ll 1$, is given by

$$\rho(\theta) = 1 + \varepsilon \log |\varepsilon| \frac{1}{2\tilde{m}} \cos \theta + \varepsilon \frac{1}{4\tilde{m}} \left(1 - 6 \log(2) - \tilde{m} - \frac{(1 + \tilde{m})}{\text{Bo}} \right) \cos \theta + \dots \quad (4.3.43)$$

$$B_n(\theta) = -\varepsilon \log |\varepsilon| \sin \theta + \varepsilon \frac{1}{2} \left(6 \log(2) - 1 + \frac{1}{\text{Bo}} \right) \sin \theta + \dots \quad (4.3.44)$$

$$H_t(\theta) = -1 - \varepsilon \frac{1}{2\text{Bo}} \cos \theta + \dots \quad (4.3.45)$$

$$P = \left(\frac{1}{\text{Bo}} - 1 \right) + \dots \quad (4.3.46)$$

The next highest order terms, which we have not included in the above, would be, at most, $O((\varepsilon \log |\varepsilon|)^2)$. Additionally, we note that ρ_1 , ρ_2 , $H_{t,1}$, and $H_{t,2}$ all satisfy (4.2.13), and both $B_{n,1}$ and $B_{n,2}$ satisfy (4.2.14), as required by the symmetry of the problem.

4.4 Comparison with Numerical Method

We have been able to find an asymptotic solution to the problem for $|\varepsilon| \ll 1$. However, the full problem can be solved numerically in a manner entirely analogous to that described in Chapter 3. We note that fixing a value of ε is equivalent to fixing a value of the fluid volume, as a result of (4.1.24).

Hence, by fixing $\tilde{m}$, Bo , and ε , we can solve the full problem numerically and compare the computed values of ρ , H_t , B_n , and P with the asymptotic solutions (4.3.43)–(4.3.46). If ε is chosen to be sufficiently small, then we would expect to see an agreement between the numerical solutions and the asymptotic solutions.

For the numerical solutions, we take N points on the ferrofluid boundary and fix equally spaced values of the polar angle θ . That is, $\theta_i = i\alpha$, where $i = 1, 2, \dots, N$, and $\alpha = 2\pi/N$. The numerical method then determines solutions for the values of ρ , H_t , and B_n at each value of θ_i , as well as a value for P . These same values of θ_i are used to evaluate the asymptotic solutions in Python, allowing for easy comparison with the numerical solutions.

Figures 4.4.1–4.4.3 compare the numerical and asymptotic solutions for ρ , H_t , and B_n , where we have taken $N = 100$, $\text{Bo} = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$. These figures demonstrate a strong qualitative agreement between the numerical solution and the asymptotic solution, with correction terms up to $O(\varepsilon)$ included, for each of the variables ρ , H_t and B_n .

To gain an understanding of the quantitative agreement between the numerical and analytical solutions, we can examine the behaviour of the error between the numerical and analytical solutions as we decrease the value of ε . Using the subscript ‘num’ to denote numerical solutions and the subscript ‘asy’ to denote the asymptotic solutions up to $O(\varepsilon)$ from (4.3.43)–(4.3.46), we define the errors

to be

$$E_P = |P_{\text{num}} - P_{\text{asy}}|, \quad (4.4.1)$$

$$E_\rho = N^{-\frac{1}{2}} \|\rho_{\text{num}} - \rho_{\text{asy}}\|, \quad (4.4.2)$$

$$E_H = N^{-\frac{1}{2}} \|H_{t,\text{num}} - H_{t,\text{asy}}\|, \quad (4.4.3)$$

$$E_B = N^{-\frac{1}{2}} \|B_{n,\text{num}} - B_{n,\text{asy}}\|, \quad (4.4.4)$$

where $\|\cdot\|$ is the Euclidean norm. Each of the quantities defined in (4.4.1)–(4.4.4) represents a quantitative measure of how close the numerical and asymptotic solutions are to each other. Equation (4.4.1) represents the absolute error between the numerical and asymptotic solutions for P , whilst (4.4.2)–(4.4.4) represent the root mean squared error between the numerical and asymptotic solutions of ρ , H_t , and B_n .

Figures 4.4.4–4.4.7 show how the errors defined in equations (4.4.1)–(4.4.4) behave as ε is varied, plotted on logarithmically scaled axes. The values of ε considered in these plots range from $\varepsilon = 0.05$ to $\varepsilon = 0.0005$. As expected, we observe that the size of the errors decreases as ε is decreased. Additionally, the behaviour of these errors behaves accordingly with the next asymptotic order not included in the expansions (this asymptotic order being anything proportional to ε^2 , including powers of $\log|\varepsilon|$). Moreover, the size of the error when $\varepsilon = 0.0005$ is $O(10^{-5})$ for B_n , $O(10^{-6})$ for P and H_t , and $O(10^{-7})$ for ρ . In all cases, the size of the error is at least two orders of magnitude smaller than the maximum size of the deviation of the numerical solution from the leading-order solution when $\varepsilon = 0$.

Finally, we consider the range of validity of the asymptotic solution in comparison to the numerical solution. To do this, we define $x_r = \rho(\theta = 0)$. Hence, x_r represents the rightmost point of the ferrofluid boundary. Figure 4.4.8 shows the behaviour of x_r as ε is varied, for both the asymptotic and numerical solutions. We observe that the asymptotic and numerical solution curves for x_r converge toward each other as ε decreases from $\varepsilon = 0.1$, and appear to coalesce around

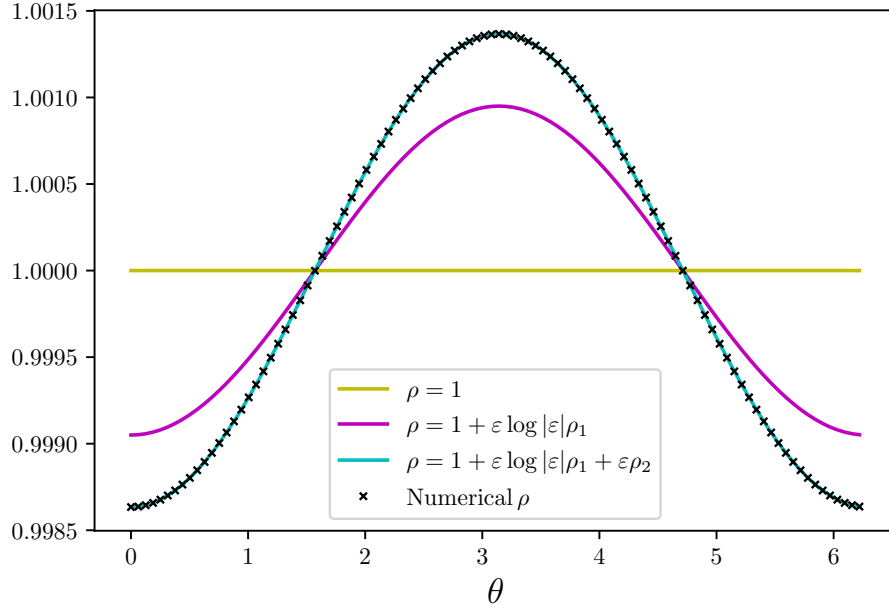


Figure 4.4.1: Comparison of asymptotic and numerical solutions for ρ , where $\text{Bo} = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$.

$\varepsilon = 0.01$. Moreover, we denote the relative error between the asymptotic and numerical solutions for x_r by E_x . That is,

$$E_x = \frac{|x_{r,\text{asy}} - x_{r,\text{num}}|}{x_{r,\text{num}}}. \quad (4.4.5)$$

A graph of E_x against ε is given in figure 4.4.9. We see that E_x is around 1% when $\varepsilon = 0.1$ and falls to approximately 0.01% when $\varepsilon = 0.01$. By the time $\varepsilon = 0.0005$, the value of E_x has fallen to approximately 0.00002%. As such, for $0.0005 < \varepsilon < 0.01$, we have that the asymptotic and numerical solutions for x_r differ by less than one-hundredth of a percent.

We have thus obtained strong evidence to suggest that the asymptotic solutions (4.3.43)–(4.3.46) agree with solutions obtained via the numerical method described in Chapter 3. We have hence been able to validate the numerical method via a direct comparison with an analytical solution obtained in the limit of a toroid with a cross-section much smaller than the radius of the circular wire-loop that it surrounds.

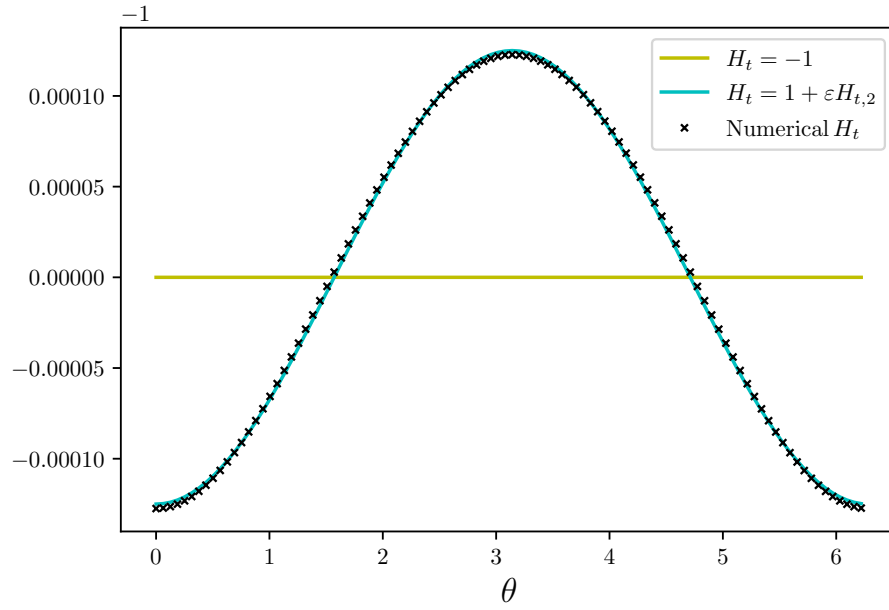


Figure 4.4.2: Comparison of asymptotic and numerical solutions for H_t , where $\text{Bo} = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$.

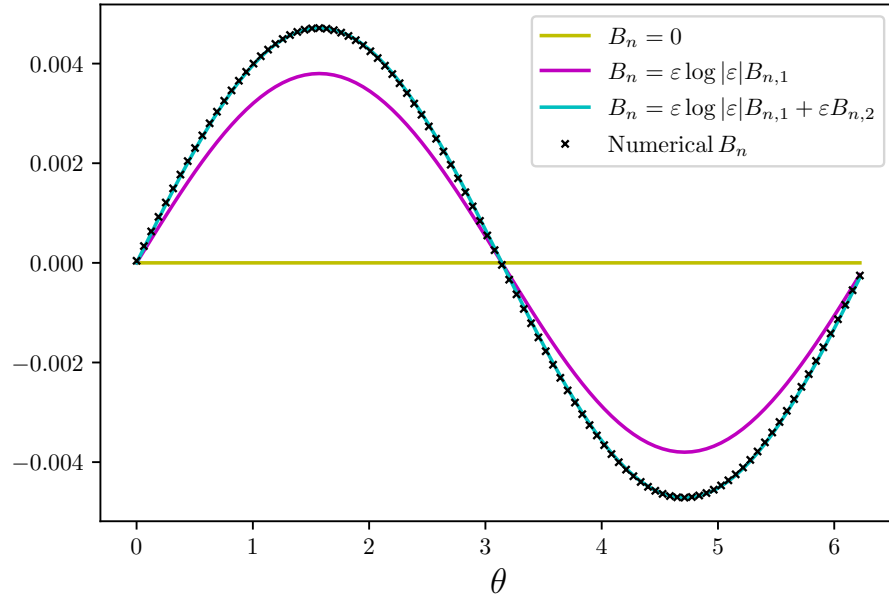


Figure 4.4.3: Comparison of asymptotic and numerical solutions for ρ , where $\text{Bo} = 2$, $\tilde{m} = 2$, and $\varepsilon = 0.0005$.

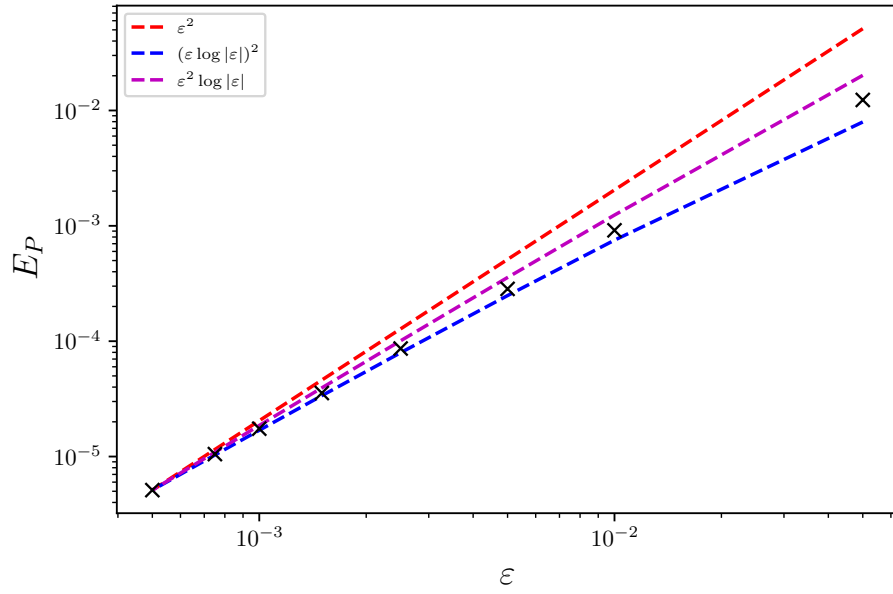


Figure 4.4.4: log-log plot of E_P (the error between the numerical and asymptotic solutions for the over-pressure P) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.

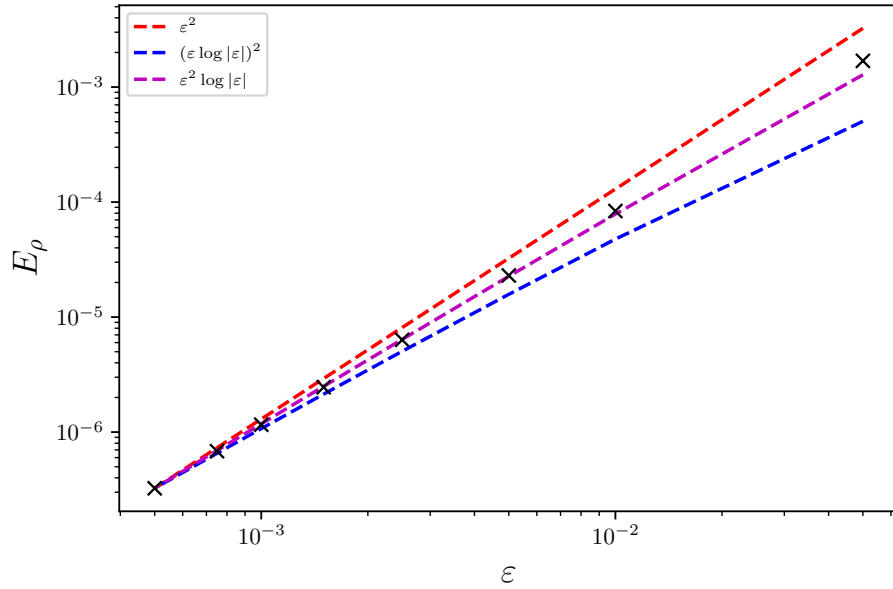


Figure 4.4.5: log-log plot of E_ρ (the error between the numerical and asymptotic solutions for the ferrofluid boundary ρ) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.

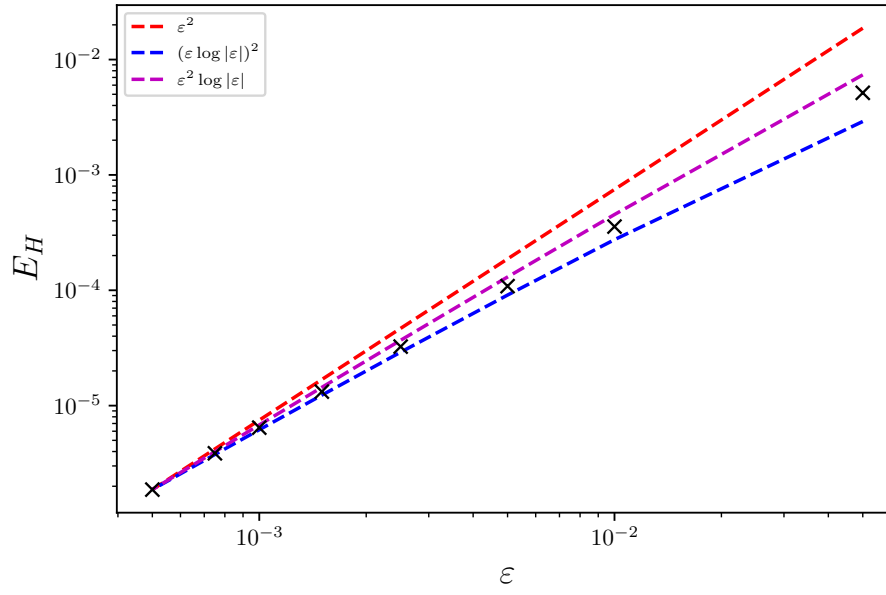


Figure 4.4.6: log-log plot of E_H (the error between the numerical and asymptotic solutions for the tangential component H_t) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.

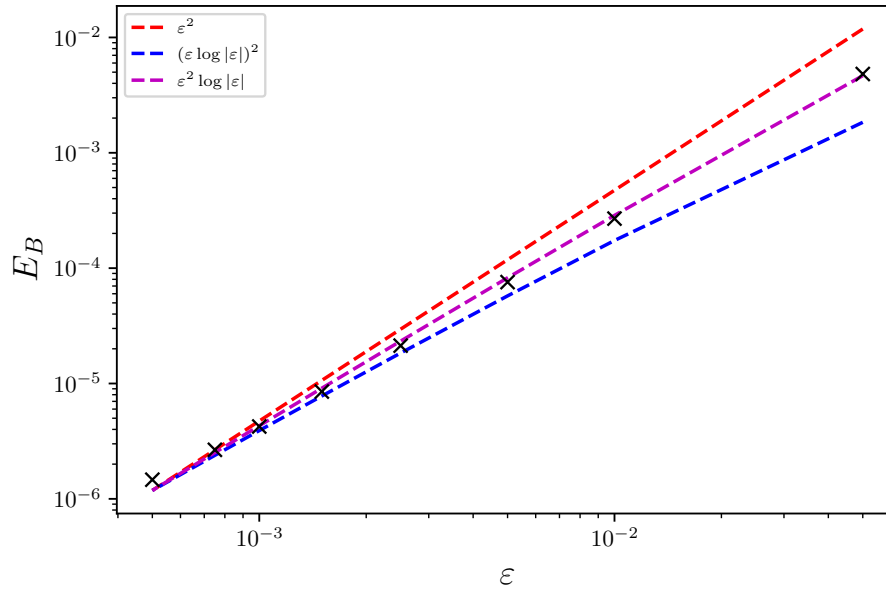


Figure 4.4.7: log-log plot of E_B (the error between the numerical and asymptotic solutions for the normal component B_n) against ε , also included are representative plots of the next three possible orders in the asymptotic expansions.

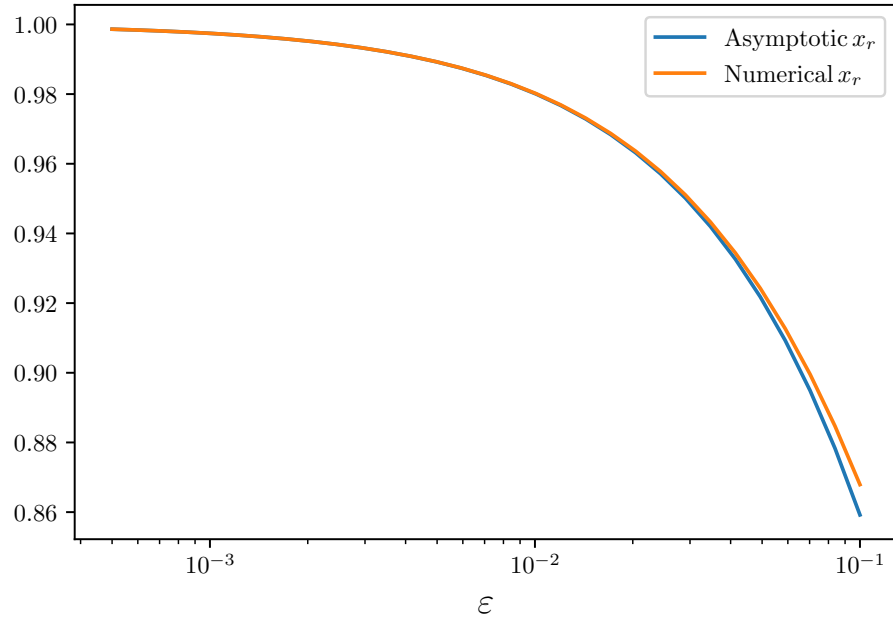


Figure 4.4.8: Semi-log plot of the asymptotic and numerical solutions for the rightmost point of the ferrofluid boundary x_r against ε . As expected, since $x_r = \rho(\theta = 0)$, with ρ as in (4.3.43), x_r approaches unity as ε tends to zero.

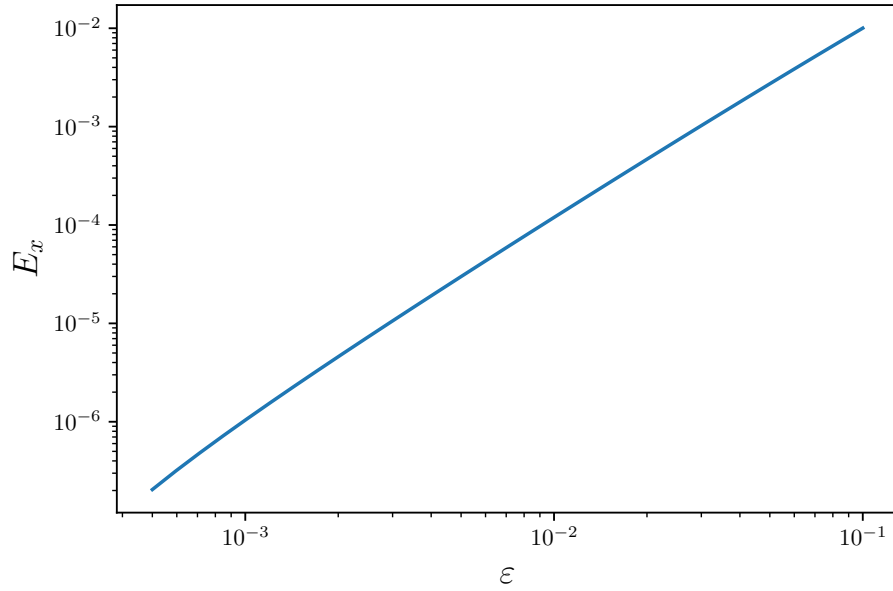


Figure 4.4.9: log-log plot of E_x (the relative error between the asymptotic and numerical solutions for the rightmost point of the ferrofluid boundary x_r) against ε .

Equilibrium Configurations of Ferrofluid Toroids

In Chapter 4, we validated the numerical method described in Chapter 3, by considering the special case where the radius of the cross-section of the ferrofluid toroid was small in comparison to the radius of the circular-wire loop that it surrounds. In this chapter, we will discuss the equilibrium configurations for ferrofluid toroids, as found numerically following the method described in Chapter 3. Specifically, we will investigate the nature of ferrofluid toroids in equilibrium, where they exist, as parameters in the problem, such as the Bond number and the fluid volume, are varied.

5.1 Problem Statement and Parameters

5.1.1 Reminder of the Dimensionless Problem

We are considering a static axisymmetric toroidal volume V_f of ferrofluid surrounding a thin wire forming a circular loop of radius R and carrying a current I . The magnetic permeability of the ferrofluid is denoted as μ_f and the coefficient of surface tension on the ferrofluid free surface is denoted by γ . The aim is to determine the position of the ferrofluid free surface, such that the toroid is in equilibrium. To do this, we need to solve a two-dimensional problem for the ferrofluid boundary, the magnetic induction $\mathbf{B}$ on the ferrofluid

boundary, and the constant over-pressure P . To non-dimensionalise the problem, we scale lengths on R , magnetic inductions on $\mu_0 I/R$, magnetic fields on I/R , and pressures on γ/R .

The points on the ferrofluid boundary are described using plane polar coordinates (r, θ) centred on the wire. We denote the position of the ferrofluid boundary by $r = \rho(\theta)$, such that a point $\mathbf{X}$ on the boundary is given by

$$\mathbf{X}(\theta) = \begin{pmatrix} \rho(\theta) \cos \theta \\ \rho(\theta) \sin \theta \end{pmatrix}. \quad (5.1.1)$$

The dimensionless centre of curvature for the wire-loop is chosen to have Cartesian coordinates $x = -1, z = 0$.

The magnetic induction at a point $\mathbf{X}$ on the ferrofluid boundary, is determined by the boundary-integral equation (3.1.2)

$$\mathbf{B}(\mathbf{X}) = -\frac{\tilde{m}}{4\pi} \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) + \frac{(1 - \tilde{m})}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) \, ds, \quad (5.1.2)$$

where H_t is the component of the magnetic field $\mathbf{H}$ tangential to the ferrofluid boundary at $\mathbf{X}$, $\tilde{\mathbf{P}}$ and $\mathbf{f}$ are the azimuthal integrals as defined in (2.7.10)–(2.7.13) and (2.7.22), $\tilde{m} = \mu_f/\mu_0$ is the dimensionless permeability ratio, s is arc-length around the ferrofluid boundary, $2C$ is the total arc-length of the ferrofluid boundary, and $\boldsymbol{\xi}_0(s)$ describes the ferrofluid boundary.

At each point $\mathbf{X}$ on the ferrofluid boundary, we need to satisfy the dynamic boundary condition (2.8.7), or explicitly

$$P + \frac{1}{2}(\tilde{m} - 1)\text{Bo} \{H_t^2 + \tilde{m}^{-1}B_n^2\} = \kappa, \quad (5.1.3)$$

where B_n is the component of $\mathbf{B}$ normal to the ferrofluid boundary at $\mathbf{X}$, κ is the total curvature of the ferrofluid toroid, and Bo is the magnetic Bond number,

defined as

$$\text{Bo} = \frac{\mu_0 I^2}{R\gamma}, \quad (5.1.4)$$

which quantifies the relative strength of magnetic effects to surface tension effects.

Additionally, we have the volume constraint, which states that the total volume V_f of ferrofluid is prescribed. Using Pappus's centroid theorem for the fluid volume, the volume constraint can be written as

$$V_f = 2\pi \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3}\rho \cos \theta \right) \rho^2 d\theta. \quad (5.1.5)$$

5.1.2 Method of Numerical Solution

We represent the ferrofluid boundary by a set of N points $\mathbf{X}_i$, where $i = 0, 1, 2, \dots, N-1$. We impose the polar angle θ_i of each point $\mathbf{X}_i$, so that the points $\mathbf{X}_i$ are characterised only by their radial coordinate ρ_i . That is,

$$\mathbf{X}_i = \begin{pmatrix} \rho_i \cos \theta_i \\ \rho_i \sin \theta_i \end{pmatrix}. \quad (5.1.6)$$

In accordance with the work in Chapter 3, the full problem can then be discretised into a system of $N + 1$ nonlinear equations in $N + 1$ unknowns. The $N + 1$ unknowns are the N values of ρ_i , describing the ferrofluid boundary, as well as the over-pressure P . The $N + 1$ equations are the application of the dynamic boundary condition (5.1.3) at each of the N points $\mathbf{X}_i$, as well as the volume constraint (5.1.5).

To solve the problem, we first set values for the Bond number Bo , permeability ratio $\tilde{m}$, and fluid volume V_f . We then prescribe values for the polar angles θ_i . The simplest choice is to take the θ_i to be equally spaced, such that $\theta_i = i\alpha$, with $\alpha = 2\pi/N$. However, this choice can lead to a ferrofluid boundary that is poorly resolved close to the axis of rotation at $x = -1$. This occurs when the

rightmost point of the ferrofluid boundary is much closer to the wire than the leftmost point of the boundary. When this is anticipated to be the case, we instead employ unevenly spaced angular coordinates that yield a higher concentration of points on the boundary close to $x = -1$. In particular, we take

$$\theta_i = i\alpha + \frac{1}{2} \sin(i\alpha). \quad (5.1.7)$$

We then make an initial guess for the unknowns ρ_i and P , and seek a solution to the problem using the nonlinear solver *fsolve*. At each iteration, the values of H_t and B_n at each point $\mathbf{X}_i$ are determined by solving the discretised form of the boundary-integral equation (5.1.2) as detailed in Chapter 3, which allows for the evaluation of the dynamic boundary condition.

Should *fsolve* converge, then we have obtained a numerical solution for the equilibrium boundary of a ferrofluid toroid, as well as the over-pressure, for the chosen values of the problem parameters.

All numerical results presented in this chapter use $N = 150$ points on the ferrofluid boundary.

5.1.3 The Value of $\tilde{m}$

Recall that the magnetic susceptibility χ_m of a ferrofluid is defined to be $\chi_m = \mu_f/\mu_0 - 1$, from which it follows that $\chi_m = \tilde{m} - 1$. In Chapter 2, we investigated the value of χ_m under the assumption of linear isotropic magnetisation. This investigation suggested typical values of χ_m in the range $0 < \chi_m < 4$, which corresponds to $1 < \tilde{m} < 5$.

Currently, we have three parameters characterising the problem. By fixing a value of $\tilde{m}$ in the range $1 < \tilde{m} < 5$, we can simplify the problem to only depend on two parameters, the fluid volume and the Bond number, whilst retaining the physical relevance of the obtained solutions. We choose to take $\tilde{m} = 2$ for the

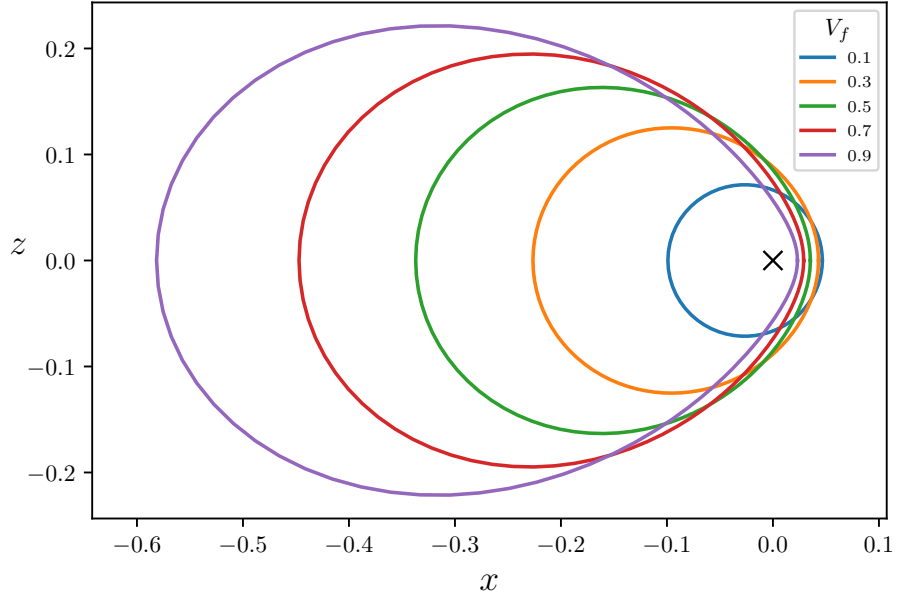


Figure 5.2.1: Equilibrium boundaries for $Bo = 0.5$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane.

remainder of this work, noting that this choice is entirely arbitrary.

5.2 Varying V_f with Fixed Bo

Having simplified the problem to depend on only two parameters, we now seek a systematic approach to study the behaviour of solutions over $Bo - V_f$ parameter space. To begin, we consider cases where the value of Bo is fixed and the value of V_f is varied.

The obtained equilibrium boundaries for $Bo = 0.5$, $Bo = 2$, and $Bo = 10$, for varying values of the fluid volume V_f , are given in figures 5.2.1, 5.2.2, and 5.2.3 respectively. In each case, we observe that the leftmost point of the equilibrium boundary, henceforth denoted as x_l , monotonically becomes more negative as the fluid volume increases. Note that x_l must lie in the range $-1 < x_l < 0$.

Additionally, in all three cases, *fsolve* failed to converge to a solution after V_f was increased above a certain value. This suggests that for a given Bond number, there exists some maximum volume beyond which equilibrium boundaries no

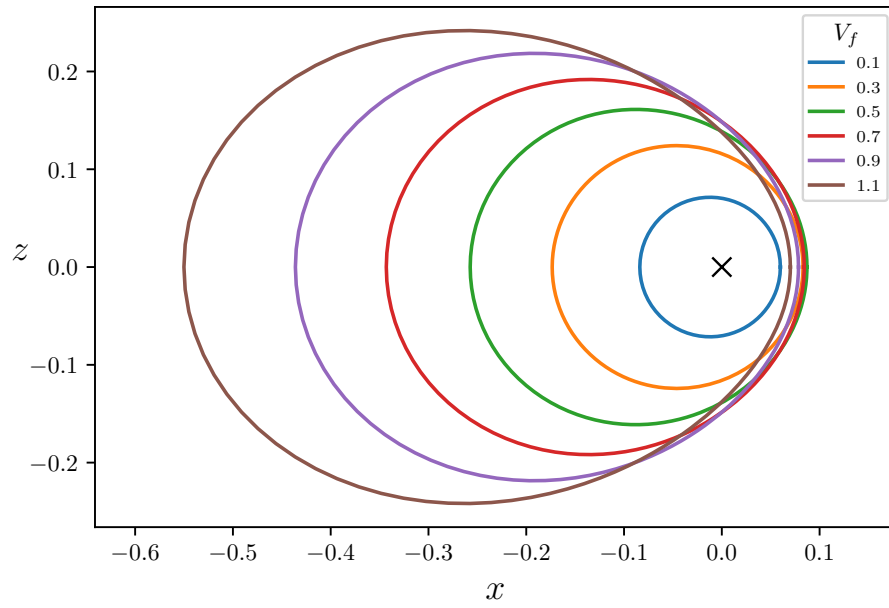


Figure 5.2.2: Equilibrium boundaries for $Bo = 2$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane.

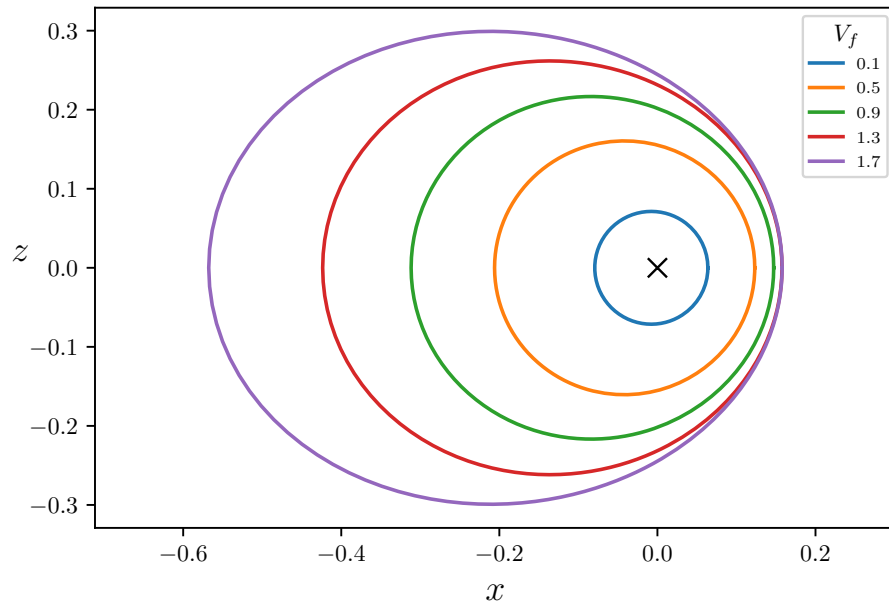


Figure 5.2.3: Equilibrium boundaries for $Bo = 10$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane.

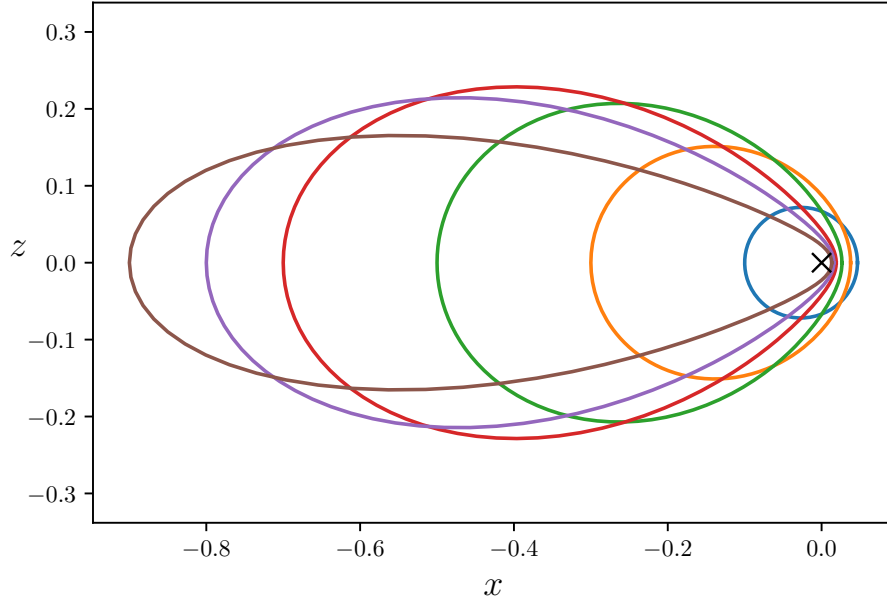


Figure 5.2.4: Equilibrium boundaries for $Bo = 0.5$, each with a different fixed value of x_l . The cross represents the location of the wire in the plane.

longer exist.

To investigate this further, we note the fact that the volume constraint can be replaced by a different physical constraint on the ferrofluid. Based on the observed behaviour of x_l as the fluid volume was increased, we choose to prescribe the value of x_l instead of V_f .

The obtained equilibrium boundaries for $Bo = 0.5$, $Bo = 2$, and $Bo = 10$, for varying values of x_l , are given in figures 5.2.4, 5.2.5, and 5.2.6 respectively. Additionally, plots of the fluid volume V_f against x_l , for $Bo = 0.5$, $Bo = 2$, and $Bo = 10$, are given in figures 5.2.7, 5.2.8, and 5.2.9 respectively. As x_l is decreased, we find that we eventually reach a critical value, $x_l = x_{crit}$, at which the fluid volume reaches a maximum value of $V_f = V_{max}$.

Furthermore, as x_l is decreased beyond x_{crit} , towards its minimum value of -1 , further equilibrium boundaries are found, with volumes appearing to decrease monotonically as $x_l \rightarrow -1$. We thus propose the categorisation of equilibrium boundaries into two distinct types: Type I boundaries, where $x_l > x_{crit}$ and so the fluid volume is increasing as x_l is decreased, and Type II boundaries, where

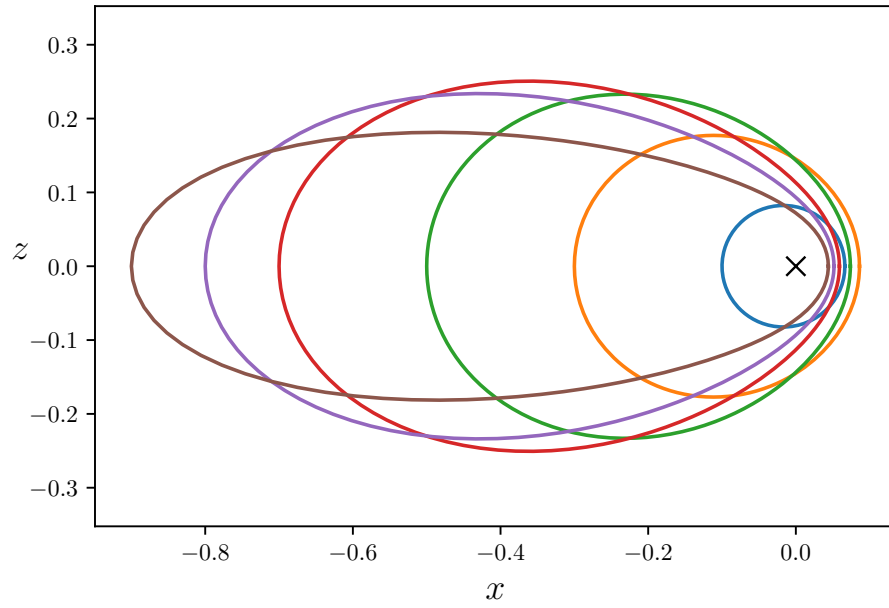


Figure 5.2.5: Equilibrium boundaries for $Bo = 2$, each with a different fixed value of x_l . The cross represents the location of the wire in the plane.

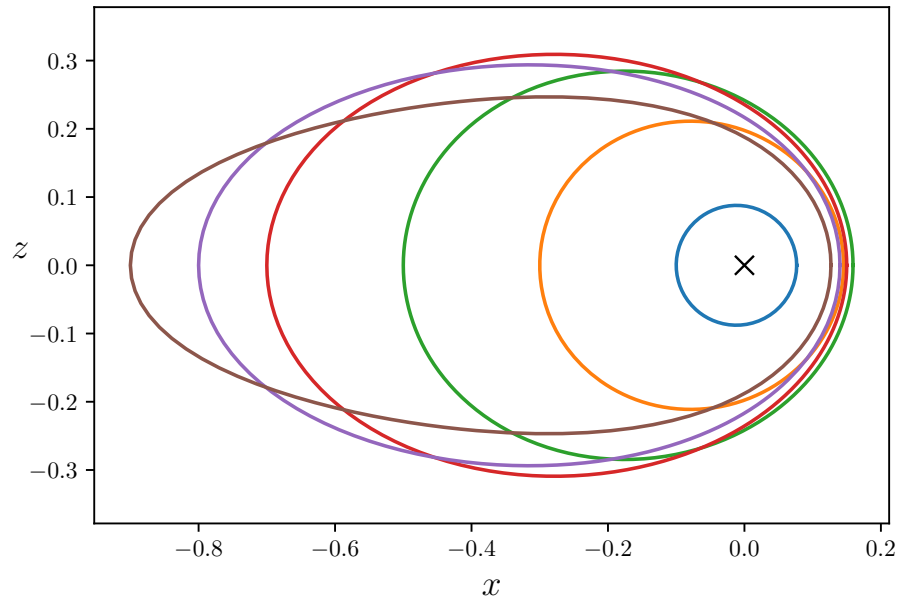


Figure 5.2.6: Equilibrium boundaries for $Bo = 10$, each with a different fixed value of x_l . The cross represents the location of the wire in the plane.

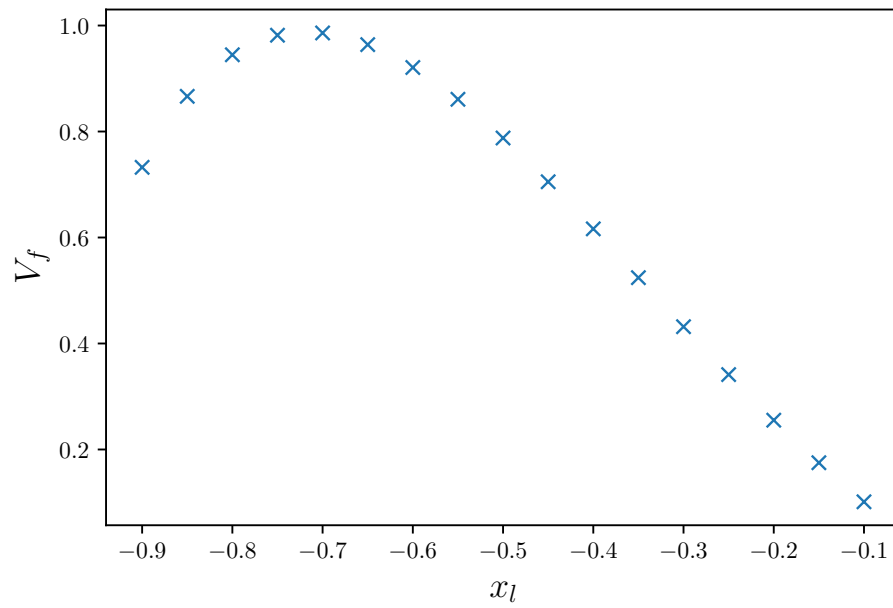


Figure 5.2.7: Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 0.5$.

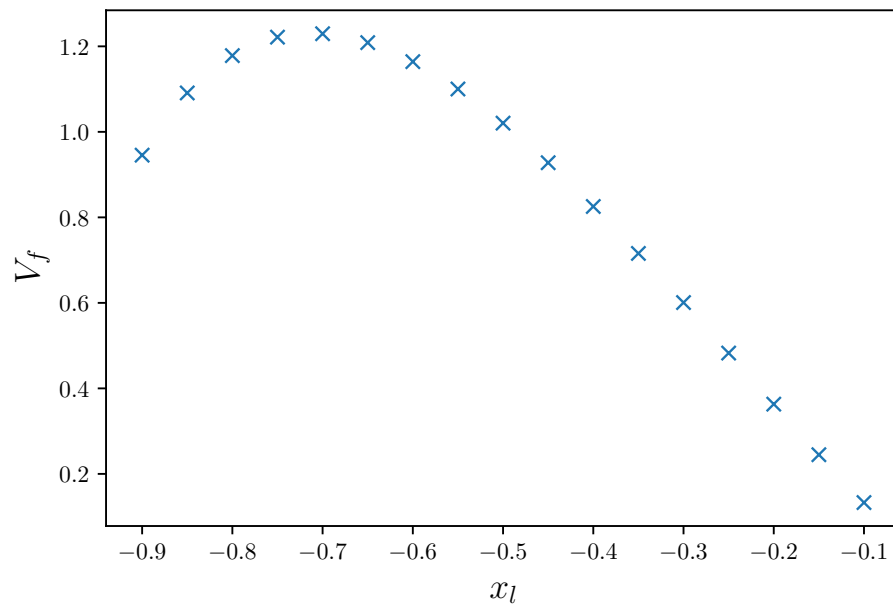


Figure 5.2.8: Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 2$.

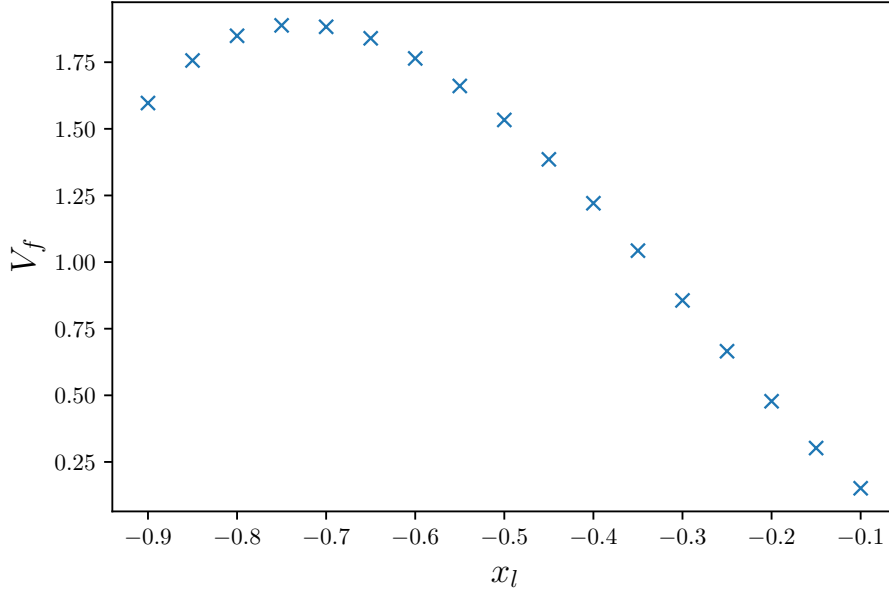


Figure 5.2.9: Plot of fluid volume V_f against the leftmost point x_l for equilibrium boundaries obtained when $Bo = 10$.

$x_l < x_{crit}$ and the fluid volume is decreasing as x_l is decreased.

The difference between Type I and Type II boundaries can be seen by fixing the values of Bo and V_f , and computing the corresponding Type I and Type II boundaries. Figures 5.2.10 and 5.2.11 show, respectively, the Type I and Type II boundaries obtained when $Bo = 2$ and $V_f = 1$. Additionally, the behaviours of H_t and B_n for the above boundaries are given in figures 5.2.12 and 5.2.13 respectively. Therefore, for given values of Bo and V_f , it may be possible to find two qualitatively different equilibrium boundaries.

In order to gain a greater understanding of where Type I and Type II boundaries exist, we can consider a large range of values of the Bond number, spanning several orders of magnitude, and vary x_l from 0 to -1 in each instance. However, there is a potential difficulty in solving for the equilibrium boundary numerically when x_l is close to either 0 or -1 .

When x_l is close to 0, the field due to the wire becomes large (specifically, the complete elliptic integral of the first kind in $\tilde{P}(\mathbf{X}, \mathbf{0})$ diverges). Fortunately, the case of x_l being close to 0, with the fluid volume being small, is an example of

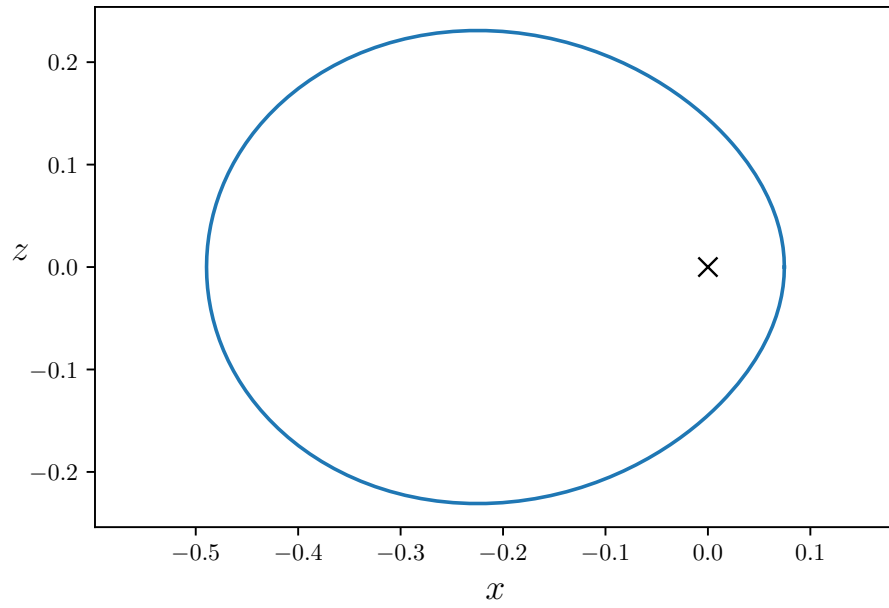


Figure 5.2.10: Type I boundary for $Bo = 2$ and $V_f = 1$. The cross represents the location of the wire in the plane.

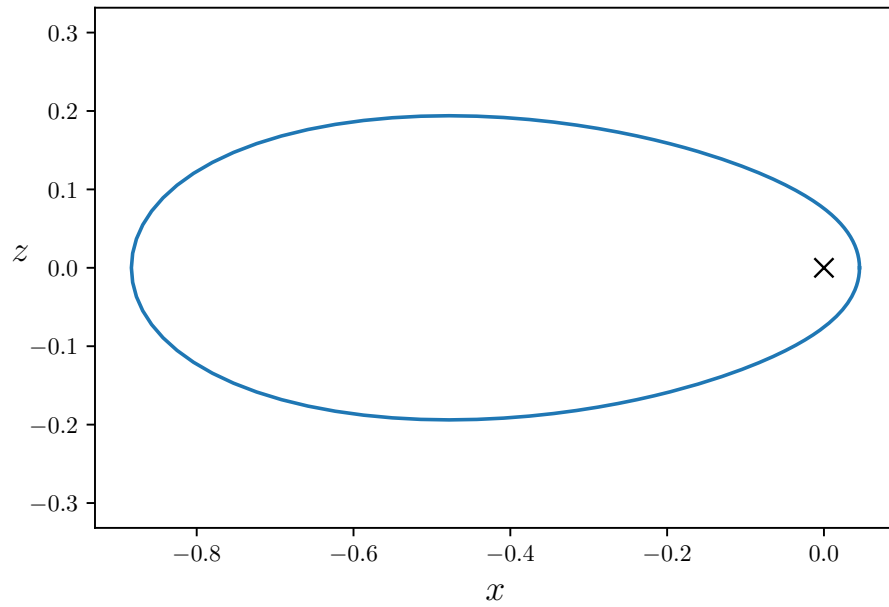


Figure 5.2.11: Type II boundary for $Bo = 2$ and $V_f = 1$. The cross represents the location of the wire in the plane.

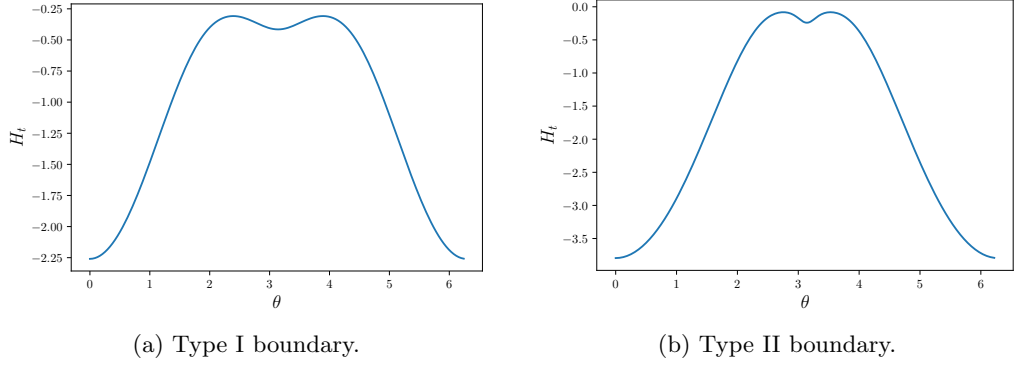


Figure 5.2.12: Numerically computed H_t for the Type I and Type II boundaries obtained when $Bo = 2$ and $V_f = 1$.

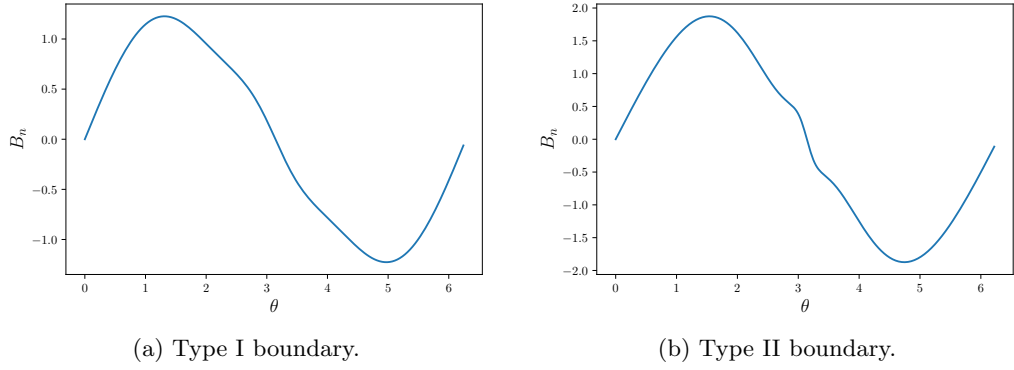


Figure 5.2.13: Numerically computed B_n for the Type I and Type II boundaries obtained when $Bo = 2$ and $V_f = 1$.

the special case studied asymptotically in Chapter 4. Therefore, we know that solutions exist in this limit, and the equilibrium boundary is, to leading-order, a circle centred on the wire (4.3.43). It follows that Type I boundaries will exist for any volume satisfying $0 < V_f \leq V_{max}$, and so determining V_{max} as a function of Bo is sufficient for determining when Type I solutions exist. Whilst it may not be possible to determine V_{max} as a function of Bo exactly, determining the value of V_{max} numerically for a sufficiently large range of values of Bo will enable us to gain an understanding of the domain in $Bo - V_f$ parameter space wherein Type I boundaries exist. This is discussed further in section 5.4.

The case where x_l is close to -1 is, unfortunately, more complicated. When x_l is close to -1 , the ferrofluid boundary is encroaching on its axis of rotation at $x = -1$. A number of quantities in the problem diverge as $x \rightarrow -1$. This

is evident from the full discretised problem described in Chapter 3, in which the quantity $x + 1$ appears in several places as a denominator, and also as the argument of a logarithm. It is not uncommon for *fsolve* to fail to converge when x_l is close to -1 . Unlike in the case of x_l being close to 0 , we do not have a well-studied limit for x_l being close to -1 . This will prove to be a limitation in determining exactly when Type II boundaries exist, however, we will consider this in more detail in section 5.4.

5.3 Extreme Values of Bo

Solving the full problem numerically requires us to take a finite, non-zero value for the Bond number. By considering a range of values for the Bond number, spanning multiple orders of magnitude, we can gain an understanding of the limiting behaviour as $Bo \rightarrow 0$ and $Bo \rightarrow \infty$. However, by considering the two extreme cases, $Bo = 0$ and $Bo = \infty$, we can gain a deeper insight into the behaviour of the equilibrium boundaries as $Bo \rightarrow 0$ and $Bo \rightarrow \infty$, and verify that it agrees with the observed limiting behaviour for the full problem.

5.3.1 Limit of Zero Magnetic Bond Number

$Bo = 0$ Problem

We first investigate the case where $Bo = 0$. This occurs when there are no magnetic effects whatsoever and the problem is entirely dominated by surface tension. Ordinarily, we would not expect to be able to find an equilibrium boundary in this limit (see the discussion of this in Chapter 1). However, if we suppose that the ferrofluid boundary is pinned to the wire at one end, which would lead to there being a kink in the shape of the boundary, then we may be able to make some progress, even though this seems unphysical.

Consider a toroidal volume of fluid, which need not be ferrofluid since magnetic

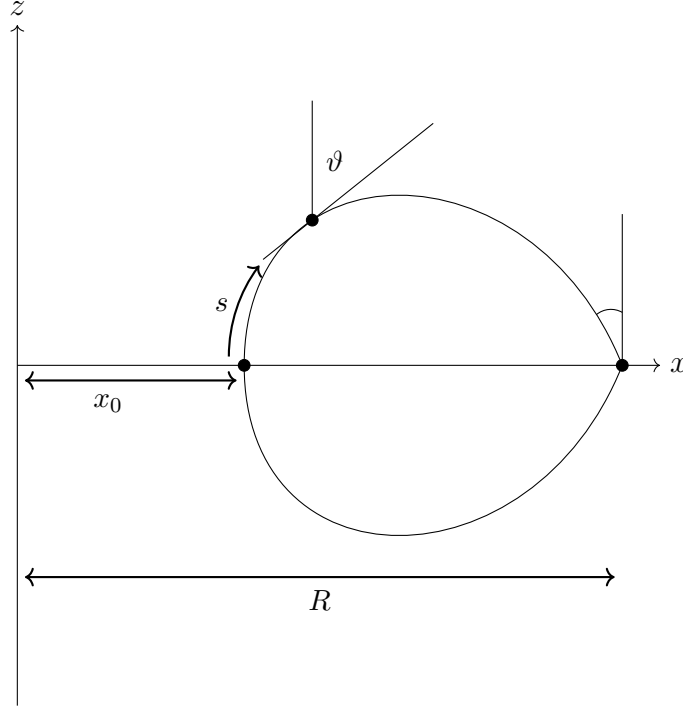


Figure 5.3.1: Diagram showing the set-up and coordinates for a toroid of fluid pinned to the wire-loop.

effects are not present, under surface tension and pinned to the circular wire-loop on the outside. We consider the two-dimensional problem for the shape of the cross-section of the toroid in a meridional plane, which we will refer to as the fluid boundary.

We work in Cartesian (x, z) coordinates, with the rightmost point of the toroid pinned to the wire at $x = R$, $z = 0$. We define s to be the arc-length around the fluid boundary, with $s = 0$ corresponding to the point $x = x_0$, $z = 0$, which is the end of the cross-section of the toroid not pinned to the wire. We denote the angle the tangent vector at a point on the boundary makes with the vertical by ϑ .

Where the toroid is pinned to the wire, we expect a kink in the shape of the fluid boundary. Hence, the cross-section comes into the wire locally as a straight line. We define β to be the angle this line makes with the vertical. A schematic for the problem is shown in figure 5.3.1.

This problem is governed by a system of ODEs. In particular, as in Chapter 1, we have that

$$\kappa_0 = \frac{d\vartheta}{ds_0} - \frac{\cos \vartheta}{x}, \quad (5.3.1)$$

$$\frac{dx}{ds} = \sin \vartheta, \quad (5.3.2)$$

$$\frac{dz}{ds} = \cos \vartheta, \quad (5.3.3)$$

$$\frac{dV_0}{dx} = 4\pi xz, \quad (5.3.4)$$

where κ_0 is the total curvature of the toroid, and V_0 is the fluid volume. From (5.3.2) and (5.3.4), we have that

$$\frac{dV_0}{ds} = 4\pi xy \sin \vartheta. \quad (5.3.5)$$

We scale lengths on x_0 , and introduce dimensionless variables, denoted by quantities with tildes, as follows

$$\begin{aligned} \kappa_0 &= \frac{\tilde{\kappa}_0}{x_0}, \quad s = x_0 \tilde{s}, \quad x = x_0 \tilde{x}, \quad z = x_0 \tilde{z}, \\ R &= x_0 \tilde{R}, \quad x_1 = x_0 \tilde{x}_1, \quad z_1 = x_0 \tilde{z}_1, \quad V_0 = x_0^3 \tilde{V}_0. \end{aligned} \quad (5.3.6)$$

The governing equations (5.3.1)–(5.3.5) are invariant under this non-dimensionalisation.

To solve for the fluid boundary, we begin by eliminating $\tilde{s}$ in favour of ϑ in the governing equations. This yields a system of three ODEs:

$$\frac{d\tilde{x}}{d\vartheta} = \frac{\sin \vartheta}{\left(\tilde{\kappa}_0 + \frac{\cos \vartheta}{\tilde{x}}\right)}, \quad (5.3.7)$$

$$\frac{d\tilde{z}}{d\vartheta} = \frac{\cos \vartheta}{\left(\tilde{\kappa}_0 + \frac{\cos \vartheta}{\tilde{x}}\right)}, \quad (5.3.8)$$

$$\frac{d\tilde{V}_0}{d\vartheta} = \frac{4\pi \tilde{x} \tilde{z} \sin \vartheta}{\left(\tilde{\kappa}_0 + \frac{\cos \vartheta}{\tilde{x}}\right)}. \quad (5.3.9)$$

For a given value of $\tilde{\kappa}_0$, we can integrate (5.3.7)–(5.3.9) from $\vartheta = 0$ to

$\vartheta = T = \frac{\pi}{2} + \Delta$, and $\Delta > 0$ is small and chosen in order to prevent a division by zero in the next step of the method. A suitable value of Δ is not necessarily known *a priori*. In the coming numerics, a value of $\Delta = \pi/30$ was chosen. This integration is carried out with initial conditions $\tilde{x}(0) = 1$, $\tilde{z}(0) = 0$, and $\tilde{V}_0(0) = 0$. Having carried out the integration, we find $\tilde{x}_1 = \tilde{x}(T)$, $\tilde{z}_1 = \tilde{z}(T)$, and $\tilde{V}_1 = \tilde{V}(T)$.

Next, we take the equations (5.3.1)–(5.3.5) and choose to eliminate $\tilde{s}_0$ in favour of $\tilde{z}$. This yields

$$\frac{d\tilde{x}}{d\tilde{z}} = \tan \vartheta, \quad (5.3.10)$$

$$\frac{d\vartheta}{d\tilde{z}} = \frac{(\tilde{\kappa}_0 + \frac{\cos \vartheta}{\tilde{x}})}{\cos \vartheta}, \quad (5.3.11)$$

$$\frac{d\tilde{V}_0}{d\tilde{z}} = 4\pi\tilde{x}\tilde{z}\tan \vartheta. \quad (5.3.12)$$

We now integrate (5.3.10)–(5.3.12) from $\tilde{z} = \tilde{z}_1$ to $\tilde{z} = 0$, with the initial conditions $\tilde{x}(\tilde{z}_1) = \tilde{x}_1$, $\tilde{\vartheta}(\tilde{z}_1) = T$, and $\tilde{V}_0(\tilde{z}_1) = \tilde{V}_1$. We then find $\tilde{R} = \tilde{x}(0)$, $\alpha = \pi - \theta(0)$, and $\tilde{V}_2 = \tilde{V}_0(0)$.

Finally, we choose to rescale the lengths on $\tilde{R}$ and perform a translation by one unit in the negative x -direction. This allows for a direct comparison with the full dimensionless problem, which has lengths scaled on R with coordinates centred on the wire. Using hats to denote rescaled variables, the position of the wire, at which the fluid boundary is pinned, is then $\hat{x} = 0$, $\hat{z} = 0$, the leftmost point of the fluid boundary has coordinates $\hat{x} = \tilde{x}_0/\tilde{R} - 1$, $\hat{z} = 0$, and the fluid volume is given by $\hat{V}_0 = \tilde{V}_0/\tilde{R}^3$.

The integration required for solving the ODEs present in this problem is carried out numerically, in Python, using the *solve_ivp* algorithm from the SciPy package. *solve_ivp* solves initial value problems for systems of ODEs using a form of Runge–Kutta integration.

Figures 5.3.2–5.3.4 show the computed equilibrium boundaries for three different

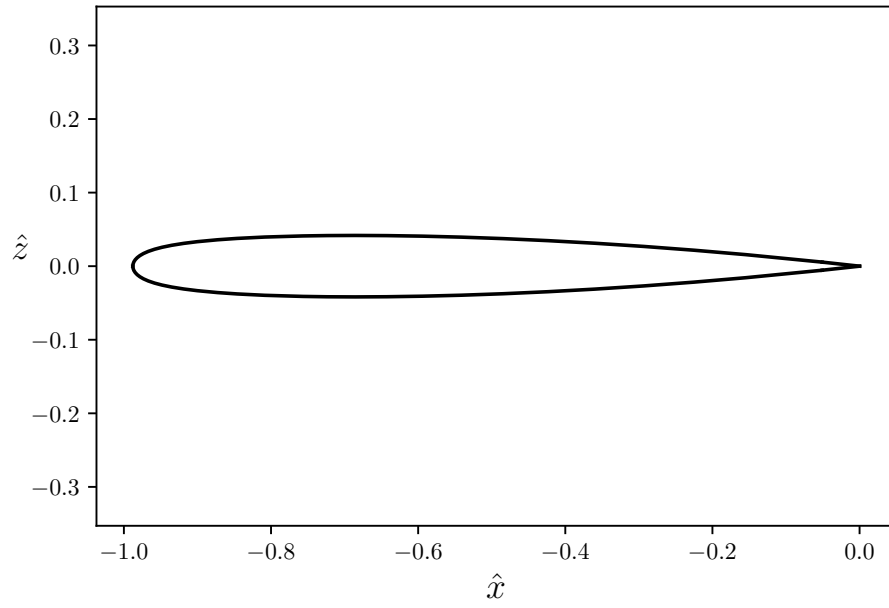


Figure 5.3.2: Equilibrium boundary obtained in the $Bo = 0$ case, with $\kappa_0 = 0.003$.

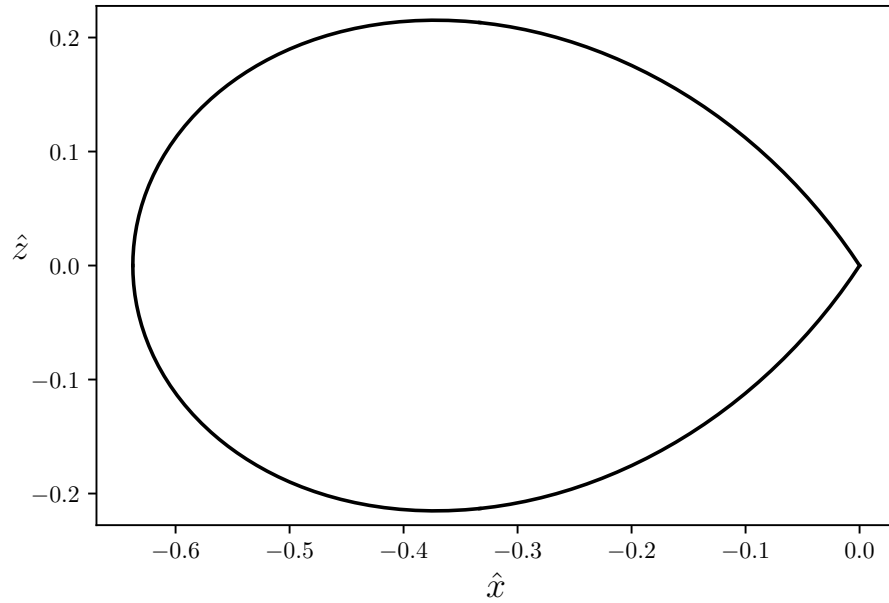


Figure 5.3.3: Equilibrium boundary obtained in the $Bo = 0$ case, with $\kappa_0 = 1$.

choices of κ_0 . Moreover, by solving for the equilibrium boundary for a range of values of κ_0 , we can plot a graph of the fluid volume $\hat{V}_0$ against the leftmost point of the fluid boundary $\hat{x}_0$. This is shown in figure 5.3.5.

Similarly to the full problem, we observe the existence of Type I and Type II boundaries, with a maximum fluid volume occurring at a critical value of $\hat{x}_0$. We

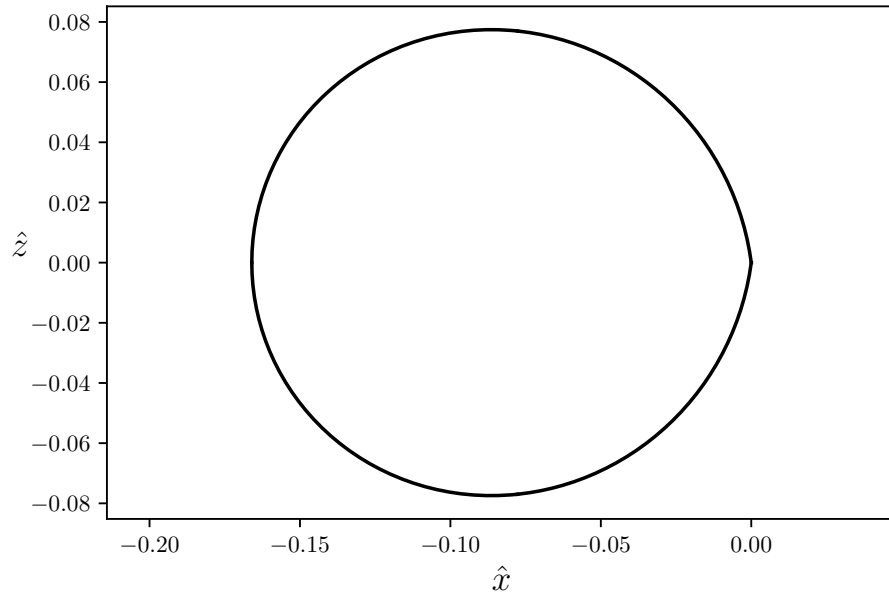


Figure 5.3.4: Equilibrium boundary obtained in the $Bo = 0$ case, with $\tilde{\kappa}_0 = 10$.

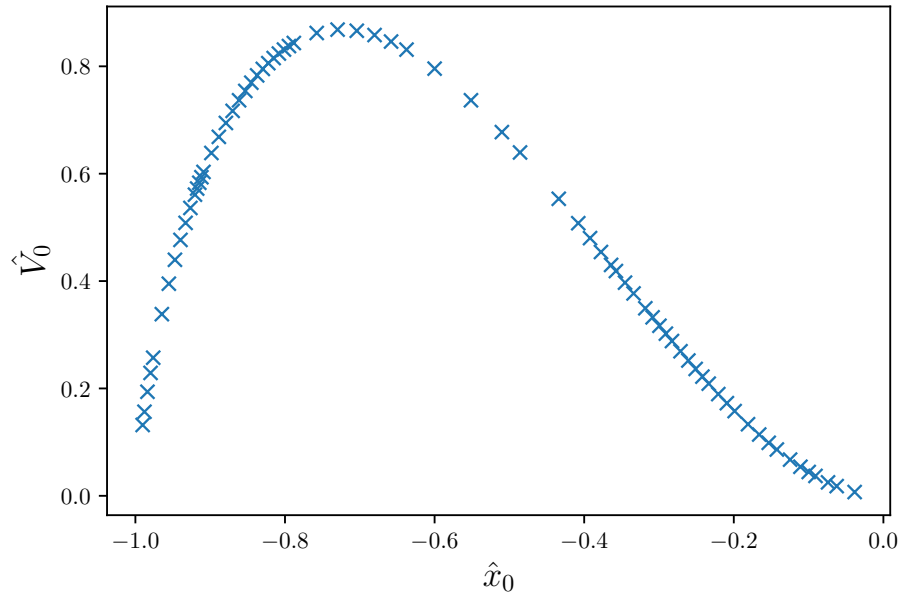


Figure 5.3.5: Plot of fluid volume $\hat{V}_0$ against $\hat{x}_0$ for equilibrium boundaries obtained when $Bo = 0$.

denote this maximum volume as $V_{0,max}$, which we have found to be $V_{0,max} \approx 0.868$. Moreover, we notice that $\hat{V}_0 \rightarrow 0$ when $\hat{x}_0 \rightarrow 0$, and when $\hat{x}_0 \rightarrow -1$. Therefore, for $Bo = 0$ and any fluid volume satisfying $0 < \hat{V}_0 \leq V_{0,max}$, there exists exactly two equilibrium boundaries, one of Type I and one of Type II.

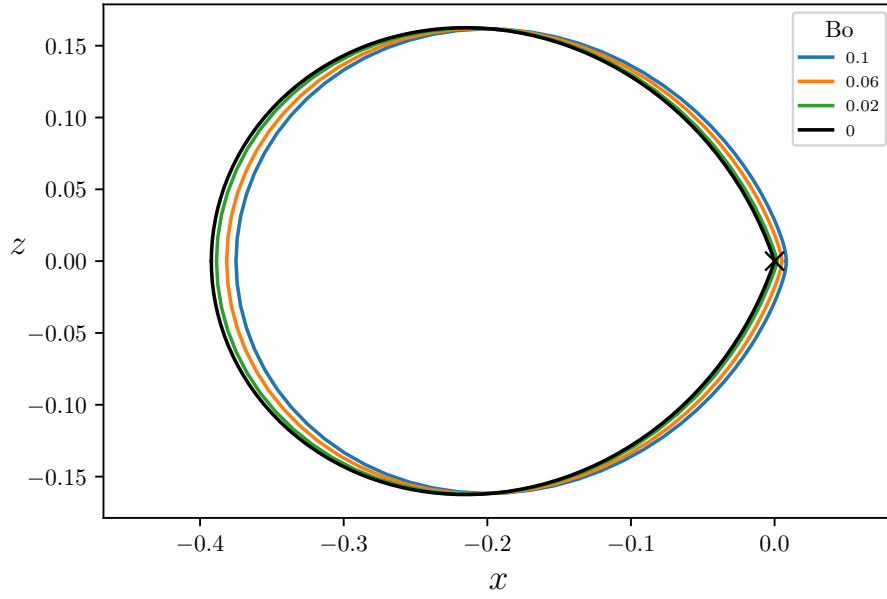


Figure 5.3.6: Comparison of equilibrium boundaries for small values of Bo with the $Bo = 0$ boundary, for $V_f = 0.48$. The cross represents the location of the wire in the plane.

Comparison with the Full Problem for Small Bo

To validate the $Bo = 0$ case as a sensible limit for the full problem as $Bo \rightarrow 0$, we can solve the full problem for a fixed fluid volume V_f and compare the obtained equilibrium boundaries for small values of Bo with the corresponding $Bo = 0$ boundaries.

Figure 5.3.6 shows the obtained Type I boundaries when $V_f = 0.48$ for some representative small values of Bo , plotted on the same axes as the corresponding Type I boundary for the $Bo = 0$ problem. The equilibrium boundaries appear to be tending towards the $Bo = 0$ boundary as Bo decreases.

In the $Bo = 0$ problem, the fluid boundary was pinned to the wire. This cannot be the case as $Bo \rightarrow 0$, as for small, but non-zero, values of Bo , magnetic effects are still present, albeit small in comparison to surface tension effects. We know that the field due to the wire becomes infinitely large as you approach the wire. This suggests that the $Bo = 0$ boundary may not accurately capture the behaviour of the equilibrium boundary everywhere as $Bo \rightarrow 0$.

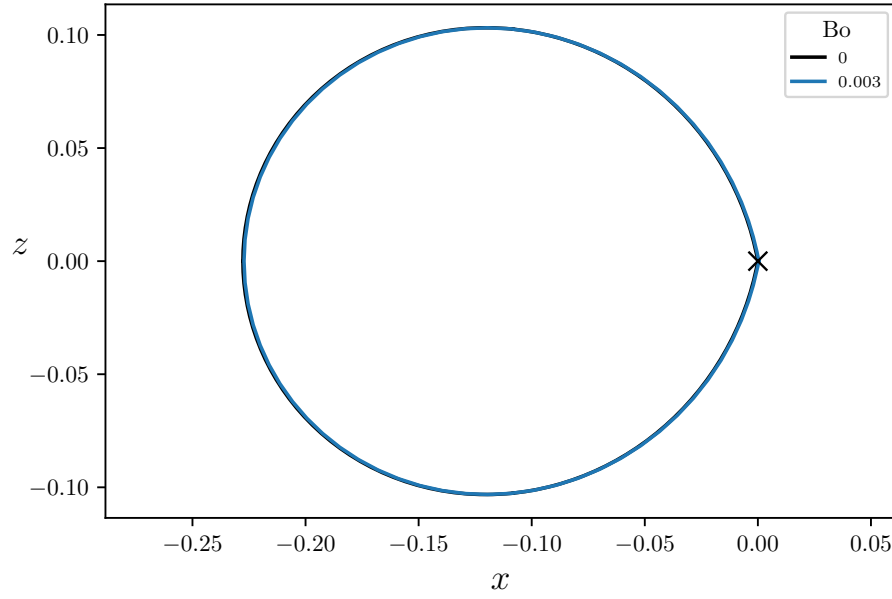


Figure 5.3.7: Comparison of equilibrium boundaries for $Bo = 0.003$ and $Bo = 0$, where $V_f = 0.2$. The cross represents the location of the wire in the plane.

Figure 5.3.7 shows the obtained Type I boundary when $V_f = 0.2$ for $Bo = 0.003$, plotted on the same axes as the corresponding Type I boundary for the $Bo = 0$ problem. Whilst we observe that the agreement between the two boundaries is good globally, close to the wire there is a qualitative difference in the behaviour of the two boundaries, as illustrated in figure 5.3.8. Whilst the $Bo = 0$ boundary is pinned at the wire, the $Bo = 0.003$ boundary deviates around the wire.

To understand what is going on here, we would need to consider this region close to the wire more carefully. We posit that the full limiting solution as $Bo \rightarrow 0$ can be found by formulating a matched asymptotics problem, in which the outer solution is the $Bo = 0$ boundary described above. The inner problem then comes from considering a region close to the wire. For $Bo \ll 1$, when lengths are $O(Bo)$ magnetic effects become $O(Bo^{-1})$. Therefore, we can see from the dynamic boundary condition (5.1.3) that magnetic effects are comparable to surface tension effects on this scale, even though the Bond number is small. The inner region is thus expected to have a length scale on the order of the Bond number.

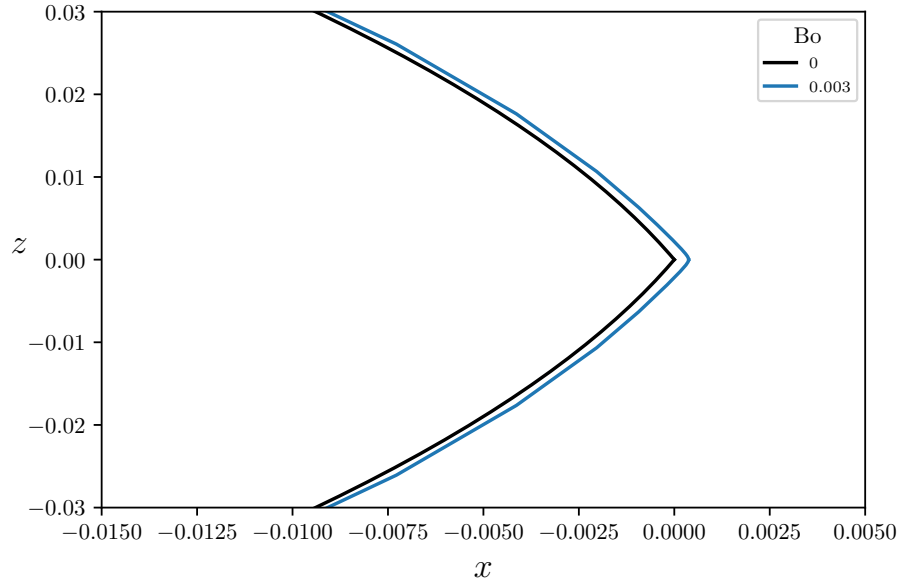


Figure 5.3.8: Comparison of equilibrium boundaries for $Bo = 0.003$ and $Bo = 0$, where $V_f = 0.2$, close to the wire at the origin.

An attempt was made to formulate and solve the inner problem for $Bo \rightarrow 0$, however, this proved to be no more straightforward than solving the full problem. The exact nature of the limiting solution as $Bo \rightarrow 0$ is thus left as an open problem. However, we have obtained evidence that the $Bo = 0$ problem captures the limiting behaviour as $Bo \rightarrow 0$, except in a region of $O(Bo)$ close to the wire.

5.3.2 Limit of Infinite Magnetic Bond Number

We now consider the case where the Bond number is infinitely large. Physically, this means that there are no surface tension effects whatsoever and the problem is entirely dominated by magnetic effects. Unlike in the case of $Bo = 0$, we have no reason to expect that equilibrium boundaries should not exist in this limit.

Consider the dynamic boundary condition (5.1.3). Clearly, if the Bond number becomes infinitely large, then either the over-pressure or the curvature must also become infinitely large, otherwise there is no way to balance the magnetic effects. Since the effects of curvature come into the problem through surface tension, we expect that it must be the over-pressure that becomes accordingly large to balance

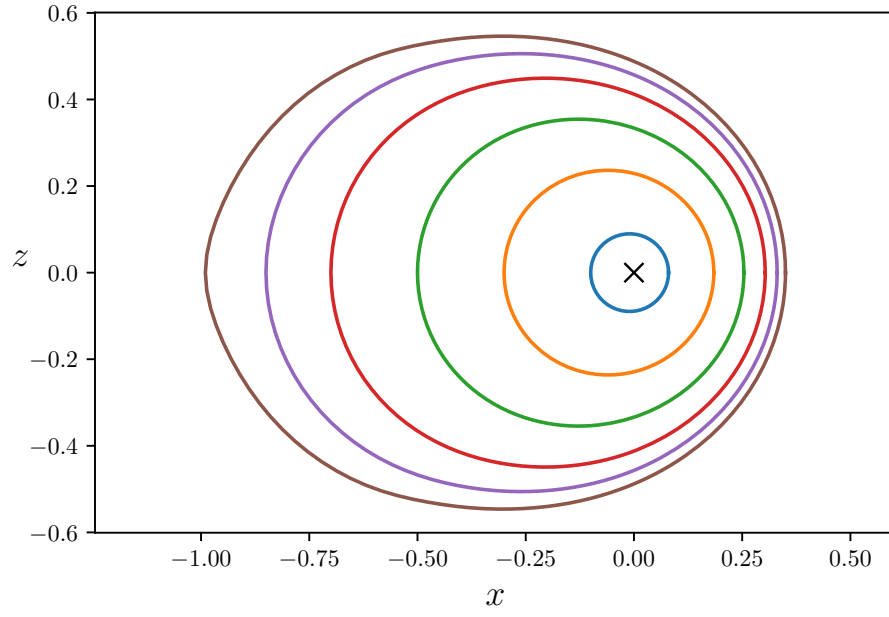


Figure 5.3.9: Equilibrium boundaries in the infinite Bond number case, each with a different fixed value of x_l . The cross represents the location of the wire in the plane.

the magnetic effects. Rescaling P on Bo , and taking the limit as $\text{Bo} \rightarrow \infty$, (5.1.3) becomes

$$\hat{P} + \frac{1}{2}(\tilde{m} - 1) \{H_t^2 + \tilde{m}^{-1}B_n^2\} = 0, \quad (5.3.13)$$

where $\hat{P}$ is the rescaled over-pressure.

To understand what happens when the Bond number is infinite, we solve the full problem numerically using (5.3.13) as the dynamic boundary condition, whilst varying the value of the leftmost point on the equilibrium boundary x_l . Figure 5.3.9 shows some obtained equilibrium boundaries for the infinite Bond number case. Additionally, figure 5.3.10 shows a plot of the fluid volume V_f against x_l , in which we observe that V_f appears to be tending towards a finite maximum value as $x_l \rightarrow -1$. Therefore, only Type I boundaries are known to exist when the Bond number is infinite. We denote the maximum volume in this case as $V_{\infty, \max}$, where $V_{\infty, \max} \approx 4.987$.

To validate the infinite Bond number case as a sensible limit for the full problem as $\text{Bo} \rightarrow \infty$, we solve the full problem for a fixed V_f and check that the obtained

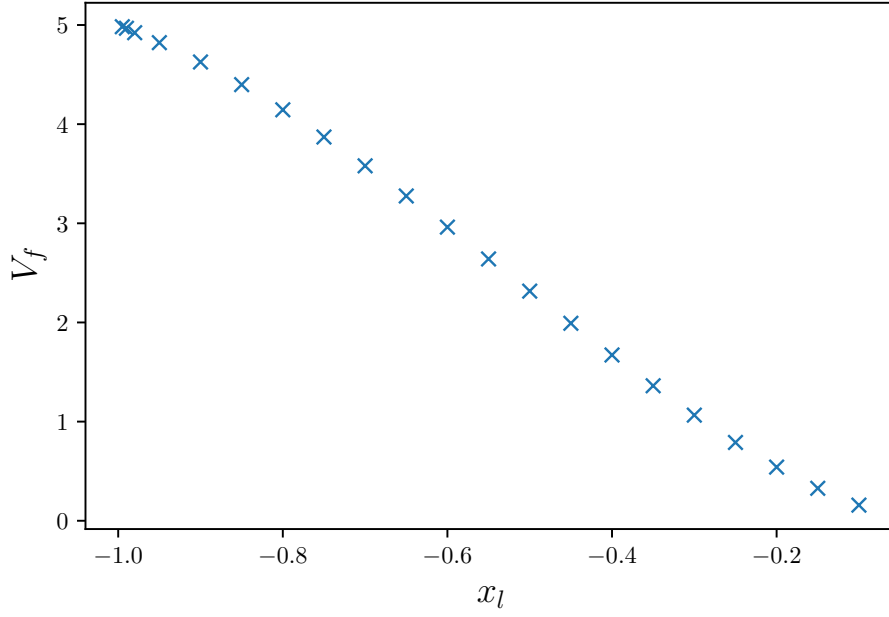


Figure 5.3.10: Plot of the fluid volume V_f against x_l for equilibrium boundaries obtained when the Bond number is infinite.

equilibrium boundaries appear to tend towards the infinite Bond number case as Bo becomes large. This is indeed seen in figure 5.3.11, which shows the obtained Type I boundaries when $V_f = 1.3$ for some representative large values of Bo , plotted on the same axes as the corresponding Type I boundary for the infinite Bond number problem.

5.4 Existence of Equilibrium Boundaries

So far, we have obtained evidence to suggest that for a given value of the Bond number, there exists a maximum fluid volume V_{max} beyond which no equilibrium boundaries exist. This maximum fluid volume occurs when the leftmost point of the equilibrium boundary, x_l , reaches a critical value x_{crit} . Moreover, we have identified two distinct types of equilibrium boundaries: Type I boundaries with $x_l > x_{crit}$, and Type II boundaries with $x_l < x_{crit}$.

In this section, we will study the results of solving the full problem numerically for fixed values of the Bond number ranging over several orders of magnitude.

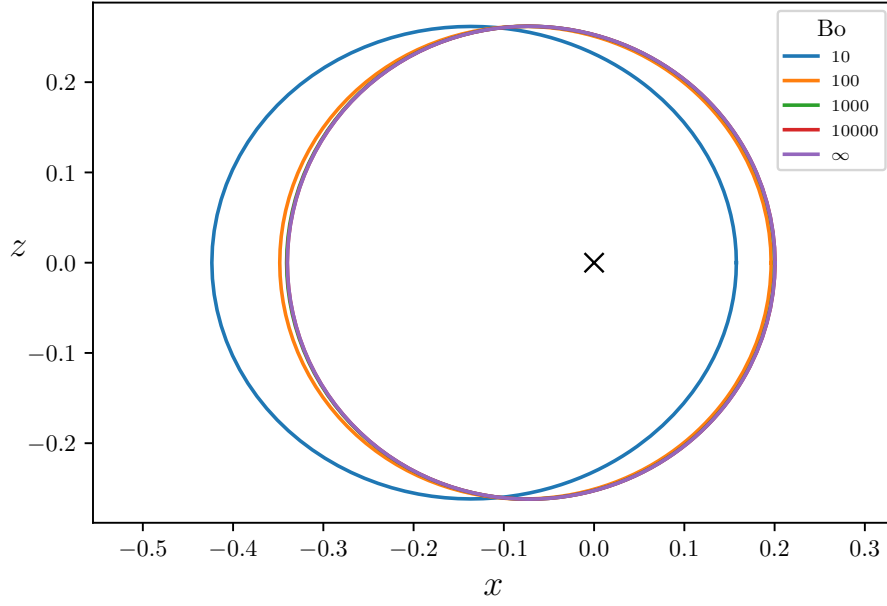


Figure 5.3.11: Type I equilibrium boundaries obtained for increasibly large values of the Bond number.

The smallest value considered is $Bo = 0.1$ and the largest value is $Bo = 10,000$. For each value of Bo , the full problem is solved for multiple values of x_l in the range $-1 < x_l < 0$.

Three sets of values of x_l are considered for each value of Bo . Firstly, the problem is solved for values of x_l decreasing from $x_l = -0.1$ to $x_l = -0.9$, with a step size of 0.1. This enables us to determine a value of L , where $L = -0.1, -0.2, \dots, -0.9$, for which $(L - 0.1) < x_{crit} < (L + 0.1)$. Secondly, we solve the problem for values of x_l decreasing from $x_l = L + 0.1$ to $x_l = L - 0.1$, with step size 0.01. We then take x_{crit} to be the value of x_l in this range for which the fluid volume is greatest. The value of V_{max} is hence taken to be the value of V_f at the calculated value of x_{crit} . Finally, we solve the problem for $x_l = x_{crit} - 0.01j$, where $j = 1, 2, \dots$, until such time that the numerical scheme fails to converge to a solution. The minimum possible value of x_l here is $x_l = -0.99$, however, the exact value of x_l at which the process terminates is dependent on the Bond number. This is discussed further in section 5.4.2.

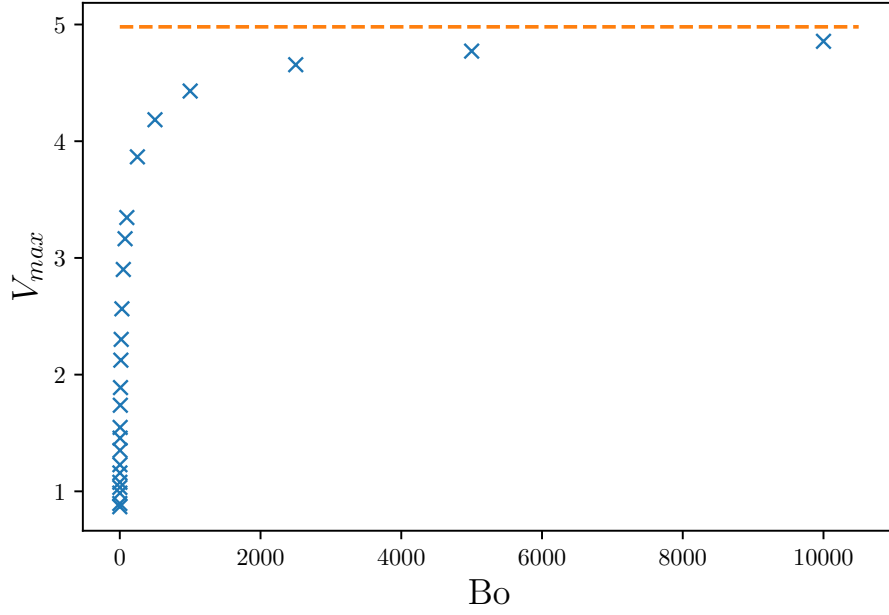


Figure 5.4.1: Plot of the maximum fluid volume V_{max} , for which a Type I equilibrium boundary can be shown to exist, against Bo . The dashed orange line corresponds to $V_{max} = V_{\infty,max}$.

5.4.1 Type I Boundaries

For each value of the Bond number considered, we are able to determine a value of x_{crit} at which the fluid volume V_f is maximal, with $V_f = V_{max}$. Therefore, for a given Bond number, a Type I boundary exists as long as $0 < V_f \leq V_{max}$. Additionally, in section 5.3, we determined a value of V_{max} for both the $Bo = 0$ and infinite Bond number cases. We can therefore plot the values of V_{max} against Bo to determine the domain in $Bo - V_f$ parameter space in which a Type I boundary exists.

The relationship between V_{max} and Bo is illustrated in figures 5.4.1–5.4.3. We observe that V_{max} increases monotonically with Bo , with an asymptote at $V_f = V_{\infty,max}$. Therefore, there exists a global maximum volume, $V_f = V_{\infty,max}$, beyond which no equilibrium boundaries exist, no matter the value of Bo .

Whilst an exact form of V_{max} as a function of Bo is unknown, we have obtained numerical evidence to suggest the existence of a smooth curve in $Bo - V_f$ parameter space, above which no equilibrium boundaries exist and below which

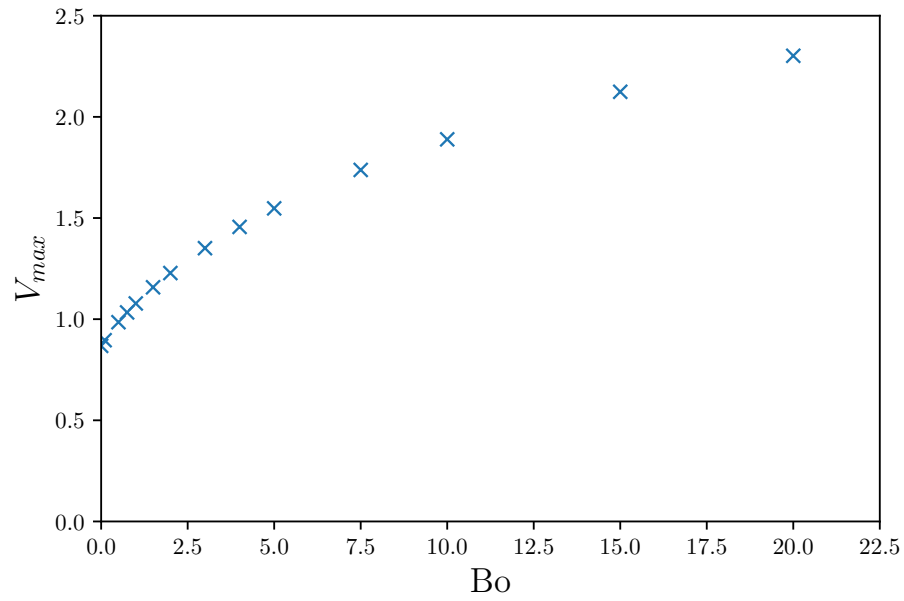


Figure 5.4.2: Plot of the maximum fluid volume V_{max} against Bo for just small values of Bo .

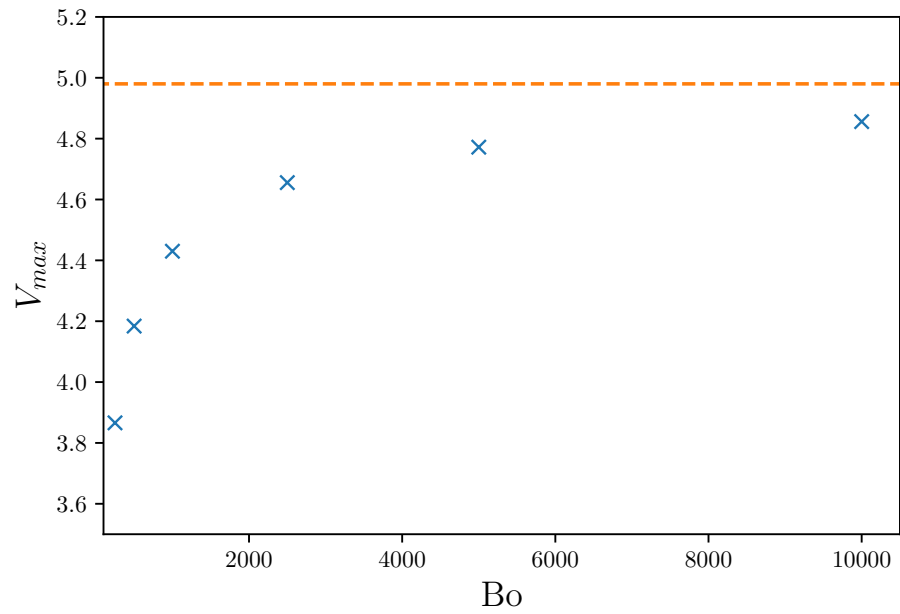


Figure 5.4.3: Plot of the maximum fluid volume V_{max} against Bo for just large values of Bo . The dashed orange line corresponds to $V_{max} = V_{\infty, max}$.

a Type I boundary exists. We make specific note of the fact that the values of V_{max} obtained for the $Bo = 0$ and infinite Bond number cases are in good agreement with the observed behaviour of V_{max} for non-zero finite Bond number.

5.4.2 Type II Boundaries

We know that for a Type II boundary to exist, the fluid volume must satisfy $V_f \leq V_{max}$. However, unlike in the case of Type I boundaries, it is not necessarily true that a Type II boundary will exist for any V_f satisfying $0 < V_f \leq V_{max}$. Instead, it may be the case that, for a given Bond number, there exists a non-zero minimum volume V_{min} below which Type II boundaries do not exist. Motivating this theory is the fact that no Type II boundaries were seen to exist in the case where the Bond number was infinite.

Determining the domain in which Type II solutions exist is then a case of determining whether this non-zero V_{min} exists, in which case we would need to determine V_{min} as a function of Bo , or showing that Type II solutions exist for any V_f such that $0 < V_f \leq V_{max}$. We do not have a well-studied limiting case as $x_l \rightarrow -1$. Instead, for each value of Bo , we solve the full problem numerically for $x_l = x_{crit} - 0.01j$, where $j = 1, 2, \dots$, until such time that *fsolve* fails to converge to a sensible solution. Once the process has terminated, we can plot the values of V_f against x_l to see how the fluid volume behaves as $x_l \rightarrow -1$. Additionally, we note the smallest value of the fluid volume for which a Type II boundary was found, denoted by V_{least} . The final value of x_l for which *fsolve* successfully converged always satisfied $x_l \leq -0.97$.

The relationship between V_{least} and Bo is shown in figures 5.4.4 and 5.4.5, in which it is compared with the relationship between V_{max} and Bo . Additionally, figures 5.4.6, 5.4.7, and 5.4.8 show the obtained Type II equilibrium boundaries with the least fluid volume, for $Bo = 0.5$, $Bo = 5$, and $Bo = 100$. Whilst the observed results suggest the presence of a non-zero V_{min} curve in parameter

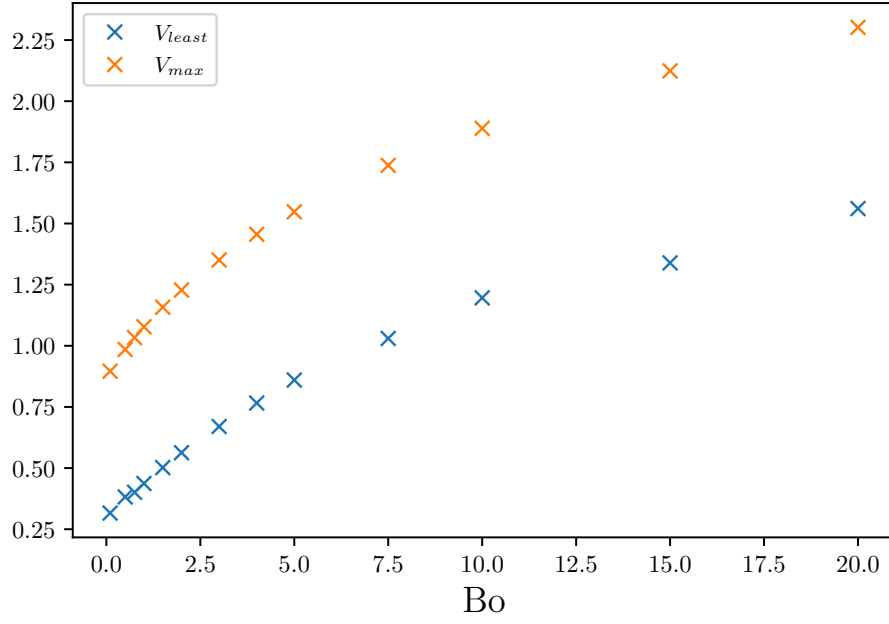


Figure 5.4.4: Plots of V_{least} and V_{max} against Bo for just small values of Bo .

space, below which Type II boundaries do not occur, it is still not definitive that such a curve exists. The possibility that the fluid volume rapidly drops to zero as $x_l \rightarrow -1$ cannot be ruled out. As such, further investigation into the limit as $x_l \rightarrow -1$ is needed to fully understand the domain in which these Type II boundaries exist.

As it stands, we have been able to determine a range $V_{least} \leq V_f \leq V_{max}$, for specific values of Bo , in which both Type I and Type II equilibrium boundaries can be seen to exist. As such, the shapes of the toroidal droplets in equilibrium for fixed values of the Bond number and the fluid volume, are not necessarily unique.

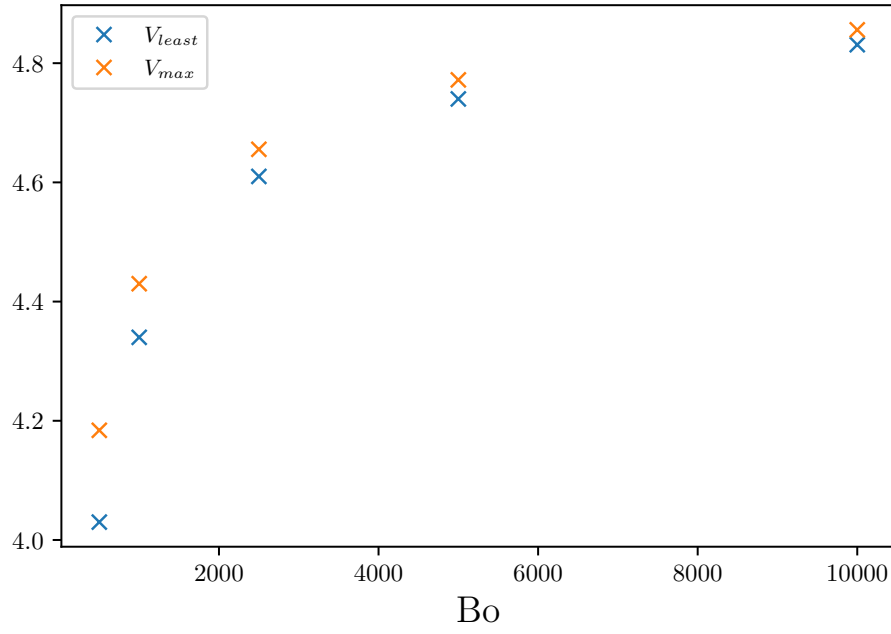


Figure 5.4.5: Plots of V_{least} and V_{max} against Bo for just large values of Bo .

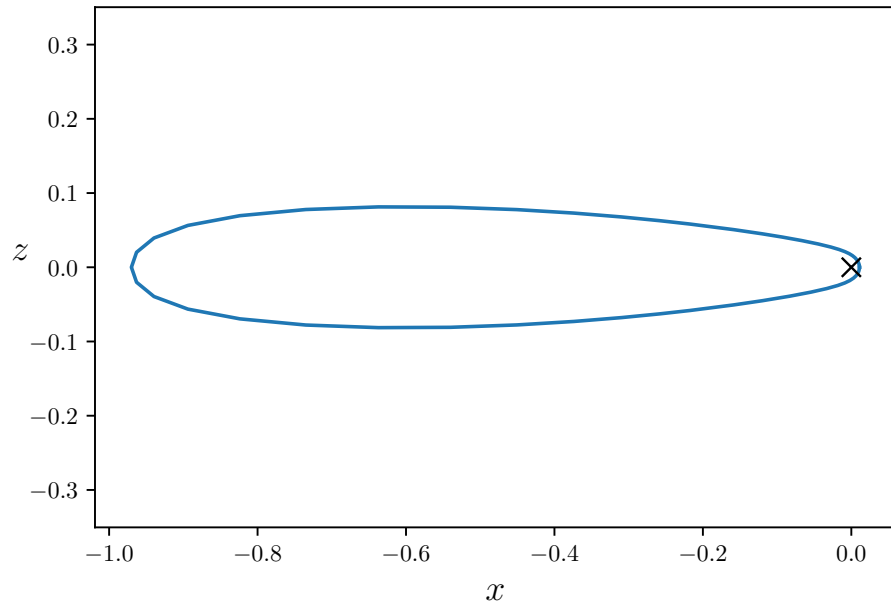


Figure 5.4.6: Type II equilibrium boundary with the smallest observed fluid volume for $Bo = 0.5$.

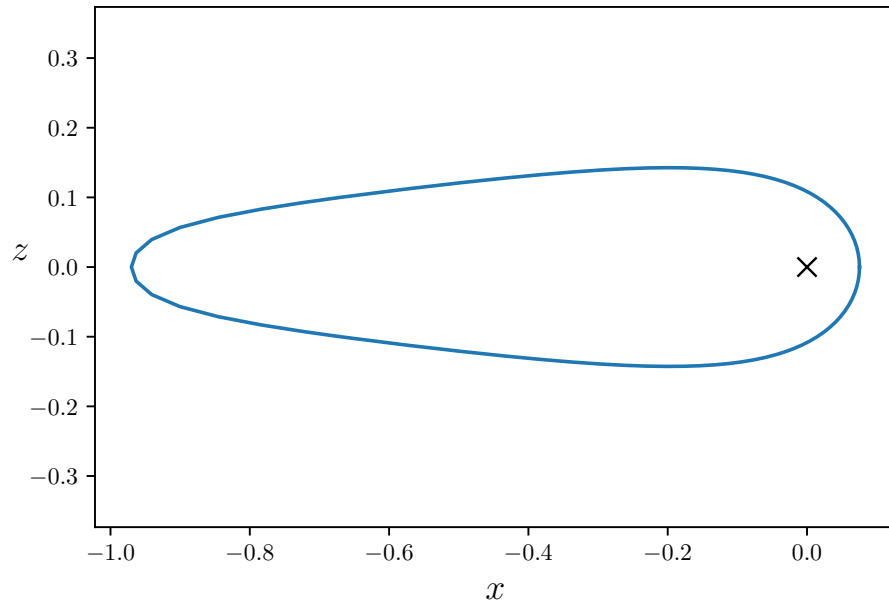


Figure 5.4.7: Type II equilibrium boundary with the smallest observed fluid volume for $Bo = 5$.

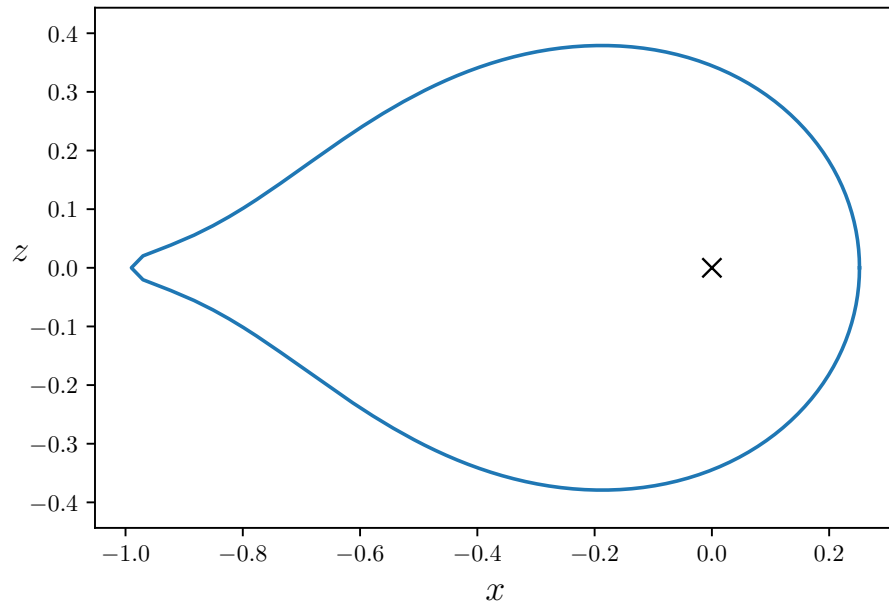


Figure 5.4.8: Type II equilibrium boundary with the smallest observed fluid volume for $Bo = 100$.

Extension to Ferrofluid Droplets

In this chapter, we seek to extend the analysis from the preceding chapters in order to study the equilibrium configurations of simply-connected ferrofluid droplets. In particular, we will consider how the mathematical model derived in Chapter 2 can be used to study static ferrofluid droplets in the presence of a circular loop of current-carrying wire. We will then consider an appropriate adaptation of the numerical method described in Chapter 3, which will enable us to solve for the shape of equilibrium configurations of such ferrofluid droplets. Finally, a brief analysis of the numerically computed equilibrium configurations will be presented.

6.1 Extension of the Mathematical Model

6.1.1 Motivation and Premise

In Chapter 5, we found that there was a maximum fluid volume beyond which ferrofluid toroids could no longer exist in equilibrium. Therefore, to study equilibrium configurations for volumes of ferrofluid greater than this maximum, it is necessary to consider non-toroidal ferrofluid domains. In particular, we seek to study the equilibrium configurations of axisymmetric ferrofluid droplets.

We consider an axisymmetric ferrofluid droplet in the presence of a thin wire forming a circular loop of radius R and carrying a current I . We denote the

permeability of the ferrofluid by μ_f and note that the surrounding air will have permeability μ_0 . The ferrofluid droplet may either surround the circular wire-loop, or, be contained within the circle formed by the wire-loop.

In Chapter 2, we derived a general equation for the magnetic induction $\mathbf{B}$ at a point $\mathbf{x}$ in space, for the general case of a domain comprising different materials that each have uniform magnetic permeability, where the only current is carried by a thin wire. In particular, we found that (2.6.14)

$$\mathbf{B}(\mathbf{x}) = -\frac{1}{4\pi} \int_w \mu_s I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi + \frac{1}{4\pi} \iint_S \frac{[\mu](\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_\xi, \quad (6.1.1)$$

where w is the centre-line of the wire, μ_s is the magnetic permeability of the material surrounding the wire, S is the surface comprising any interfaces between materials not including that between the wire and its surroundings, $\mathbf{H}$ is the magnetic field, $\hat{\mathbf{n}}$ is the unit normal to S , and $[\mu]$ denotes the change in the permeability μ across S , defined as the value of μ on the side of the interface $\hat{\mathbf{n}}$ points into minus the value of μ on the other side. Note that both $\hat{\mathbf{n}}$ and $\mathbf{H}$ in the surface integral of (6.1.1) are evaluated at the point $\boldsymbol{\xi}$.

For the problem we are considering, we note that either $\mu_s = \mu_f$, if the ferrofluid droplet surrounds the wire-loop, or $\mu_s = \mu_0$, if the ferrofluid droplet is entirely contained within the circle formed by the wire-loop. Additionally, the surface S is simply the free surface of the ferrofluid droplet. Therefore, we have that

$$\mathbf{B}(\mathbf{x}) = -\frac{1}{4\pi} \int_w \mu_s I \frac{(\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \wedge d\mathbf{l}_\xi + \frac{1}{4\pi} (\mu_0 - \mu_f) \iint_S \frac{(\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_\xi, \quad (6.1.2)$$

where S now corresponds to the free surface of the ferrofluid droplet.

In Chapter 2, we showed that in the case of a ferrofluid toroid surrounding the circular wire-loop, the integral representation (6.1.2) could be written as

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_f I}{4\pi} \mathbf{P}(\mathbf{x}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) ds, \quad (6.1.3)$$

where $\boldsymbol{\xi}_0(s)$ describes the cross-section of the ferrofluid toroid in a meridional plane, which we refer to as the ferrofluid boundary. Additionally, s is an arc-length coordinate around the ferrofluid boundary, $2C$ is the total arc-length around the ferrofluid boundary, H_t is the component of $\mathbf{H}$ tangential to the ferrofluid boundary, and $\mathbf{P}$ and $\mathbf{f}$ are the azimuthal integrals defined in (2.7.10)–(2.7.13), (2.7.17), and (2.7.22).

Although framed as a derivation for the boundary-integral representation for $\mathbf{B}$ for a ferrofluid toroid, it so happens that the methodology employed to obtain (6.1.3) from (6.1.2) is valid for any ferrofluid volume that has the same axisymmetry as the wire-loop. This is because there is no requirement in the derivation that the intersection of the ferrofluid with a meridional plane needs to form a closed curve.

Therefore, the magnetic induction $\mathbf{B}$ for a static ferrofluid droplet in the presence of a circular loop of current-carrying wire, where the ferrofluid droplet and the wire share the same axisymmetry, is given by

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_s I}{4\pi} \mathbf{P}(\mathbf{x}, \mathbf{0}) + \frac{(\mu_0 - \mu_f)}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{x}, \boldsymbol{\xi}_0(s)) \, ds, \quad (6.1.4)$$

where here, $\boldsymbol{\xi}_0(s)$ describes the cross-section of the ferrofluid droplet in a meridional plane. We refer to this cross-section as the ferrofluid boundary, with it now taken to be understood that the term ‘ferrofluid boundary’ can refer to the cross-section of any axisymmetric volume of ferrofluid in a meridional plane.

Moreover, the dynamic boundary condition (2.7.27) is derived without any specific reference to the geometry of the ferrofluid. It follows that the governing equations for a static axisymmetric ferrofluid droplet in the presence of a circular loop of current-carrying wire, are entirely analogous to those for a static ferrofluid toroid surrounding the circular loop of current-carrying wire.

6.1.2 Dimensionless Problem Statement

We work in Cartesian coordinates and use dimensionless variables, with lengths scaled on R , magnetic inductions scaled on $\mu_0 I/R$, magnetic fields on I/R , and pressures on γ/R . The aim is to solve for the position of the ferrofluid boundary, such that the corresponding ferrofluid droplet is in equilibrium.

In terms of dimensionless variables, the boundary-integral equation (6.1.4) for a point $\mathbf{X}$ on the ferrofluid boundary, becomes

$$\mathbf{B}(\mathbf{X}) = -\frac{M}{4\pi} \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) + \frac{(1-\tilde{m})}{4\pi} \int_{-C}^C H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) \, ds, \quad (6.1.5)$$

where $M = \tilde{m}$, with $\tilde{m} = \mu_f/\mu_0$, if the ferrofluid droplet surrounds the wire-loop, and $M = 1$ if the ferrofluid droplet is contained within the circle formed by the wire-loop, and $\tilde{\mathbf{P}}$ is the azimuthal integral defined in (2.7.10). The ferrofluid boundary is described by

$$\boldsymbol{\xi}_0(s) = \begin{pmatrix} \xi(s) \\ \zeta(s) \end{pmatrix}. \quad (6.1.6)$$

The dimensionless dynamic boundary condition (2.8.7), is given by

$$P + \frac{1}{2}(\tilde{m} - 1)\text{Bo} \{H_t^2 + \tilde{m}^{-1}B_n^2\} = \kappa, \quad (6.1.7)$$

where B_n is the component of $\mathbf{B}$ normal to the ferrofluid boundary, κ is the total curvature of the ferrofluid droplet, P is the dimensionless over-pressure, and

$$\text{Bo} = \frac{\mu_0 I^2}{R\gamma}, \quad (6.1.8)$$

is the magnetic Bond number.

In addition to the boundary-integral equation (6.1.5) and the dynamic boundary condition (6.1.7), we require that the total volume of ferrofluid, V_f , be prescribed.

We choose the centre of curvature for the wire-loop to be located at $x = -1$. We

therefore work in the $(x > -1, z)$ half-plane, where the center of the cross-section of the wire-loop is located at the point $x = 0, z = 0$, and the line $x = -1$ serves as the axis of rotational symmetry for the ferrofluid droplet.

We describe the ferrofluid boundary using plane polar coordinates (r, θ) , which are centred on the centre of curvature of the wire-loop at $x = -1, z = 0$. If the position of the ferrofluid boundary is given by $r = \rho(\theta)$, then the Cartesian coordinates of a point $\mathbf{X}$ on the ferrofluid boundary, are given by

$$\mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -1 + \rho \cos \theta \\ \rho \sin \theta \end{pmatrix}. \quad (6.1.9)$$

Moreover, for a ferrofluid droplet, the ferrofluid boundary must meet the axis of rotation twice, once when $z > 0$ and once when $z < 0$. We take $s = -C$ at the point where the ferrofluid boundary meets the axis of rotation when $z < 0$, where $\theta = -\pi/2$. Therefore, the point where the ferrofluid boundary meets the axis of rotation for $z > 0$ has an arc-length coordinate of $s = C$, with $\theta = \pi/2$. We denote the arc-length coordinate of the point $\mathbf{X}$ by $s = s_X$. The angle that the ferrofluid boundary makes with the vertical at the point $\mathbf{X}$ is denoted by ϑ . Finally, we define $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ to be, respectively, the unit tangent and normal vectors to the ferrofluid boundary at the point $\mathbf{X}$. A schematic for the setup of this problem is shown in figure 6.1.1.

6.1.3 Analytic Treatment of the Singularity

Similarly to the case of the problem for a toroidal volume of ferrofluid, the integrand of the arc-length integral in (6.1.5) contains an s^{-1} singularity, the integral of which is formally interpreted by a Cauchy principal value. Since the arc-length coordinate of the point $\mathbf{X}$ is taken to be $s = s_X$, the singularity occurs at $s = s_X$ for this problem, as opposed to $s = 0$ for the toroidal problem.

The method for dealing with the singularity in the integrand numerically is

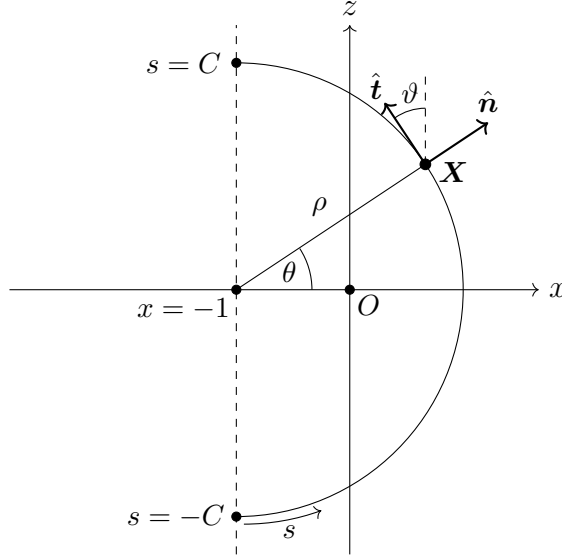


Figure 6.1.1: The setup for the dimensionless problem of an axisymmetric ferrofluid droplet

entirely analogous to the method used for the toroidal problem, described in Chapter 2. Specifically, we seek to add and subtract a simple function $\mathbf{f}^*$ that shares the same singularity as $H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$ at $s = s_X$. Following the work in Chapter 2, we obtain

$$\mathbf{f}^*(\mathbf{X}, s) = \frac{h_0 \mathbf{a}_{-1}}{s - s_X} + h_0 \mathbf{b}_0 \log |s - s_X| + (h_1 \mathbf{b}_0 + h_0 \mathbf{b}_1) (s - s_X) \log |s - s_X|, \quad (6.1.10)$$

where $h_0 = H_t(\mathbf{X})$, $h_1 = H'_t(\mathbf{X})$, and we have

$$\mathbf{a}_{-1} = 2\hat{\mathbf{n}}, \quad (6.1.11)$$

$$\mathbf{b}_0 = \frac{1}{x+1} (\hat{\mathbf{n}} \sin \vartheta + \hat{\mathbf{t}} \cos \vartheta), \quad (6.1.12)$$

$$\mathbf{a}_0 = \left(\kappa_m + \frac{\cos \vartheta}{x+1} \right) \hat{\mathbf{t}} - \frac{\log(8(x+1))}{x+1} (\hat{\mathbf{n}} \sin \vartheta + \hat{\mathbf{t}} \cos \vartheta), \quad (6.1.13)$$

$$\mathbf{b}_1 = \frac{1}{4(x+1)^2} (\hat{\mathbf{n}} (1 + 2 \cos^2 \vartheta) - 2\hat{\mathbf{t}} \sin \vartheta \cos \vartheta), \quad (6.1.14)$$

where κ_m is the plane curvature of the fluid boundary at the point $\mathbf{X}$. The subtracted copy of $\mathbf{f}^*$, as given in (6.1.10), yields an integrand which is well behaved as $s \rightarrow s_X$, whilst the added copy is replaced by its analytic integral.

For convenience in the numerical method, we decompose $\mathbf{f}^*$ as

$$\mathbf{f}^*(\mathbf{X}, s) = h_0 \mathbf{f}_0^*(\mathbf{X}, s) + h_1 \mathbf{f}_1^*(\mathbf{X}, s), \quad (6.1.15)$$

where we have

$$\mathbf{f}_0^*(\mathbf{X}, s) = \frac{\mathbf{a}_{-1}}{s - s_X} + \mathbf{b}_0 \log |s - s_X| + \mathbf{b}_1 (s - s_X) \log |s - s_X|, \quad (6.1.16)$$

and

$$\mathbf{f}_1^*(\mathbf{X}, s) = \mathbf{b}_0 (s - s_X) \log |s - s_X|. \quad (6.1.17)$$

Additionally, we need to compute the arc-length integral of $\mathbf{f}^*$ from $s = -C$ to $s = C$. The integration is straightforward and, after some algebra, we obtain

$$\mathbf{I}(\mathbf{X}) = \int_{-C}^C \mathbf{f}^*(\mathbf{X}, s) \, ds = h_0 \mathbf{I}_0(\mathbf{X}) + h_1 \mathbf{I}_1(\mathbf{X}), \quad (6.1.18)$$

where

$$\begin{aligned} \mathbf{I}_0(\mathbf{X}) = & \frac{\mathbf{b}_1}{2} (C - s_X)^2 \left(\log |C - s_X| - \frac{1}{2} \right) \\ & - \frac{\mathbf{b}_1}{2} (C + s_X)^2 \left(\log |-C - s_X| - \frac{1}{2} \right) + \{(C + s_X) \mathbf{b}_0 - \mathbf{a}_{-1}\} \log |-C - s_X| \\ & + \{(C - s_X) \mathbf{b}_0 + \mathbf{a}_{-1}\} \log |C - s_X| - 2\mathbf{b}_0 C, \end{aligned} \quad (6.1.19)$$

and

$$\begin{aligned} \mathbf{I}_1(\mathbf{X}) = & \frac{\mathbf{b}_0}{2} (C - s_X)^2 \left(\log |C - s_X| - \frac{1}{2} \right) \\ & - \frac{\mathbf{b}_0}{2} (C + s_X)^2 \left(\log |-C - s_X| - \frac{1}{2} \right). \end{aligned} \quad (6.1.20)$$

Finally, we require the limiting value of $H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)$ as $s \rightarrow s_X$, to be used in the numerical method at $s = s_X$. As in Chapter 2, the

limiting value $L(\mathbf{X})$, is given by

$$L(\mathbf{X}) = h_0 \mathbf{a}_0 + h_1 \mathbf{a}_{-1} \quad (6.1.21)$$

6.2 Numerical Method for the Droplet Problem

The numerical method used to solve for the equilibrium boundary of a ferrofluid droplet is very similar to the method used to solve for the equilibrium boundary of a ferrofluid toroid, described in Chapter 3. However, a key difference in the two problems is the fact that the ferrofluid boundary of a droplet must touch the axis of rotation. This was not the case for a ferrofluid toroid.

We will run into trouble numerically if we take points on the axis of rotation when discretising the ferrofluid boundary. If $x = -1$, a number of quantities appear to diverge. This is evident in (6.1.12)–(6.1.14), for example. Additionally, $\tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0})$ and $\mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$ both encounter ‘division by zero’ errors when evaluated numerically at $x = -1$. Taking $\xi = -1$ in $\mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$ also leads to a ‘division by zero’ error.

One may be concerned that $\mathbf{B}(\mathbf{X})$ is not well-defined for this problem. However, a series expansion of $\tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0})$, computed in Maple, verifies that $\tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0})$ has a finite limit as $x \rightarrow -1$. Additionally, we note that the axisymmetry of the problem necessitates that $H_t = 0$ on the axis of rotation. Therefore, by considering a Taylor series expansion of $H_t(\boldsymbol{\xi}_0(s))$ about $\xi = -1$, we see that $H_t = O(\xi + 1)$ as $\xi \rightarrow -1$.

Moreover, we can study the behaviour of the integrand in (6.1.5), both when $x \rightarrow -1$ and $\xi \rightarrow -1$, by considering appropriate series expansions of $\mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s))$, computed in Maple. In particular, when $\xi \rightarrow -1$, we find that

$$H_t(\boldsymbol{\xi}_0(s)) \mathbf{f}(\mathbf{X}, \boldsymbol{\xi}_0(s)) \rightarrow 0, \quad \text{if } \mathbf{X} \neq \boldsymbol{\xi}_0, \quad (6.2.1)$$

and

$$H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) = O(1), \quad \text{if } \mathbf{X} = \xi_0. \quad (6.2.2)$$

Additionally, for $\xi \neq -1$, we find that $H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s))$ has a finite limit as $x \rightarrow -1$. It follows that the integrand is always well-behaved when $x = -1$ or $\xi = -1$, even when $\mathbf{X} = \xi_0$, where we would usually expect a singularity.

Therefore, $\mathbf{B}(\mathbf{X})$ is well-defined here, and the problem can be handled numerically as long as we avoid placing any points on the axis of rotation when discretising the ferrofluid boundary.

6.2.1 Variables

We discretise the problem by representing the ferrofluid boundary by a set of N points $\mathbf{X}_i$. We impose the polar angle θ_i of each point so that the $\mathbf{X}_i$ are characterised by their radial coordinates ρ_i . In particular, we have that

$$\mathbf{X}_i = \begin{pmatrix} x_i \\ z_i \end{pmatrix} = \begin{pmatrix} -1 + \rho_i \cos \theta_i \\ \rho_i \sin \theta_i \end{pmatrix}, \quad (6.2.3)$$

where the θ_i are prescribed and the ρ_i need to be determined. The magnetic field at each point $\mathbf{X}_i$ is described by the normal component $B_{n,i}$ of $\mathbf{B}$ and the tangential component $H_{t,i}$ of $\mathbf{H}$.

To avoid placing points on the axis of rotation, we impose the following equally spaced angular coordinates

$$\theta_i = -\frac{\pi}{2} + \left(i + \frac{1}{2}\right) \alpha, \quad (6.2.4)$$

where $\alpha = \pi/N$. Based on the symmetry of the problem, we also introduce two fictitious points: $\mathbf{X}_-$, which is the reflection of $\mathbf{X}_0$ in the line $x = -1$, and $\mathbf{X}_+$, which is the reflection of $\mathbf{X}_{N-1}$ in the line $x = -1$. These fictitious points enable us to compute needed quantities relating to the surface geometry of the ferrofluid

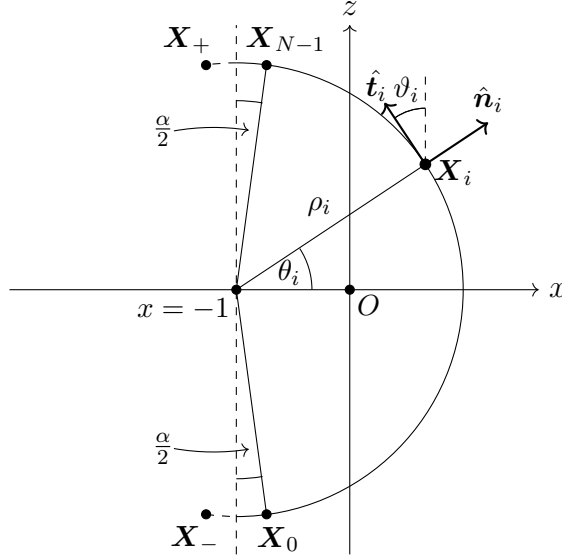


Figure 6.2.1: The setup for the discretised problem of an axisymmetric ferrofluid droplet

boundary, without requiring any knowledge of the points on the axis of rotation.

Finally, we denote the angle between the tangent to the ferrofluid boundary and the vertical, at the point $\mathbf{X}_i$, by ϑ_i and the unit tangent and normal vectors to the ferrofluid boundary, at the point $\mathbf{X}_i$, by $\hat{\mathbf{t}}_i$ and $\hat{\mathbf{n}}_i$ respectively. A schematic of the discretised problem is shown in figure 6.2.1.

6.2.2 Surface Geometry

Noting that we are using equally spaced angular coordinates, with consecutive angles separated by an amount α , we define the finite-difference angular derivative approximations as

$$\rho_i'^+ = \frac{\rho_{i+1} - \rho_i}{\alpha}, \quad \rho_i'^- = \frac{\rho_i - \rho_{i-1}}{\alpha}, \quad \rho_i' = \frac{\rho_{i+1} - \rho_{i-1}}{2\alpha}; \quad (6.2.5)$$

and

$$\rho_i'' = \frac{\rho_{i+1} - 2\rho_i + \rho_{i-1}}{\alpha^2}. \quad (6.2.6)$$

The fictitious points, $\mathbf{X}_+$ and $\mathbf{X}_-$, allow us to calculate the finite-difference approximations at the endpoints $\mathbf{X}_0$ and $\mathbf{X}_{N-1}$. In particular, we have

$$\rho'_{N-1} = 0, \quad \rho'_0 = 0, \quad \rho'_{N-1} = \frac{\rho_{N-1} - \rho_{N-2}}{2\alpha}, \quad \rho'_0 = \frac{\rho_1 - \rho_0}{2\alpha}; \quad (6.2.7)$$

$$\rho''_{N-1} = \frac{\rho_{N-2} - \rho_{N-1}}{\alpha^2}, \quad \rho''_0 = \frac{\rho_1 - \rho_0}{\alpha^2}. \quad (6.2.8)$$

We then approximate the arc-distance along the ferrofluid boundary between neighbouring points either side of $\mathbf{X}_i$, denoted d_i^+ for the distance to the point $\mathbf{X}_{i+1}$ or d_i^- for the distance to the point $\mathbf{X}_{i-1}$, using the polar coordinate arc-length formula, as in (3.3.5). Therefore, we have

$$d_i^+ = \alpha \left((\rho'_i)^2 + \frac{1}{4} (\rho_{i+1} + \rho_i)^2 \right)^{\frac{1}{2}}, \quad \text{for } i = 0, 1, \dots, N-2, \quad (6.2.9)$$

$$d_i^- = \alpha \left((\rho'_i)^2 + \frac{1}{4} (\rho_i + \rho_{i-1})^2 \right)^{\frac{1}{2}}, \quad \text{for } i = 1, \dots, N-1. \quad (6.2.10)$$

We also need to define the quantities d_{N-1}^+ and d_0^- , such that they compute the arc-length distance along the ferrofluid boundary from the endpoints, $\mathbf{X}_{N-1}$ and $\mathbf{X}_0$, to the axis of rotation, respectively. Noting the fact that the fictitious points are the reflections of the endpoints in the axis of rotation, we have

$$d_{N-1}^+ = \frac{\alpha}{2} \left((\rho'_{N-1})^2 + (\rho_{N-1})^2 \right)^{\frac{1}{2}}, \quad (6.2.11)$$

$$d_0^- = \frac{\alpha}{2} \left((\rho'_0)^2 + (\rho_0)^2 \right)^{\frac{1}{2}}. \quad (6.2.12)$$

We also define

$$d_i = \frac{1}{2} (d_i^- + d_i^+), \quad (6.2.13)$$

which will be used as the cell width in the trapezium rule.

The total surface arc-length of the ferrofluid boundary is hence given by

$$2C = d_0^- + d_{N-1}^+ + \sum_{j=0}^{N-2} d_j^+. \quad (6.2.14)$$

Moreover, by construction, the arc-length coordinate s_0 of the point $\mathbf{X}_0$ is given by

$$s_0 = -C + d_0^-. \quad (6.2.15)$$

The arc-length coordinate s_i of the point $\mathbf{X}_i$, for $i = 1, 2, \dots, N-1$, is then given by

$$s_i = -C + d_0^- + \sum_{j=0}^{i-1} d_j^+. \quad (6.2.16)$$

The unit normal and tangent vectors, at the point $\mathbf{X}_i$, are given by

$$\hat{\mathbf{n}}_i = \begin{pmatrix} \cos \vartheta_i \\ \sin \vartheta_i \end{pmatrix}, \quad \hat{\mathbf{t}}_i = \begin{pmatrix} -\sin \vartheta_i \\ \cos \vartheta_i \end{pmatrix}, \quad (6.2.17)$$

with the sine and cosine of the surface angle ϑ_i approximated as in (3.3.19) and (3.3.20), such that

$$\sin \vartheta_i = \frac{-\rho'_i \cos \theta_i + \rho_i \sin \theta_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{1}{2}}}, \quad (6.2.18)$$

$$\cos \vartheta_i = \frac{\rho'_i \sin \theta_i + \rho_i \cos \theta_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{1}{2}}}. \quad (6.2.19)$$

The expressions for the curvatures $\kappa_{m,i}$ (the curvature in the meridional plane) and κ_i (the total curvature), at the point $\mathbf{X}_i$, are unchanged from Chapter 3. Therefore, we have

$$\kappa_{m,i} = \frac{\rho_i^2 + 2(\rho'_i)^2 - \rho_i \rho''_i}{((\rho'_i)^2 + \rho_i^2)^{\frac{3}{2}}}, \quad (6.2.20)$$

$$\kappa_i = \kappa_{m,i} + \frac{\cos \vartheta_i}{(x_i + 1)}, \quad (6.2.21)$$

We also note that the expressions for the magnetic field coefficients h_0 and h_1 , at the point $\mathbf{X}_i$, are also unchanged from Chapter 3. Therefore, we have

$$h_0 = H_{t,i}, \quad (6.2.22)$$

$$h_1 = \frac{-d_i^+ H_{t,i-1}}{d_i^- (d_i^+ + d_i^-)} + \frac{(d_i^+ - d_i^-) H_{t,i}}{d_i^+ d_i^-} + \frac{d_i^- H_{t,i+1}}{d_i^+ (d_i^+ + d_i^-)}. \quad (6.2.23)$$

6.2.3 Governing Equations

The equations to be satisfied on the ferrofluid boundary are the boundary-integral equation (6.1.5) and the dynamic boundary condition (6.1.7). Additionally, we have the constraint that the volume of the ferrofluid must be prescribed.

We note the result from appendix C, which says that for a point $\mathbf{X}$ on the ferrofluid boundary, $\mathbf{B}(\mathbf{X})$ is given by

$$\mathbf{B}(\mathbf{X}) = \frac{1}{2} (1 + \tilde{m}) H_t(\mathbf{X}) \hat{\mathbf{t}} + B_n(\mathbf{X}) \hat{\mathbf{n}}. \quad (6.2.24)$$

Moreover, following the work in section 6.1.3, we write the boundary-integral equation (6.1.5) as

$$\begin{aligned} \mathbf{B}(\mathbf{X}) = & -\frac{M}{4\pi} \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1 - \tilde{m})}{4\pi} \int_{-C}^C (H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + \frac{(1 - \tilde{m})}{4\pi} (h_0 \mathbf{I}_0(\mathbf{X}) + h_1 \mathbf{I}_1(\mathbf{X})). \end{aligned} \quad (6.2.25)$$

Tangential Field Equation

For each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we take the component of the boundary-integral equation (6.2.25) in the direction of the tangent $\hat{\mathbf{t}}_i$ at $\mathbf{X}_i$.

Using (6.2.24) and (6.2.25), we have that

$$\begin{aligned} \frac{1}{2} (1 + \tilde{m}) H_t(\mathbf{X}) = & -\frac{M}{4\pi} \hat{\mathbf{t}} \cdot \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1 - \tilde{m})}{4\pi} \hat{\mathbf{t}} \cdot \int_{-C}^C (H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + \frac{(1 - \tilde{m})}{4\pi} \hat{\mathbf{t}} \cdot (h_0 \mathbf{I}_0(\mathbf{X}) + h_1 \mathbf{I}_1(\mathbf{X})), \end{aligned} \quad (6.2.26)$$

where $\mathbf{I}_0$ and $\mathbf{I}_1$ are as in (6.1.19) and (6.1.20) respectively. We also note that, as a result of (6.2.1), the function $H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s))$ goes to zero at $s = \pm C$.

We then discretise (6.2.26), using the trapezium rule for the integral, to obtain

$$\frac{(1-\tilde{m})}{4\pi} \sum_{j=0}^{N-1} (\mathbb{A}_{ij}^t + \mathbb{B}_{ij}^t - \mathbb{C}_{ij}^t - \mathbb{D}_{ij}^t - \mathbb{E}_{ij}^t + \mathbb{F}_{ij}^t) H_{t,j} = b_i^t, \quad (6.2.27)$$

where

$$\mathbb{A}_{ij}^t = \begin{cases} -2\pi \frac{(1+\tilde{m})}{(1-\tilde{m})} + \mathbf{I}_0(\mathbf{X}_i) \cdot \hat{\mathbf{t}}_i + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{I}_1(\mathbf{X}_i) \cdot \hat{\mathbf{t}}_i & : i = j \\ \frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{I}_1(\mathbf{X}_i) \cdot \hat{\mathbf{t}}_i & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.28)$$

$$\mathbb{B}_{ij}^t = \begin{cases} 0 & : i = j \\ \hat{\mathbf{t}}_i \cdot \mathbf{f}(\mathbf{X}_i, \mathbf{X}_j) d_j & : \text{otherwise} \end{cases} \quad (6.2.29)$$

$$\mathbb{C}_{ij}^t = \begin{cases} \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, s_k) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, s_k) \right) d_k & : j = i \\ \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, s_k) \right) d_k & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.30)$$

$$\mathbb{D}_{ij}^t = \begin{cases} \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, -C) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, -C) \right) \frac{d_0^-}{2} & : j = i \\ \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, -C) \right) \frac{d_0^-}{2} & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.31)$$

$$\mathbb{E}_{ij}^t = \begin{cases} \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, C) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, C) \right) \frac{d_{N-1}^+}{2} & : j = i \\ \sum_{k \neq i} \hat{\mathbf{t}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, C) \right) \frac{d_{N-1}^+}{2} & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.32)$$

$$\mathbb{F}_{ij}^t = \begin{cases} \left(\kappa_{m,i} + \frac{\cos \vartheta_i}{x_i+1} - \frac{\log(8(x_i+1)) \cos \vartheta_i}{x_i+1} \right) d_i & : i = j \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.33)$$

and

$$b_i^t = \frac{M}{4\pi} \hat{\mathbf{t}}_i \cdot \tilde{\mathbf{P}}(\mathbf{X}_i, \mathbf{0}). \quad (6.2.34)$$

In deriving these matrices, we have used the decomposition of $\mathbf{f}^*$ given in (6.1.15), and the expressions for h_0 and h_1 given by (6.2.22) and (6.2.23) respectively. Here, $\mathbb{A}_{ij}^t$ contains the direct contributions from $H_t(\mathbf{X}_i)$, $\mathbb{B}_{ij}^t$ and $\mathbb{C}_{ij}^t$ are the two terms in the integral excluding the singular point and the physical endpoints of the ferrofluid boundary at $s = \pm C$, $\mathbb{D}_{ij}^t$ is the contribution from the endpoint at $s = -C$, $\mathbb{E}_{ij}^t$ is the contribution from the endpoint at $s = C$, $\mathbb{F}_{ij}^t$ is the contribution from the singular point, and b_i^t is the forcing from the wire loop. The contribution from the singular point is obtained by taking the component of $\mathbf{L}(\mathbf{X}_i)$ in the direction of $\hat{\mathbf{t}}_i$, where $\mathbf{L}(\mathbf{X})$ is as in (6.1.21).

This discretisation of the tangential field equation sets up a system of N implicit linear equations for the $H_{t,i}$ values in terms of the ρ_i values.

Normal Field Equation

For each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we take the component of the boundary-integral equation (6.2.25) in the direction of the normal $\hat{\mathbf{n}}_i$ at $\mathbf{X}_i$. Using (6.2.24) and (6.2.25), we have that

$$\begin{aligned} B_n(\mathbf{X}) = & -\frac{M}{4\pi} \hat{\mathbf{n}} \cdot \tilde{\mathbf{P}}(\mathbf{X}, \mathbf{0}) \\ & + \frac{(1 - \tilde{m})}{4\pi} \hat{\mathbf{n}} \cdot \int_{-C}^C (H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s)) - \mathbf{f}^*(\mathbf{X}, s)) \, ds \\ & + \frac{(1 - \tilde{m})}{4\pi} \hat{\mathbf{n}} \cdot (h_0 \mathbf{I}_0(\mathbf{X}) + h_1 \mathbf{I}_1(\mathbf{X})), \end{aligned} \quad (6.2.35)$$

where $\mathbf{I}_0$ and $\mathbf{I}_1$ are as in (6.1.19) and (6.1.20) respectively. We also note that, as a result of (6.2.1), the function $H_t(\xi_0(s)) \mathbf{f}(\mathbf{X}, \xi_0(s))$ goes to zero at $s = \pm C$.

We then discretise (6.2.35), using the trapezium rule for the integral, to obtain

$$B_{n,i} = \frac{(1 - \tilde{m})}{4\pi} \sum_{j=0}^{N-1} (\mathbb{A}_{ij}^n + \mathbb{B}_{ij}^n - \mathbb{C}_{ij}^n - \mathbb{D}_{ij}^n - \mathbb{E}_{ij}^n + \mathbb{F}_{ij}^n) H_{t,j} - b_i^n, \quad (6.2.36)$$

where

$$\mathbb{A}_{ij}^n = \begin{cases} \mathbf{I}_0(\mathbf{X}_i) \cdot \hat{\mathbf{n}}_i + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{I}_1(\mathbf{X}_i) \cdot \hat{\mathbf{n}}_i & : i = j \\ \frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{I}_1(\mathbf{X}_i) \cdot \hat{\mathbf{n}}_i & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.37)$$

$$\mathbb{B}_{ij}^n = \begin{cases} 0 & : i = j \\ \hat{\mathbf{n}}_i \cdot \mathbf{f}(\mathbf{X}_i, \mathbf{X}_j) d_j & : \text{otherwise} \end{cases} \quad (6.2.38)$$

$$\mathbb{C}_{ij}^n = \begin{cases} \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, s_k) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, s_k) \right) d_k & : j = i \\ \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, s_k) \right) d_k & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.39)$$

$$\mathbb{D}_{ij}^n = \begin{cases} \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, -C) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, -C) \right) \frac{d_0^-}{2} & : j = i \\ \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, -C) \right) \frac{d_0^-}{2} & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.40)$$

$$\mathbb{E}_{ij}^n = \begin{cases} \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\mathbf{f}_0^*(\mathbf{X}_i, C) + \frac{(d_i^+ - d_i^-)}{d_i^- d_i^+} \mathbf{f}_1^*(\mathbf{X}_i, C) \right) \frac{d_{N-1}^+}{2} & : j = i \\ \sum_{k \neq i} \hat{\mathbf{n}}_i \cdot \left(\frac{\pm d_i^\mp}{d_i^\pm (d_i^+ + d_i^-)} \mathbf{f}_1^*(\mathbf{X}_i, C) \right) \frac{d_{N-1}^+}{2} & : j \equiv i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.41)$$

$$\mathbb{F}_{ij}^n = \begin{cases} \left(-\frac{\log(8(x_i+1)) \sin \vartheta_i}{x_i+1} + \frac{2(d_i^+ - d_i^-)}{d_i^- d_i^+} \right) d_i & : i = j \\ \frac{\pm 2d_i^\mp}{d_i^\pm (d_i^- + d_i^+)} d_i & : j = i \pm 1 \pmod{N} \\ 0 & : \text{otherwise} \end{cases} \quad (6.2.42)$$

and

$$b_i^n = \frac{M}{4\pi} \hat{\mathbf{n}}_i \cdot \tilde{\mathbf{P}}(\mathbf{X}_i, \mathbf{0}). \quad (6.2.43)$$

In deriving these matrices, we have used the decomposition of $\mathbf{f}^*$ given in (6.1.15), and the expressions for h_0 and h_1 given by (6.2.22) and (6.2.23) respectively. Here, $\mathbb{A}_{ij}^n$ contains the direct contributions from $H_t(\mathbf{X}_i)$, $\mathbb{B}_{ij}^n$ and $\mathbb{C}_{ij}^n$ are the two terms in the integral excluding the singular point and the physical endpoints of the ferrofluid boundary at $s = \pm C$, $\mathbb{D}_{ij}^n$ is the contribution from the endpoint at $s = -C$, $\mathbb{E}_{ij}^n$ is the contribution from the endpoint at $s = C$, $\mathbb{F}_{ij}^n$ is the contribution from the singular point, and b_i^n is the forcing from the wire loop. The contribution from the singular point is obtained by taking the component of $\mathbf{L}(\mathbf{X}_i)$ in the direction of $\hat{\mathbf{n}}_i$, where $\mathbf{L}(\mathbf{X})$ is as in (6.1.21).

This discretisation of the normal field equation sets up a system of N explicit equations for the $B_{n,i}$ values in terms of the ρ_i and $H_{t,i}$ values.

Dynamic Boundary Condition

At each point $\mathbf{X} = \mathbf{X}_i$ on the ferrofluid boundary, we apply the dynamic boundary condition (6.1.7). This yields a system of N nonlinear equations involving the ρ_i , $H_{t,i}$, and $B_{n,i}$ values. In particular, we have

$$P + \frac{1}{2}(\tilde{m} - 1)\text{Bo} \{H_{t,i}^2 + \tilde{m}^{-1}B_{n,i}^2\} = \kappa_i, \quad (6.2.44)$$

where κ_i is the total curvature of the ferrofluid toroid at the point $\mathbf{X}_i$, as in (6.2.21).

Volume Constraint

We also note the condition that the volume V_f of ferrofluid be prescribed. Moreover, since an axisymmetric ferrofluid droplet is a volume of revolution, the volume of the ferrofluid droplet can be computed using Pappus's centroid

theorem. Therefore, the volume constraint is given by

$$V_f = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\rho r \cos \theta r \, dr d\theta. \quad (6.2.45)$$

Therefore, we have that

$$V_f = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} \rho^3 \cos \theta \, d\theta. \quad (6.2.46)$$

Finally, we can discretise the volume constraint by using the trapezium rule for the integral, to obtain

$$V_f = 2\pi \left(\frac{\rho_0^3 \cos(\theta_0) \alpha}{4} + \frac{\rho_{N-1}^3 \cos(\theta_{N-1}) \alpha}{4} + \sum_{i=1}^{N-2} \frac{1}{3} \rho_i^3 \cos(\theta_i) \alpha \right). \quad (6.2.47)$$

6.2.4 Method of Solution

The method for solving the discretised problem numerically is entirely analogous to the method used for the toroidal problem, described in Chapter 3. As a reminder, we are seeking the position of a ferrofluid boundary such that the corresponding ferrofluid droplet is in equilibrium, referred to as an equilibrium boundary.

As in Chapter 3, we choose to set $\tilde{m} = 2$, such that the problem depends on just two parameters, Bo and V_f .

6.3 Results

A full analysis and discussion of the equilibrium boundaries of ferrofluid droplets is not presented as part of this work. Instead, we will present some preliminary results demonstrating the nature of the equilibrium boundaries obtained, both for droplets that surround the wire-loop and for droplets that are contained within the circle formed by the wire-loop.

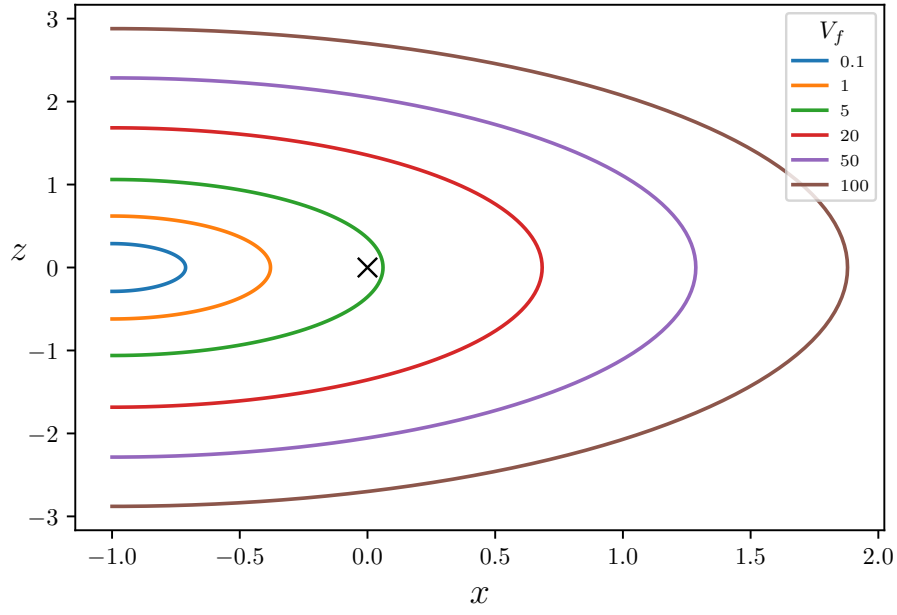


Figure 6.3.1: Equilibrium boundaries for $Bo = 0$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane.

6.3.1 The Case of $Bo = 0$

For the toroidal problem, we were not able to solve the problem numerically with $Bo = 0$, without introducing a pinned contact line along the wire. This is not the case for a ferrofluid droplet. If $Bo = 0$, then we have a droplet influenced only by surface tension, with no magnetic effects at all. In this case, it is well-known that stable equilibrium configurations of such droplets exist. Specifically, these equilibria take the form of spherical droplets, as spheres are the shape that minimises surface area for a given volume.

Therefore, if we set $Bo = 0$ and solve the problem numerically, we expect that we would obtain semi-circular equilibrium boundaries, corresponding to the cross-section of a spherical droplet. Indeed, setting $Bo = 0$ and varying the value of the fluid volume V_f , we obtain a set of semi-circular equilibrium boundaries, as expected, and shown in figure 6.3.1. For any given equilibrium boundary, the error in the boundary being a perfect semi-circle was found to be, at most, $O(10^{-9})$.

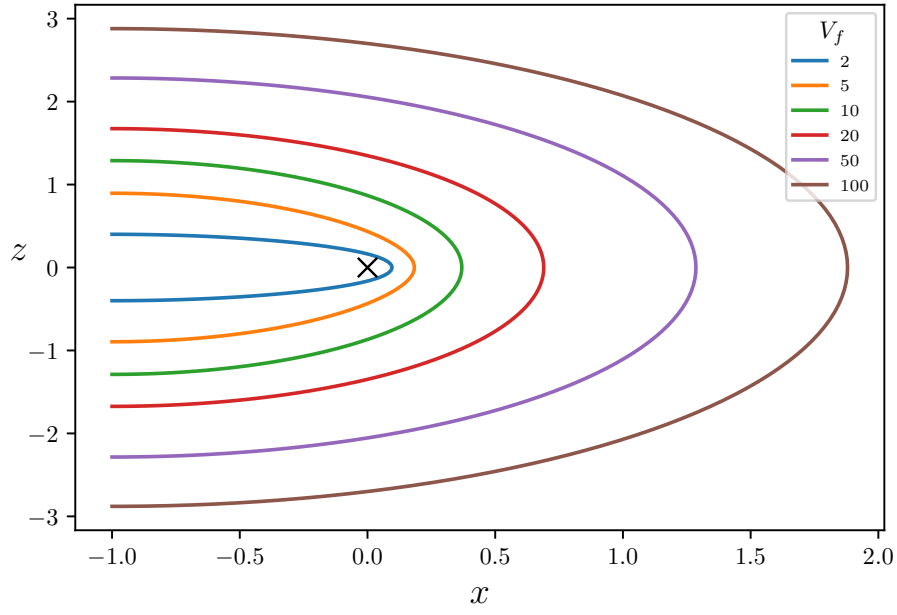


Figure 6.3.2: Equilibrium boundaries for $Bo = 5$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane. In this instance, the fluid volume is becoming increasingly large.

6.3.2 Droplets Surrounding the Wire-Loop

One of the key motivations for extending the analysis to consider ferrofluid droplets was the fact that there exists a global maximum fluid volume beyond which ferrofluid toroids no longer exist in equilibrium. This maximum volume is denoted $V_{\infty, max}$ and in Chapter 5 we found $V_{\infty, max} \approx 5$. In the case of $Bo = 0$, we observe the existence of ferrofluid droplets in equilibrium, having larger fluid volumes than the maximum volume for which ferrofluid toroids can exist in equilibrium.

We can also consider the case of ferrofluid droplets with a non-zero Bond number. For example, figure 6.3.2 shows the obtained equilibrium boundaries for $Bo = 5$, for varying values of the fluid volume V_f . Again, we observe the existence of ferrofluid droplets in equilibrium, which can have a much larger fluid volume than the maximum fluid volume for ferrofluid toroids.

Additionally, we consider briefly another case of fixed Bond number with varying fluid volume, shown in figure 6.3.3, and also the case of fixed fluid volume with

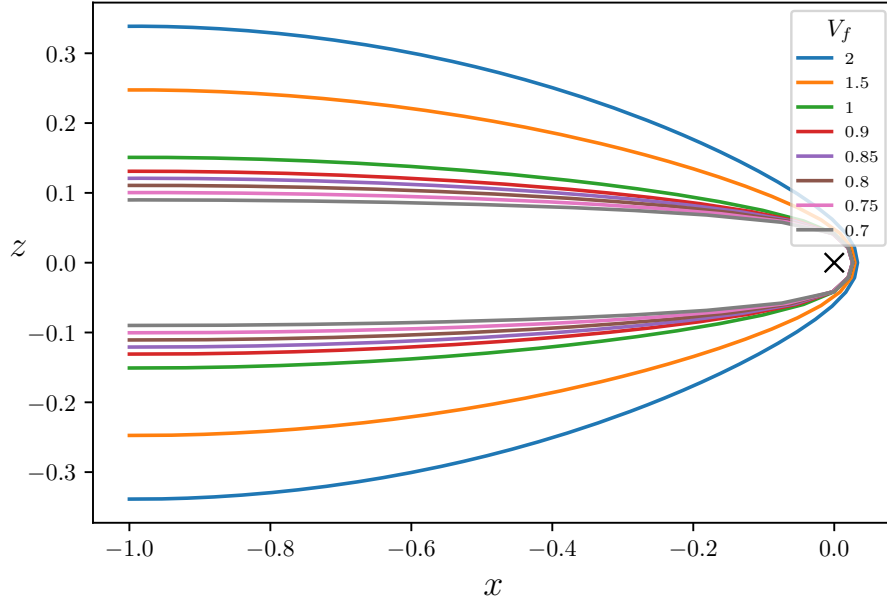


Figure 6.3.3: Equilibrium boundaries for $Bo = 5$, each with a different fluid volume V_f . The cross represents the location of the wire in the plane. In this instance, the fluid volume is being decreased.

varying Bond number, shown in figures 6.3.4 and 6.3.5. In figure 6.3.3, we see that the equilibrium boundary seems to be tending towards a long finger-like shape as V_f is decreased. However, as the fluid volume was decreased below a value of $V_f = 0.7$, the numerical scheme failed to converge.

In figure 6.3.4, we see that there are the signs of a pinching occurring close to the axis of rotation as the Bond number gets large. The numerical scheme failed to converge when Bo was increased further. This possible pinching could suggest a transition toward a toroidal equilibrium boundary in the limit of large Bond number.

Finally, in figure 6.3.5, we see that the equilibrium boundary seems to be tending towards a limiting solution as $Bo \rightarrow 0$. However, we can plot the $Bo = 0$ solution for reference, and in doing so we observe that this $Bo = 0$ solution is not the limiting solution as $Bo \rightarrow 0$. Instead, we suspect that the limiting solution could comprise two spherical caps joined along the wire-loop.

As such, we have opened up a variety of areas of possible further study for

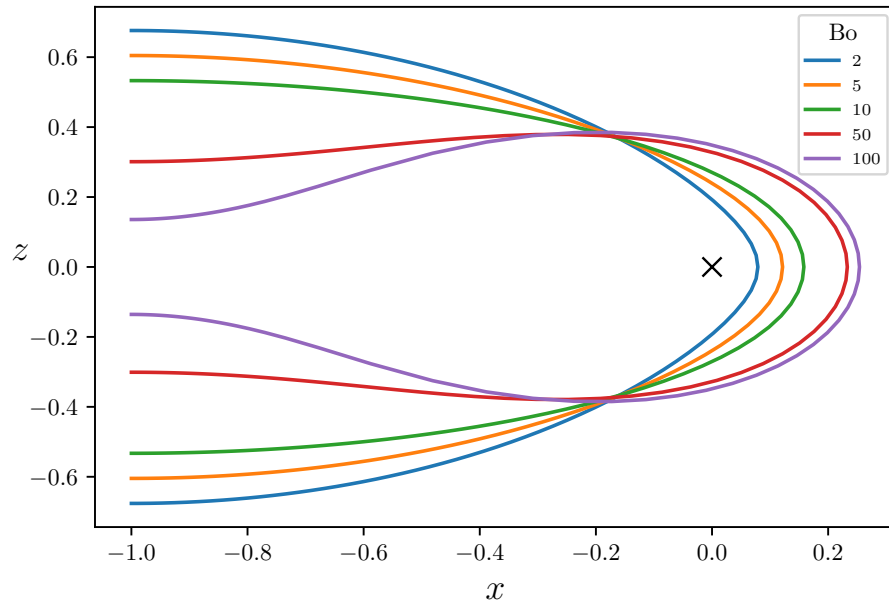


Figure 6.3.4: Equilibrium boundaries for $V_f = 3$, each with a different value of the Bond number Bo . The cross represents the location of the wire in the plane. Here the Bond number is becoming increasingly large.

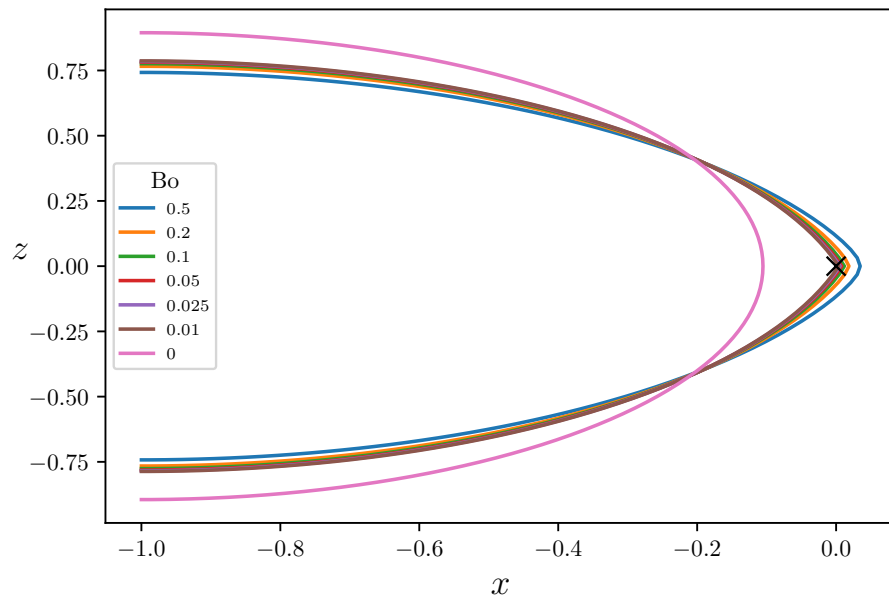


Figure 6.3.5: Equilibrium boundaries for $V_f = 3$, each with a different value of the Bond number Bo . The cross represents the location of the wire in the plane. Here, the Bond number is becoming increasingly small. The $Bo = 0$ solution is also plotted.

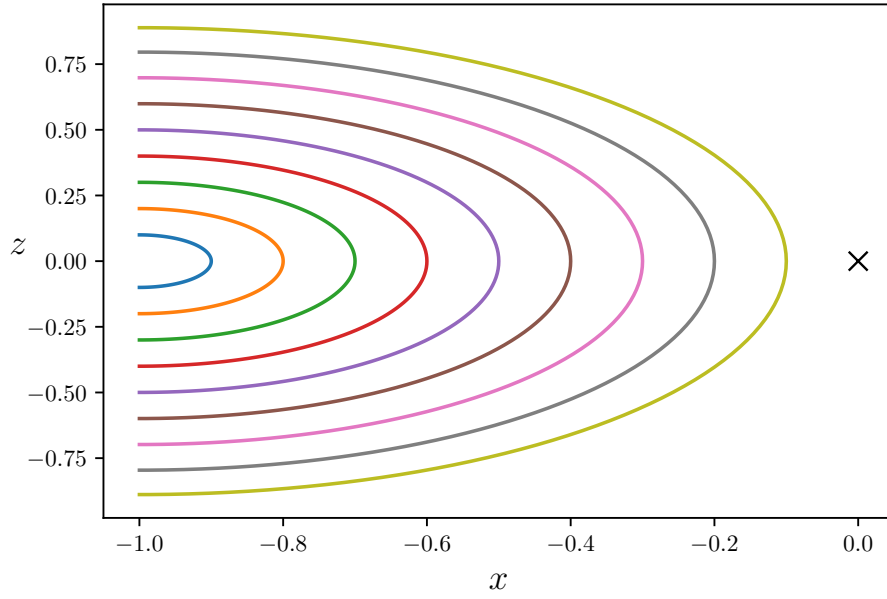


Figure 6.3.6: Equilibrium boundaries for $Bo = 0.1$ for varying values of the rightmost point x_r .

ferrofluid droplets of this type.

6.3.3 Case Where the Wire-Loop Lies Outside of the Droplet

Finally, we consider the case of droplets lying within the wire-loop, such that the circle formed by the circular loop of current-carrying wire lies entirely outside the droplet. The most interesting observation made during this analysis occurred when the volume constraint was replaced by a constraint fixing the right most point of the ferrofluid boundary, hereafter denoted as x_r . The equilibrium boundaries obtained for $Bo = 0.1$, $Bo = 5$, and $Bo = 10$ as x_r increases towards 1, are presented in figures 6.3.6, 6.3.7, and 6.3.8 respectively.

Additionally, plots of the fluid volume V_f against the value of x_r for each of the considered values of the Bond number are given in figures 6.3.9, 6.3.10, and 6.3.11 respectively. We hence see that two possible equilibrium boundaries, analogous to the Type I and Type II boundaries seen for toroidal droplets, may exist for the simply-connected droplets. This is not studied further in the present work, but is an interesting link to the toroidal problem and is worthy of further investigation.

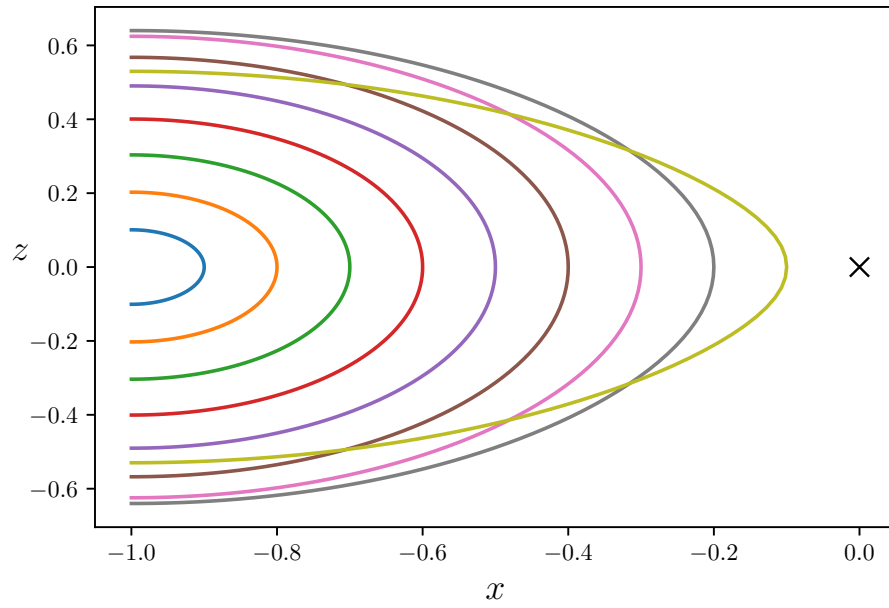


Figure 6.3.7: Equilibrium boundaries for $Bo = 5$ for varying values of the rightmost point x_r .

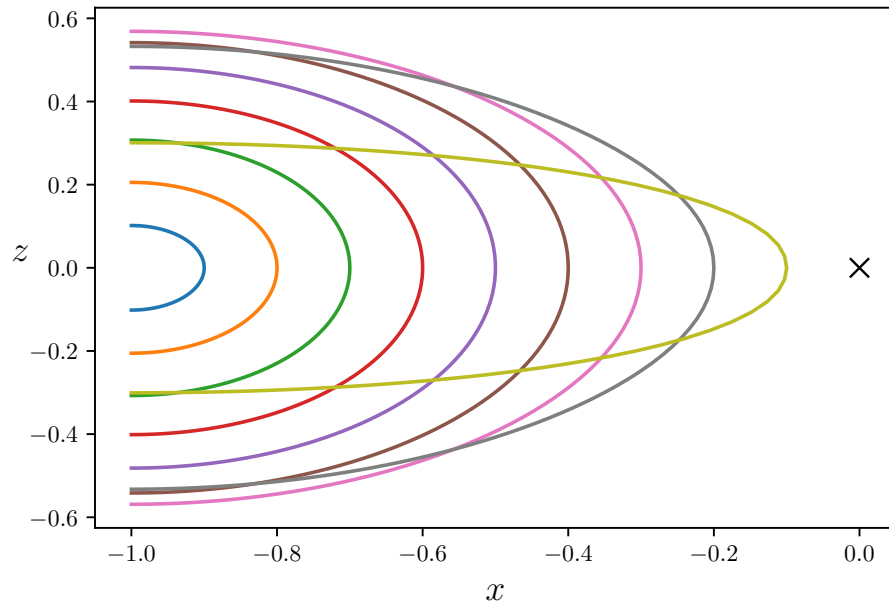


Figure 6.3.8: Equilibrium boundaries for $Bo = 10$ for varying values of the rightmost point x_r .

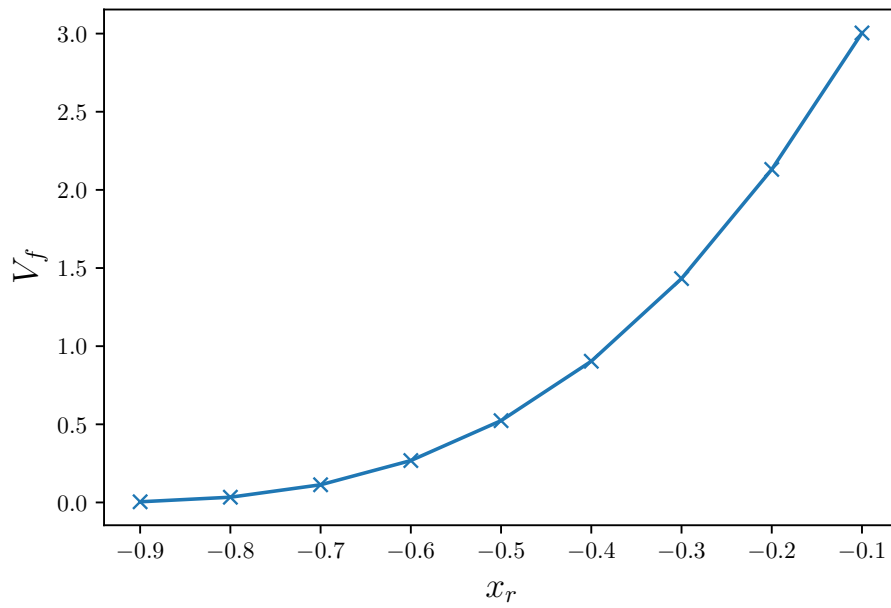


Figure 6.3.9: Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the righthmost point x_r for $Bo = 0.1$.

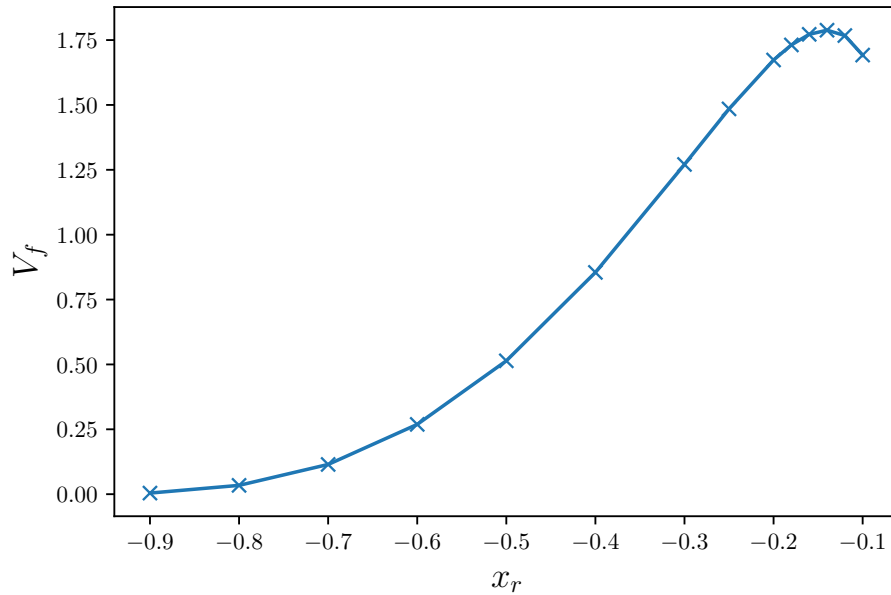


Figure 6.3.10: Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the righthmost point x_r for $Bo = 5$.

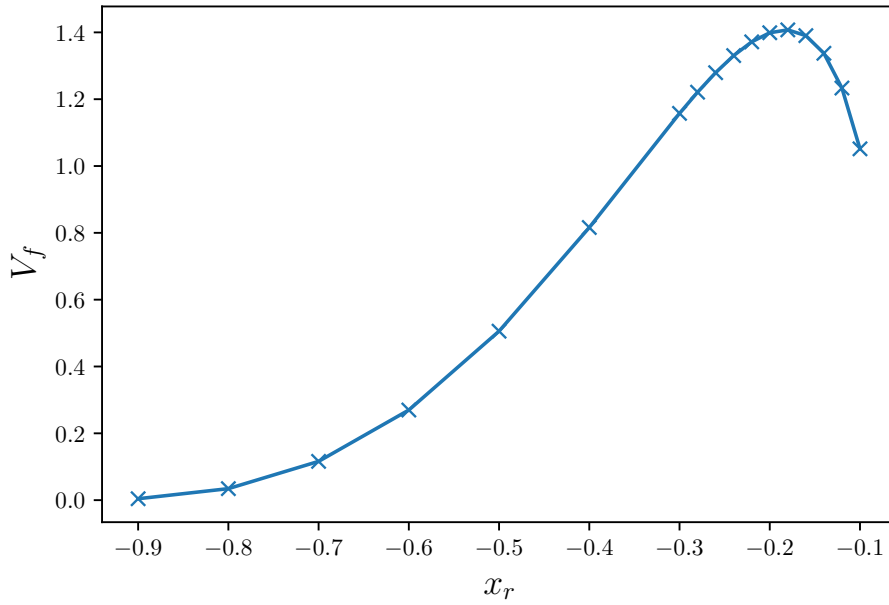


Figure 6.3.11: Plot of the fluid volume V_f of the obtained equilibrium boundary against the value of the rightmost point x_r for $Bo = 10$.

6.4 Concluding Remarks

To briefly summarise the findings of this chapter, we have shown that the mathematical model governing the equilibrium shapes of ferrofluid toroids subject to the magnetic field produced by a circular loop of current-carrying wire, can be readily extended to consider the equilibrium shapes of simply-connected ferrofluid droplets subject to the same field.

A preliminary investigation into such droplets has yielded two possible connections between these droplets and toroidal droplets. In particular, the possibility simply-connected droplets can pull apart to form a toroidal droplet when the Bond number is large, and the existence of two solutions for the equilibrium boundary, analogous to the Type I and Type II solutions observed for ferrofluid toroids, when the wire-loop lies outside of the droplet.

Conclusion

7.1 Summary

In this thesis, we successfully derived and validated a mathematical model describing the equilibrium shape of toroidal droplets of ferrofluid in the presence of an applied magnetic field generated by a circular loop of current-carrying wire. The model was discretised and solved numerically, allowing for an investigation into equilibrium configurations of static toroidal droplets of ferrofluid.

The obtained static toroidal solutions to the equilibrium equations represent the first known instance of static steady-state liquid toroids with a surface entirely open to the surrounding air. Moreover, our model allowed for the numerical analysis of the existence and shape of such toroids.

We found that, for any given value of the magnetic Bond number (ratio of magnetic effects to surface tension effects), there is a local maximum dimensionless fluid volume beyond which ferrofluid toroids cannot exist in equilibrium. This fluid volume is non-dimensionalised using lengths scaled on the radius of the circular wire-loop. Moreover, this local maximum volume increases monotonically with the Bond number. However, there exists a global maximum fluid volume beyond which no ferrofluid toroids exist in equilibrium, no matter how large the Bond number. For permissible fluid volumes, at least one equilibrium configuration was observed, with a second qualitatively

different solution also observed in certain cases.

Finally, we considered how the model that was developed for toroidal droplets could be extended to include simply-connected ferrofluid droplets. The analysis of these droplets was not considered in great detail, however, some preliminary results were presented.

7.2 Future Work

Throughout the course of this thesis, a number of areas meriting further investigation were identified. Firstly, we left open the problem of the limiting toroidal droplet as the Bond number tends to zero. Additionally, the domain of existence for Type II equilibrium boundaries in $Bo - V_f$ parameter space is not yet well understood. Further study is required to elucidate the exact range of values of the fluid volume for which toroidal droplets with a Type II equilibrium boundary exist.

Having determined the existence of toroidal droplets of ferrofluid in equilibrium, the obvious question can be asked regarding their stability. No stability analysis was conducted as part of this work. However, we note the interesting possibility that the magnetic body force may have both stabilising and destabilising effects on the toroids. The force is anticipated to be stabilising to the shrinking and Rayleigh-Plateau instabilities affecting toroidal droplets. However, for sufficiently strong magnetic fields, it may be the case that the normal field instability manifests as a result of the magnetic field having a non-zero component normal to the free surface of the toroid. The full nature of the magnetic field required for static toroidal droplets was not considered as part of this work.

In parts of the mathematical development, it was assumed that the fluid was inviscid and the viscous terms were neglected. This was not strictly necessary for this work, since all the problems considered have stationary fluid making the

viscous terms zero anyway. However, this would not be the case when considering dynamical perturbations to analyse the stability of equilibrium configurations. Introducing a time-dependent wave-like perturbation to the ferrofluid free surface would introduce some flow into the problem. For a viscous fluid, this would result in an additional term that depends on the fluid flow being introduced into the dynamic boundary condition. Solving for the fluid flow in such a scenario is not likely to be straightforward. It may still be possible to use a boundary-integral technique for the stability analysis if the Stokes flow limit of small Reynolds number is considered, removing the need to compute the fluid flow everywhere. However, the general case will likely require calculation of the full fluid flow. As such, studying the stability of a viscous ferrofluid toroid has the potential to be particularly complicated.

We also note that there is a lot of scope for further study of the droplet problem. Whilst we were able to show that our model could extend to the case of axisymmetric simply-connected droplets, only preliminary results concerning the equilibrium of such droplets were obtained.

Finally, we consider where the results of this thesis may fit into the larger picture of manipulable liquid surfaces. The equilibrium surfaces constructed herein, if stable, can be realised by precisely controlling the magnetic field. This is the case for both the toroidal and axisymmetric simply-connected droplets. As such, the results of this thesis support the possibility that liquid droplets of precise volume and shape can be engineered using a controlled magnetic field.

A

Surface Geometry and Curvatures

In this appendix, we consider formulations for the mean and in-plane curvatures of a surface of revolution, of which a toroid is an example. We work in Cartesian (x, y, z) coordinates. Without loss of generality, we suppose that the surface is axisymmetric about the z -axis, with the cross-section lying in the (x, z) plane.

If points on the plane curve describing the cross-section of the surface are described by the parameterisation $(x, z) = (a(u), b(u))$, then the standard parameterisation for the coordinates of the surface of revolution is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a(u) \cos v \\ a(u) \sin v \\ b(u) \end{pmatrix}, \quad (\text{A.0.1})$$

where v is the azimuthal angle around the z -axis.

The meridional, in-plane, curvature κ_m , and the mean curvature $\bar{\kappa}$, are given by (Gray et al., 2006)

$$\kappa_m = \frac{a'b'' - a''b'}{\left((a')^2 + (b')^2\right)^{\frac{3}{2}}}, \quad (\text{A.0.2})$$

$$\bar{\kappa} = \frac{a(a''b' - b''a') - b'\left((a')^2 + (b')^2\right)}{2|a|\left((a')^2 + (b')^2\right)^{\frac{3}{2}}}, \quad (\text{A.0.3})$$

where primes denote derivatives with respect to u .

We now consider the case where the plane curve defining the cross-section is

described using plane polar coordinates (r, θ) , centred on $x = c$, $z = 0$, where $c \geq 0$ is a constant. Now, suppose that the cross-section is specified by $r = \rho(\theta)$. Then, $u = \theta$, $a = c + \rho(\theta) \cos \theta$, $b = \rho(\theta) \sin \theta$. A point $\mathbf{X}$ on the plane curve is thus given by

$$\mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} c + \rho \cos \theta \\ \rho \sin \theta \end{pmatrix}. \quad (\text{A.0.4})$$

In such case, the meridional curvature, as given by (A.0.2) can be seen to reduce, unsurprisingly, to the standard polar coordinate representation

$$\kappa_m = \left\{ \rho^2 + 2 \left(\frac{d\rho}{d\theta} \right)^2 - 2\rho \frac{d^2\rho}{d\theta^2} \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{3}{2}}. \quad (\text{A.0.5})$$

Additionally, after some manipulations, the mean curvature, as given by (A.0.3) can be written as

$$\begin{aligned} \bar{\kappa} = & -\frac{1}{2} \left\{ \rho^2 + 2 \left(\frac{d\rho}{d\theta} \right)^2 - 2\rho \frac{d^2\rho}{d\theta^2} \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{3}{2}} \\ & - \frac{1}{2(c + \rho \cos \theta)} \left\{ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{1}{2}}. \end{aligned} \quad (\text{A.0.6})$$

We note that the first term in (A.0.6) is precisely $-\frac{1}{2}\kappa_m$. As such,

$$\bar{\kappa} = -\frac{1}{2}\kappa_m - \frac{1}{2(c + \rho \cos \theta)} \left\{ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{1}{2}}. \quad (\text{A.0.7})$$

We now consider, more specifically, the cross-sectional shape in the (x, z) plane. We begin by defining an arc-length s around the plane curve that defines the cross-section. Additionally, we denote the unit tangent to the shape by $\hat{\mathbf{t}}$ and the unit normal to the shape by $\hat{\mathbf{n}}$. We define $\hat{\mathbf{t}}$ such that it points in the direction of increasing arc-length, and $\hat{\mathbf{n}}$ such that it points outwards for a closed cross-section. A closed cross section can correspond to either the plane curve that defines the cross section being a closed curve, or both ends of the plane curve meeting the z -axis. The angle that the tangent to the curve makes with the vertical is denoted by ϑ . A diagram of this setup, in the case of an anticlockwise

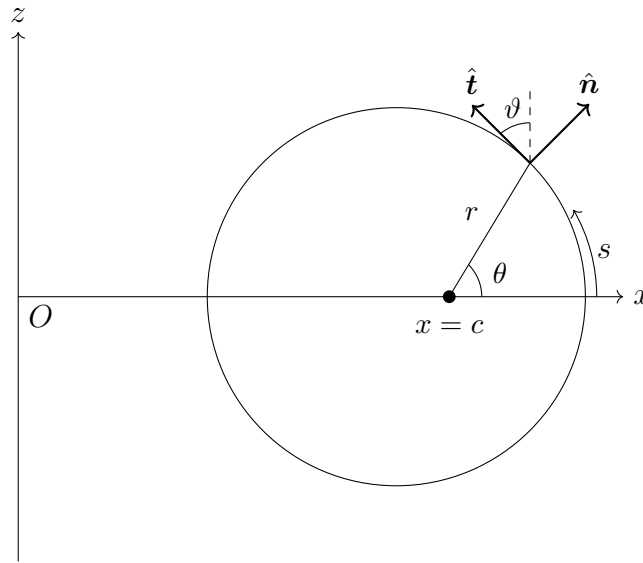


Figure A.0.1: Coordinate system for the closed cross-section of a surface of revolution, parameterised using plane polar coordinates.

choice for the direction of increasing s , is given in figure A.0.1. In this instance, the unit normal and tangent vectors are given by

$$\hat{\mathbf{n}} = \begin{pmatrix} \cos \vartheta \\ \sin \vartheta \end{pmatrix}, \quad \hat{\mathbf{t}} = \begin{pmatrix} -\sin \vartheta \\ \cos \vartheta \end{pmatrix}. \quad (\text{A.0.8})$$

As the unit tangent to any curve is given by the derivative of the position vector with respect to arc-length, we have that

$$\frac{d\mathbf{X}}{ds} = \hat{\mathbf{t}}, \quad (\text{A.0.9})$$

It follows that

$$\sin \vartheta = -\frac{dx}{ds}, \quad \cos \vartheta = \frac{dz}{ds}. \quad (\text{A.0.10})$$

Since

$$x = c + \rho \cos \theta, \quad z = \rho \sin \theta, \quad (\text{A.0.11})$$

and noting that, via the chain rule, we can write

$$\sin \vartheta = -\frac{dx}{d\theta} \frac{d\theta}{ds}, \quad \cos \vartheta = \frac{dz}{d\theta} \frac{d\theta}{ds}, \quad (\text{A.0.12})$$

we obtain

$$\sin \vartheta = \left\{ -\frac{d\rho}{d\theta} \cos \theta + \rho \sin \theta \right\} \frac{d\theta}{ds}, \quad (\text{A.0.13})$$

and

$$\cos \vartheta = \left\{ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta \right\} \frac{d\theta}{ds}. \quad (\text{A.0.14})$$

Moreover, it can be shown from (A.0.13) and (A.0.14), that

$$\frac{d\theta}{ds} = \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{1}{2}}. \quad (\text{A.0.15})$$

Therefore,

$$\cos \vartheta = \left\{ \frac{d\rho}{d\theta} \sin \theta + \rho \cos \theta \right\} \left(\left(\frac{d\rho}{d\theta} \right)^2 + \rho^2 \right)^{-\frac{1}{2}}. \quad (\text{A.0.16})$$

Comparison of (A.0.16) with (A.0.7), yields the fact that

$$\bar{\kappa} = -\frac{1}{2}\kappa_m - \frac{\cos \vartheta}{2(c + \rho \cos \theta)}, \quad (\text{A.0.17})$$

or, more simply

$$\bar{\kappa} = -\frac{1}{2}\kappa_m - \frac{\cos \vartheta}{2x}. \quad (\text{A.0.18})$$

We now note that the total curvature κ of the surface, is given by

$$\kappa = \pm 2\bar{\kappa}, \quad (\text{A.0.19})$$

where the choice of sign is such that the total curvature is positive when the surface curves away from the unit normal. As such, the negative sign is taken, such that

$$\kappa = \kappa_m + \frac{\cos \vartheta}{x}. \quad (\text{A.0.20})$$

We now note that, by definition,

$$\kappa_m = \frac{d\vartheta}{ds}, \quad (\text{A.0.21})$$

since ϑ is the angle that the unit tangent to the curve makes with a fixed direction (in this case the vertical). Therefore, the total curvature is also given by

$$\kappa = \frac{d\vartheta}{ds} + \frac{\cos \vartheta}{x}. \quad (\text{A.0.22})$$

We now make three final observations. Firstly, if we consider the arc-length s to be going clockwise around the curve, then an analogous argument to that presented for the anticlockwise case yields

$$\kappa = \frac{d\vartheta}{ds} - \frac{\cos \vartheta}{x}, \quad (\text{A.0.23})$$

where we note that ϑ here is not the same as ϑ in the anticlockwise case (in fact, $\vartheta_{\text{clockwise}} = \pi - \vartheta_{\text{anticlockwise}}$.)

Secondly, we have that

$$\frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{t}}}{d\vartheta} \frac{d\vartheta}{ds}. \quad (\text{A.0.24})$$

By computing the derivative of $\hat{\mathbf{t}}$ with respect to ϑ , and noting (A.0.21), we find that

$$\frac{d\hat{\mathbf{t}}}{d\vartheta} = -\kappa_m \hat{\mathbf{n}}, \quad (\text{A.0.25})$$

a result analogous to one of the Frenet-Serret formulae.

Finally, we note that if the axis of rotation is not the z -axis, but instead a different line parallel to the z -axis in the (x, z) plane, then a simple translation will yield the appropriate expression for the total curvature.

B

The Magnetic Stress Tensor

In this appendix, we present an abridged version of Cowley and Rosensweig's derivation of the magnetic stress tensor (Cowley and Rosensweig, 1967), which aims to capture the key features of their argument.

Consider an isolated system comprising a layer of ferrofluid bound between two rigid parallel flat plates separated by a distance a , in the presence of a magnetic field. The magnetic field and magnetic induction are given by $\mathbf{H}$ and $\mathbf{B}$ respectively and the magnetisation of the ferrofluid is given by $\mathbf{M}$.

First, suppose that the ferrofluid is magnetised whilst the bounding plates are held fixed to ensure no deformation of the fluid. Then, assuming the process is reversible, the change in Helmholtz free energy during magnetisation is given by

$$dF = -SdT + \mathbf{H} \cdot d\mathbf{B}, \quad (\text{B.0.1})$$

where S is the entropy of the ferrofluid, dT is the change in absolute temperature, and $\mathbf{H} \cdot d\mathbf{B}$ is the work done in magnetising the ferrofluid (Landau and Lifshitz, 1984). In general, the free energy will depend on the temperature, the magnetic field, and the fluid density ρ .

If the magnetisation is assumed to occur isothermally, then (B.0.1) will simply become

$$dF = \mathbf{H} \cdot d\mathbf{B}. \quad (\text{B.0.2})$$

Moreover, if the $\mathbf{H}$ and $\mathbf{B}$ fields are collinear, the scalars $H = |\mathbf{H}|$ and $B = |\mathbf{B}|$ can be used instead of $\mathbf{H}$ and $\mathbf{B}$. Therefore, we obtain

$$dF = H dB. \quad (\text{B.0.3})$$

Supposing that during the magnetisation of the ferrofluid, B is raised from an initial value of zero to a final value of B , we have that

$$F(\rho, T, H) = F_0(\rho, T) + \int_0^B H dB, \quad (\text{B.0.4})$$

or equivalently

$$F(\rho, T, H) = F_0(\rho, T) + HB - \int_0^H B dH, \quad (\text{B.0.5})$$

where F_0 is the free energy of the ferrofluid in the absence of a magnetic field. Note that the integral in (B.0.5) is to be carried out at constant density and temperature.

Now suppose that the ferrofluid layer is allowed to deform due to an infinitesimal displacement $\boldsymbol{\xi}$ of one of the bounding plates. The displacement need not be parallel to the unit normal of the plate $\hat{\mathbf{n}}$, hence the separation of the plates increases by an amount $\delta a = \boldsymbol{\xi} \cdot \hat{\mathbf{n}}$. During deformation, both the temperature and the applied magnetic field are held constant.

If the stress tensor of the ferrofluid is $\mathbf{T}'_m$, then a force of $-\hat{\mathbf{n}} \cdot \mathbf{T}'_m$ is exerted by the fluid per unit area of the bounding plate. Moreover, the mechanical work done by the ferrofluid on the displaced plate, per unit area, is given by

$$\delta W = -\hat{\mathbf{n}} \cdot \mathbf{T}'_m \cdot \boldsymbol{\xi}. \quad (\text{B.0.6})$$

The mechanical work done during an isothermal process is given by the decrease in the free energy. On a per unit area basis, this is given by (Cowley and Rosensweig, 1967; Rosensweig, 1985)

$$\delta W = -d(Fa) = -Fda - adF. \quad (\text{B.0.7})$$

Introducing the specific volume $\nu = \rho^{-1}$, we note that conservation of mass requires $a\nu^{-1}$ to remain constant, therefore

$$\frac{da}{a} = \frac{d\nu}{\nu}. \quad (\text{B.0.8})$$

Using (B.0.5) and (B.0.8) in (B.0.7), then substituting into (B.0.6), we obtain

$$\hat{\mathbf{n}} \cdot \mathbf{T}'_m \cdot \boldsymbol{\xi} = da \left\{ \left[\frac{\partial(\nu F_0)}{\partial \nu} \right]_T + \int_0^B H \, dB - \int_0^H \nu \left(\frac{\partial B}{\partial \nu} \right)_{H,T} dH \right\} + aH \, dB, \quad (\text{B.0.9})$$

where subscripts of a given variable are used to indicate that differentiation is carried out with the variable in question held constant. Through a series of mathematical manipulations omitted for brevity, (B.0.9) can be written as

$$\hat{\mathbf{n}} \cdot \mathbf{T}'_m \cdot \boldsymbol{\xi} = \hat{\mathbf{n}} \cdot \left(\left\{ \left[\frac{\partial(\nu F_0)}{\partial \nu} \right]_T - \int_0^H \left(\frac{\partial(\nu B)}{\partial \nu} \right)_{H,T} dH \right\} \mathbf{I} \right) \cdot \boldsymbol{\xi} + \hat{\mathbf{n}} \cdot (\mathbf{B}\mathbf{H}) \cdot \boldsymbol{\xi}, \quad (\text{B.0.10})$$

where $\mathbf{I}$ is the identity tensor. From (B.0.10), it is clear that the stress tensor is given by

$$\mathbf{T}'_m = \left\{ \left[\frac{\partial(\nu F_0)}{\partial \nu} \right]_T - \int_0^H \left(\frac{\partial(\nu B)}{\partial \nu} \right)_{H,T} dH \right\} \mathbf{I} + \mathbf{B}\mathbf{H}. \quad (\text{B.0.11})$$

The first term in (B.0.11) can be rewritten using the thermodynamic identity

$$\left[\frac{\partial(\nu F)}{\partial \nu} \right]_T = -p, \quad (\text{B.0.12})$$

where p is the ordinary thermodynamic pressure of the ferrofluid. Additionally, using the relation

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad (\text{B.0.13})$$

the integrand in the second term of (B.0.11) can be rewritten as

$$\left(\frac{\partial(\nu B)}{\partial \nu} \right)_{H,T} = \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} + \mu_0 H, \quad (\text{B.0.14})$$

where $M = |\mathbf{M}|$, providing $\mathbf{M}$ is also collinear to $\mathbf{B}$ and $\mathbf{H}$. Using (B.0.12) and

(B.0.14) in (B.0.11), and carrying out a trivial integration, we obtain

$$\mathbf{T}'_m = - \left\{ p_0(\rho, T) + \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B}\mathbf{H}. \quad (\text{B.0.15})$$

Since, p_0 corresponds to the pressure of the ferrofluid in the absence of a magnetic field, the purely magnetic part of the stress tensor $\mathbf{T}_m$, can be expressed as

$$\mathbf{T}_m = - \left\{ \int_0^H \mu_0 \left(\frac{\partial(\nu M)}{\partial \nu} \right)_{H,T} dH + \frac{1}{2} \mu_0 H^2 \right\} \mathbf{I} + \mathbf{B}\mathbf{H}, \quad (\text{B.0.16})$$

which is the magnetic stress tensor as obtained by Rosensweig (1985).

C

Evaluation of the Boundary Integral Equation for a Point on the Ferrofluid Boundary

In the boundary-integral equation derived for the magnetic induction in Chapter 2, we have a surface integral of the form

$$\mathbf{I}(\mathbf{x}) = \frac{1}{4\pi} \iint_S \frac{[\mu](\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} dS_{\boldsymbol{\xi}}, \quad (\text{C.0.1})$$

where S is the interface between two materials of different magnetic permeability μ . As such, there is a discontinuity in the tangential component of $\mathbf{B}$ across this interface. Herein, we seek to understand how this discontinuity manifests in (C.0.1), and determine how $\mathbf{B}(\mathbf{x})$ computed by the boundary-integral representation depends on the limiting values of $\mathbf{B}$ either side of the interface, when $\mathbf{x}$ is a point on this interface. Moreover, the component of $\mathbf{H}$ tangential to the interface is continuous, and so $\hat{\mathbf{n}} \wedge \mathbf{H}$ is well-defined on S

We take $\mathbf{x}_0 \in S$ to be a point on the interface and let $\hat{\mathbf{n}}_0$ be the unit normal there. We then define $\hat{\mathbf{t}}_1$ and $\hat{\mathbf{t}}_2$ to be two orthogonal tangent vectors to the surface. We now consider two neighbouring points either side of the interface, denoted by $\mathbf{x}_{\pm} = \mathbf{x}_0 \pm \varepsilon \hat{\mathbf{n}}_0$, where $\varepsilon \ll 1$. Local to $\mathbf{x}_0$, we can write $\boldsymbol{\xi} = \mathbf{x}_0 + r\hat{\mathbf{r}}(\theta) + e(r, \theta)\hat{\mathbf{n}}_0$, where $\hat{\mathbf{r}}(\theta) = \hat{\mathbf{t}}_1 \cos \theta + \hat{\mathbf{t}}_2 \sin \theta$ and $e(r, \theta) = O(r^2)$ as $r \rightarrow 0$.

We also introduce $\mathbf{H}^\pm$, $\mathbf{B}^\pm$ and $\mu^\pm$, be the well-defined limiting values of $\mathbf{H}$, $\mathbf{B}$, and μ on either side of the interface. The normal and tangential components of the limiting values are then introduced by

$$\mathbf{H}^\pm = H_n^\pm \hat{\mathbf{n}}_0 + \mathbf{H}_t^\pm, \quad \mathbf{B}^\pm = B_n^\pm \hat{\mathbf{n}}_0 + \mathbf{B}_t^\pm, \quad (\text{C.0.2})$$

where $\mathbf{H}_t^\pm \cdot \hat{\mathbf{n}}_0 = \mathbf{B}_t^\pm \cdot \hat{\mathbf{n}}_0 = 0$. We now define $\mathbf{I}_\pm = \mathbf{I}(\mathbf{x}_\pm)$ and $\mathbf{I}_0 = \mathbf{I}(\mathbf{x}_0)$, and consider the quantity

$$\mathbf{I}_\pm - \mathbf{I}_0 = \frac{1}{4\pi} \iint_S [\mu] (\hat{\mathbf{n}} \wedge \mathbf{H}) \wedge \left(\frac{(\mathbf{x}_\pm - \boldsymbol{\xi})}{|\mathbf{x}_\pm - \boldsymbol{\xi}|^3} - \frac{(\mathbf{x}_0 - \boldsymbol{\xi})}{|\mathbf{x}_0 - \boldsymbol{\xi}|^3} \right) dS_\xi. \quad (\text{C.0.3})$$

Through a series of expansions in ε , omitted for brevity, the significant contribution to the integral in (C.0.3) comes from $\boldsymbol{\xi}$ in the neighbourhood of $\mathbf{x}_0$. Moreover, the surface of integration can be approximated as being the infinite tangent plane to the point $\mathbf{x}_0$.

As such, to leading order, it follows that

$$\mathbf{I}_\pm - \mathbf{I}_0 \sim \frac{1}{4\pi} \int_0^{2\pi} \int_0^\infty [\mu] (\hat{\mathbf{n}}_0 \wedge \mathbf{H}_t) \wedge \frac{(\varepsilon \hat{\mathbf{n}}_0 - r \hat{\mathbf{r}})}{(\varepsilon^2 + r^2)^{\frac{3}{2}}} r dr d\theta + o(\varepsilon^0). \quad (\text{C.0.4})$$

The $r \hat{\mathbf{r}}$ term integrates to zero by symmetry. Therefore, we are left with

$$\mathbf{I}_\pm - \mathbf{I}_0 \sim \pm \frac{1}{2} [\mu] \mathbf{H}_t \int_0^\infty \frac{\varepsilon r}{(\varepsilon^2 + r^2)^{\frac{3}{2}}} dr + o(\varepsilon^0), \quad (\text{C.0.5})$$

$$\sim \pm \frac{1}{2} [\mu] \mathbf{H}_t + o(\varepsilon^0). \quad (\text{C.0.6})$$

The change in $\mathbf{B}$ across the interface is given by the difference between $\mathbf{I}_+$ and $\mathbf{I}_-$. Hence,

$$[\mathbf{B}] = \mathbf{I}_+ - \mathbf{I}_- = [\mu] \mathbf{H}_t. \quad (\text{C.0.7})$$

Moreover, since $\mathbf{B} = \mu \mathbf{H}$, as we assume linear isotropic magnetisation, we can

also write

$$[\mathbf{B}] = [\mu \mathbf{H}_t] + [B_n] \hat{\mathbf{n}}_0 = \left(\frac{1}{2} [\mu] \mathbf{H}_t^+ + \frac{1}{2} [\mu] \mathbf{H}_t^- + \bar{\mu} [\mathbf{H}_t] \right) + [B_n] \hat{\mathbf{n}}_0. \quad (\text{C.0.8})$$

Comparison of the normal and tangential components of (C.0.7) and (C.0.8), verifies that the interface conditions

$$[B_n] = 0 \quad [\mathbf{H}_t] = 0. \quad (\text{C.0.9})$$

Finally, the value of $\mathbf{B}$ computed by the boundary integral representation when $\mathbf{x} \in S$ is given by $\mathbf{I}_0$. By summing the two signs in (C.0.6), we find that

$$\mathbf{I}_+ + \mathbf{I}_- - 2\mathbf{I}_0 = o(\varepsilon^0). \quad (\text{C.0.10})$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\mathbf{I}_0 = \frac{1}{2} (\mathbf{B}_+ + \mathbf{B}_-). \quad (\text{C.0.11})$$

Therefore, the value of the integral when $\mathbf{x} \in S$ is the arithmetic mean of the limiting values of $\mathbf{B}$ from either side. Therefore, we have that

$$\mathbf{B} = \frac{1}{2} (\mathbf{B}^+ + \mathbf{B}^-) = \bar{\mu} \mathbf{H}_t + B_n \hat{\mathbf{n}}_0. \quad (\text{C.0.12})$$

D

Analytic Evaluation of the Ring Integrals

For the field induced by the current in the circular wire-loop, and for the surface integral over the ferrofluid boundary, we require the analytic evaluation of integrals of the form

$$\tilde{P}(\mathbf{x}, \boldsymbol{\xi}_0(s)) = \int_0^{2\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}(s, \phi)) \wedge \hat{\boldsymbol{\phi}}}{|\mathbf{x} - \boldsymbol{\xi}(s, \phi)|^3} d\phi, \quad (\text{D.0.1})$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}, \quad \boldsymbol{\xi}_0 = \begin{pmatrix} \xi \\ 0 \\ \zeta \end{pmatrix}, \quad \boldsymbol{\xi} = \begin{pmatrix} -D + (\xi + D) \cos \phi \\ \pm(\xi + D) \sin \phi \\ \zeta \end{pmatrix}, \quad \hat{\boldsymbol{\phi}} = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}. \quad (\text{D.0.2})$$

Such integrals can be evaluated in terms of complete elliptic integrals. The complete elliptic integrals of the first and second kind respectively, are given by (Abramowitz and Stegun, 1965)

$$\text{Ek}(m) = \int_0^{\pi/2} (1 - m \sin \theta)^{-\frac{1}{2}} d\theta, \quad (\text{D.0.3})$$

$$\text{Ee}(m) = \int_0^{\pi/2} (1 - m \sin \theta)^{\frac{1}{2}} d\theta, \quad (\text{D.0.4})$$

where $\text{Ek}(m)$ is well-defined for $0 \leq m < 1$ and $\text{Ee}(m)$ is well-defined for

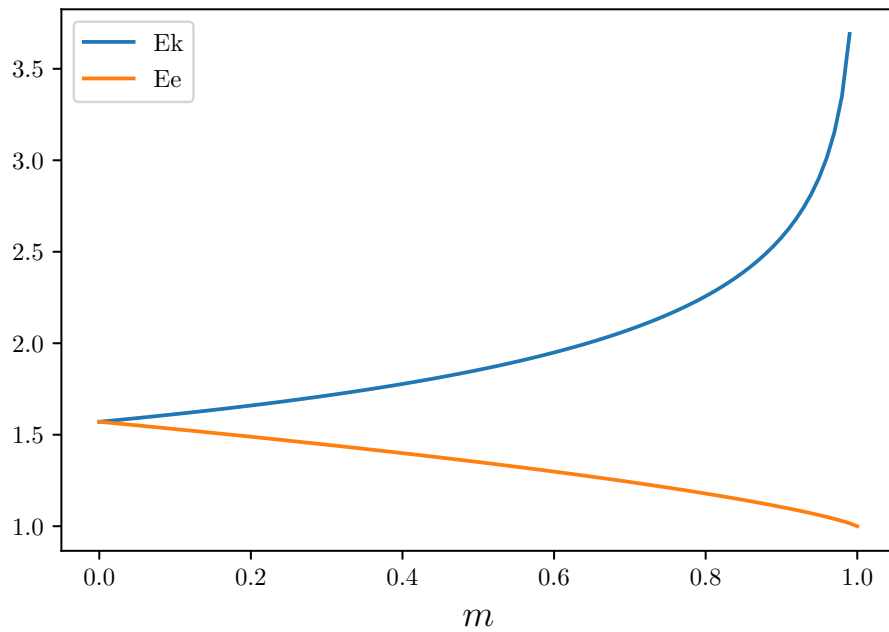


Figure D.0.1: Plots of the complete elliptic integrals of the first and second kind, denoted as E_k and E_e respectively.

$0 \leq m \leq 1$. Plots of E_k and E_e against m are given in figure D.0.1. We note that both E_k and E_e are well-defined at $m = 0$. In particular, we have that $E_k(0) = E_e(0) = \pi/2$. We also have that E_e is well defined at $m = 1$, with $E_e(1) = 1$, whilst E_k diverges to infinity as $m \rightarrow 1$.

To proceed, we note that

$$\mathbf{x} - \boldsymbol{\xi} = \begin{pmatrix} (x + D) - (\xi + D) \cos \phi \\ \mp (\xi + D) \sin \phi \\ (z - \zeta) \end{pmatrix}, \quad (\text{D.0.5})$$

and so

$$|\mathbf{x} - \boldsymbol{\xi}|^2 = \left(X + \tilde{\xi}\right)^2 - 4X\tilde{\xi} \cos^2\left(\frac{\phi}{2}\right) + (z - \zeta)^2, \quad (\text{D.0.6})$$

where $X = x + D$ and $\tilde{\xi} = \xi + D$.

It can also be shown that

$$(\mathbf{x} - \boldsymbol{\xi}) \wedge \hat{\boldsymbol{\phi}} = \begin{pmatrix} -(z - \zeta) \left(1 - 2 \sin^2 \left(\frac{\phi}{2}\right)\right) \\ \mp (z - \zeta) \sin \left(\frac{\phi}{2}\right) \cos \left(\frac{\phi}{2}\right) \\ (X - \tilde{\xi}) - 2X \sin^2 \left(\frac{\phi}{2}\right) \end{pmatrix}, \quad (\text{D.0.7})$$

or equivalently

$$(\mathbf{x} - \boldsymbol{\xi}) \wedge \hat{\boldsymbol{\phi}} = \begin{pmatrix} (z - \zeta) \left(1 - 2 \cos^2 \left(\frac{\phi}{2}\right)\right) \\ \mp (z - \zeta) \sin \left(\frac{\phi}{2}\right) \cos \left(\frac{\phi}{2}\right) \\ - (X + \tilde{\xi}) + 2X \cos^2 \left(\frac{\phi}{2}\right) \end{pmatrix}, \quad (\text{D.0.8})$$

Now, using (D.0.5)–(D.0.8), we can write

$$\tilde{\mathbf{P}}(\mathbf{x}, \boldsymbol{\xi}_0) = \int_0^{2\pi} \frac{(\mathbf{x} - \boldsymbol{\xi}(s, \phi)) \wedge \hat{\boldsymbol{\phi}}}{|\mathbf{x} - \boldsymbol{\xi}(s, \phi)|^3} d\phi \quad (\text{D.0.9})$$

$$= (z - \zeta) (I_1 - 2I_2) \hat{\mathbf{x}} + (-(x + \xi + 2D) I_1 + 2(x + D) I_2) \hat{\mathbf{z}}, \quad (\text{D.0.10})$$

where

$$I_1 = \int_0^{2\pi} \left(A - B \cos^2 \left(\frac{\phi}{2} \right) \right)^{-\frac{3}{2}} d\phi, \quad (\text{D.0.11})$$

$$I_2 = \int_0^{2\pi} \cos^2 \left(\frac{\phi}{2} \right) \left(A - B \cos^2 \left(\frac{\phi}{2} \right) \right)^{-\frac{3}{2}} d\phi, \quad (\text{D.0.12})$$

and

$$A = (x + \xi(s) + 2D)^2 + (z - \zeta(s))^2, \quad B = 4(x + D)(\xi(s) + D). \quad (\text{D.0.13})$$

Note that there is no component in the $\hat{\mathbf{y}}$ direction, since the component of the integrand in the $\hat{\mathbf{y}}$ direction is odd in ϕ and so integrates to zero.

We now seek to express the integrals I_1 and I_2 in terms of the complete elliptic integrals. Firstly, we define $m = B/A$ and perform the change of variables

$\phi = \pi - 2t$, to obtain

$$I_1 = \frac{4}{A^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} (1 - m \sin^2 t)^{-\frac{3}{2}} dt, \quad (\text{D.0.14})$$

$$I_2 = \frac{4}{A^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \sin^2 t (1 - m \sin^2 t)^{-\frac{3}{2}} dt. \quad (\text{D.0.15})$$

Now, for I_1 , it can be observed that

$$\frac{m}{1-m} \frac{d}{dt} \left(\frac{\sin t \cos t}{(1 - m \sin^2 t)^{\frac{1}{2}}} \right) \equiv \frac{(1 - m \sin^2 t)^{\frac{1}{2}}}{1-m} - \frac{1}{(1 - m \sin^2 t)^{\frac{3}{2}}}. \quad (\text{D.0.16})$$

Therefore, we have that

$$I_1 = \frac{4}{(1-m)A^{\frac{3}{2}}} \text{Ee}(m). \quad (\text{D.0.17})$$

For I_2 , we observe that

$$\frac{1}{(1 - m \sin^2 t)^{\frac{1}{2}}} \equiv \frac{1}{(1 - m \sin^2 t)^{\frac{3}{2}}} - \frac{m \sin^2 t}{(1 - m \sin^2 t)^{\frac{3}{2}}}. \quad (\text{D.0.18})$$

As such,

$$I_1 - mI_2 = \frac{4}{A^{\frac{3}{2}}} \text{Ek}(m). \quad (\text{D.0.19})$$

Therefore, we have

$$I_2 = \frac{4}{mA^{\frac{3}{2}}} \left(\frac{\text{Ee}(m)}{(1-m)} - \text{Ek}(m) \right). \quad (\text{D.0.20})$$

We conclude this appendix by briefly considering the limiting behaviour of I_1 and I_2 as $m \rightarrow 0$ and $m \rightarrow 1$, whilst A remains fixed.

Plots of I_1 and I_2 against m , taking $A = 1$, are given in figure D.0.2. We first note that both I_1 and I_2 diverge to infinity as $m \rightarrow 1$. However, both I_1 and I_2 appear to have finite limits as $m \rightarrow 0$. In the case of I_1 , as defined in (D.0.17), everything is well-defined at $m = 0$. Therefore, using the fact that $\text{Ee}(0) = \pi/2$, it is easy to see that $I_2 = 2\pi/A^{\frac{3}{2}}$ when $m = 0$.

Determining the limit of I_2 as $m \rightarrow 0$ is slightly more complicated. Looking at

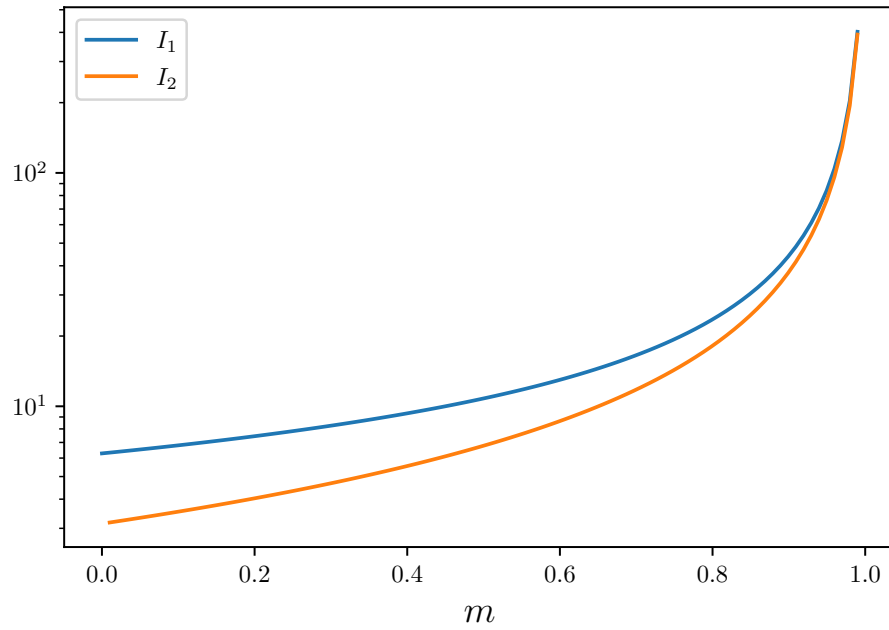


Figure D.0.2: Semi-log plots of the functions I_1 and I_2 against m , taking $A = 1$.

the definition of I_2 given by (D.0.20), we see that we cannot simply set $m = 0$. Instead, we consider series expansions of E_k and E_e about $m = 0$. Computing these series expansions in Maple, we find that

$$E_k(m) = \frac{\pi}{2} + \frac{\pi m}{8} + O(m^2), \quad (\text{D.0.21})$$

$$E_e(m) = \frac{\pi}{2} - \frac{\pi m}{8} + O(m^2). \quad (\text{D.0.22})$$

Substituting (D.0.21) and (D.0.22) into (D.0.20), and using the standard geometric series expansion, it is straightforward to show that, about $m = 0$,

$$I_2 = \frac{\pi}{A^{\frac{3}{2}}} + O(m). \quad (\text{D.0.23})$$

It follows that $I_2 \rightarrow \pi/A^{\frac{3}{2}}$ as $m \rightarrow 0$.

E

Integrals Needed for the Asymptotic Analysis

When determining the asymptotic correction terms in Chapter 4, a number of non-trivial integrals need to be evaluated. These integrals are evaluated in this appendix.

The first result we seek to prove is that for any $\theta \in [0, 2\pi)$,

$$I_1 = \int_0^{2\pi} \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} d\theta' = 0. \quad (\text{E.0.1})$$

We first consider an analysis of the integrand. We note that the denominator is zero for two values of $\theta' \in [0, 2\pi)$. These values are $\theta' = \theta$ and $\theta' = 2\pi - \theta$. We consider here the case where $\theta < \pi$, noting that an analogous argument can be used to show the desired result for $\theta > \pi$. The numerator of the integrand is then zero when $\theta' = \pi + \theta$ and $\theta' = 2\pi - \theta$.

The integrand is hence zero at $\theta' = \pi + \theta$ and divergent when $\theta' = \theta$. Further analysis shows that the integrand is well-defined at $\theta' = 2\pi - \theta$ (in particular, the integrand takes a value of $-\cot \theta$ when $\theta' = 2\pi - \theta$).

Since the integrand in (E.0.1) is periodic, with period 2π , we can write

$$I_1 = \int_0^{2\pi} \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} d\theta' = \int_{\theta-\pi}^{\pi+\theta} \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} d\theta'. \quad (\text{E.0.2})$$

Moreover, using the change of variables $\theta'' = \theta' + \theta$, we obtain

$$I_1 = \int_{-\pi}^{\pi} f(\theta, \theta'') \, d\theta'', \quad (\text{E.0.3})$$

where

$$f(\theta, \theta'') = \frac{\sin \theta + \sin(\theta'' + \theta)}{\cos \theta - \cos(\theta'' + \theta)}. \quad (\text{E.0.4})$$

With a little algebra, it can be shown that $f(\theta, \theta'') + f(\theta, -\theta'') = 0$, for all θ .

Hence, $f(\theta, \theta'')$ is odd in θ''

Taking the principal value in (E.0.4) then yields $I_1 = 0$. Therefore, we have obtained the desired result

$$\int_0^{2\pi} \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} \, d\theta' = 0. \quad (\text{E.0.5})$$

The second integral of interest is

$$I_2 = \int_0^{2\pi} \log(1 - \cos(\theta' - \theta)) \, d\theta'. \quad (\text{E.0.6})$$

We first make the substitution $\varphi = \theta' - \theta$. Due to the periodicity of the integrand, we obtain

$$I_2 = \int_0^{2\pi} \log(1 - \cos \varphi) \, d\varphi. \quad (\text{E.0.7})$$

From here, straightforward trigonometric manipulations yield

$$I_2 = \int_0^{2\pi} \log\left(2 \sin^2\left(\frac{\varphi}{2}\right)\right) \, d\varphi, \quad (\text{E.0.8})$$

from which it follows that

$$I_2 = 2\pi \log 2 + 2 \int_0^{2\pi} \log\left(\sin\left(\frac{\varphi}{2}\right)\right) \, d\varphi. \quad (\text{E.0.9})$$

We then make the change of variables $t = \varphi/2$, from which it can be shown that

$$I_2 = 2\pi \log 2 + 4 \int_0^\pi \log |\sin t| \, dt, \quad (\text{E.0.10})$$

$$= 2\pi \log 2 + 2 \int_0^{2\pi} \log |\sin t| \, dt. \quad (\text{E.0.11})$$

Moreover, we note that

$$\int_0^{2\pi} \log |\sin t| \, dt = \int_0^{2\pi} \log |\cos t| \, dt. \quad (\text{E.0.12})$$

As such,

$$\begin{aligned} 2 \int_0^{2\pi} \log |\sin t| \, dt &= \int_0^{2\pi} \log |\sin t| + \log |\cos t| \, dt = \int_0^{2\pi} \log |\sin t \cos t| \, dt \\ &= \int_0^{2\pi} \log \left(\frac{1}{2} \right) \, dt + \int_0^{2\pi} \log |\sin(2t)| \, dt = -2\pi \log 2 + \int_0^{2\pi} \log |\sin t| \, dt, \end{aligned} \quad (\text{E.0.13})$$

where the change of variables $x = 2t$ is used to obtain the final equality in (E.0.13). It follows that

$$\int_0^{2\pi} \log (\sin t) \, dt = -2\pi \log 2, \quad (\text{E.0.14})$$

and so

$$I_2 = -2\pi \log 2. \quad (\text{E.0.15})$$

The final, and most significant, integral results we seek to prove here, are

$$\int_0^{2\pi} I(\cos(n\theta)) \, d\theta' = -2\pi(n-1) \cos(n\theta), \quad (\text{E.0.16})$$

and

$$\int_0^{2\pi} I(\sin(n\theta)) \, d\theta' = -2\pi(n-1) \sin(n\theta), \quad (\text{E.0.17})$$

where $n = 1, 2, \dots$, and

$$I(h) = \left[1 - \frac{1 - \cos(\theta + \theta')}{(\cos \theta - \cos \theta')^2} \right] (h(\theta) - h(\theta')) - \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} h'(\theta), \quad (\text{E.0.18})$$

for a general function h . If we consider $I(e^{in\theta})$, then using Euler's identity $e^{i\theta} = \cos \theta + i \sin \theta$, it is clear that we need to show

$$\int_0^{2\pi} I(e^{in\theta}) d\theta' = -2\pi(n-1) \cos(n\theta) - 2\pi(n-1) \sin(n\theta) i. \quad (\text{E.0.19})$$

Firstly, we note that

$$\int_0^{2\pi} (e^{in\theta} - e^{-in\theta'}) d\theta' = 2\pi e^{in\theta} = 2\pi \cos(n\theta) + 2\pi i \sin(n\theta). \quad (\text{E.0.20})$$

It remains to show that

$$\int_0^{2\pi} -\frac{1 - \cos(\theta + \theta')}{(\cos \theta - \cos \theta')^2} (e^{in\theta} - e^{in\theta'}) - in \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} e^{in\theta} d\theta' = -2\pi n e^{in\theta}. \quad (\text{E.0.21})$$

Moreover, it can be shown that (E.0.21) can be written as

$$-\frac{\partial U}{\partial \theta} = -2\pi n e^{in\theta}, \quad (\text{E.0.22})$$

where

$$U = \int_0^{2\pi} \frac{\sin \theta + \sin \theta'}{\cos \theta - \cos \theta'} (e^{in\theta} - e^{-in\theta'}) d\theta'. \quad (\text{E.0.23})$$

Therefore, we can prove (E.0.16) and (E.0.17) by showing that

$$U = -2\pi i e^{in\theta}. \quad (\text{E.0.24})$$

We begin by making the substitution $z = e^{i\theta'}$. With this substitution, we have

$$e^{in\theta'} = z^n, \quad d\theta' = \frac{1}{iz} dz, \quad \sin \theta' = \frac{z - z^{-1}}{2i}, \quad \cos \theta' = \frac{z + z^{-1}}{2}. \quad (\text{E.0.25})$$

As such, we have

$$U = \int_{|z|=1} \frac{\left(\sin \theta + \frac{z-z^{-1}}{2i}\right) (e^{in\theta} - z^n)}{iz \left(\cos \theta - \frac{z+z^{-1}}{2}\right)} dz. \quad (\text{E.0.26})$$

We can factorise the denominator of the integrand, noting Euler's identity, to obtain

$$U = \int_{|z|=1} \frac{\left(\sin \theta + \frac{z-z^{-1}}{2i}\right) (e^{in\theta} - z^n)}{\frac{-i}{2} (z - e^{i\theta}) (z - e^{-i\theta})} dz. \quad (\text{E.0.27})$$

Multiplying the top and bottom of the integrand in (E.0.27) by z , enables a factorisation of the term in the numerator involving $\sin \theta$. Once again, noting Euler's identity, we find that

$$U = \int_{|z|=1} \frac{\frac{1}{2i} (z - e^{-i\theta}) (z + e^{i\theta}) (e^{in\theta} - z^n)}{\frac{-i}{2} z (z - e^{i\theta}) (z - e^{-i\theta})} dz. \quad (\text{E.0.28})$$

Or, more simply

$$U = \int_{|z|=1} \frac{(z + e^{i\theta}) (e^{in\theta} - z^n)}{z (z - e^{i\theta})} dz. \quad (\text{E.0.29})$$

Additionally, we have that

$$(e^{in\theta} - z^n) = (e^{i\theta} - z) (e^{i(n-1)\theta} + e^{i(n-2)\theta}z + \dots + z^{n-1}), \quad (\text{E.0.30})$$

and so

$$U = \int_{|z|=1} -\frac{(z + e^{i\theta})}{z} (e^{i(n-1)\theta} + e^{i(n-2)\theta}z + \dots + z^{n-1}) dz. \quad (\text{E.0.31})$$

We note finally, that the integrand of (E.0.31) is holomorphic on and inside the unit circle, except for at a simple pole at $z = 0$. As such, the integral can be computed via Cauchy's residue theorem. The coefficient of z^{-1} in the integrand, which is the residue at $z = 0$, can easily be read off as $-e^{in\theta}$.

As such, Cauchy's residue theorem states that

$$U = 2\pi i (-e^{in\theta}) = 2\pi \sin(n\theta) - 2\pi i \cos(n\theta), \quad (\text{E.0.32})$$

as required.

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