Quasiminimality for holomorphic functions

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Abstract

In [Zil97] Zilber conjectured that the complex exponential field is quasiminimal, i.e., all definable subsets are countable or cocountable. This conjecture led to a more general conjecture that extending the complex field with any unary entire function, or even all of them simultaneously, makes it quasiminimal. We present two examples of additional analytical structure on the complex field which indeed turn it into a quasiminimal structure.

Our first example considers two elliptic curves and a correspondence between them. Then we show via the blurring method from [Kir19] that the complex field with the correspondence is quasiminimal under certain restrictions on the two elliptic curves. This result can be seen as a progress towards the Quasiminimality Conjecture for two Weierstrass functions.

For the second example we investigate the theory of a generic function as introduced by Zilber in [Zil02]. We provide a classification of types in this theory and deduce that the complex field with an entire generic function is quasiminimal. Furthermore, we show that any two such structures are isomorphic.

Additionally, we establish a general construction for a pregeometry given by a family of analytic functions on the real or the complex field. We give several characterizations of this pregeometry and conclude that whenever the family in question is countable, the corresponding pregeometry has the countable closure property. We provide an application of this construction by using these results to show quasiminimality of our first example.

Contents

To the brave people of Ukraine, who motivated me when nothing else would

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Introduction

One of the most successful and well-known accomplishments of model theory is the study of the complex field $\mathbb{C}_{\text{field}} = (\mathbb{C}; +, \cdot)$ as this structure turns out to be extremely well-behaved. Its complete theory coincides with the theory of algebraically closed fields of characteristic 0, usually abbreviated as ACF_0, and thus there is a comprehensive first-order axiomatization of $\mathbb{C}_{\text{field}}$. Moreover, the celebrated result of Tarski [Tar52] shows that ACF_0 is decidable and admits quantifier elimination. As quantifier-free formulas in one variable in the language of fields define boolean combinations of zeros of a polynomial, every definable subset of a model of ACF_0 has to be finite or cofinite, i.e., ACF_0 is strongly minimal and $\mathbb{C}_{\text{field}}$ is *minimal*.

Given that $\mathbb{C}_{\text{field}}$ appears to be an exceptionally tame structure, it is natural to ask whether some natural additional structure would keep this tameness. Remarkably, one of such questions was mentioned by Tarski himself in the same work [Tar52], where he considers introducing the exponentiation to the complex field. As he points out, the theory of $\mathbb{C}_{\exp} = (\mathbb{C}; +, \cdot, \exp)$ is undecidable. Indeed, the ring of integers is definable in $\mathbb{C}_{\exp}$ as the multiplicative stabilizer of the kernel, i.e., $\mathbb{Z} = \{a \in \mathbb{C} : \forall z \ z \in \ker \exp \to az \in \ker \exp\}$. Moreover, as $\mathbb{C}_{\exp}$ interprets first-order arithmetic, it lacks all the other properties listed before: it is not recursively axiomatizable, does not admit quantifier elimination, and, since we can define $\mathbb{Z}$, is not minimal. Nevertheless, there is still hope that model-theoretic properties of $\mathbb{C}_{\exp}$ are sufficiently nice. In [Zil97] Zilber conjectured that every subset of $\mathbb{C}$ definable in $\mathbb{C}_{\exp}$ is either countable or co-countable. By analogy with minimality, he named this property quasiminimality.

Definition 1.1. A structure $\mathcal{M}$ is *quasiminimal* if every definable (with parameters) subset of $\mathcal{M}$ is either countable or cocountable.

Note that, unlike strong minimality, quasiminimality is not a property of a first-order theory, but solely of a structure. In particular, whenever $\mathcal{M}$ is quasiminimal (but not minimal), there always exists an elementary extension of $\mathcal{M}$ which is not quasiminimal.

As a strategy for solving the Quasiminimality Conjecture, Zilber constructed a field $\mathbb{B}$, called the *pseudo-exponential field*, with a group homomorphism $\exp : \mathbb{B} \to \mathbb{B}^\times$ on it. The pseudo-exponential field can be obtained via the Hrushovski amalgamation-with-predimension method, which we use in Section 4.2 to construct a specific countable model of the theory of a generic function. The field $\mathbb{B}$ is a unique model of size continuum of an uncountably categorical theory T, all models of which are quasiminimal. As mentioned above, such a theory T cannot be first-order, more precisely it is a theory in the infinitary logic $\mathcal{L}_{\omega_1,\omega}$, extended by a quantifier Q, where $Qx\ \phi(x)$ is interpreted as "there exist uncountably many x such that $\phi(x)$". Thus, if $\mathbb{C}_{\exp} \models T$, we get $\mathbb{C}_{\exp} \cong \mathbb{B}$ and $\mathbb{C}_{\exp}$ is also quasiminimal.

We separate the axioms of T into four parts and list them below, explaining which ones have already been proven for $\mathbb{C}_{\exp}$.

1. The first part consists of the first-order axiomatization of an algebraically closed field F of characteristic zero with a surjective homomorphism $\exp : F \to F^\times$ between the additive and the multiplicative groups of F and an infinitary axiom stating that the kernel of exp is an infinite cyclic group generated by a transcendental element. Note that these axioms clearly hold in $\mathbb{C}_{\exp}$ as $\ker(\exp) = 2\pi i\mathbb{Z}$ and $2\pi i$ is transcendental.

2. The second part of axioms says that for any $z_1, \ldots, z_n \in F$ we have $\operatorname{td}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)/\mathbb{Q}) - \operatorname{ldim}_{\mathbb{Q}}(z_1, \ldots, z_n) \geqslant 0$. Note that this axiom is equivalent to $\operatorname{td}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)/\mathbb{Q}) \geqslant n$ whenever $z_1, \ldots, z_n$ are $\mathbb{Q}$-linearly independent. We call it the *Schanuel property* since in the case of $\mathbb{C}_{\exp}$ it coincides with Schanuel's Conjecture. First published in [Lan66, p. 30], Schanuel's Conjecture seems far out of reach for now, as, for example, Schanuel's Conjecture implies that π and e are algebraically independent, which is on its own a long-standing and unapproachable problem. However, there are some partial results towards the conjecture. The Lindemann–Weierstrass theorem [Wei97] proves Schanuel's Conjecture for the case $z_1, \ldots, z_n$ algebraic, and Ax's theorem roughly gives Schanuel's Conjecture for power series. We state Ax's theorem in its geometric form in Theorem 3.18.

3. The third axiom is called *Strong Exponential-Algebraic closedness* and states that algebraic subvarieties $V \subseteq F^n \times (F^\times)^n$, which have dimension n and are not degenerate in some specific sense, have a generic point in the intersection with the graph of the exponentiation. Let us now define the restrictions on the algebraic subvarieties in order to formulate Strong Exponential-Algebraic closedness properly. We denote the first n coordinates of $F^n \times (F^\times)^n$ by $x_1, \ldots, x_n$ and the second n coordinates by $y_1, \ldots, y_n$.

Definition 1.2. An irreducible algebraic subvariety $V \subseteq F^n \times (F^\times)^n$ is *additively free* if V does not lie inside an algebraic subvariety defined by an equation of the form $\sum_{j=1}^{n} m_j x_j = c$, where $m_j \in \mathbb{Z}$ (not all zero) and $c \in F$. Similarly, V is *multiplicatively free* if V does not lie inside an algebraic subvariety defined by an equation of the form $\prod_{j=1}^{n} y_j^{m_j} = c$, where $m_j \in \mathbb{Z}$ (not all zero) and $c \in F^\times$. V is *free* if it is both additively and multiplicatively free.

Note that V being free is equivalent to the projection of V onto F^n not being contained in any $\mathbb{Q}$-linear subspace of F^n and the projection of V onto $(F^\times)^n$ not being contained in any coset of an algebraic subgroup of $(F^\times)^n$. Now we view F and $F^\times$ as additive and multiplicative groups of F and therefore $\mathbb{Z}$-modules. Thus, matrices $\mathrm{Mat}_n(\mathbb{Z})$ act on $F^n \times (F^\times)^n$ and for $M \in \mathrm{Mat}_n(\mathbb{Z})$ and a subvariety $V \subseteq F^n \times (F^\times)^n$ we denote the image of V under M by $M \cdot V$. Note that $M \cdot V$ is an algebraic subvariety as well.

Definition 1.3. An irreducible algebraic subvariety $V \subseteq F^n \times (F^\times)^n$ is *rotund* if for every matrix $M \in \mathrm{Mat}_n(\mathbb{Z})$ we have $\dim(M \cdot V) \geqslant \mathrm{rk}\, M$.

Finally, we say that F is Strongly Exponentially-Algebraically closed if for any free and rotund irreducible subvariety $V \subseteq F^n \times (F^\times)^n$ of dimension n defined over a finitely generated subfield $A \subseteq F$, there exist $z_1, \ldots, z_n \in F$ such that $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \in V$ and is generic in V over A. The question of whether $\mathbb{C}_{\exp}$ is Strongly Exponentially-Algebraically closed is still open, though there is some progress in this regard, for example, assuming Schanuel's Conjecture, the case of $n = 1$ was proved in [Mar06, Theorem 1.6]. Even more progress has been made towards a weaker property called *Exponential-Algebraic closedness*, which we discuss later in this chapter.

4. The final fourth axiom considers a special pregeometry on the field F. First note that we can define formal derivatives on exponential polynomials, i.e., functions of the form $P(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$ with $P \in F[x_1, \ldots, x_n, y_1, \ldots, y_n]$. That allows us to introduce a formal Jacobian $\mathrm{Jac}\ F(z_1, \ldots, z_n)$ for a system of exponential polynomials $F(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$. We specify a closure operator ecl_F on F by letting $a_1 \in \mathrm{ecl}_F(C)$ if there are $a_2, \ldots, a_n \in F$ and a system of exponential polynomials $F(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))$ such that $F(a_1, \ldots, a_n) = 0$ but $\mathrm{Jac}\ F(a_1, \ldots, a_n) \neq 0$. By [Kir10, Theorem 1.1] ecl_F is a pregeometry. Then the fourth axiom states that ecl_F has the countable closure property, that is $\mathrm{ecl}_F(C)$ is countable whenever $C \subseteq F$

is countable. Note that the quantifier Q is required for this axiom. The countable closure property for $\mathbb{C}_{\exp}$ was proven in [Zil05a] via Ax's theorem.

As we have noted, the only two axioms not yet proven for $\mathbb{C}_{\exp}$ are Schanuel's Conjecture and Strong Exponential-Algebraic closedness. Thus, showing these two properties would immediately imply $\mathbb{C}_{\exp} \cong \mathbb{B}$ and quasiminimality of $\mathbb{C}_{\exp}$. However, as Schanuel's Conjecture is unapproachable for the time being, there was a need for a different tactic. In [BK18], Bays and Kirby generalized Zilber's strategy to a wider class of structures, which they called Γ-fields. This approach also allows to improve Zilber's result by, roughly speaking, choosing the base field of the construction to contain all possible essential counterexamples to Schanuel's Conjecture. Thus, the field constructed this way does not necessarily coincide with $\mathbb{B}$ but is still quasiminimal. Moreover, the only axiom left to prove for $\mathbb{C}_{\exp}$ is Exponential-Algebraic closedness, a weaker version of Strong Exponential-Algebraic closedness, which states that for any free and rotund irreducible subvariety $V \subseteq F^n \times (F^\times)^n$ of dimension n, there exists $z_1, \ldots, z_n \in F$ such that $(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \in V$. As quasiminimality of $\mathbb{C}_{\exp}$ has been reduced to Exponential-Algebraic closedness, it emerged as the primary focus of the research within the field, with multiple specific cases being resolved in recent years. Among others, in [BM17] the instance when the projection of V onto $\mathbb{C}^n$ has dimension n is proved, while in [Gal24] the instance when V splits as the product of a linear subspace of the additive group and an algebraic subvariety of the multiplicative group is solved.

Alternatively, since quasiminimality of a structure implies quasiminimality of its reducts, one can try proving quasiminimality of the reducts of $\mathbb{C}_{\exp}$, seeing it as progress towards Zilber's Quasiminimality Conjecture. An easy exercise shows that the reduct $\mathbb{C}_{\mathbb{Z}} = (\mathbb{C}; +, \cdot, \mathbb{Z})$ is quasiminimal since it has the same automorphisms as $\mathbb{C}_{\text{field}}$. One of the recent results of this nature is the paper [GK24], where the authors consider graphs $\Gamma_\lambda = \{(\exp(z), \exp(\lambda z)) : z \in \mathbb{C}\}$ for $\lambda \in \mathbb{C}$ of the multi-valued functions $w \mapsto w^\lambda$. Then the structure $\mathbb{C}_{\mathbb{C}-\text{powers}} = (\mathbb{C}; +, \cdot, (\Gamma_\lambda)_{\lambda \in \mathbb{C}})$ is quasiminimal. Another example of a quasiminimal reduct is introduced in [Kir19]. Take the graph of the exponential map, approximated by the group $\mathbb{Q} + 2\pi i\mathbb{Q}$, i.e., $\Gamma_{\text{AE}} = \{(x, y) \in \mathbb{C}^2 : y = e^{x+q+2\pi i r} \text{ for some } q, r \in \mathbb{Q}\}$, and let $\mathbb{C}_{\text{AE}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{AE}})$. This structure fits the definition of a Γ-field mentioned before and thus by [BK18, Corollary 11.7] its quasiminimality can be reduced to Γ-closedness, a property generalizing Exponential-Algebraic closedness, and the countable closure property of some particular pregeometry. While the later follows from the countable closure property of $\mathbb{C}_{\exp}$, Γ-closedness is proved by so-called blurring method which uses density and countability of $\mathbb{Q} + 2\pi i\mathbb{Q}$.

One can also formulate the Quasiminimality Conjecture with the exponential map replaced by another function or a function-like object. In some cases Zilber's strategy outlined above goes through, for example, if the obtained structure is a Γ-field in the sense of [BK18, Definition 3.8], we can immediately reduce its quasiminimality to a countable closure property and an analogue of Exponential-Algebraic closedness. This is the case for the exponential map of an elliptic curve, as shown in [BK18, Theorem 9.3]. The properties we obtain instead of Exponential-Algebraic closedness can be seen as a form of Existential closedness (with respect to certain embeddings) and is of interest in its own right, see [Asl24] for exposition. For functions with domain neither countable nor cocountable, such as a proper open subset of $\mathbb{C}$, the quasiminimal question is trivially negative, but the Existential closedness question could still be relevant. One such example is the j-function for which we can formulate a corresponding Existential closedness property and prove multiple results analogous to the exponential ones (e.g. [AK22], [AEK23]), despite it being defined only on the upper-half plane.

Since functions defined on a proper open subset of $\mathbb{C}$ cannot produce quasiminimal structures, let us restrict to unary and entire, i.e., holomorphic on $\mathbb{C}$, ones. It is conjectured then that for any such f the structure $\mathbb{C}_f = (\mathbb{C}; +, \cdot, f)$ is quasiminimal. Furthermore, as pointed out by Koiran and Wilkie, it even remains open whether adding all unary entire functions to the complex field would make it quasiminimal or non-quasiminimal. Note that as meromorphic functions can be represented as a ratio of two holomorphic functions, we can formulate an equivalent conjecture for all functions meromorphic on $\mathbb{C}$. Though we suspect that $\mathbb{C}_f$ is quasiminimal for any unary entire f, it remains difficult to show that for most f. Bear in mind that restricting ourselves to unary functions is crucial here as there exist holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^n$ for $n \geqslant 2$ with the image open but not dense. Such functions were introduced by Fatou and Bieberbach, some examples can be found in [RR88, Theorem 7.1]. Remarkably, they define sets which are neither countable or cocountable as explained in [Zil97, Section 1] and thus cannot produce quasiminimal structures.

In this thesis we present the only known non-polynomial example of a unary entire function f that makes $\mathbb{C}_f$ quasiminimal. To be more precise, we consider a class of entire functions axiomatized by a first-order theory. Then each function of this class induces a quasiminimal structure with all acquired structures isomorphic to each other. We also provide a more natural example of a quasiminimal structure on the complex field, arising from a correspondence between two elliptic curves. This result is obtained through the blurring method and can be seen as partial progress towards a variant of the

Quasiminimality Conjecture. Additionaly, we lay out a general construction for certain pregeometries on $\mathbb{R}$ or $\mathbb{C}$, applying this framework to the correspondence structure mentioned above.

1.1 Main results

In Chapter 2 we consider pregeometries coming from families of functions on the real or the complex field. Studied in [Wil08] and [JKS16] in connection to local definability, they arise in the quasiminimal setting as we have seen in the exponential case. We explore various characterizations of these pregeometries, keeping track of the conditions required from the functions in question. As the countable closure property of these pregeometries is of special interest in the context of quasiminimality, we also establish that for countable families of analytic functions the pregeometry given by derivations always has the countable closure property. Summarizing different characterizations and connecting to Wilkie's result [Wil08, Theorem 1.10], we obtain the following theorem. Here the structure $\mathbb{R}(PR(\mathcal{S}))$ is the expansion of the ordered field $\mathbb{R}$ by restrictions of functions in $\mathcal{S}$ to rational open boxes, while derivations and implicitly defined functions are introduced in Section 2.1 and Section 2.2 correspondingly.

Theorem 1.4. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

1. *Let $\mathcal{S}$ be a set of differentiable functions on $\mathbb{K}$ and $C_0 \subseteq \mathbb{K}$ a subfield. Then for any $B \subseteq \mathbb{K}$ the following two subsets of $\mathbb{K}$ coincide.*
   - *The constants of all the derivations on $\mathbb{K}$ which respect $\mathcal{S}$ and vanish on $B \cup C_0$.*
   - *The kernel of the universal derivation $d : \mathbb{K} \to \Omega_{\mathcal{S},C_0}(\mathbb{K}/B)$.*
2. *Let $\mathcal{S}$ be a set of continuously differentiable functions on $\mathbb{K}$ and $C_0 \subseteq \mathbb{K}$ a subfield. Then for any $B \subseteq \mathbb{K}$ the following two subsets of $\mathbb{K}$ coincide.*
   - *The coordinates of isolated zeroes of a map with component functions being polynomials over C_0 of the functions in $\mathcal{S}$ applied to elements of B.*
   - *Images of B under functions implicitly defined from $\mathcal{S}$.*
3. *Furthermore, whenever all functions in $\mathcal{S}$ are analytic and $\mathcal{S}$ is closed under differentiation, the four subsets described above coincide.*
4. *If additionally in case of $\mathbb{K} = \mathbb{C}$ we have $\mathcal{S}$ closed under Schwarz reflection, then for any function F locally definable in $\mathbb{R}(PR(\mathcal{S}))$ and any generic point $\bar{a} \in \operatorname{dom}(F)$ there is a function G implicitly defined from $\mathcal{S}$ coinciding with F on some neighbourhood of $\bar{a}$.*

In Chapter 3 we consider two elliptic curves E_1 and E_2 with the corresponding exponential maps $\exp_1$ and $\exp_2$. As mentioned above, methods developed for Zilber's Quasiminimality Conjecture typically work when dealing with the analogous conjecture for an exponential map of an elliptic curve. Thus, in [Kir19, Conjecture 8.1] Kirby proposes that the blurring method can be also used in this setting to prove the quasiminimality of a certain reduct of the structure $\mathbb{C}_{\exp_1,\exp_2}$ given by a correspondence between the two corresponding elliptic curves. This result can be seen then as progress towards quasiminimality of $\mathbb{C}_{\exp_1,\exp_2}$. Following the strategy of [Kir19] we establish the conditions under which the structure in question is a Γ-field as introduced in [BK18, Definition 3.8] and then apply [BK18, Corollary 11.7]. Thus we reduce quasiminimality to the countable closure property and a Γ-closedness condition. Rather then deducing the countable closure property from another structure holding this property as was done in [Kir19], we use this opportunity to showcase the framework developed in Chapter 2 and apply it instead. We formulate Γ-closedness for the considered structure and prove it using the techniques of [Kir19, Proposition 6.2], thus obtaining the following quasiminimality result.

Theorem 1.5. *Let E_1, E_2 be two non-isogenous elliptic curves without complex multiplication defined over a number field. Let Λ_1 and Λ_2 be the corresponding lattices on $\mathbb{C}$ and $\exp_1$, $\exp_2$ their corresponding exponential maps. Assume $\Lambda_1 + \Lambda_2$ is dense in $\mathbb{C}$ and consider the subgroup $\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$. Then $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{corr}})$ is quasiminimal.*

In Chapter 4 we take a look at the theory of generic functions T_{gf} introduced in [Zil02]. This theory is the result of implementation of Zilber's construction in a simpler context, working with a first-order theory rather than one in infinitary logic. The axioms of T_{gf} are similar to the $\mathcal{L}_{\omega_1,\omega}(Q)$-axioms of $\mathbb{B}$ but lack the countable closure property as we show that any model of the form $\mathbb{C}_g = (\mathbb{C}; +, \cdot, g)$ for an entire $g : \mathbb{C} \to \mathbb{C}$ already satisfies it. Thus, T_{gf} consists of the axiomatization of an algebraically closed field of characteristic zero with a function g on it, a generalized Schanuel property and a form of Existential closedness. The first property roughly says that the algebraic dependencies on the complex numbers do not transfer into algebraic dependencies on their images, while the second one establishes that under certain conditions systems of equations involving g have solutions. In [Zil02] Zilber has established that T_{gf} is complete, ω-stable and possesses quantifier elimination after extending the language. We take the study of the theory of a generic function further.

Suppose we have a function $g : \mathbb{C} \to \mathbb{C}$ such that $\mathbb{C}_g \models T_{\text{gf}}$. We call such a function *generic*. Then since g already satisfies some forms of the Schanuel property and Existential closedness, we should be able to prove quasiminimality of $\mathbb{C}_g$ using Zilber's method for the exponential map. Indeed, such a strategy is outlined as an example in [Zil05b]. We follow it to produce the following two results.

Theorem 1.6. *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire generic function. Then $\mathbb{C}_g$ is quasiminimal.*

Theorem 1.7. *Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be entire generic functions. Then $\mathbb{C}_{g_1} \cong \mathbb{C}_{g_2}$.*

Some examples of entire generic functions on $\mathbb{C}$ are known. In [Wil05] Wilkie suggested a construction for a class of entire generic functions, inspired by Liouville numbers; while Koiran finished the proof in [Koi03], showing that these functions indeed satisfy the theory of a generic function and, moreover, coincide with the limit theory of generic polynomials developed in [Koi05].

We end this section with a final note on the main results listed above. Though we define quasiminimality by looking at the definable subsets, there is a stronger definition of quasiminimality in the sense of automorphisms.

Definition 1.8. A structure $\mathcal{M}$ is *quasiminimal in the sense of automorphisms* if for every countable subset $A \subseteq \mathcal{M}$ (of "parameters") we have that if $S \subseteq \mathcal{M}$ is invariant under $Aut(\mathcal{M}/A)$, then S is countable or its complement is countable.

Despite the quasiminimality in the sense of automorphisms being stronger than the regular quasiminimality, most results work for this stronger version. Moreover, there is an even stronger notion of quasiminimal excellence, serving as an analogue of strong minimality. We avoid giving the full definition here and refer the reader to [Zil05c] and [BHH$^+$14] for more details. However, we note that Theorem 1.5 works for this stronger notion of quasiminimal excellence, because as a main step it uses [BK18, Corollary 11.7]. Moreover, one can show that Theorem 1.6 can also be improved by using [Hay16, Section 3.5] and proving that $\mathbb{C}_g$ is quasiminimal excellent. Nevertheless, we stick to the quasiminimality as described in Definition 1.1, as it keeps the methods rather elementary.

1.2 Overview

We conclude the introduction with a brief overview of the thesis.

Chapter 2 introduces a pregeometry given by a family of functions on $\mathbb{R}$ or $\mathbb{C}$ through two different approaches. In Section 2.1 we consider derivations and differential forms and show how they give rise to the corresponding pregeometry. We also recall the definitions of a pregeometry and the countable closure property. Then in Section 2.2 we establish another characterization of the pregeometry through implicit functions and show that they coincide under certain conditions on the family of functions. We finish the chapter by proving the countable closure property for countable families.

In Chapter 3 we consider the correspondence between two elliptic curves and prove quasiminimality of the arising structure. We start by giving necessary background on elliptic curves in Section 3.1 and explaining the required restrictions on the correspondence in Section 3.2. Section 3.3 deals with the countable closure property, defining the pregeometry in question and using results of Chapter 2. Finally, in Section 3.4 we show Γ-closedness of the structure and derive the quasiminimality result.

Chapter 4 investigates the theory of a generic function. In Section 4.1 we establish the axiomatization of the theory and introduce Liouville functions as examples of functions on $\mathbb{C}$ which are entire and generic. Section 4.2 displays the Hrushovski's amalgamation-with-predimension construction, obtaining a special model of T_{gf}. We provide a classification of types in Section 4.3, deducing Zilber's results from it. In Section 4.4 we define a pregeometry on models of T_{gf} and prove that in case of $\mathbb{C}_g$ with g entire and generic this pregeometry has the countable closure property. Then quasiminimality of $\mathbb{C}_g$ follows immediately. Moreover, as shown in Section 4.5, models of the form $\mathbb{C}_g$ are prime over some independent set and thus have to be isomorphic. The last section of the chapter (Section 4.6) provides a slightly more general construction of the theory of a generic function, stating the main the results in this setting.

2

Closure for analytic functions

The question of quasiminimality has been closely connected to the pregeometry arising from Hrushovski's predimension function and, more specifically, its countable closure property. In the classical exponential case Zilber used the countable closure property as one of the axioms for the pseudoexponential field in [Zil05a]. Notably, the countable closure property is one of the axioms proven to be satisfied by the complex exponential field $\mathbb{C}_{\exp}$. In the generalisation of this method to Γ-fields by Bays and Kirby [BK18], an analogous pregeometry is considered and quasiminimality is reduced to the countable closure property and Γ-closedness.

Besides characterizing the pregeometry via a predimension function, one can define it through derivations or implicit definitions, as shown in [Kir10]. This also holds when considering some other functions, such as the j-function in [AEK23]. However, there has not been developed a way to define the predimension for an arbitrary function. Moreover, proving that the pregeometry coming from the predimension coincides with the pregeometry coming from derivations requires a corresponding Ax-Schanuel theorem. On the other hand, given any collection of differentiable functions on $\mathbb{R}$ or $\mathbb{C}$, we can define a corresponding pregeometry arising from derivations. Similarly, given any collection of continuously differentiable functions on $\mathbb{R}$ or $\mathbb{C}$, we can use implicit functions to define a closure operator, which turns into a pregeometry whenever the functions are analytic and the collection in question is closed under differentiation. Under these conditions the two pregeometries coincide, as already shown by Wilkie in [Wil08].

The main interest in such pregeometries is their connection to local definability in the structure $\mathbb{R}(PR(\mathcal{S}))$, that is the expansion of the ordered field $\mathbb{R}$ by restrictions of functions in $\mathcal{S}$ to rational open boxes. In [Wil08, Conjecture 1.8] Wilkie conjectured that given a set $\mathcal{S}$ of holomorphic functions closed under differentiation and Schwarz reflection, if a function F is locally definable in $\mathbb{R}(PR(\mathcal{S}))$ then for any $\bar{a} \in \mathrm{dom}(F)$ there is a function G implicitly defined from $\mathcal{S}$ which coincides with F on some neighbourhood of $\bar{a}$. As shown in [Wil08, Theorem 1.10], this conjecture holds if we restrict to $\bar{a}$ being generic, i.e., independent with respect to the pregeometry mentioned above. Note that the same result for real analytic functions (without the restriction of being closed under Schwarz reflection) follows from Gabrielov's theorem (see [Gab96]). Although Wilkie's conjecture fails in general as seen in [JKLGS19, Theorems A, B, C], it can be proved in some important contexts, such as exponentiation and Weierstrass elliptic functions, see [JKS16, Theorem 1.1]. We mention the result of Wilkie in Theorem 1.4 connecting it to the pregeometries studied in this chapter.

In this chapter we consider a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a set $\mathcal{S}$ of differentiable functions on it and a subfield $C_0 \subseteq \mathbb{K}$. In Section 2.1 we define derivations on $\mathbb{K}$ that respect S and vanish on C_0. These derivations give rise to a pregeometry $\mathrm{Dcl}_{\mathcal{S},C_0}$ on $\mathbb{K}$. Following [JKS16] we also give an alternative description of $\mathrm{Dcl}_{\mathcal{S},C_0}$ in terms of the dual space of differential forms. In Section 2.2 we require $\mathcal{S}$ to consist of continuously differentiable functions in order to define a closure operator $\mathrm{icl}_{\mathcal{S},C_0}$ on $\mathbb{K}$ via Khovanskii systems. We also provide an alternative characterization through implicitly defined functions. Moreover, whenever the functions in $\mathcal{S}$ are analytic and $\mathcal{S}$ is closed under differentiation, $\mathrm{icl}_{\mathcal{S},C_0}$ is a pregeometry and we have $\mathrm{Dcl}_{\mathcal{S},C_0} = \mathrm{icl}_{\mathcal{S},C_0}$. We can summarize all of this in the following theorem from Section 1.1.

Theorem (Theorem 1.4)**.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

1. Let $\mathcal{S}$ be a set of differentiable functions on $\mathbb{K}$ and $C_0 \subseteq \mathbb{K}$ a subfield. Then for any $B \subseteq \mathbb{K}$ the following two subsets of $\mathbb{K}$ coincide.
   - The constants of the derivations on $\mathbb{K}$ which respect $\mathcal{S}$ and vanish on $B \cup C_0$.
   - The kernel of the universal derivation $d : \mathbb{K} \to \Omega_{\mathcal{S},C_0}(\mathbb{K}/B)$.
2. Let $\mathcal{S}$ be a set of continuously differentiable functions on $\mathbb{K}$ and $C_0 \subseteq \mathbb{K}$ a subfield. Then for any $B \subseteq \mathbb{K}$ the following two subsets of $\mathbb{K}$ coincide.
   - The coordinates of isolated zeroes of a map with component functions being polynomials over C_0 of the functions in $\mathcal{S}$ applied to elements of B.
   - Images of B under functions implicitly defined from $\mathcal{S}$.
3. Furthermore, whenever all functions in $\mathcal{S}$ are analytic and $\mathcal{S}$ is closed under differentiation, the four subsets described above coincide.
4. If additionally in case of $\mathbb{K} = \mathbb{C}$ we have $\mathcal{S}$ closed under Schwarz reflection, then for any function F locally definable in $\mathbb{R}(PR(\mathcal{S}))$ and any generic point $\bar{a} \in \mathrm{dom}(F)$ there is a function G implicitly defined from $\mathcal{S}$ coinciding with F on some neighbourhood of $\bar{a}$.

This approach allows to prove the countable closure property using countability of the number of isolated zeroes of a continuously differentiable map. Note that we obtain the countable closure property for $\mathrm{Dcl}_{\mathcal{S},C_0}$ whenever the functions in $\mathcal{S}$ are analytic, even if $\mathcal{S}$ is not closed under differentiation.

2.1 Derivations and differential forms

Let us begin with the characterization of the pregeometry through derivations. We introduce derivations and differential forms, which respect a set of analytic functions and vanish on a subset of constants, and explain the connection between them. Then we recall the definition of a pregeometry and show how derivations give rise to one. We also obtain a dual formulation in terms of differential forms. First recall the standard definition of a derivation on an abstract field of characteristic zero.

Definition 2.1. Let F be a field of characteristic zero and M an F-vector space. A map $D : F \to M$ is a *derivation* if for all $x, y \in F$

- $D(x + y) = Dx + Dy$,
- $D(xy) = xDy + yDx$.

The *constants* of a set $\mathcal{D}$ of derivations is the subset of F consisting of all a such that any $D \in \mathcal{D}$ vanishes on a. Note that constants always form a relatively algebraically closed subfield of F.

In the case $M = F$, we denote by $\mathrm{Der}(F/C)$ for some subset $C \subseteq F$ the set of all derivations on F that vanish on C. Note that $\mathrm{Der}(F/C)$ has a structure of a vector space over F.

Now we need to establish some conventions and the general set up. Fix $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a differentiable function on $\mathbb{K}$ we mean a function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ differentiable with respect to every coordinate at every point in some open subset U of $\mathbb{K}^n$ for some n. Fix a set $\mathcal{S}$ of differentiable functions on $\mathbb{K}$ and a subfield $C_0 \subseteq \mathbb{K}$. Note that the functions in $\mathcal{S}$ may have different U and n. For a differentiable function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ we denote by $\frac{\partial f}{\partial x_i}$ the partial derivative of f with respect to the i-th variable.

Definition 2.2. Let M be a $\mathbb{K}$-vector space. A derivation $D : \mathbb{K} \to M$ *respects* the set $\mathcal{S}$ of differentiable functions if for any function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{S}$ and any point $\bar{a} = (a_1, \ldots, a_n) \in U$ we have

$$Df(\bar{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a}) Da_i.$$

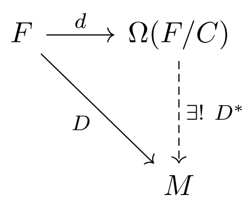
In case $M = \mathbb{K}$, we denote by $\mathrm{Der}_{\mathcal{S},C_0}(\mathbb{K}/C)$ the set of all derivations on $\mathbb{K}$ which respect $\mathcal{S}$ and vanish on $C \cup C_0$. It forms a vector subspace of $\mathrm{Der}(\mathbb{K}/C)$. Furthermore, for a subfield $F \subseteq \mathbb{K}$ we denote by $\mathrm{Der}_{\mathcal{S},C_0}(F/C)$ the set of derivations on F which vanish on $C \cup C_0$ and for any function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{S}$ and any point $\bar{a} = (a_1, \dots, a_n) \in U$ we have

$$Df(\bar{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a}) Da_i$$

in case a_i, $f(a_i)$ and $\frac{\partial f}{\partial x_i}(\bar{a})$ belongs to F for all $i = 1, \dots, n$.

Note that we can also treat C_0 as constant functions and add them to $\mathcal{S}$. For a shorter notation we are going to omit C_0 and just write $\mathrm{Der}_{\mathcal{S}}$. In some situations it is more handy to look at the dual spaces of $\mathrm{Der}(F/C)$ and $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$, that is the space of differential forms. We follow the exposition from [Kir10, Section 4] and [JKS16, Section 4].

Let F be a field of characteristic zero and $C \subseteq F$ a subset. Then there is an F-vector space $\Omega(F/C)$ generated by symbols $\{da : a \in F\}$ and quotiented by the relations $d(a + b) = da + db$, $d(ab) = adb + bda$ and $dc = 0$ for all $a, b \in F$ and $c \in C$. Elements of $\Omega(F/C)$ are called *differential forms*. Thus, $d : F \to \Omega(F/C)$ has to be a derivation vanishing on C, called the *universal derivation*. The universal property in question states that for any derivation $D : F \to M$ vanishing on C there exists a unique F-linear map $D^* : \Omega(F/C) \to M$ making the following diagram commute. Note that this makes $\mathrm{Der}(F/C)$ into the dual space of $\Omega(F/C)$. For a deeper exposition on derivatives and differential forms, see [Eis95].



Similarly we can construct a $\mathbb{K}$-vector space $\Omega_{\mathcal{S}}(\mathbb{K}/C)$ such that $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$ is dual to it by quotiening $\Omega(\mathbb{K}/C)$ with relations $da = 0$ for $a \in C_0$ and $df(\bar{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a}) da_i$ for $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{S}$ and $\bar{a} \in U$. The space $\Omega_{\mathcal{S}}(\mathbb{K}/C)$ satisfies an analogous universal property and $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$ is the dual space of $\Omega_{\mathcal{S}}(\mathbb{K}/C)$. We denote by $\Omega_{\mathcal{S}}(\mathbb{K})$ the space $\Omega_{\mathcal{S}}(\mathbb{K}/\varnothing)$.

We are now ready to define a pregeometry on $\mathbb{K}$ using the space $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$, or alternatively the space $\Omega_{\mathcal{S}}(\mathbb{K}/C)$. Let us first recall the definitions of a closure operator and a pregeometry.

Definition 2.3. Let A be a set and let $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$ be an operator on the power set of A. We say that cl is a *closure* if the following conditions are satisfied.

1. If $B \subseteq A$, then $B \subseteq \mathrm{cl}(B)$.
2. If $C \subseteq B \subseteq A$, then $\mathrm{cl}(C) \subseteq \mathrm{cl}(B)$.
3. If $B \subseteq A$, then $\mathrm{cl}(\mathrm{cl}(B)) \subseteq \mathrm{cl}(B)$.

Moreover, a closure cl is *a pregeometry* if it satisfies the following two additional conditions.

4. (finite character) If $B \subseteq A$ and $b \in \mathrm{cl}(B)$, then there is a finite $B_0 \subseteq B$ such that $b \in \mathrm{cl}(B_0)$.
5. (exchange) If $B \subseteq A$, $a, c \in A$, and $c \in \mathrm{cl}(B \cup \{a\})$, then $c \in \mathrm{cl}(B)$ or $a \in \mathrm{cl}(B \cup \{c\})$.

Given a pregeometry cl we say that a subset $B \subseteq A$ is *independent with respect to* cl if for every $b \in B$ we have $b \notin \mathrm{cl}(B \setminus \{b\})$. Then the *dimension* of a subset $B \subseteq A$ is the size of the maximal independent subset of B. For more details see [Mar02, Section 8.1]. We are particularly interested in a property of a closure called the countable closure property, introduced by Zilber in [Zil05a] as an axiom for the pseudoexponential field. It can be formulated for any operator on a powerset.

Definition 2.4. An operator $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$ has the *countable closure property* (CCP) if whenever $B \subseteq A$ is countable, the set $\mathrm{cl}(B)$ is also countable.

Note that whenever cl is a closure with finite character, CCP is equivalent to $\mathrm{cl}(\bar{b})$ being countable for any finite tuple $\bar{b} \in A$.

We are interested in the pregeometry given by the constants of $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$. By the end of the chapter we will see that it has the countable closure property whenever $\mathcal{S}$ consists of analytic functions and both $\mathcal{S}$ and C_0 are countable.

Definition 2.5. Let $\mathrm{Dcl}_{\mathcal{S}} : \mathcal{P}(\mathbb{K}) \to \mathcal{P}(\mathbb{K})$ be defined by $a \in \mathrm{Dcl}_{\mathcal{S}}(C)$ for some $C \subseteq \mathbb{K}$ if $Da = 0$ for any $D \in \mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$, i.e $\mathrm{Dcl}_{\mathcal{S}}(C)$ is exactly the constants of $\mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$.

Recall that the vector space $\Omega_{\mathcal{S}}(\mathbb{K})$ has a natural pregeometry given by the linear span. We can pull it back via $d : \mathbb{K} \to \Omega_{\mathcal{S}}(\mathbb{K})$ and obtain a pregeometry on $\mathbb{K}$. This pregeometry coincides with $\mathrm{Dcl}_{\mathcal{S}}$ and thus $\mathrm{Dcl}_{\mathcal{S}}$ is also a pregeometry. Let us explain this in more details.

Definition 2.6. Let A be a set with an operator $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$, A' a set and $\alpha : A' \to A$ a function. Then the *pullback operator* $\mathrm{cl}' : \mathcal{P}(A') \to \mathcal{P}(A')$ is defined by $\mathrm{cl}'(B') = \alpha^{-1}(\mathrm{cl}(\alpha(B')))$ for $B' \subseteq A'$.

Proposition 2.7. *Let A be a set with an operator* $\mathrm{cl} : \mathcal{P}(A) \to \mathcal{P}(A)$*, A' a set and $\alpha : A' \to A$ a function. Then if* cl *is a closure, then the pullback operator* $\mathrm{cl}' : \mathcal{P}(A') \to \mathcal{P}(A')$ *is also a closure. Moreover, if* cl *is a pregeometry, then the pullback operator* $\mathrm{cl}' : \mathcal{P}(A') \to \mathcal{P}(A')$ *is also a pregeometry.*

Proof. We start by assuming that cl is a closure and proving that cl' satisfies each of the closure properties from Definition 2.3.

1. Let $B' \subseteq A'$. Then $\alpha(B') \subseteq \mathrm{cl}(\alpha(B'))$, so $B' \subseteq \alpha^{-1}(\mathrm{cl}(\alpha(B'))) = \mathrm{cl}'(B')$.

2. Let $C' \subseteq B' \subseteq A'$. Then $\alpha(C') \subseteq \alpha(B')$ and $\mathrm{cl}(\alpha(C')) \subseteq \mathrm{cl}(\alpha(B'))$. Thus, $\mathrm{cl}'(C') = \alpha^{-1}(\mathrm{cl}(\alpha(C'))) \subseteq \alpha^{-1}(\mathrm{cl}(\alpha(B'))) = \mathrm{cl}'(B')$.

3. Let $B' \subseteq A'$. Denote $C = \mathrm{cl}(\alpha(B'))$ and $D = \mathrm{cl}(\alpha(\alpha^{-1}(C)))$. Then $\alpha(\alpha^{-1}(C)) \subseteq C$ and therefore $D = \mathrm{cl}(\alpha(\alpha^{-1}(C))) \subseteq \mathrm{cl}(C) = C$. Finally, $\mathrm{cl}'(\mathrm{cl}'(B')) = \alpha^{-1}(D) \subseteq \alpha^{-1}(C) = \mathrm{cl}'(B')$.

Now assume cl is a pregeometry. We show that cl' satisfies finite character and exchange.

For finite character let $B' \subseteq A'$ and $b' \in \mathrm{cl}'(B')$. Then $\alpha(b') \in \mathrm{cl}(\alpha(B'))$, so by finite character of cl there exist $b_1, \ldots, b_n \in \alpha(B')$ such that $\alpha(b') \in \mathrm{cl}(\{b_1, \ldots, b_n\})$. Since $b_1, \ldots, b_n \in \alpha(B')$ there are also $b'_1, \ldots, b'_n \in B'$ such that for each $i = 1, \ldots, n$ we have $\alpha(b'_i) = b_i$. Thus, $\alpha(b') \in \mathrm{cl}(\alpha(B'_0))$ for $B'_0 = \{b'_1, \ldots, b'_n\}$ and $b' \in \mathrm{cl}'(B_0)$.

For exchange let $B' \subseteq A'$, $a', c' \in A'$ and $c' \in \mathrm{cl}'(B' \cup \{a'\})$. Then $\alpha(c') \in \mathrm{cl}(\alpha(B') \cup \{\alpha(a')\})$, so by exchange for cl either $\alpha(c') \in \mathrm{cl}(\alpha(B'))$ or $\alpha(a') \in \mathrm{cl}(\alpha(B') \cup \{\alpha(c')\})$. In the first case we get $c' \in \mathrm{cl}'(B')$ and in the second case $a' \in \mathrm{cl}'(B' \cup \{c'\})$. ☐

Consider the pregeometry on $\Omega_{\mathcal{S}}(\mathbb{K})$ given by the linear span. Then its pullback via $d : \mathbb{K} \to \Omega_{\mathcal{S}}(\mathbb{K})$ maps a subset $C \subseteq \mathbb{K}$ to the set of points $a \in \mathbb{K}$ such that da belongs to the linear span generated by $\{dc : c \in C\}$. In other words, $da = 0$ in $\Omega_{\mathcal{S}}(\mathbb{K}/C)$. We finish this section by showing that this pregeometry coincides with $\mathrm{Dcl}_{\mathcal{S}}$, thus making it a pregeometry

Proposition 2.8. *Let $C \subseteq \mathbb{K}$ and $a \in \mathbb{K}$. Then $a \in \mathrm{Dcl}_{\mathcal{S}}(C)$ if and only if $da = 0$ in $\Omega_{\mathcal{S}}(\mathbb{K}/C)$. Therefore, $\mathrm{Dcl}_{\mathcal{S}}$ is a pregeometry.*

Proof. By the universal property, if $da = 0$ in $\Omega_{\mathcal{S}}(\mathbb{K}/C)$ then for any $D \in \mathrm{Der}_{\mathcal{S}}(\mathbb{K}/C)$ we have $Da = 0$ and thus $a \in \mathrm{Dcl}_{\mathcal{S}}(C)$. For the other direction assume $da \neq 0$ in $\Omega_{\mathcal{S}}(\mathbb{K}/C)$. Consider a $\mathbb{K}$-basis of $\Omega_{\mathcal{S}}(\mathbb{K}/C)$ extending da (possibly infinite). Then we can define a $\mathbb{K}$-linear map $\phi : \Omega_{\mathcal{S}}(\mathbb{K}/C) \to \mathbb{K}$ which maps da to 1 and all the other basis vectors to 0. The composition $\phi \circ d : \mathbb{K} \to \mathbb{K}$ is a derivation preserving $\mathcal{S}$ and vanishing on $C \cup C_0$, so $\phi(da) \neq 0$ gives us $a \notin \mathrm{Dcl}_{\mathcal{S}}(C)$.

As the operator on the vector space $\Omega_{\mathcal{S}}(\mathbb{K})$ given by the linear span is a pregeometry, so is $\mathrm{Dcl}_{\mathcal{S}}$ by Proposition 2.7. □

2.2 Implicit definitions

In the second part of the chapter we look at a closure defined via the Implicit Function Theorem. Generalized to numerous settings and successfully applied to various areas of mathematics such as the theory of differential equations and geometric analysis, the Implicit Function Theorem has been a powerful tool since its first rigorous formulation by Augustin-Louis Cauchy. For a detailed account of the history, theory and applications of the Implicit Function Theorem we recommend [KP13].

As the original aim of [Wil08] was the characterization of locally definable holomorphic functions, Wilkie looked at analytic functions implicitly defined from other analytic functions. One can also formulate this in terms of germs, as has been done in [JKLGS19, Definition 2.2]. Then the implicit closure of a set A is the set of images of A under functions implicitly defined from the fixed set $\mathcal{S}$ of analytic functions. On the other hand, we can omit working with implicitly definable functions and define the implicit closure directly using Khovanskii systems, as described in [Kir10, Definition 3.2] for the exponential case and in [AK22, Section 6.1] for the j-function. In the case of $\mathcal{S}$ closed under differentiation, these give equivalent definitions of the pregeometry $\mathrm{icl}_{\mathcal{S}}$, which coincides with $\mathrm{Dcl}_{\mathcal{S}}$ introduced in the previous section. Furthermore, we show that this pregeometry has countable closure property.

We start by fixing a set $\mathcal{S}$ of continuously differentiable functions on $\mathbb{K}$ and a subfield $C_0 \subseteq \mathbb{K}$. By continuously differentiable function we mean a differentiable function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ such that each partial derivative $\frac{\partial f}{\partial x_i}$ is continuous on U. Denote by $\mathcal{S}^{\mathrm{poly}}$ the closure of $\mathcal{S}$ under polynomials over C_0. More specifically, given $\mathcal{S}$ we add functions of the form $P(\bar{x}, f_1(\bar{x}), \ldots, f_k(\bar{x}))$ for every polynomial $P \in C_0[\bar{x}, t_1, \ldots, t_k]$ with coefficients in C_0 and functions $f_1, \ldots, f_k \in \mathcal{S}$ to obtain a new set denoted by $\mathcal{S}^{\mathrm{poly}}$. Note that $f_1, \ldots, f_n$ are allowed to not use all of the variables $\bar{x}$, i.e., can be defined on subsets of different dimensions. Moreover, that means $\mathcal{S}^{\mathrm{poly}}$ is closed under adding dummy variables, i.e., given a function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{S}^{\mathrm{poly}}$, every function of the form $g : U \times \mathbb{K}^m \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}$ defined by $(\bar{x}, \bar{y}) \mapsto f(\bar{x})$ also belongs to $\mathcal{S}^{\mathrm{poly}}$.

Now let us define the implicit closure $\mathrm{icl}_{\mathcal{S},C_0}$ on the field $\mathbb{K}$. We first use Khovanskii systems although we avoid defining Khovanskii systems per se and rather define the closure directly. Recall that a continuously differentiable map $F : U \subseteq \mathbb{K}^n \to \mathbb{K}^m$ is a map whose components are continuously differentiable functions and given a continuously differentiable map $F : U \subseteq \mathbb{K}^n \to \mathbb{K}^n$ we denote by $\mathrm{Jac}F(\bar{x}) \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ the Jacobian matrix of F. Correspondingly,

$\det(\mathrm{Jac}F(\bar{x}))$ is the Jacobian determinant. Note that $\det(\mathrm{Jac}F) : U \subseteq \mathbb{K}^n \to \mathbb{K}$ is a continuous function. Finally, given a continuously differentiable map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ and $\bar{a} \in \mathbb{K}^m$ we denote by $F_{\bar{a}}$ the map $\bar{x} \mapsto F(\bar{x}, \bar{a})$. The following definition is an analogue of [Kir10, Definition 3.2] and coincides with the definition of $\mathrm{ecl}_{\mathbb{K}}$ introduced in Chapter 1 in the case $\mathcal{S} = \{\exp\}$.

Definition 2.9. Let $\mathrm{icl}_{\mathcal{S},C_0} : \mathcal{P}(\mathbb{K}) \to \mathcal{P}(\mathbb{K})$ be defined by $a_1 \in \mathrm{icl}_{\mathcal{S},C_0}(B)$ for some $B \subseteq \mathbb{K}$ if there are $a_2, \dots, a_n \in \mathbb{K}$, $b_1, \dots, b_m \in B$ and a map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\mathrm{poly}}$ such that $F_{\bar{b}}(\bar{a}) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}}(\bar{a})) \neq 0$.

Note that the conditions on F, $\bar{a}$ and $\bar{b}$ imply that $\bar{a}$ is a simple and thus isolated zero of $F_{\bar{b}}$. This observation gives rise to the countable closure property for $\mathrm{icl}_{\mathcal{S},C_0}$ in case $\mathcal{S}$ and C_0 are countable.

Proposition 2.10. *Suppose $\mathcal{S}$ and C_0 are countable. Then $\mathrm{icl}_{\mathcal{S},C_0}$ has the countable closure property, i.e., for any countable subset $B \subseteq \mathbb{K}$, the set $\mathrm{icl}_{\mathcal{S},C_0}(B)$ is also countable.*

Proof. Every element of $\mathrm{icl}_{\mathcal{S},C_0}(B)$ is a coordinate of an isolated zero of a map $F_{\bar{b}}$. Since $\mathcal{S}$ and C_0 are countable, $\mathcal{S}^{\mathrm{poly}}$ is countable, so there are countably many maps $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\mathrm{poly}}$. As B is countable, there are also countably many maps of the form $F_{\bar{b}}$ with $\bar{b} \in B$. Each such map has countably many isolated zeros, so $\mathrm{icl}_{\mathcal{S},C_0}(B)$ has to be countable. □

Just as in Section 2.1 we omit C_0 from the notation and write $\mathrm{icl}_{\mathcal{S}}$. We postpone the proof of $\mathrm{icl}_{\mathcal{S}}$ being a pregeometry for later and first explain its connection to implicit functions. Let us start by recalling the Implicit Function Theorem. We need the real continuously differentiable, the real infinitely differentiable and the real analytic versions, which can be found in [KP13, Theorem 3.3.1] and [KP02, Theorem 1.8.3]. Note that in the complex case the three versions coincide and can be found in [FG02, Theorem 7.6].

Fact 2.11 (Implicit Function Theorem)**.** *Let $F = (f_1, \dots, f_m) : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ be a continuously differentiable map and $(\bar{a}, \bar{b}) \in U$ a point with $F(\bar{a}, \bar{b}) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}}(\bar{a})) \neq 0$. Then there are open neighbourhoods W of $\bar{a}$ and V of $\bar{b}$ with $W \times V \subseteq U$ and a continuously differentiable map $G : V \subseteq \mathbb{K}^m \to W \subseteq \mathbb{K}^n$ such that for any $\bar{a}' \in W$ and $\bar{b}' \in V$ we have $F(\bar{a}', \bar{b}') = 0$ if and only if $G(\bar{b}') = \bar{a}'$. Moreover, such G is unique as a function on V and if F is infinitely differentiable or analytic, then G is also infinitely differentiable or analytic correspondingly.*

The Implicit Function Theorem gives rise to the notion of an implicitly defined function. It also provides an alternative characterization of $\mathrm{icl}_{\mathcal{S}}$. This is a generalization of [Wil08, Definition 1.4], where the case of $\mathbb{K} = \mathbb{C}$ and $\mathcal{S}$ consisting of one function is considered.

Definition 2.12. A function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ is *implicitly defined from $\mathcal{S}$ at a point* $\bar{b} \in V$ if there exist an open neighbourhood $V' \subseteq V$ of $\bar{b}$, a map $G : V' \subseteq \mathbb{K}^m \to \mathbb{K}^n$ with the first component being $g \restriction_{V'}$ and a map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ defined on an open neighbourhood U of $(G(\bar{b}), \bar{b})$ with component functions in $\mathcal{S}^{\text{poly}}$ such that for any $\bar{b}' \in V'$ we have $F_{\bar{b}'}(G(\bar{b}')) = 0$ and $\det(\text{Jac}F_{\bar{b}'}(G(\bar{b}'))) \neq 0$.

In case we want to specify the maps F and G used in the definition, we say that g is implicitly defined from $\mathcal{S}$ at $\bar{b}$ *via F and G*. By Implicit Function Theorem, if g is implicitly defined from a set of infinitely differentiable or analytic functions $\mathcal{S}$, then g is itself infinitely differentiable or analytic correspondingly. Note that by permuting the component functions of F and G we see that if one component function of G is implicitly defined from $\mathcal{S}$ at a point $\bar{b}$, then each of them is. Implicitly defined functions are also closed under domain restriction, i.e., given a function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at a point $\bar{b} \in V$ and an open neighbourhood $V' \subseteq V$ of $\bar{b}$, the function $g \restriction_{V'}$ is also implicitly defined from $\mathcal{S}$ at $\bar{b}$. Furthermore, since $\mathcal{S}^{\text{poly}}$ is closed under adding dummy variables, so are the functions implicitly defined from $\mathcal{S}$, i.e., if $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ is implicitly defined from $\mathcal{S}$ at a point $\bar{b} \in V$, then a function of the form $h : V \times \mathbb{K}^n \to \mathbb{K}$ defined by $h(\bar{x}, \bar{y}) = g(\bar{x})$ is implicitly defined from $\mathcal{S}$ at $(\bar{b}, \bar{c})$ for any $\bar{c} \in \mathbb{K}^n$. We also see that given a algebraic over C_0, it is implicitly defined as a constant function from any set of functions at any point via its minimal polynomial. Some of less obvious properties of implicitly defined functions are listed in the following lemma.

Lemma 2.13. *1. Let $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ be a function in $\mathcal{S}^{\text{poly}}$ and $\bar{b} \in V$. Then g is implicitly defined from $\mathcal{S}$ at $\bar{b}$.*

2. *Let $g_1, \dots, g_k : V \subseteq \mathbb{K}^m \to \mathbb{K}$ be functions implicitly defined from $\mathcal{S}$ at a point $\bar{b} \in V$ and $h : W \subseteq \mathbb{K}^k \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $(g_1(\bar{b}), \dots, g_k(\bar{b})) \in W$. Then the function $h(g_1(\bar{y}), \dots, g_k(\bar{y})) : V \subseteq \mathbb{K}^m \to \mathbb{K}$ is also implicitly defined from $\mathcal{S}$ at $\bar{b}$.*

3. *Let $h : W \subseteq \mathbb{K}^r \to \mathbb{K}$ be a function implicitly defined from analytic functions $g_1, \dots, g_k$ at a point $\bar{c} \in W$ via a map $H : W' \subseteq \mathbb{K}^r \to \mathbb{K}^k$ and a map $G : V \subseteq \mathbb{K}^k \times \mathbb{K}^r \to \mathbb{K}^k$ with component functions $g_1, \dots, g_k$. Suppose for each $i = 1, \dots, k$ the function $g_i : V \subseteq \mathbb{K}^k \times \mathbb{K}^r \to \mathbb{K}^k$ is implicitly defined from $\mathcal{S}$ at the point $\bar{b} = (H(\bar{c}), \bar{c})$. Then h is implicitly defined from $\mathcal{S}$ at $\bar{c}$.*

Proof. 1. Let $f : \mathbb{K} \times V \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ be defined by $f(x, \bar{y}) = x - g(\bar{y})$. Note that $f \in \mathcal{S}^{\text{poly}}$. Then for any $\bar{b}' \in V$ we have $f(g(\bar{b}'), \bar{b}') = 0$ and $\frac{\partial f}{\partial x}(g(\bar{b}'), \bar{b}') = 1$. Thus, g is implicitly defined from $\mathcal{S}$ at $\bar{b}$ (via f and g).

2. For each $i = 1, \ldots, k$ let $G_i : V_i' \subseteq \mathbb{K}^m \to \mathbb{K}^{n_i}$ be a map with $V_i' \subseteq V$ an open neighbourhood of $\bar{b}$ acting by $\bar{y} \mapsto (g_{i1}(\bar{y}), \ldots, g_{in_i}(\bar{y}))$ with $g_{i1} = g_i \restriction_{V_i'}$ and $F^{(i)} : U_i \subseteq \mathbb{K}^{n_i} \times \mathbb{K}^m \to \mathbb{K}^{n_i}$ a map with U_i an open neighbourhood of $(G_i(\bar{b}), \bar{b})$ acting by $(\bar{x}_i, \bar{y}) \mapsto (f_{i1}(\bar{x}_i, \bar{y}), \ldots, f_{in_i}(\bar{x}_i, \bar{y}))$ with $f_{i1}, \ldots, f_{in_i} \in \mathcal{S}^{\text{poly}}$ such that for any $\bar{b}' \in V_i'$ we have $F^{(i)}_{\bar{b}'}(G_i(\bar{b}')) = 0$ and $\det(\text{Jac}F^{(i)}_{\bar{b}'}(G_i(\bar{b}'))) \neq 0$.

   Similarly, let $H : W' \subseteq \mathbb{K}^k \to \mathbb{K}^r$ be a map with $W' \subseteq W$ an open neighbourhood of $\bar{c} = (g_1(\bar{b}), \ldots, g_k(\bar{b}))$ and first component function $h \restriction_{W'}$, and $L : O \subseteq \mathbb{K}^r \times \mathbb{K}^k \to \mathbb{K}^r$ with O an open neighbourhood of $(H(\bar{c}), \bar{c})$ and component functions in $\mathcal{S}^{\text{poly}}$ such that for any $\bar{c}' \in W'$ we have $L_{\bar{c}'}(H(\bar{c}')) = 0$ and $\det(\text{Jac}L_{\bar{c}'}(H(\bar{c}'))) \neq 0$.

   Let $V' \subseteq \bigcap_{i=1}^{k} V_i' \cap V$ be a neighbourhood of $\bar{b}$ such that for any $\bar{b}' \in V'$ we have $(g_1(\bar{b}'), \ldots, g_k(\bar{b}')) \in W'$. We define a map

$$P : V' \subseteq \mathbb{K}^m \to \mathbb{K}^r \times \prod_{i=1}^{k} \mathbb{K}^{n_i}$$

   by

$$P(\bar{y}) = (H(g_1(\bar{y}), \ldots, g_k(\bar{y})), G_1(\bar{y}), \ldots, G_k(\bar{y})).$$

   By narrowing V' if needed, let $E \subseteq \mathbb{K}^r \times \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^m$ be an open neighbourhood of $(P(\bar{b}), \bar{b})$ such that for any $\bar{b}' \in V'$ we have $(P(\bar{b}'), \bar{b}') \in E$. Then we define a map

$$Q : E \subseteq \mathbb{K}^r \times \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^m \to \mathbb{K}^r \times \prod_{i=1}^{k} \mathbb{K}^{n_i}$$

   by

$$Q(\bar{t}, \bar{x}_1, \ldots, \bar{x}_k, \bar{y}) = (L(\bar{t}, x_{11}, \ldots, x_{k1}), F^{(1)}(\bar{x}_1, \bar{y}), \ldots, F^{(k)}(\bar{x}_k, \bar{y})),$$

   where $\bar{x}_i = (x_{i1}, \ldots, x_{in_i})$ for $i = 1, \ldots, k$. Note that the component functions of Q belong to $\mathcal{S}^{\text{poly}}$ and the first component function of P is $h(g_1(\bar{y}), \ldots, g_k(\bar{y})) \restriction_{V'}$. Moreover, by construction we have $Q_{\bar{b}'}(P(\bar{b}')) = 0$ for any $\bar{b}' \in V'$. Denoting $\bar{c}' = (g_1(\bar{b}'), \ldots, g_k(\bar{b}'))$ we get the following matrix $\mathrm{Jac}Q_{\bar{b}'}(P(\bar{b}'))$.

$$\begin{bmatrix} \mathrm{Jac}L_{\bar{c}'}(H(\bar{c}')) & & * & & \\ 0 & \mathrm{Jac}F^{(1)}_{\bar{b}'}(G_1(\bar{b}')) & 0 & \dots & 0 \\ 0 & 0 & \mathrm{Jac}F^{(2)}_{\bar{b}'}(G_2(\bar{b}')) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mathrm{Jac}F^{(k)}_{\bar{b}'}(G_k(\bar{b}')) \end{bmatrix}$$

Thus, it has non-zero determinant and $h(g_1(\bar{y}),\dots,g_k(\bar{y}))$ is implicitly defined from $\mathcal{S}$ at $\bar{b}$ via maps P and Q.

3. For each $i = 1,\dots,k$ let $G_i : V_i' \subseteq \mathbb{K}^k \times \mathbb{K}^r \to \mathbb{K}^{n_i}$ be a map with $V_i' \subseteq V$ an open neighbourhood of $\bar{b}$ and acting by $(\bar{y},\bar{z}) \mapsto (g_{i1}(\bar{y},\bar{z}),\dots,g_{in_i}(\bar{y},\bar{z}))$ with $g_{i1} = g_i \restriction_{V_i'}$, and $F^{(i)} : U_i \subseteq \mathbb{K}^{n_i} \times \mathbb{K}^k \times \mathbb{K}^r \to \mathbb{K}^{n_i}$ with U_i an open neighbourhood of the point $\bar{a}_i = G_i(\bar{b})$ and acting by $(\bar{x}_i,\bar{y},\bar{z}) \mapsto (f_{i1}((\bar{x}_i,\bar{y},\bar{z}),\dots,f_{in_i}(\bar{x}_i,\bar{y},\bar{z}))$ with $f_{i1},\dots,f_{in_i} \in \mathcal{S}^{\mathrm{poly}}$ be such that for any $\bar{b}' \in V_i'$ we have $F^{(i)}_{\bar{b}'}(G_i(\bar{b}')) = 0$ and $\det(\mathrm{Jac}F^{(i)}_{\bar{b}'}(G_i(\bar{b}'))) \neq 0$.

   For each $i = 1,\dots,k$ let W_i' to be a projection of V_i' onto $\mathbb{K}^r$ and $W_0 = \bigcap_{i=1}^{k} W_i' \cap W'$. We define a map

$$P : W_0 \subseteq \mathbb{K}^r \to \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^k$$

   by

$$P(\bar{y}) = (G_1(H(\bar{z}),\bar{z}),\dots,G_k(H(\bar{z}),\bar{z}),H(\bar{z})).$$

   By narrowing W_0 if needed, let $E \subseteq \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^k \times \mathbb{K}^r$ be an open neighbourhood of $(P(\bar{c}),\bar{c})$ such that for any $\bar{c}' \in W_0$ we have $(P(\bar{c}'),\bar{c}') \in E$. Then we define a map

$$Q : E \subseteq \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^k \times \mathbb{K}^r \mapsto \prod_{i=1}^{k} \mathbb{K}^{n_i} \times \mathbb{K}^k$$

   by

$$Q(\bar{x}_1,\dots,\bar{x}_k,\bar{y},\bar{z}) = (F^{(1)}(\bar{x}_1,\bar{y},\bar{z}),\dots,F^{(k)}(\bar{x}_k,\bar{y},\bar{z}),x_{11},\dots,x_{k1}),$$

   where $\bar{x}_i = (x_{i1},\dots,x_{in_i})$ for $i = 1,\dots,k$. Note that the component functions of Q belong to $\mathcal{S}^{\mathrm{poly}}$ and one of the component functions of P is $h \restriction_{W_0}$. Moreover, for any $\bar{c}' \in W_0$ we have $Q_{\bar{c}'}(P(\bar{c}')) = 0$ since $F^{(i)}_{(H(\bar{c}'),\bar{c})}(G_i(H(\bar{c}'),\bar{c}) = 0$ and $g_i(H(\bar{c}'),\bar{c}') = 0$ for each $i = 1,\dots,k$. For each $i = 1,\dots,k$ denote by e_i the vector $(1,0,\dots,0)$ of size n_i and by

$\frac{\partial F^{(i)}}{\partial \bar{y}} : U_i \subseteq \mathbb{K}^{n_i} \times \mathbb{K}^k \times \mathbb{K}^r \to \mathbb{K}^{n_i}$ the partial derivative of $F^{(i)}$ with respect to $\bar{y}$. Finally, let $\bar{c}' \in W_0$ and denote $\bar{b}' = (H(\bar{c}'), \bar{c}')$. Then we get the following matrix $\mathrm{Jac}Q_{\bar{c}'}(P(\bar{c}'))$.

$$\begin{bmatrix} \mathrm{Jac}F^{(1)}_{\bar{b}'}(G_1(\bar{b}')) & 0 & \dots & 0 & \frac{\partial F^{(1)}}{\partial \bar{y}}(G_1(\bar{b}'), \bar{b}') \\ 0 & \mathrm{Jac}F^{(2)}_{\bar{b}'}(G_2(\bar{b}')) & \dots & 0 & \frac{\partial F^{(2)}}{\partial \bar{y}}(G_2(\bar{b}'), \bar{b}') \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathrm{Jac}F^{(k)}_{\bar{b}'}(G_k(\bar{b}')) & \frac{\partial F^{(k)}}{\partial \bar{y}}(G_k(\bar{b}'), \bar{b}') \\ e_1 & 0 & \dots & 0 & 0 \\ 0 & e_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e_k & 0 \end{bmatrix}$$

Fix $i = 1, \dots, k$. In order to calculate $\frac{\partial F^{(i)}}{\partial \bar{y}}(G_i(\bar{b}'), \bar{b}')$ recall that for any $\bar{y} \in V_i'$ we have $F^{(i)}(G_i(\bar{y}), \bar{y}) = 0$. Differentiating this equality by $\bar{y}$ at the point $\bar{b}'$ we obtain $\frac{\partial F^{(i)}}{\partial \bar{y}}(G_i(\bar{b}'), \bar{b}') + \frac{\partial F^{(i)}}{\partial \bar{x}_i}(G_i(\bar{b}'), \bar{b}')\frac{\partial G_i}{\partial \bar{y}}(\bar{b}') = 0$. Note that $\frac{\partial F^{(i)}}{\partial \bar{x}_i}(G_i(\bar{b}'), \bar{b}') = \mathrm{Jac}F^{(i)}_{\bar{b}'}(G_i(\bar{b}'))$ and recall that its determinant is non-zero. Let us denote this Jacobian by J_i and write $\frac{\partial F^{(i)}}{\partial \bar{y}}(G_i(\bar{b}'), \bar{b}') = -J_i\frac{\partial G_i}{\partial \bar{y}}(\bar{b}')$.

Now decompose $\mathrm{Jac}Q_{\bar{c}'}(P(\bar{c}'))$ as a block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

$$A = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{bmatrix}, B = \begin{bmatrix} -J_1\frac{\partial G_1}{\partial \bar{y}}(\bar{b}') \\ -J_2\frac{\partial G_2}{\partial \bar{y}}(\bar{b}') \\ \vdots \\ -J_k\frac{\partial G_k}{\partial \bar{y}}(\bar{b}') \end{bmatrix}, C = \begin{bmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e_k \end{bmatrix},$$

$D \in \mathrm{Mat}_{k\times n}(\mathbb{K})$ is a zero matrix for $n = \prod_{i=1}^{k} n_i$. By a formula from [AM05, Exercise 5.30], the determinant of $\mathrm{Jac}Q_{\bar{c}'}(P(\bar{c}'))$ is equal to $\det(A)\det(D - CA^{-1}B)$. As $\det(A) \neq 0$ and $D = 0$, it suffices to show $\det(CA^{-1}B) \neq 0$. Multiplying A^{-1} and B we get $A^{-1}B = \begin{bmatrix} -\frac{\partial G_1}{\partial \bar{y}}(\bar{b}') \\ -\frac{\partial G_2}{\partial \bar{y}}(\bar{b}') \\ \vdots \\ -\frac{\partial G_k}{\partial \bar{y}}(\bar{b}') \end{bmatrix}$.

Recall that for each $i = 1, \dots, k$, the function g_{1i} coincides with g_i on V_i' and $G = (g_1, \dots, g_k)$. Thus, $CA^{-1}B = -\mathrm{Jac}G_{\bar{c}'}(H(\bar{c}'))$, which has non-zero determinant. Therefore, h is implicitly defined from $\mathcal{S}$ at $\bar{c}$ via maps P and Q. □

Using the Implicit Function Theorem we connect $\mathrm{icl}_{\mathcal{S}}$ to the functions implicitly defined from $\mathcal{S}$.

Proposition 2.14. *Let $a_1 \in \mathbb{K}$ and $B \subseteq \mathbb{K}$. Then $a_1 \in \mathrm{icl}_{\mathcal{S}}(B)$ if and only if there exist $b_1, \dots, b_m \in B$ and a function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at the point $\bar{b}$ such that $g(\bar{b}) = a_1$. Moreover, if $\mathcal{S}$ consists of infinitely differentiable or analytic functions, then we can choose g to also be infinitely differentiable or analytic correspondingly.*

Proof. ($\Rightarrow$) Suppose $a_1 \in \mathrm{icl}_{\mathcal{S}}(B)$. Then there are $a_2, \dots, a_n \in \mathbb{K}$, $b_1, \dots, b_m \in B$ and $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\mathrm{poly}}$ such that $F_{\bar{b}}(\bar{a}) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}}(\bar{a})) \neq 0$. As the component functions of $\mathcal{S}$ are continuously differentiable, there are open neighbourhoods W of $\bar{a}$ and V of $\bar{b}$ such that for any $\bar{a}' \in W$ and $\bar{b}' \in V$ we have $\det(\mathrm{Jac}F_{\bar{b}'}(\bar{a}')) \neq 0$.

By the Implicit Function Theorem (Fact 2.11), there are open neighbourhoods W' of $\bar{a}$ and V' of $\bar{b}$ and a map $G : V' \subseteq \mathbb{K}^m \to W' \subseteq \mathbb{K}^n$ such that for any $\bar{a}' \in W$ and $\bar{b}' \in V'$ we have $F(\bar{a}', \bar{b}') = 0$ if and only if $G(\bar{b}') = \bar{a}'$. If $\mathcal{S}$ consists of infinitely differentiable or analytic functions, G is also infinitely differentiable or analytic correspondingly.

By narrowing neighbourhoods we can assume $V' \subseteq V$ and $W' \subseteq W$. Let $g : V' \subseteq \mathbb{K}^m \to \mathbb{K}$ be the first component function of G. Then for any $\bar{b}' \in V'$ we have $F_{\bar{b}'}(G(\bar{b}')) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}'}(G(\bar{b}'))) \neq 0$. Thus, g is implicitly defined from $\mathcal{S}$ at $\bar{b}$ and is analytic in case $\mathcal{S}$ consists of analytic functions.

($\Leftarrow$) Follows immediately from the definitions. □

Let us now show that $\mathrm{icl}_{\mathcal{S}}$ is a pregeometry. We start by proving all the properties besides exchange.

Proposition 2.15. *The operator* $\mathrm{icl}_{\mathcal{S}}$ *is a closure and satisfies finite character.*

Proof. We go through the properties listed in Definition 2.3 one by one.

1. Let $B \subseteq \mathbb{K}$ and $b \in B$. Consider the map $f : (x, y) \mapsto x - y$ on $\mathbb{K}$ with $f_b : x \mapsto x - b$. It belongs to $\mathcal{S}^{\mathrm{poly}}$, $f_b(b) = 0$ and $\mathrm{Jac}f_b = I_{1\times 1}$, the identity matrix. Hence, $b \in \mathrm{icl}_{\mathcal{S}}(B)$ and $B \subseteq \mathrm{icl}_{\mathcal{S}}(B)$.

2. Let $B \subseteq B' \subseteq \mathbb{K}$ and $a_1 \in \mathrm{icl}_{\mathcal{S}}(B)$. Then there are $a_2, \dots, a_n \in \mathbb{K}$, $b_1, \dots, b_m \in B$ and a map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\mathrm{poly}}$, such that $F_{\bar{b}}(\bar{a}) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}}(\bar{b})) \neq 0$. Since the tuple $\bar{b}$ also belongs to B', we get $a_1 \in \mathrm{icl}_{\mathcal{S}}(B')$ by definition. Hence, $\mathrm{icl}_{\mathcal{S}}(B) \subseteq \mathrm{icl}_{\mathcal{S}}(B')$.

3. Let $C \subseteq \mathbb{K}$ and $a \in \mathrm{icl}_{\mathcal{S}}(\mathrm{icl}_{\mathcal{S}}(C))$. Then by Proposition 2.14, there are $b_1, \dots, b_m \in \mathrm{icl}_{\mathcal{S}}(C)$ and a function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined

from $\mathcal{S}$ at $\bar{b}$ such that $g(\bar{b}) = a$. Moreover, by Proposition 2.14, for every $i = 1, \ldots, m$ there are $c_{i1}, \ldots, c_{ir_i} \in C$ and a map $h_i : W_i \subseteq \mathbb{K}^{r_i} \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $\bar{c}_i = (c_{i1}, \ldots, c_{ir_i})$ such that $h_i(\bar{c}_i) = b_i$. We add dummy variables to each h_i to get functions $\tilde{h}_i : \prod_{i=1}^m W_i \subseteq \prod_{i=1}^m \mathbb{K}^{r_i} \to \mathbb{K}$ defined on the same set and implicitly defined from $\mathcal{S}$ at $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_m)$. By Lemma 2.13 the function $g(h_1(\bar{z}), \ldots, h_m(\bar{z}))$ is implicitly defined from $\mathcal{S}$ at $\bar{c} \in C$, while $g(h_1(\bar{c}), \ldots, h_m(\bar{c})) = a$. Thus, $a \in \mathrm{icl}_{\mathcal{S}}(C)$ and $\mathrm{icl}_{\mathcal{S}}(\mathrm{icl}_{\mathcal{S}}(C)) \subseteq \mathrm{icl}_{\mathcal{S}}(C)$.

Finally, let $B \subseteq \mathbb{K}$ and $a_1 \in \mathrm{icl}_{\mathcal{S}}(B)$. Then there are $a_2, \ldots, a_n \in \mathbb{K}$, $b_1, \ldots, b_m \in B$ and a map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\mathrm{poly}}$, such that $F_{\bar{b}}(\bar{a}) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}}(\bar{a})) \neq 0$. Thus, $a_1 \in \mathrm{icl}_{\mathcal{S}}(\bar{b})$ by definition. That demonstrates finite character for $\mathrm{icl}_{\mathcal{S}}$. □

For the exchange property we assume $\mathcal{S}$ consists of analytic functions and is closed under differentiation, i.e., for any function $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{S}$, the functions $\frac{\partial f}{\partial x_i} : U \subseteq \mathbb{K}^n \to \mathbb{K}$ for $i = 1, \ldots, n$ also belong to $\mathcal{S}$. Note that whenever $\mathcal{S}$ is closed under differentiation, so is $\mathcal{S}^{\mathrm{poly}}$. We follow the proof from [Wil08] and first show that whenever $\mathcal{S}$ is closed under differentiation, the set of functions implicitly defined from $\mathcal{S}$ is also closed under differentiation.

Lemma 2.16. *Suppose $\mathcal{S}$ consists of infinitely differentiable functions and is closed under differentiation. Then for any function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at some point $\bar{b} \in V$, the functions $\frac{\partial g}{\partial y_i} : V \subseteq \mathbb{K}^m \to \mathbb{K}$ for $i = 1, \ldots, m$ are also implicitly defined from $\mathcal{S}$ at $\bar{b}$.*

Proof. Suppose $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ is implicitly defined from $\mathcal{S}$ at $\bar{b} \in V$ and fix $i = 1, \ldots, m$. Then there are maps $G : V' \subseteq \mathbb{K}^m \to \mathbb{K}^n$ acting by $\bar{y} \mapsto (g_1(\bar{y}), \ldots, g_n(\bar{y})$ with $g_1 = g \restriction_{V'}$ and $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ acting by $(\bar{x}, \bar{y}) \mapsto (f_1((\bar{x}, \bar{y}), \ldots, f_n(\bar{x}, \bar{y}))$ with $f_1, \ldots, f_n \in \mathcal{S}^{\mathrm{poly}}$ such that for any $\bar{b}' \in V'$ we have $F_{\bar{b}'}(G(\bar{b}')) = 0$ and $\det(\mathrm{Jac}F_{\bar{b}'}(G(\bar{b}'))) \neq 0$. Note that both F and G consist of infinitely differentiable functions. By differentiating the equation $F_{\bar{y}}(G(\bar{y})) = 0$ at some point $\bar{b}' \in V'$ with respect to y_i we get the following equality for each $j = 1, \ldots, n$.

$$\sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}(G(\bar{b}'), \bar{b}') \frac{\partial g_k}{\partial y_i}(\bar{b}') + \frac{\partial f_j}{\partial y_i}(G(\bar{b}'), \bar{b}') = 0.$$

For each $j = 1, \ldots, n$ let $h_j : \mathbb{K}^n \times V' \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}$ be defined by $h_j(\bar{z}, \bar{y}) = \sum\limits_{k=1}^{n} \frac{\partial f_j}{\partial x_k}(G(\bar{y}), \bar{y}) z_k + \frac{\partial f_j}{\partial y_i}(G(\bar{y}), \bar{y})$. Then since component functions $g_1, \ldots, g_n$ of G are implicitly defined from $\mathcal{S}$ at $\bar{b}$, each h_j is implicitly defined from $\mathcal{S}$ at $(\frac{\partial G}{\partial y_i}(\bar{b}), \bar{b})$ by Lemma 2.13. Combine $h_1, \ldots, h_n$ into one map $H : \mathbb{K}^n \times V' \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$, note that it is infinitely differentiable. Then for the map $\frac{\partial G}{\partial y_i} : V \subseteq \mathbb{K}^m \to \mathbb{K}^n$ with component functions $\frac{\partial g_k}{\partial y_i}$ for $k = 1, \dots, n$ we have $H_{\bar{b}'}(\frac{\partial G}{\partial y_i}(\bar{b}')) = 0$ for any $\bar{b}' \in V'$ by the equality above. Differentiating $H_{\bar{b}'}$ gives us $\mathrm{Jac}H_{\bar{b}'}(\frac{\partial G}{\partial y_i}(\bar{b}')) = \mathrm{Jac}F_{\bar{b}'}(G(\bar{b}'))$, thus the determinant is non-zero. Therefore, $\frac{\partial g}{\partial y_i}$ is implicitly defined from $\mathcal{S}$ at $\bar{b}$ by Lemma 2.13. □

Now we are ready to prove the exchange property for $\mathrm{icl}_{\mathcal{S}}$ under the assumption of $\mathcal{S}$ consisting of analytic functions and being closed under differentiation.

Proposition 2.17. *Suppose $\mathcal{S}$ consists of analytic functions and is closed under differentiation. Then the operator* $\mathrm{icl}_{\mathcal{S}}$ *is a pregeometry.*

Proof. By Proposition 2.15 it suffices to prove the exchange property. Suppose $c \in \mathrm{icl}_{\mathcal{S}}(\{a\} \cup B)$. Then by Proposition 2.14 there is a finite tuple $\bar{b} \in B$ and an analytic function $g : V \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ acting by $(x, \bar{y}) \mapsto g(x, \bar{y})$ implicitly defined from $\mathcal{S}$ at the point $(a, \bar{b})$ such that $g(a, \bar{b}) = c$. If $\frac{\partial g}{\partial x}(a, \bar{b}) \neq 0$, consider a function $f(x, \bar{y}, z) = g(x, \bar{y}) - z$. It is also implicitly defined from $\mathcal{S}$ at the point $(a, \bar{b}, c)$ by Lemma 2.13, while $f(a, \bar{b}, c) = 0$ and $\frac{\partial f}{\partial x}(a, \bar{b}, c) \neq 0$. Thus we can use Implicit Function Theorem (Fact 2.11) to construct a function $h : W \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ with $f(h(\bar{b}', c'), \bar{b}', c') = 0$ for any $(\bar{b}', c') \in W$. Since $\frac{\partial f}{\partial x}(a, \bar{b}, c) \neq 0$, by continuity of partial derivatives there is an open neighbourhood U of $(a, \bar{b}, c)$ with $\frac{\partial f}{\partial x}(a', \bar{b}', c') \neq 0$ for any $(a', \bar{b}', c') \in U$. Choose an open set $W' \subseteq W$ such that whenever $(\bar{b}', c') \in W'$, we have $(h(\bar{b}', c'), \bar{b}', c') \in U$ and let $\tilde{h} = h \restriction_{W'}$. Then $\tilde{h}$ is implicitly defined from $\mathcal{S}$ at $(\bar{b}, c)$ by definition and Lemma 2.13. Hence, $a \in \mathrm{icl}_{\mathcal{S}}(\{c\} \cup B)$ by Proposition 2.14.

If $\frac{\partial g}{\partial x}(a, \bar{b}) = 0$, we can assume $\frac{\partial^k g}{\partial x^k}(a, \bar{b}) = 0$ for all $k \in \mathbb{N}$ as otherwise we can replace g with $\tilde{g} = g + \frac{\partial^k g}{\partial x^k}$, where k is the largest natural number with $\frac{\partial^k g}{\partial x^k}(a, \bar{b}) = 0$. Indeed, then the function $\tilde{g}$ is implicitly defined from $\mathcal{S}$ at $(a, \bar{b})$ by Lemma 2.16 and Lemma 2.13 with $\tilde{g}(a, \bar{b}) = c$ and $\frac{\partial \tilde{g}}{\partial x}(a, \bar{b}) \neq 0$.

Thus, we assume $\frac{\partial^k g}{\partial x^k}(a, \bar{b}) = 0$ for all $k \in \mathbb{N}$ and since g is analytic, there exists an open neighbourhood O of a with $g(a', \bar{b}) = c$ for any $a' \in O$. Choose $a_0 \in O$ algebraic over C_0. Then as a constant function a_0 is implicitly defined from $\mathcal{S}$ at any point. Then by Lemma 2.13 the function $\bar{y} \mapsto g(a_0, \bar{y})$ is implicitly defined from $\mathcal{S}$ at $\bar{b}$. Thus, $c \in \mathrm{icl}_{\mathcal{S}}(B)$ by Proposition 2.14. □

Note that the only property of an analytic function that we used is that whenever all derivatives of an analytic function (defined on a connected neighbourhood) are zero at one of its roots, the function has to be zero on all of its domain. Thus, we can generalize Proposition 2.17 as well as Lemma 2.20 and Proposition 2.21 to a class of quasi-analytic functions [Inf16], i.e., infinitely differentiable functions possessing exactly this property. For the purpose of clarity, we rather stick to the more well-known class of analytic functions.

Finally, we show that the pregeometries $\mathrm{Dcl}_{\mathcal{S}}$ and $\mathrm{icl}_{\mathcal{S}}$ coincide. Note that this result is a generalization of [Wil08, Theorem 3.4] as we do not assume $\mathcal{S}^{\mathrm{poly}}$ to be closed under Schwarz Reflection. Nevertheless, we use the same strategy throughout. We start by proving that derivations respecting $\mathcal{S}$ (and vanishing on C) also respect functions from $\mathcal{S}^{\mathrm{poly}}$ in the following sense.

Lemma 2.18. *Let $a_1, \dots, a_n \in \mathbb{K}$, $B \subseteq \mathbb{K}$ and $b_1, \dots, b_m \in B$. Suppose that all component functions of the map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ belong to $\mathcal{S}^{\mathrm{poly}}$. Then for any $D \in \mathrm{Der}_{\mathcal{S}}(\mathbb{K}/B)$ we have $D(F_{\bar{b}}(\bar{a})) = JD\bar{a}$, where $J = \mathrm{Jac}F_{\bar{b}}(\bar{a})$ and $D\bar{x} = (Dx_1, \dots, Dx_n)$.*

Proof. We show the equality of two tuples component-wise. Let $f(\bar{x}, \bar{y})$ be the i-th component functions of F for $i = 1, \dots, n$. Then we need to show

$$Df(\bar{a}, \bar{b}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a}, \bar{b}) Da_i.$$

As $f(\bar{x}, \bar{y}) \in \mathcal{S}^{\mathrm{poly}}$, it suffices that this equality holds for all functions in $\mathcal{S}$ and is preserved by composition with polynomials over C_0.

In case $f \in \mathcal{S}$, we have

$$Df(\bar{a}, \bar{b}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a}, \bar{b}) Da_i + \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(\bar{a}, \bar{b}) Db_j.$$

But since D vanishes on B, the second summand in the equality is zero. Now suppose $f(\bar{x}, \bar{y}) = P(\bar{x}, \bar{y}, g_1(\bar{x}, \bar{y}), \dots, g_k(\bar{x}, \bar{y}))$ for $P \in C_0[\bar{x}, \bar{y}, t_1, \dots, t_k]$ and $g_1, \dots, g_k \in \mathcal{S}$. Since D respects polynomials over C_0, we have

$$\begin{aligned} Df(\bar{a}, \bar{b}) = DP(\bar{a}, \bar{b}, g_1(\bar{a}, \bar{b}), \dots, g_k(\bar{a}, \bar{b})) &= \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(\bar{a}, \bar{b}, g_1(\bar{a}, \bar{b}), \dots, g_k(\bar{a}, \bar{b})) Da_i \\ &+ \sum_{j=1}^{m} \frac{\partial P}{\partial y_j}(\bar{a}, \bar{b}, g_1(\bar{a}, \bar{b}), \dots, g_k(\bar{a}, \bar{b})) Db_j \\ &+ \sum_{l=1}^{k} \frac{\partial P}{\partial t_l}(\bar{a}, \bar{b}, g_1(\bar{a}, \bar{b}), \dots, g_k(\bar{a}, \bar{b})) Dg_l(\bar{a}, \bar{b}). \end{aligned}$$

Once again the second summand disappears, while for the third one we have

$$Dg_l(\bar{a}, \bar{b}) = \sum_{i=1}^{n} \frac{\partial g_l}{\partial x_i}(\bar{a}, \bar{b}) Da_i$$

for every $l = 1, \dots, k$. By the chain rule $\frac{\partial f}{\partial x_i} = \frac{\partial P}{\partial x_i} + \sum_{l=1}^{k} \frac{\partial P}{\partial t_l} \frac{\partial g_l}{\partial x_i}$, so we are done. □

Now we can show one of the inclusions. Note that this direction does not require $\mathcal{S}$ to consist of analytic functions or be closed under differentiation.

Proposition 2.19. *Let $B \subseteq \mathbb{K}$. Then* $\text{icl}_{\mathcal{S}}(B) \subseteq \text{Dcl}_{\mathcal{S}}(B)$.

Proof. Take $a_1 \in \text{icl}_{\mathcal{S}}(B)$. Then there are $a_2, \ldots, a_n \in \mathbb{K}$, $b_1, \ldots, b_m \in B$ and a map $F : U \subseteq \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^n$ with component functions in $\mathcal{S}^{\text{poly}}$ such that $F_{\bar{b}}(\bar{a}) = 0$ and $\det(\text{Jac}F_{\bar{b}}(\bar{a})) \neq 0$. Let $J = \text{Jac}F_{\bar{b}}(\bar{a})$, note that this is an invertible matrix. Then by Lemma 2.18 we have $D(F_{\bar{b}}(\bar{a})) = JD\bar{a}$ for any $D \in \text{Der}_{\mathcal{S}}(\mathbb{K}/B)$. Since the left hand side is zero, we ought to have $JD\bar{a} = 0$. But J is invertible, thus $D\bar{a} = 0$ and $a_1 \in \text{Dcl}_{\mathcal{S}}(B)$. □

For the other direction we need the following lemma about the implicit closure and implicit definitions. Following [Wil08], we assume $\mathcal{S}$ to consist of analytic functions and be closed under differentiation.

Lemma 2.20. *Suppose $\mathcal{S}$ consists of analytic functions and is closed under differentiation. Let $a_1, \ldots, a_m \in \mathbb{K}$. Then $a_1, \ldots, a_m$ are dependent with respect to* $\text{icl}_{\mathcal{S}}$ *if and only if there exists an analytic function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $(a_1, \ldots, a_m)$ with $g(a_1, \ldots, a_m) = 0$ such that there is no open neighbourhood of $(a_1, \ldots, a_m)$ with g identical zero on it.*

Proof. First suppose $a_1, \ldots, a_m$ are dependent with respect to $\text{icl}_{\mathcal{S}}$. Without the loss of generality, we can assume $a_1 \in \text{icl}_{\mathcal{S}}(a_2, \ldots, a_m)$. Hence, by Proposition 2.14 there exists an analytic function $h : W \subseteq \mathbb{K}^{m-1} \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $(a_2, \ldots, a_m)$ with $h(a_2, \ldots, a_m) = a_1$. Note that by adding dummy variables, we can assume that h is implicitly defined at $(a_2, \ldots, a_m)$ rather than some subtuple.

Let $g : \mathbb{K} \times W' \subseteq \mathbb{K}^m \to \mathbb{K}$ be defined by $g(x_1, \ldots, x_m) = h(x_2, \ldots, x_m) - x_1$. Note that g is an analytic function with $g(a_1, \ldots, a_m) = 0$ and implicitly defined from $\mathcal{S}$ at $(a_1, \ldots, a_m)$ by Lemma 2.13. Suppose it vanishes on some open neighbourhood $V \subseteq \mathbb{K} \times W'$ of $(a_1, \ldots, a_m)$. Then take a point of the form $(a'_1, a_2, \ldots, a_m) \in V$ with $a_1 \neq a'_1$ and note that $g(a'_1, a_2, \ldots, a_m) = 0$ contradicts h being a function. This concludes one of the directions.

Now suppose there exists an analytic function $g : V \subseteq \mathbb{K}^m \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $(a_1, \ldots, a_m)$ with $g(a_1, \ldots, a_m) = 0$ and g does not vanish on any open neighbourhood of $(a_1, \ldots, a_m)$. Then as g is analytic, there is $i = 1, \ldots, m$ and $k \in \mathbb{N}$ such that $\frac{\partial^k g}{\partial x_i^k}(a_1, \ldots, a_m) \neq 0$. Take the smallest such k and replace g with $\frac{\partial^{k-1} g}{\partial x_i^{k-1}}$. Note that $\frac{\partial^{k-1} g}{\partial x_i^{k-1}}(a_1, \ldots, a_m) = 0$ and $\frac{\partial^{k-1} g}{\partial x_i^{k-1}} : V \subseteq \mathbb{K}^m \to \mathbb{K}$ is analytic and implicitly defined from $\mathcal{S}$ at $(a_1, \ldots, a_m)$ by Lemma 2.16. After replacing g if necessary, we get $\frac{\partial g}{\partial x_i}(a_1, \ldots, a_m) \neq 0$ for some $i = 1, \ldots, m$. To simplify notation let $i = 1$. Using $\frac{\partial g}{\partial x_1}$ being continuous, let $V' \subseteq V$ be an open neighbourhood of $(a_1, \ldots, a_m)$ such that for any $(a'_1, \ldots, a'_m) \in V'$ we have $\frac{\partial g}{\partial x_1}(a'_1, \ldots, a'_m) \neq 0$.

By Implicit Function Theorem (Fact 2.11) there exists a function $h : W \subseteq \mathbb{K}^{n-1} \to U \subseteq \mathbb{K}$ with an open neighbourhood U of a_1 such that for any

$(a_2',\dots,a_m') \in W$ we have $g(h(a_2',\dots,a_n'),a_2',\dots,a_m') = 0$. We can assume $U \times W \subseteq V'$ and thus h is implicitly defined from $\mathcal{S}$ at $(a_2,\dots,a_m)$ by definition and Lemma 2.13. So $a_1 \in \mathrm{icl}_{\mathcal{S}}(a_2,\dots,a_m)$ by Proposition 2.14. □

Finally we show the second inclusion.

Proposition 2.21. *Suppose $\mathcal{S}$ consists of analytic functions and is closed under differentiation and let $C \subseteq \mathbb{K}$. Then $\mathrm{Dcl}_{\mathcal{S}}(C) \subseteq \mathrm{icl}_{\mathcal{S}}(C)$.*

Proof. Let $a \notin \mathrm{icl}_{\mathcal{S}}(C)$. Then there is a basis of $\mathbb{K}$ with respect to $\mathrm{icl}_{\mathcal{S}}$ of the form $\{a\} \cup B$ with $C \subseteq \mathrm{icl}_{\mathcal{S}}(B)$. Note that by Proposition 2.19, $\mathrm{icl}_{\mathcal{S}}(B) \subseteq \mathrm{Dcl}_{\mathcal{S}}(B)$ and any $D \in \mathrm{Der}_{\mathcal{S}}(B)$ vanishes on C. Thus, it suffices to construct a derivation $D \in \mathrm{Der}_{\mathcal{S}}(B)$ with $Da \neq 0$.

Consider some $\alpha \in \mathbb{K}$. Then $\alpha \in \mathrm{icl}_{\mathcal{S}}(\{a\} \cup B)$, so by Proposition 2.14 there are $b_1,\dots,b_m \in B$ and an analytic function $g : V \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ acting by $(x,\bar{y}) \mapsto g(x,\bar{y})$ and implicitly defined from $\mathcal{S}$ at the point $(a,\bar{b})$ such that $g(a,\bar{b}) = \alpha$. Define $D\alpha = \frac{\partial g}{\partial x}(a,\bar{b})$. In order to show that this is well-defined suppose we have another analytic function $h : W \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ acting by $(x,\bar{y}) \mapsto h(x,\bar{y})$ implicitly defined from $\mathcal{S}$ at $(a,\bar{b})$ such that $g(a,\bar{b}) = \alpha$. Note that we can assume that both functions are defined on the same tuple $(a,\bar{b})$ by adding dummy variables. Then $g-h$ is a function implicitly defined from $\mathcal{S}$ at $(a,\bar{b})$ by Lemma 2.13 such that $(g-h)(a,\bar{b}) = 0$. By Lemma 2.20 and since $a,\bar{b}$ are independent, there is a neighbourhood V' of $(a,\bar{b})$, where $g-h$ vanishes. Then $\frac{\partial g}{\partial x}(a,\bar{b}) = \frac{\partial h}{\partial x}(a,\bar{b})$ and $D\alpha$ is well-defined.

Note that by taking g to be a projection to one of the coordinates, we get $Da = 1$ and $Db = 0$ for any $b \in B$. Hence, the only thing left is to show that D is a derivation respecting $\mathcal{S}$ and vanishing on C_0. Indeed, D is a derivation since differentiation respects addition and admits Leibniz rule. As the constant function c for $c \in C_0$ belongs to $\mathcal{S}^{\mathrm{poly}}$, we have $Dc = 0$. Finally, let $f : U \subseteq \mathbb{K}^n \to \mathbb{K}$ be an analytic function in $\mathcal{S}$ acting by $(t_1,\dots,t_n) \mapsto f(t_1,\dots,t_n)$ and $(\alpha_1,\dots,\alpha_n) \in U$. Then by Proposition 2.14 for every $i = 1,\dots,n$ there is a tuple $\bar{b}_i \in B$ and an analytic function $g_i : V_i \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ implicitly defined from $\mathcal{S}$ at $(a,\bar{b}_i)$ such that $g_i(a,\bar{b}_i) = \alpha_i$. Let $V = \bigcap_{i=1}^{m} V_i$. Then the function $f(g_1(x,\bar{y}),\dots,g_n(x,\bar{y})) : V \subseteq \mathbb{K}^{m+1} \to \mathbb{K}$ is analytic and implicitly defined from $\mathcal{S}$ at $(a,\bar{b}_i)$ by Lemma 2.13. Thus,

$$Df(\bar{\alpha}) = \frac{\partial f(g_1(x,\bar{y}),\dots,g_n(x,\bar{y}))}{\partial x}(a,\bar{b}_i).$$

By the chain rule and $D\alpha_i = \frac{\partial g_i}{\partial x}(a,\bar{b}_i)$, we get $Df(\bar{\alpha}) = \sum_{i=1}^{n} \frac{\partial f}{\partial t_i}(\bar{\alpha}) D\alpha_i$. □

Now we can combine the results so far into the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $C_0 \subseteq \mathbb{K}$ a subfield, $B \subseteq \mathbb{K}$ a subset and $\mathcal{S}$ a set of of differentiable functions on $\mathbb{K}$. Then we have the following for each item of Theorem 1.4.

1. $\mathrm{Dcl}_{\mathcal{S},C_0}(B)$ coincides with the kernel of $d : \mathbb{K} \to \Omega_{\mathcal{S},C_0}(\mathbb{K}/B)$ by Proposition 2.8.

2. If $\mathcal{S}$ consists of continuously differentiable functions, $\mathrm{icl}_{\mathcal{S},C_0}(B)$ coincides with images of B under functions implicitly defined from $\mathcal{S}$ by Proposition 2.14.

3. If $\mathcal{S}$ consists of analytic functions and is closed under differentiation, $\mathrm{Dcl}_{\mathcal{S},C_0}(B) = \mathrm{icl}_{\mathcal{S},C_0}(B)$ by Propositions 2.19 and 2.21.

4. If $\mathcal{S}$ is closed under differentiation and Schwarz reflection (for $\mathbb{K} = \mathbb{C}$), then for any function F locally definable in $\mathbb{R}(PR(\mathcal{S}))$ and any generic point $\bar{a} \in \mathrm{dom}(F)$ there is a function G implicitly defined from $\mathcal{S}$ coinciding with F on some neighbourhood of $\bar{a}$ by [Wil08, Theorem 1.10].

□

We have seen that whenever $\mathcal{S}$ consists of analytic functions and is closed under differentiation, $\mathrm{icl}_{\mathcal{S}}$ is a pregeometry coinciding with $\mathrm{Dcl}_{\mathcal{S}}$. As closing $\mathcal{S}$ under differentiation does not change the cardinality of $\mathcal{S}$ and $\mathrm{Dcl}_{\mathcal{S}} \subseteq \mathrm{Dcl}_{\mathcal{S}'}$ whenever $\mathcal{S} \subseteq \mathcal{S}'$, we obtain countable closure property for $\mathrm{Dcl}_{\mathcal{S}}$ with any countable $\mathcal{S}$ consisting of analytic functions and countable C_0.

Corollary 2.22. *Suppose $\mathcal{S}$ consists of analytic functions and both $\mathcal{S}$ and C_0 are countable. Then $\mathrm{Dcl}_{\mathcal{S},C_0}$ has countable closure property, i.e., for any countable subset $B \subseteq \mathbb{K}$, the set $\mathrm{Dcl}_{\mathcal{S},C_0}(B)$ is also countable*

Proof. Let $\mathcal{S}^{\mathrm{dif}}$ consist of all the functions in $\mathcal{S}$ together with all their derivatives. Note that $\mathcal{S}^{\mathrm{dif}}$ is still countable, consists of analytic functions and is closed under differentiation. Thus, by Proposition 2.21 and Proposition 2.19, pregeometries $\mathrm{icl}_{\mathcal{S}^{\mathrm{dif}}}$ and $\mathrm{Dcl}_{\mathcal{S}^{\mathrm{dif}}}$ coincide. So by Proposition 2.10, $\mathrm{Dcl}_{\mathcal{S}^{\mathrm{dif}}}$ has countable closure property. Let $B \subseteq \mathbb{K}$ be a countable subset. Then $\mathrm{Dcl}_{\mathcal{S}}(B) \subseteq \mathrm{Dcl}_{\mathcal{S}^{\mathrm{dif}}}(B)$, so $\mathrm{Dcl}_{\mathcal{S}}(B)$ is countable and $\mathrm{Dcl}_{\mathcal{S}}$ has CCP. □

3

Correspondence between elliptic curves

In [BK18] Bays and Kirby considered a conjecture analogous to Zilber's Quasiminimality Conjecture, where the exponential map is replaced with the exponential map of an elliptic curve. In this case the methods of reducing quasiminimality to a form of Existential closedness can be replicated as explained in [BK18, Section 9.3]. Furthermore, in [Kir19, Conjecture 8.2] Kirby combines both conjectures into a conjecture stating that the complex field with the exponentiation and countably many exponential maps of elliptic curves is quasiminimal.

In [Kir19] Kirby considers a reduct of $\mathbb{C}_{\exp}$, given by a blurring of the exponential map by a subgroup $\mathbb{Q} + 2\pi i\mathbb{Q}$. To be more precise, let $\Gamma_{\mathrm{AE}} = \{(x, y) \in \mathbb{C}^2 : y = e^{x+q+2\pi ir} \text{ for some } q, r \in \mathbb{Q}\}$. Then $\mathbb{C}_{\mathrm{AE}} = (\mathbb{C}; +, \cdot, \Gamma_{\mathrm{AE}})$ is a reduct of $\mathbb{C}_{\exp}$ and fits the definition of a Γ-field from [BK18, Definition 3.8]. Thus, we can reduce its quasiminimality to the countable closure property and so-called Γ-closedness by [BK18, Corollary 11.7]. While CCP for $\mathbb{C}_{\mathrm{AE}}$ follows from CCP for $\mathbb{C}_{\exp}$, Γ-closedness is obtained by using countability and density of Γ_{AE}. Therefore, we obtain quasiminimality of $\mathbb{C}_{\mathrm{AE}}$. Note that as $\mathbb{C}_{\mathrm{AE}}$ is a reduct of $\mathbb{C}_{\exp}$, this result can be seen as a partial result towards Zilber's Quasiminimality Conjecture.

As suggested in [Kir19, Conjecture 8.1], the same approach can be used in the elliptic case by considering two elliptic curves E_1, E_2 and a set $\Gamma_{\mathrm{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$, where $\exp_1$ and $\exp_2$ are the exponential maps of E_1 and E_2. Thus we obtain a structure $\mathbb{C}_{\mathrm{corr}} = (\mathbb{C}; +, \cdot, \Gamma_{\mathrm{corr}})$, which is a reduct of a structure $\mathbb{C}_{\exp_1,\exp_2} = (\mathbb{C}; +, \cdot, \exp_1, \exp_2)$. Then whenever $\mathbb{C}_{\mathrm{corr}}$ is a Γ-field, we can apply [BK18, Corollary 11.7] and reduce quasiminimality to CCP and Γ-closedness. The obtained quasiminimality then constitutes a partial result towards the quasiminimiality of $\mathbb{C}_{\exp_1,\exp_2}$.

We start the chapter by looking into the restrictions needed for $\mathbb{C}_{\mathrm{corr}}$ to fit the definition of a Γ-field. As explained in Section 3.2 it requires E_1 and E_2 to be defined over a number field, non-isogenous and not have complex multiplication. Under these restrictions we can use [BK18, Corollary 11.7] and see that in order to prove quasiminimality of $\mathbb{C}_{\text{corr}}$, it suffices to show CCP and Γ-closedness. For the countable closure property we demonstrate in Section 3.3 that the pregeometry in question is weaker than a certain example of the pregeometry Dcl as considered in Section 2.1 and thus has CCP by Corollary 2.22. We could have also used [BK18, Theorem 1.8] to immediately obtain the countable closure property, but we find it more beneficial to connect the pregeometry given by a predimension to the pregeometry given by derivations. Regarding the Γ-closedness, we aim to use the methods of [Kir19]. In Section 3.2 we explain that it requires to assume the density of the set $\Lambda_1 + \Lambda_2$, where Λ_1, Λ_2 are the corresponding lattices of E_1, E_2. Note that this is analogous to the set $\mathbb{Q} + 2\pi i\mathbb{Q}$ being dense in case of $\mathbb{C}_{\text{AE}}$.

Alternatively, we could deduce the Γ-closedness of $\mathbb{C}_{\text{corr}}$ directly from [Gal24, Theorem 4.1]. Indeed, Gallinaro considers an abelian variety A of dimension g and a free and rotund variety $L \times W$, where $L \subseteq \mathbb{C}^g$ is a linear space and $W \subseteq A$ is an algebraic variety. Taking $A = E_1^n \times E_2^n$ with $g = 2n$ and L to be the diagonal $\{(\bar{x}, \bar{x}) : \bar{x} \in \mathbb{C}^n\}$, we see that the condition $\exp_A(L) \cap W \neq \varnothing$ is equivalent to the Γ-closedness of $\mathbb{C}_{\text{corr}}$. Nevertheless, we stick to the restriction of $\Lambda_1 + \Lambda_2$ being dense and provide a proof of the Γ-closedness analogous to [Kir19, Proposition 6.2], as the proof of [Gal24, Theorem 4.1] requires much heavier techniques.

Thus, the main result of this chapter, as already stated in Section 1.1, is the following.

Theorem (Theorem 1.5)**.** Let E_1, E_2 be two non-isogenous elliptic curves without complex multiplication defined over a number field. Let Λ_1 and Λ_2 be the corresponding lattices on $\mathbb{C}$ and $\exp_1$, $\exp_2$ their corresponding exponential maps. Assume $\Lambda_1 + \Lambda_2$ is dense in $\mathbb{C}$ and consider the subgroup $\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$. Then $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{corr}})$ is quasiminimal.

The provided proof of Γ-closedness also goes through for elliptic curves not defined over a number field, thus producing a partial result for this case. On the other hand, the restriction to non-isogenous curves without complex multiplications is essential to the proof. We require the algebraic subgroups of $E_1^n \times E_2^n$ to split as shown in [JKS16, Fact 5.2], which fails when E_1 and E_2 are isogenous. We then use the symmetry of Γ_{corr} as well as End $E_1 \cong$ End E_2 in order to calculate the dimensions of the algebraic subgroups. Thus, as elliptic curves ought to be non-isogenous and have isomorphic endomorphism ring for the proof to go through, they cannot have complex multiplication.

Moreover, the isogenous case is interdefinable with the structure induced on the complex field by a set $\{(\exp_2(z), \exp_2(\alpha z)) : z \in \mathbb{C}\} \subseteq E_2 \times E_2$ for some $\alpha \in \mathbb{C}^\times$. Therefore, it seems likely to require more convoluted techniques, similar to the ones used in [GK24] for proving the quasiminimality of complex powers, i.e., considering $\Gamma_\lambda = \{(\exp(z), \exp(\lambda z)) : z \in \mathbb{C}\}$ for $\lambda \in \mathbb{C}$.

3.1 Background on elliptic curves

In this section we give the necessary background on Weierstrass elliptic functions and elliptic curves as well as define the structure we are going to work with. The following exposition is mostly based on [Sil86]. We start by considering elliptic curves over $\mathbb{C}$ and later also talk about elliptic curves over some subfields of $\mathbb{C}$.

Let $\Lambda \subset \mathbb{C}$ be a lattice, i.e., $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ for some $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$. Then Λ induces the *Weierstrass elliptic function*, defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

The function $\wp_\Lambda(z) : \mathbb{C} \setminus \Lambda \to \mathbb{C}$ is surjective, doubly periodic with the periods ω_1 and ω_2 and meromorphic with poles at Λ of order 2. Moreover, it satisfies a differential equation of the form $\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3$, where $g_2, g_3 \in \mathbb{C}$ with $4g_2^3 \neq 27g_3^2$. Consider the map $\exp_\Lambda : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$, called the *exponential map* and defined by

$$\exp_\Lambda(z) = \begin{cases} (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1) & \text{if } z \notin \Lambda, \\ (0 : 1 : 0) & \text{if } z \in \Lambda. \end{cases}$$

As the image of $\wp_\Lambda$ is all of $\mathbb{C}$ and it satisfies a differential equation above, the image of $\exp_\Lambda$ is a projective variety. We call it the *elliptic curve (over* $\mathbb{C}$*)* generated by the lattice Λ and denote by E_Λ.

Note that $E_\Lambda = \{(X : Y : Z) \in \mathbb{P}^2 \mid Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3\}$. We say that E_Λ is *defined over a subfield* $K \subseteq \mathbb{C}$ if $g_2, g_3 \in K$. Note that E_Λ is defined over a number field if and only if g_2 and g_3 are algebraic numbers. For any $g_2, g_3 \in \mathbb{C}$ with $4g_2^3 \neq 27g_3^2$ there is a lattice Λ such that the elliptic curve E_Λ generated by it corresponds to the equation $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$. The condition $4g_2^3 \neq 27g_3^2$ is equivalent to the curve E_Λ being non-singular.

An elliptic curve E_Λ also has a group structure inherited from the additive structure on $\mathbb{C}$, i.e., $\exp_\Lambda$ provides the group isomorphism between $\mathbb{C}/\Lambda$ and E_Λ. Furthermore, the group operation is given by an algebraic map. We denote by O_Λ the identity element of E_Λ and by Tor_Λ the torsion subgroup of E_Λ. Note that O_Λ has coordinates $(0 : 1 : 0)$ in $\mathbb{P}^2(\mathbb{C})$.

Given an elliptic curve E_Λ with the corresponding equation $Y^2Z = 4X^3 - g_2XZ^2 - g_3Z^3$, we can consider its intersection with the affine chart defined by $Z = 1$. Then we get an algebraic variety in $\mathbb{C}^2$ defined by the equation $Y^2 = 4X^3 - g_2X - g_3$. We call this variety *the affine part of* E_Λ. We say that it is defined by the polynomial $f(X) = 4X^3 - g_2X - g_3$.

Let F_0 be the field generated by the coefficients g_2 and g_3. Then given a field

$F \supseteq F_0$, we can consider a homogeneous polynomial $Y^2Z - 4X^3 - g_2XZ^2 - g_3Z^3$ on the projective space $\mathbb{P}^2(F)$. The zero set of the former polynomial forms an *elliptic curve over* F, denoted by $E_\Lambda(F)$. As the group operation on E_Λ is algebraic, it is well-defined on $E_\Lambda(F)$ and Tor_Λ is an algebraic set, i.e., $\mathrm{Tor}_\Lambda \subseteq E_\Lambda(F_0^{\mathrm{alg}})$, where F_0^{alg} is the algebraic closure of F_0. Therefore, Tor_Λ coincides with the torsion subgroup $\mathrm{Tor}_\Lambda(F_0^{\mathrm{alg}})$ of the group $E_\Lambda(F_0^{\mathrm{alg}})$.

Note that there are three 2-torsion points of E_Λ and their preimage is $\frac{1}{2}\Lambda \setminus \Lambda$. The 2-torsion points are also exactly the points of E_Λ with $Y = 0$, thus $\frac{1}{2}\Lambda \setminus \Lambda$ is exactly the set of points $z \in \mathbb{C}$ with $\wp'_\Lambda(z) = 0$. Therefore, the function $\wp_\Lambda$ is locally invertible on $\mathbb{C} \setminus \frac{1}{2}\Lambda$.

Given two elliptic curves E_1 and E_2, we can consider them as projective varieties and look at the morphisms $\phi : E_1 \to E_2$ between them. Let O_1 and O_2 be the corresponding identity elements of E_1 and E_2. Then whenever $\phi(O_1) = O_2$ and ϕ is not a constant map, we call the morphism ϕ an *isogeny.* If such a map between E_1 and E_2 exists, we say that the E_1 and E_2 are *isogenous.* By [Sil86, Theorem III.6.1], this is an equivalence relation and by [Sil86, Theorem III.4.8], every isogeny is a group homomorphism. Moreover, isogenies are exactly the maps induced by the scalar multiplication of the lattices in the following sense.

Let Λ_1 and Λ_2 be lattices in $\mathbb{C}$ corresponding to E_1 and E_2, and suppose that for some $\alpha \in \mathbb{C}^\times$ we have $\alpha\Lambda_1 \subseteq \Lambda_2$. Then scalar multiplication by α gives a well-defined homomorphism

$$f_\alpha : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 \qquad f_\alpha(z) = \alpha z \pmod{\Lambda_2}.$$

As the corresponding exponential maps $\exp_1$ and $\exp_2$ provide group isomorphisms, we obtain a map $\phi_\alpha : E_1 \to E_2$ defined by $\exp_1(z) \mapsto \exp_2(\alpha z)$. While ϕ_α is by construction a group homomorphism, by [Sil86, IV.Theorem 4.1] it is also isogeny and, furthermore, every isogeny between E_1 and E_2 is of a form ϕ_α for some $\alpha \in \mathbb{C}^\times$. Thus, E_1 and E_2 are isogenous if and only if there exists $\alpha \in \mathbb{C}^\times$ such that $\alpha\Lambda_1 \subseteq \Lambda_2$.

Now let us look at the isogenies from an elliptic curve E_Λ to itself. Together with the zero map (i.e., a constant map sending all points of E_Λ to the point O_Λ) they coincide with the *endomorphism ring* $\mathrm{End}\ E_\Lambda$ of all algebraic group endomorphisms of E_Λ, where the addition is given by the group structure on E_Λ and the multiplication is given by the composition. Then the map $m \mapsto \phi_m$ (with the isogeny $\phi_m : E_\Lambda \to E_\Lambda$ constructed above) defines a ring embedding $\mathbb{Z} \hookrightarrow \mathrm{End}\ E_\Lambda$. In case all the isogenies in $\mathrm{End}\ E_\Lambda$ are of the form ϕ_m for $m \in \mathbb{Z}$, we get $\mathrm{End}\ E_\Lambda \cong \mathbb{Z}$. Otherwise $\mathrm{End}\ E_\Lambda \cong \mathbb{Z}[\tau]$ for some imaginary quadratic τ, as shown in [Sil86, Corollary III.9.4], and we say that E_Λ has *complex multiplication.* Reformulating in terms of the lattice, E_Λ has complex multiplication if and only if there exists $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ such that $\alpha\Lambda \subseteq \Lambda$; and in this case α is an imaginary quadratic. Finally, it is not hard to see that isogenous elliptic curves have isomorphic endomorphism rings and, furthermore, if two elliptic curves with complex multiplication have isomorphic endomorphism rings, then they are isogenous.

3.2 Restrictions

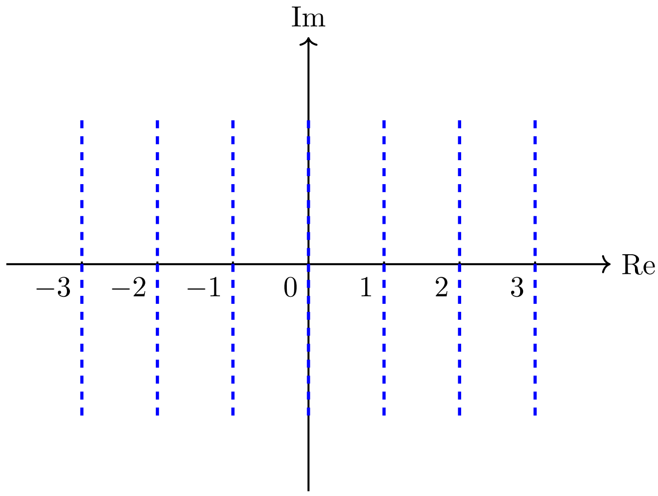
With the background established we can properly define the structure $\mathbb{C}_{\text{corr}}$ and explain the restrictions required in Theorem 1.5. Let Λ_1 and Λ_2 be two lattices which induce two elliptic curves E_1 and E_2 and two exponential maps $\exp_1$ and $\exp_2$. Consider the subgroup $\Gamma_{\text{corr}} \subseteq E_1 \times E_2$ consisting of pairs of the form $(\exp_1(z), \exp_2(z))$ for some $z \in \mathbb{C}$. We call this subgroup the *correspondence* between E_1 and E_2. By realizing each elliptic curve as a subset of $\mathbb{C}^2$ with an additional point, we can interpret the correspondence Γ_{corr} as a relation on the complex numbers and define the structure $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{corr}})$. As a reduct of $\mathbb{C}_{\exp_1, \exp_2}$, the structure $\mathbb{C}_{\text{corr}}$ is conjecturally quasiminimal. However, the techniques required to prove quasiminimality of $\mathbb{C}_{\text{corr}}$ heavily depend on the interrelation between the two elliptic curves.

Let $\Lambda_1 = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then in case Λ_2 is generated by $\mathbb{Q}$-linear combinations of ω_1 and ω_2, the sum of lattices $\Lambda_1 + \Lambda_2$ is also a discrete lattice. Furthermore, there is $\frac{m}{n} \in \mathbb{Q}$ such that $\frac{m}{n}\Lambda_1 \subseteq \Lambda_2$, i.e., there is an isogeny of the form $\phi_{\frac{m}{n}} : E_1 \to E_2$ in the notation of Section 3.1. Since isogenies are morphisms of projective varieties, the graph of $\phi_{\frac{m}{n}}$ is definable in $\mathbb{C}_{\text{field}}$ and consists of pairs of the form $(\exp_1(z), \exp_2(\frac{m}{n}z))$ for $z \in \mathbb{C}$. Thus we can define the correspondence Γ_{corr} in the language of fields as pairs (P, Q) such that $Q^{\bigoplus m} = \phi_{\frac{m}{n}}(P)^{\bigoplus n}$, using definability of the group structure on E_2, denoted here by $Q^{\bigoplus m}$. Therefore, in this case $\mathbb{C}_{\text{corr}}$ is definable in $\mathbb{C}_{\text{field}}$ and has to be strongly minimal.

Now let us have a look at other options. If Λ_2 is not generated by $\mathbb{Q}$-linear combinations of ω_1 and ω_2, the sum $\Lambda_1 + \Lambda_2$ cannot be discrete. It could either form a set of points dense on a family of lines, or form a set of points dense everywhere on the complex plane. Let us give some concrete examples, demonstrating that both isogenous and non-isogenous as well as with or without complex multiplication configurations can take place.

Let $\Lambda = \mathbb{Z} + \alpha i \mathbb{Z}$ for $\alpha \in \mathbb{R}$. Then End $E_\Lambda \cong \mathbb{Z}$ if $\alpha^2 \notin \mathbb{Q}$ and if $\alpha^2 = \frac{k}{m}$ for $k, m \in \mathbb{Z}$, we have End $E_\Lambda \cong \mathbb{Z}[m\alpha i]$. Furthermore, taking two lattices of this form, i.e. $\Lambda_1 = \mathbb{Z} + \alpha i \mathbb{Z}$ and $\Lambda_2 = \mathbb{Z} + \beta i \mathbb{Z}$, we get isogenous elliptic curves if and only if $\alpha\beta \in \mathbb{Q}$.

Now if $\frac{\alpha}{\beta} \notin \mathbb{Q}$, the set $\Lambda_1 + \Lambda_2$ consists of points dense on the lines corresponding to integer real part. Choosing different α and β we can ensure that the elliptic curves are isogenous or non-isogenous and have or not have complex multiplication.



It is also easy to obtain examples of $\Lambda_1 + \Lambda_2$ being dense everywhere on the complex plane. One can consider an example above, i.e. $\Lambda_1 = \mathbb{Z} + \alpha i\mathbb{Z}$, $\Lambda_2 = \mathbb{Z} + \beta i\mathbb{Z}$ and multiply the second lattice by some $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ such that $\frac{\gamma\beta}{\alpha} \notin \mathbb{Q}$. Then for $\Lambda_2' = \gamma\mathbb{Z} + \gamma\beta i\mathbb{Z}$ the sum $\Lambda_1 + \Lambda_2'$ is dense in $\mathbb{C}$ while E_2' has the same endomorphism ring and isogeny class as E_2. An even easier example for isogenous elliptic curves is taking Λ_1 to be any lattice and $\Lambda_2 = \alpha\Lambda_1$ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

For the purposes of this work we restrict ourselves to the case of non-isogenous elliptic curves without complex multiplication and with $\Lambda_1 + \Lambda_2$ dense in $\mathbb{C}$. This ensures that the strategy of [Kir19] goes through and thus we are able to prove Γ-closedness. Then whenever elliptic curves are also defined over a number field, we can apply [BK18, Corollary 11.7] and obtain quasiminimality. The restriction to number fields here is due to the use of the Kummer theory of semiabelian varieties over number fields in [BK18]. Alternatively, we could use [Gal24, Theorem 4.1] and get Γ-closedness without any restrictions on E_1 and E_2. Although [Gal24] only deals with the questions of Γ-closedness and does not consider quasiminimality, for non-isogenous E_1 and E_2 without complex multiplication and defined over a number field we can get quasiminimality via [BK18, Corollary 11.7]. However, we find it beneficial to have an alternative proof that uses a simpler method with a trade-off of being slightly less general.

We use the restrictions on the elliptic curves E_1 and E_2 in various parts of the proof of Theorem 3.20. Just as [Kir19, Proposition 6.2] uses the density of the subgroup $\mathbb{Q} + 2\pi i\mathbb{Q}$, we use the density of $\Lambda_1 + \Lambda_2$ to find a point in the intersection of the correspondence structure and an algebraic variety. In order to use this method we apply Ax's Theorem and calculate dimensions via Fact 3.17 and the symmetry given by the correspondence. As these require the elliptic curves to be non-isogenous and have isomorphic endomorphism rings, we are forced to restrict to elliptic curves without complex multiplication.

Furthermore, the case of isogenous elliptic curves is closely connected to the case of complex powers, handled in [GK24]. To see this, let $\Lambda \subseteq \mathbb{C}$ be a lattice, E_Λ the corresponding elliptic curve, $\exp_\Lambda : \mathbb{C} \to E_\Lambda$ the exponential map of E_Λ and $\beta \in \mathbb{C}^\times$. Then the set

$$\Gamma_\beta^\Lambda = \{(\exp_\Lambda(z), \exp_\Lambda(\beta z)) : z \in \mathbb{C}\} \subseteq E_\Lambda \times E_\Lambda$$

is an abelian analogue of the set

$$\Gamma_\beta = \{(\exp(z), \exp(\beta z)) : z \in \mathbb{C}\} \subseteq \mathbb{C}^\times \times \mathbb{C}^\times$$

considered in [GK24]. Now let us have a look at the isogenous case of our setup. Suppose Λ_1 and Λ_2 are lattices on $\mathbb{C}$ such that $\alpha\Lambda_1 \subseteq \Lambda_2$ for some $\alpha \in \mathbb{C}^\times$. Then there is an isogeny $f_\alpha : E_1 \to E_2$ between the corresponding elliptic curves with a graph

$$\Gamma_{f_\alpha} = \{(\exp_1(z), \exp_2(\alpha z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2.$$

Since isogenies are regular maps, the graph Γ_f is algebraic and definable in $\mathbb{C}_{\text{field}}$. Thus, recalling that

$$\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$$

and taking $\Lambda = \Lambda_2$, $\beta = \alpha^{-1}$ we get $\Gamma_{\text{corr}} = \{(t, s) : (f_\alpha(t), s) \in \Gamma_{\alpha^{-1}}^{\Lambda_2}\}$ and $\Gamma_{\alpha^{-1}}^{\Lambda_2} = \{(f_\alpha(t), s) : (t, s) \in \Gamma_{\text{corr}}\}$. Thus, the correspondence graph Γ_{corr} is interdefinable in $\mathbb{C}_{\text{field}}$ with the set Γ_β^Λ.

3.3 Gamma-fields and pregeometries

As one of the components of [BK18, Corollary 11.7] is the countable closure property, we need to define a particular pregeometry Γcl on $\mathbb{C}$. Furthermore, we show CCP by comparing Γcl to a pregeometry of the form $\text{Dcl}_{\mathcal{S},C_0}$, as introduced in Definition 2.5. We start by establishing a version of Ax's theorem for the correspondence group. Next we pick a specific countable $\mathcal{S}$ and C_0 and obtain an analogue of [Kir10, Corollary 5.2] for the corresponding $\text{Dcl}_{\mathcal{S},C_0}$. This allows us to introduce the pregeometry Γcl via so-called Γ-fields and show that it is weaker than the chosen $\text{Dcl}_{\mathcal{S},C_0}$ and thus has the countable closure property by Corollary 2.22. Note that this approach differs from the approach in [BK18], where the countable closure property for $\mathbb{C}_{\exp}$ was used. Instead we adapt results of [Kir10] for the case of a correspondence between elliptic curves and make use of the general framework developed in Chapter 2 which does not depend on the specific holomorphic functions in question.

Let us fix the setting for the rest of the chapter. Fix two lattices Λ_1 and Λ_2 with $\Lambda_1 + \Lambda_2$ dense in $\mathbb{C}$ and such that the corresponding elliptic curves E_1 and

E_2 are non-isogenous and do not have complex multiplication. Let f_1 and f_2 be the polynomials defining the affine part of E_1 and E_2 and let $K_0 \supseteq \mathbb{Q}$ be the field generated by the coefficients of f_1 and f_2. We assume that K_0 is a number field, i.e., the coefficients of f_1 and f_2 are algebraic numbers.

In order to get a suitable version of Ax's theorem, we also for the time being fix a field $F \supseteq K_0$ with commuting derivations $D_1, \dots, D_s$ and let $C = \bigcap_{k=1}^{s} \ker(D_k)$ its common field of constants containing K_0. Note that we do not require these derivations to preserve any functions. We get the version of Ax's theorem for a correspondence from the following stronger version for the Weierstrass elliptic functions as proved in [Kir05, Proposition 3.2].

Fact 3.1 (Proposition 3.2, [Kir05])**.** *Let $x_1, y_1, z_1, \dots, x_n, y_n, z_n \in F$ be such that $x_1, \dots x_n$ are $\mathbb{Q}$-linearly independent over C. Suppose for each x_j, y_j, z_j, D_k we have $f_1(y_j) \neq 0$, $f_2(z_j) \neq 0$ and*

$$\frac{(D_k y_j)^2}{f_1(y_j)} = (D_k x_j)^2, \frac{(D_k z_j)^2}{f_2(z_j)} = (D_k x_j)^2.$$

Let r be the rank of the Jacobian matrix $(D_k x_j)_{j,k}$ and L be the field generated over C by all the x_j, y_j and z_j. Then $\operatorname{td}(L/C) \geqslant 2n + r$.

In order to formulate a similar result for the correspondence we need a notion of linear independence for points on the elliptic curve. We say that points $p_1, \dots, p_n \in E_i(F)$ are *linearly independent over C* if they are $\mathbb{Q}$-linearly independent in the vector space $E_i(F)/E_i(C)$, i.e., there do not exist integer coefficients $k_1, \dots, k_n \in \mathbb{Z}$ and a C-point $c \in E_i(C)$ such that $\bigoplus_{i=1}^{n} k_i p_i = c$, where the sum is taken with respect to the group law on E_i. The linear independence of points on the elliptic curves is connected to the $\mathbb{Q}$-linear independence in the following way.

Lemma 3.2. *Let $x_1, \dots, x_n \in F$ and $p_1, \dots, p_n \in E_1(F)$ with affine coordinates (y_i, t_i) be such that $t_i D_k x_i = D_k y_i$ for all derivations $D_k \in \{D_1, \dots, D_s\}$. If $p_1, \dots, p_n$ are linearly independent over C, then $x_1, \dots, x_n$ are $\mathbb{Q}$-linearly independent over C.*

Proof. Suppose $p_1, \dots, p_n$ are linearly independent over C, but there is a $\mathbb{Q}$-linear dependence over $\mathbb{C}$ between $x_1, \dots, x_n$, i.e. $\sum_{i=1}^{n} k_i x_i = c$ for $k_i \in \mathbb{Q}$ and $c \in C$. Multiplying by denominators we can assume that $k_1, \dots, k_n$ are integers. Then for any $D_k \in \{D_1, \dots, D_s\}$ we have $D_k(\sum_{i=1}^{n} k_i x_i) = 0$. By [Kir09, Proposition 3.5], we know that the subset $\{(x, p) \in F \times E_1(F) : tD_k x = D_k y\}$, where (y, t) are the affine coordinates of p, forms a subgroup of $F \times E_1(F)$. Therefore, if we denote by p the sum $\bigoplus_{i=1}^{n} k_i p_i$ and by (y, t) its affine coordinates, we get $D_k y = 0$. Thus,

$D_k y = 0$ for any $D_k \in \{D_1, \dots, D_s\}$ and $y \in C$. But that again contradicts linear independence of $p_1, \dots, p_n$. □

Now we are ready to derive the version of Ax's theorem needed for our applications.

Theorem 3.3. *Let $p_1, \dots, p_n \in E_1(F)$ and $q_1, \dots, q_n \in E_2(F)$ be points linearly independent over C. Let the affine coordinates of p_i be (y_i, t_i) and the affine coordinates of q_i be (z_i, s_i) and suppose for each p_i, q_i, D_k we have $s_i D_k y_i = t_i D_k z_i$. Let r be the rank of the Jacobian matrix $(D_k y_i)_{i,k}$ and L be the field generated over C by $y_1, \dots, y_n$ and $z_1, \dots, z_n$. Then $\mathrm{td}(L/C) \geqslant n + r$.*

Proof. Take a differential field extension K of F by adding $x_1, \dots, x_n$ with $D_k x_i = \frac{D_k y_i}{t_i} = \frac{D_k z_i}{s_i}$. Note that all t_i and s_i are non-zero, since otherwise p_i or q_i would be a 2-torsion point, which cannot be a part of a linearly independent set.

Then $\mathrm{rk}(D_k x_i) = \mathrm{rk}(D_k y_i)$. Since $p_1, \dots, p_n$ are linearly independent over C, $x_1, \dots, x_n$ are $\mathbb{Q}$-linearly independent over C by Lemma 3.2. Since f_1 and f_2 define the affine part of E_1 and E_2 correspondingly, we have $t_i^2 = f_1(y_i)$ and $s_i^2 = f_2(z_i)$, thus $\frac{(D_k y_j)^2}{f_1(y_j)} = (D_k x_j)^2 = \frac{(D_k z_j)^2}{f_2(z_j)}$. Hence, we can apply Fact 3.1 to K and get $\mathrm{td}(L'/C) \geqslant 2n + \mathrm{rk}(D_k x_j)_{j,k} = 2n + r$ for L' the field generated over C by all the x_j, y_j and z_j. Note that L' can also be seen as the field generated over L by $x_1, \dots, x_n$, so $\mathrm{td}(L'/L) \leqslant n$. Therefore, $\mathrm{td}(L/C) = \mathrm{td}(L'/C) - \mathrm{td}(L'/L) \geqslant 2n + r - n = n + r$. □

Next we introduce functions which will define a corresponding pregeometry Dcl_γ. We move from working in a field F to working in $\mathbb{C}$. We imitate the correspondence Γ_{corr} by countably many functions which locally represent branches of a multi-valued map $\wp_2 \circ \wp_1^{-1}$. Recall that $\wp_1$ is locally invertible on $\mathbb{C} \setminus \frac{1}{2}\Lambda_1$. Let $U = \mathbb{C} \setminus (\frac{1}{2}\Lambda_1 \cup \Lambda_2)$. Since $\frac{1}{2}\Lambda_1 \cup \Lambda_2$ is discrete, U is open and $\wp_1$ is locally invertible on U. Consider a covering Ξ of U by rational boxes, i.e., open sets of the form

$$\xi = \{x + iy \in \mathbb{C} : q_1 < x < q_2,\ q_3 < y < q_4\} \text{ for } q_1, q_2, q_3, q_4 \in \mathbb{Q},$$

such that $\wp_1$ is invertible on each of them. Then for each $\xi \in \Xi$ we have a function $(\wp_1 \restriction_\xi)^{-1}$ defined on an open subset $V_\xi \subseteq \mathbb{C}$. Furthermore, each image $(\wp_1 \restriction_\xi)^{-1}(V_\xi)$ does not contain points in Λ_2, so $\gamma_\xi = \wp_2 \circ (\wp_1 \restriction_\xi)^{-1}$ is well-defined on V_ξ. Note that each γ_ξ is holomorphic on V_ξ and satisfies the differential equation $(\gamma_\xi'(z))^2 = \frac{f_2(\gamma_\xi(z))}{f_1(z)}$. Let C_0 be the field extension of K_0 generated by elements of $\wp_1(\Lambda_2 \setminus \Lambda_1) \cup \wp_2(\Lambda_1 \setminus \Lambda_2)$ and $\mathrm{Dcl}_{\gamma, C_0}$ the pregeometry on $\mathbb{C}$ specified by the subfield C_0 and the set of holomorphic functions $\{\gamma_\xi : \xi \in \Xi\}$. Note that since Ξ and C_0 are countable, $\mathrm{Dcl}_{\gamma, C_0}$ has CCP by Corollary 2.22.

Just as we did in Section 2.1, we are going to omit C_0 and write Dcl_γ, Der_γ and Ω_γ. Recall from Section 2.1 that $a \in \mathrm{Dcl}_\gamma(C)$ if for every $D \in \mathrm{Der}_\gamma(\mathbb{C}/C)$ we have $Da = 0$. Furthermore, Dcl_γ-closed sets are the sets $C \subseteq \mathbb{C}$ such that for every $a \in \mathbb{C}$ if for every $D \in \mathrm{Der}_\gamma(\mathbb{C}/C)$ we have $Da = 0$ then $a \in C$. Now we are ready to obtain an analogue of [Kir10, Corollary 5.2] for the correspondence between elliptic curves. Notably, just as [Kir10, Corollary 5.2] is a corollary of Ax's theorem [Ax71, Theorem 3], we derive our result from the version of Ax's theorem formulated and proved in Theorem 3.3.

Corollary 3.4. *Assume $C \subseteq \mathbb{C}$ is Dcl_γ-closed. Take $p_1, \dots, p_n \in E_1(\mathbb{C})$ and $q_1, \dots, q_n \in E_2(\mathbb{C})$ linearly independent over C and with $(p_i, q_i) \in \Gamma_{\mathrm{corr}}$ for $i = 1, \dots, n$. Let the affine coordinates of p_i be (y_i, t_i) and the affine coordinates of q_i be (z_i, s_i).*

Then $\mathrm{td}(\bar{y}, \bar{z}/C) \geqslant n + d(\bar{y}/C)$, where $d(\bar{y}/C)$ is the dimension of $\bar{y}$ with respect to the pregeometry Dcl_γ.

Proof. Note that since $p_1, \dots, p_n$ and $q_1, \dots, q_n$ are linearly independent over C, none of them can be the identity of the corresponding group. Let $L \subseteq \mathbb{C}$ be the algebraic closure of the field generated over C by $y_1, \dots, y_n$ and $z_1, \dots, z_n$. Then $t_1, \dots, t_n$ and $s_1, \dots, s_n$ also belong to L. Consider the set of derivations $\mathrm{Der}_\gamma(L/C)$ as introduced in Definition 2.2. Let $D \in \mathrm{Der}_\gamma(L/C)$ and $i = 1, \dots, n$. If y_i is a root of f_1, then $Dy_i = 0$ and $t_i = 0$. Thus, we have $s_i Dy_i = t_i Dz_i$. Otherwise there exists $\xi \in \Xi$ such that $y_i \in V_\xi$. Moreover, $\gamma_\xi(y_i) = z_i$ and $\gamma'_\xi(y_i) = \frac{s_i}{t_i}$. Note that $t_i \neq 0$. Thus, $y_i, \gamma_\xi(y_i), \gamma'_\xi(y_i) \in L$ and since $D \in \mathrm{Der}_\gamma(L/C)$ we have $Dz_i = \frac{s_i}{t_i} Dy_i$, which is equivalent to $s_i Dy_i = t_i Dz_i$. Therefore, $s_i Dy_i = t_i Dz_i$ for all $D \in \mathrm{Der}_\gamma(L/C)$ and $i = 1, \dots, n$.

Find $y_{i_1}, \dots, y_{i_m}$ among $y_1, \dots, y_n$, where $m = d(\bar{y}/C)$, and derivations $D_1, \dots, D_m \in \mathrm{Der}_\gamma(L/C)$ such that $D_j y_{i_k} = \delta_{j,k}$, the Kronecker delta. Then $y_{i_1}, \dots, y_{i_m}$ are linearly independent over $\mathbb{C}$. Since each $D \in \mathrm{Der}_\gamma(L/C)$ is determined by the action on $y_1, \dots, y_n$ and $z_1, \dots, z_n$, the vector space $\mathrm{Der}_\gamma(L/C)$ has finite dimension, so we can extend $D_1, \dots, D_m$ to a finite basis $D_1, \dots, D_s$. Thus we get $\mathrm{rk}(D_k y_i)_{i,k} = m$. We also have $C = \bigcap_k \ker D_k$ because C is Dcl_γ-closed, so we can apply Theorem 3.3, acquiring $\mathrm{td}(L/C) \geqslant n + m$. $\square$

We finish the section by introducing a closure operator $\Gamma\mathrm{cl}$ as described in [BK18, Section 4.3]. We use Corollary 3.4 to show that $\Gamma\mathrm{cl}$ is indeed a pregeometry and it is weaker than the pregeometry Dcl_γ. Thus we derive CCP for $\Gamma\mathrm{cl}$.

Defining pregeometry $\Gamma\mathrm{cl}$ and applying [BK18, Corollary 11.7] to get quasiminimality requires a notion of a Γ-field. As the general definition [BK18, Definition 3.8] is too vast for our purposes, we solely look at Γ-subfields as defined in [BK18, Definition 3.10] of the structure $\mathbb{C}_{\mathrm{corr}}$ seen as an analytic

Γ-field of type (COR) (see [BK18, Definitions 3.2]). We refer to these Γ-subfields simply as Γ-fields. Recall that $\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\}$ and for a field $K \supseteq K_0$, we denote by $E_i(K)$ the zero set of the corresponding polynomial in $\mathbb{P}^2(K)$ and by Tor_i be the torsion subgroup of E_i. The following is a reformulation of [BK18, Sections 3, 4] adapted to our setting.

Definition 3.5. A Γ-*field* is a pair $(K, \Gamma(K))$, where K is a field with $K_0 \subseteq K \subseteq \mathbb{C}$ and $\Gamma(K) \subseteq \Gamma_{\text{corr}} \cap E_1(K) \times E_2(K)$ is a divisible subgroup satisfying the following.

- The field generated over K_0 by the affine coordinates of points in $\Gamma(K)$ is exactly K.
- For each $i = 1, 2$ we have $\text{Tor}_i \subseteq \Gamma_i(K)$, where $\Gamma_i(K)$ is the projection of $\Gamma(K)$ onto $E_i(K)$.
- Points of the form $(\exp_1(\lambda_2), O_2)$ and $(O_1, \exp_2(\lambda_1))$, where $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$ and O_1, O_2 are the identity elements of E_1, E_2 respectively, belong to $\Gamma(K)$.

Note that $(\mathbb{C}, \Gamma_{\text{corr}})$ is a Γ-field. Also for any Γ-field $(K, \Gamma(K))$, the subgroup $\Gamma(K)$ contains the whole torsion subgroup $\text{Tor}_1 \times \text{Tor}_2 \subseteq \Gamma_{\text{corr}}$.

We show that every Dcl_γ-closed set can be seen as a Γ-field.

Proposition 3.6. *Let C be a Dcl_γ-closed subset of $\mathbb{C}$ and $\Gamma(C) = \Gamma_{\text{corr}} \cap E_1(C) \times E_2(C)$. Then $(C, \Gamma(C))$ is a Γ-field.*

Proof. First note that setting $\Gamma(C) = \Gamma_{\text{corr}} \cap E_1(C) \times E_2(C)$ automatically provides the first condition of Definition 3.5. As C is a Dcl_γ-closed subset, $C_0 \subseteq C$, hence $K_0 \subseteq C$ and we get the third condition. Moreover, being a Dcl_γ-closed subset implies that C is an algebraically closed field, which provides the second condition as well as $\Gamma(C)$ being divisible.

□

We define extensions of Γ-fields in the obvious way.

Definition 3.7. An *extension* of a Γ-field $(K, \Gamma(K))$ is a Γ-field $(L, \Gamma(L))$ with an inclusion of fields $K \subseteq L$ over K_0 such that $\Gamma(K) \subseteq \Gamma(L)$. We also say that $(K, \Gamma(K))$ is a Γ-*subfield* of $(L, \Gamma(L))$.

Thus, every Γ-field is a Γ-subfield of $(\mathbb{C}, \Gamma_{\text{corr}})$. Consider a Γ-field $(K, \Gamma(K))$ and its extension $(L, \Gamma(L))$. Since $\Gamma(K)$ contains all the torsion points, $\Gamma(L)/\Gamma(K)$ forms a vector space over $\mathbb{Q}$. We say that $(L, \Gamma(L))$ is *finitely generated* over $(K, \Gamma(K))$ if $\Gamma(L)/\Gamma(K)$ has finite dimension as a vector space over $\mathbb{Q}$. In this case we also have finite $\text{td}(L/K)$, as L is generated over K_0 by the affine coordinates of points in $\Gamma(L)$ and the group operation on $\Gamma(L)$ is algebraic. Now we can introduce the predimension function.

Definition 3.8. Let $(L, \Gamma(L))$ be a Γ-field finitely generated over a Γ-subfield $(K, \Gamma(K))$. Then we define the *predimension function* as $\delta(L/K) = \mathrm{td}(L/K) - \mathrm{ldim}_{\mathbb{Q}}(\Gamma(L)/\Gamma(K))$. Note that for the sake of brevity we write $\delta(L/K)$ rather than $\delta((L, \Gamma(L))/(K, \Gamma(K)))$.

Using the predimension function, we introduce a special kind of Γ-fields called strong Γ-fields.

Definition 3.9. We say that a Γ-field $(K, \Gamma(K))$ is *strong* if for every Γ-field $(L, \Gamma(L))$ finitely generated over $(K, \Gamma(K))$, we have $\delta(L/K) \geqslant 0$.

We show that any Dcl_γ-closed set C with a Γ-field structure as defined in Proposition 3.6 is strong.

Proposition 3.10. *Let C be a Dcl_γ-closed subset of $\mathbb{C}$ and $\Gamma(C) = \Gamma_{\mathrm{corr}} \cap E_1(C) \times E_2(C)$. Then $(C, \Gamma(C))$ is a strong Γ-field.*

Proof. $(C, \Gamma(C))$ is a Γ-field by Proposition 3.6. Suppose a Γ-field $(K, \Gamma(K))$ is finitely generated over $(C, \Gamma(C))$ by the points $(p_1, q_1), \ldots, (p_n, q_n) \in \Gamma_{\mathrm{corr}}$, i.e., $(p_1, q_1), \ldots, (p_n, q_n) \in \Gamma(K)$ are representatives of some basis for $\Gamma(K)/\Gamma(C)$. Thus these points are linearly independent over C. Using Corollary 3.4, we get $\delta(K/C) \geqslant d(\bar{y}/C)$. Since $d(\bar{y}/C) \geqslant 0$, we get $\delta(K/C) \geqslant 0$ and $(C, \Gamma(C))$ is strong. □

We define a pregeometry $\Gamma\mathrm{cl}$ on $\mathbb{C}$ by specifying its closed subsets. For simplicity we just call these subsets closed. Let $K_{\mathrm{base}} = \mathrm{Dcl}_\gamma(\varnothing)$. Note that we have $K_0 \subseteq C_0^{\mathrm{alg}} \subseteq K_{\mathrm{base}}$. Denote $\Gamma_{\mathrm{base}} = \Gamma_{\mathrm{corr}} \cap E_1(K_{\mathrm{base}}) \times E_2(K_{\mathrm{base}})$. Then by Proposition 3.10, $(K_{\mathrm{base}}, \Gamma_{\mathrm{base}})$ is a strong Γ-field.

Definition 3.11. A Γ-field $(K, \Gamma(K))$ extending $(K_{\mathrm{base}}, \Gamma_{\mathrm{base}})$ is *closed* if whenever a Γ-field $(L, \Gamma(L))$ is finitely generated over $(K, \Gamma(K))$ with $\delta(L/K) \leqslant 0$, we have $K = L$.

Note that closed Γ-fields are strong. Since $(K_{\mathrm{base}}, \Gamma_{\mathrm{base}})$ is strong, closed Γ-fields form a closure system and define a pregeometry due to [BK18, Proposition 4.14]. We denote this pregeometry by $\Gamma\mathrm{cl}$. As [BK18, Corollary 11.7] requires the countable closure property of $\Gamma\mathrm{cl}$, we show that any Dcl_γ-closed set with the Γ-field structure as defined in Proposition 3.6 is a closed Γ-field and derive CCP of $\Gamma\mathrm{cl}$ from CCP of Dcl_γ.

Proposition 3.12. *Let C be a Dcl_γ-closed subset of $\mathbb{C}$ and $\Gamma(C) = \Gamma_{\mathrm{corr}} \cap E_1(C) \times E_2(C)$. Then $(C, \Gamma(C))$ is a closed Γ-field.*

Proof. Note that $(C, \Gamma(C))$ is a Γ-field by Proposition 3.6. Similar to Proposition 3.10, suppose a Γ-field $(K, \Gamma(K))$ is finitely generated over $(C, \Gamma(C))$ by the points $(p_1, q_1), \ldots, (p_n, q_n) \in \Gamma_{\mathrm{corr}}$ and $\delta(K/C) \leqslant 0$. Using

Corollary 3.4, we get $\delta(K/C) \geqslant d(\bar{y}/C)$. Hence, $d(\bar{y}/C) = 0$ and since C is Dcl_γ-closed, $(K, \Gamma(K))$ and $(C, \Gamma(C))$ coincide. □

Thus, for any subset $A \subseteq \mathbb{C}$ we have $\Gamma\mathrm{cl}(A) \subseteq \mathrm{Dcl}_\gamma(A)$ and we can derive CCP.

Corollary 3.13. *The pregeometry* $\Gamma\mathrm{cl}$ *has the countable closure property.*

Proof. Let $A \subseteq \mathbb{C}$ be a countable subset. Then by Proposition 3.12 we have $\Gamma\mathrm{cl}(A) \subseteq \mathrm{Dcl}_\gamma(A)$. The subset $\mathrm{Dcl}_\gamma(A)$ is countable by Corollary 2.22. Hence, $\Gamma\mathrm{cl}(A)$ is also countable. □

3.4 Gamma-closedness and Quasiminimality

We want to show that $\mathbb{C}_{\mathrm{corr}}$ is quasiminimal by applying [BK18, Corollary 11.7], which states that a full Γ-field with the countable closure property which is Γ-closed has to be quasiminimal. In our context the property of a Γ-field $(K, \Gamma(K))$ being full means that projections of $\Gamma(K)$ onto $E_1(K)$ and $E_2(K)$ are surjective, which holds whenever $\Gamma(K) = \Gamma_{\mathrm{corr}} \cap E_1(K) \times E_2(K)$. The countable closure property in question is with respect to the pregeometry introduced in Section 3.3, and has thus been established in Corollary 3.13. This leaves us with only Γ-closedness left to prove. While our approach to CCP differed from the proof of CCP in [Kir19, Corollary 3.10], the proof of Γ-closedness is almost identical to that in [Kir19, Proposition 6.2]. As it requires certain symmetry and the considered subgroup to be dense, we put a restriction of E_1 and E_2 non-isogenous and without complex multiplication as well as $\Lambda_1 + \Lambda_2$ being dense. We start this section by defining Γ-closedness, then state lemmas needed for the proof and finish the chapter by showing that $\mathbb{C}_{\mathrm{corr}}$ is indeed Γ-closed and therefore obtaining quasiminimality.

Analogously to other Existential closedness type of properties, such as Exponential-Algebraic closedness and Modular Existential closedness (see [Asl24] for an exposition), Γ-closedness, which is a generalization of Exponential-Algebraic closedness, considers intersections of algebraic varieties with the structure in question. More specifically, we fix some $n > 0$ and a Γ-field $(K, \Gamma(K))$ and consider an irreducible subvariety $V \subseteq E_1(K)^n \times E_2(K)^n$ of dimension n, which satisfies certain conditions, and the subgroup $\Gamma(K)^n \subseteq E_1(K)^n \times E_2(K)^n$. Then Γ-closedness states that V intersects $\Gamma(K)^n$. Let us now define the conditions of being free and rotund which make sure that the subvariety V is general enough to intersect with $\Gamma(K)^n$. Recall that we have already defined these conditions for the exponential case in Chapter 1. We keep using the same naming as both cases can be captured by a more general Γ-field setting. We denote the first n coordinates of $E_1(K)^n \times E_2(K)^n$ by $x_1, \ldots, x_n$ and the second n coordinates by $y_1, \ldots, y_n$.

Definition 3.14 (Definition 7.1, [BK18])**.** Let V be an irreducible subvariety of $E_1(K)^n \times E_2(K)^n$. Then V is *left-free* if V does not lie inside an algebraic subvariety defined by a group equation of the form $\bigoplus_{j=1}^{n} r_j x_j = c$, where $r_j \in \mathbb{Z}$ (not all zero) and $c \in E_1(K)$. Similarly, V is *right-free* if V does not lie inside an algebraic subvariety defined by a group equation of the form $\bigoplus_{j=1}^{n} r_j y_j = c$, where $r_j \in \mathbb{Z}$ (not all zero) and $c \in E_2(K)$. V is *free* if it is both left-free and right-free.

Note that V being free is equivalent to both projections not being contained in any coset of a proper algebraic subgroup of the corresponding $E_i(K)$.

Since $E_1(K)$ and $E_2(K)$ are $\mathbb{Z}$-modules, matrices in $\mathrm{Mat}_n(\mathbb{Z})$ act on both $E_1(K)^n$ and $E_2(K)^n$. That gives us an action on $E_1(K)^n \times E_2(K)^n$. For $M \in \mathrm{Mat}_n(\mathbb{Z})$ and a subvariety $V \subseteq E_1(K)^n \times E_2(K)^n$, we denote the image of V under M by $M \cdot V$. The action of M is by regular maps, so $M \cdot V$ is an algebraic variety of $E_1(K)^n \times E_2(K)^n$ and has dimension.

Definition 3.15. Let V be an irreducible subvariety of $E_1(K)^n \times E_2(K)^n$. Then V is *rotund* if for every matrix $M \in \mathrm{Mat}_n(\mathbb{Z})$ we have $\dim(M \cdot V) \geqslant \mathrm{rk}\ M$, where dim is the dimension as an algebraic variety and rk M is the rank of the matrix M.

In particular, taking M to be the identity matrix, we see that rotund varieties have dimension at least n. Having defined the properties of being rotund and free, we can state the definition of Γ-closedness.

Definition 3.16 (Definition 10.3, [BK18])**.** A Γ-field $(K, \Gamma(K))$ is Γ-closed if for every free and rotund irreducible subvariety V of $E_1(K)^n \times E_2(K)^n$ of dimension n, the set $V \cap \Gamma(K)^n$ is non-empty.

Before proving that $\mathbb{C}_{\mathrm{corr}}$ is Γ-closed let us introduce some useful facts and lemmas. The first result is a version of a well-known Goursat-Kolchin-Ribet lemma in the case of two non-isogenous elliptic curves without complex multiplication. We refer to [JKS16, Fact 5.2] where it is proven in a slightly greater generality.

Fact 3.17 (Fact 5.2, [JKS16])**.** *Let E_1, E_2 be non-isogenous elliptic curves without complex multiplication. Consider the algebraic group $G = E_1^n \times E_2^n$, and let H be a connected algebraic subgroup of G. Then the following hold.*

- *H is of the form $H_1 \times H_2$, where H_1 is a subgroup of E_1^n and H_2 is a subgroup of E_2^n.*
- *There are subgroups H_1^* of E_1^n and H_2^* of E_2^n defined by a system of linear equations of the form $\bigoplus_{i=1}^{n} a_i x_i = 0$ for $a_i \in \mathbb{Z}$, such that H_1 and H_2 are subgroups of the corresponding H_1^* and H_2^* of finite index.*

Another component of the proof is Ax's theorem. We have already seen versions of it for Weierstrass elliptic functions (Fact 3.1) and for the correspondence group (Theorem 3.3). The following is a more general statement, formulated in terms of measuring the dimension of an intersection of an analytic subgroup and an algebraic subvariety.

Fact 3.18 (Corollary 1, [Ax72])**.** *Let G be an algebraic group defined over $\mathbb{C}$. Let D be a connected analytic subgroup of G, and K be an irreducible analytic subvariety of an open subset of G with $K \subseteq D$. Let W be the Zariski closure of K and H the smallest algebraic subgroup of G containing W. Then*

$$\dim W \geqslant \dim HD - \dim D + \dim K.$$

Finally, we need a technical lemma analogous to [Kir19, Lemma 6.1]. Note that exactly the same proof goes through and we copy it below.

Lemma 3.19. *Suppose $V \subseteq E_1^n \times E_2^n$ is irreducible and rotund, and* $\dim V = n$. *Let J be a connected algebraic subgroup of E_1^n given by a matrix equation $M\bar{x} = 0$ and $\widetilde{J}$ the algebraic subgroup of $E_1^n \times E_2^n$ given by matrix equations $M\bar{x} = 0$ and $M\bar{y} = 0$. Then there is a Zariski-open dense subset V_J of V such that if $\gamma \in V_J$ then* $\dim(V \cap \gamma \oplus \widetilde{J}) \leqslant \dim J$.

Proof. The intersection $V \cap \gamma \oplus \widetilde{J}$ is a fibre of a regular map $f : V \to M \cdot V$. So by the fibre dimension theorem [BMZ07, Section 2] there is a Zariski-open dense V_J such that for $\gamma \in V_J$ we have $\dim(V \cap \gamma \oplus \widetilde{J}) = n - \dim M \cdot V$. Since V is rotund, $\dim M \cdot V \geqslant \operatorname{rk} M = n - \dim J$. Thus for $\gamma \in V_J$ we have $\dim(V \cap \gamma \oplus \widetilde{J}) \leqslant \dim J$ as required. □

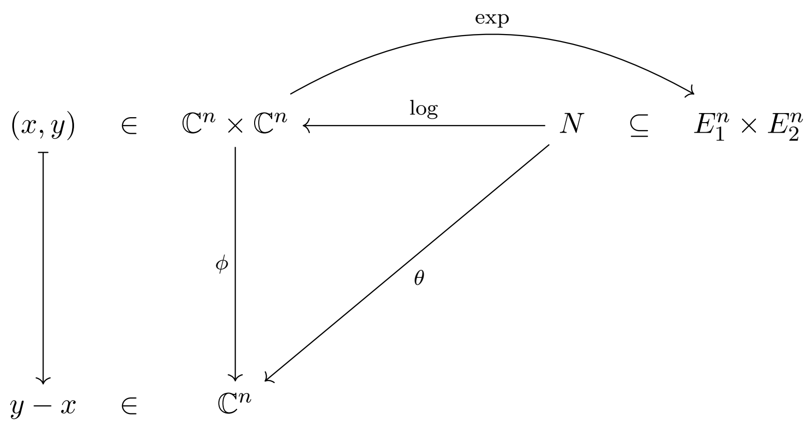
We finish this chapter by proving that $\mathbb{C}_{\text{corr}}$ is Γ-closed and therefore quasiminimal. The proof follows the proof of [Kir19, Proposition 6.2], adopting it to the elliptic setting. The map $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (\exp_1(x_1), \dots, \exp_1(x_n), \exp_2(y_1), \dots, \exp_2(y_n))$ serves as an analogue for the map $(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto (\frac{y_1}{\exp(x_1)}, \dots, \frac{y_n}{\exp(x_n)})$, while the group $\Lambda_1 + \Lambda_2$ serves as an analogue for a dense subgroup, thus requiring us to assume its density. The main difference between our proof and that of the blurred exponentiation is in the use of the Ax's theorem. While [Kir19, Proposition 6.2] cites [Ax71, Theorem 3] with the theorem formulated via derivations and specific to the exponentiation, we apply [Ax72, Corollary 1] as restated in 3.1 and calculate dimensions directly from there. Note that we do not assume E_1 and E_2 to be defined over a number field, though we restrict to E_1 and E_2 non-isogenous, without complex multiplication and with $\Lambda_1 + \Lambda_2$ dense.

Theorem 3.20. *The Γ-field $\mathbb{C}_{\text{corr}}$ is Γ-closed.*

Proof. Let V be a free and rotund irreducible subvariety of $E_1^n \times E_2^n$ of dimension n. Consider the map $\exp : \mathbb{C}^n \times \mathbb{C}^n \to E_1^n \times E_2^n$ acting by

$$(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (\exp_1(x_1), \ldots, \exp_1(x_n), \exp_2(y_1), \ldots, \exp_2(y_n)).$$

Take any regular point $a \in V$ and let N be a neighbourhood of a in $E_1^n \times E_2^n$ on which exp is invertible. Then we get an inverse $\log : N \to \mathbb{C}^n \times \mathbb{C}^n$. Let $\phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ be the map $(x, y) \mapsto y - x$ and consider the map $\theta : N \to \mathbb{C}^n$ defined as $\phi \circ \log$.



Let $U = N \cap V$, a neighbourhood of a in V. Note that $\dim U = n$. We want to find a point $b \in U$ such that b also belongs to Γ^n_{corr}, i.e. there is $\bar{z} \in \mathbb{C}^n$ such that $b = \exp(\bar{z}, \bar{z})$. If we denote by $(\bar{x}_0, \bar{y}_0)$ the image $\log(b)$, then this condition is equivalent to $\theta(b) = \bar{y}_0 - \bar{x}_0 \in \Lambda_1^n + \Lambda_2^n$. Note that this set is dense in $\mathbb{C}^n$ since $\Lambda_1 + \Lambda_2$ is dense in $\mathbb{C}$.

Let $\bar{t} = \theta(a)$ and $A = \theta^{-1}(\bar{t}) \cap U$. Note that $a \in A$ and $A \subseteq a \oplus \Gamma^n_{\mathrm{corr}}$. Furthermore, A is an analytic subset of U, so, by taking U sufficiently small, we may assume that A is connected. We first show that when $\dim A = 0$, finding b as above can be done via the Remmert open mapping theorem. The rest of the proof is dedicated to proving that there exists a regular point a with $\dim A = 0$.

Thus suppose that $\dim A = 0$. Then, since it is connected, A is the singleton $\{a\}$, and a is an isolated point of $\theta^{-1}(\bar{t})$. So, shrinking U if necessary, $\theta \restriction_U$ is finite-to-one [Łoj91, p 257]. Then, by the Remmert open mapping theorem [Łoj91, p 297], $\theta(U)$ is open in $\mathbb{C}^n$. Hence, it intersects with the set $\Lambda_1^n + \Lambda_2^n$, so we have found the point b such as above and $\mathbb{C}_\Gamma$ is Γ-closed.

It remains to show that we can find a regular point $a \in V$ such that $\dim A = 0$. Suppose that $\dim A = d > 0$. In order to apply Ax's Theorem (Fact 3.18), we translate all considered structures by a. Let A' be the set $A \ominus a = \{x \ominus a : x \in A\}$, where $\ominus$ is taken with respect to the group structure. Define V' in the same way as $V \ominus a$ and U' as $U \ominus a$. Then we have $A' \subseteq \Gamma^n_{\mathrm{corr}}$ and $O \in A'$, where O is the zero of the group $E_1^n \times E_2^n$ (and its subgroup Γ^n_{corr}). Moreover, $\dim A' = d$.

Assume A' is irreducible, otherwise take the irreducible component of A' containing O. Let W be the Zariski closure of A' and H the smallest algebraic subgroup containing W. Note that since A' is irreducible, W is also irreducible.

Apply Ax's theorem (Theorem 3.18) from above to $G = E_1^n \times E_2^n$, $D = \Gamma^n_{\text{corr}}$, $K = A'$ and H, W as before. As these satisfy the required conditions, we get $\dim W \geqslant \dim(H \oplus \Gamma^n_{\text{corr}}) - n + d > \dim(H \oplus \Gamma^n_{\text{corr}}) - n$. In order to calculate $\dim(H \oplus \Gamma^n_{\text{corr}})$, let H' be the connected component of H containing the zero of the group. Then H' is an algebraic subgroup of $E_1^n \times E_2^n$ and we can use Fact 3.17 to obtain corresponding H_1, H_2 and H_1^*, H_2^*. Note that $\dim(H' \oplus \Gamma^n_{\text{corr}}) = \dim(H \oplus \Gamma^n_{\text{corr}})$. Since $A' \subseteq \Gamma^n_{\text{corr}}$ and a group equation is also an algebraic equation, for any group equation on E_1^n of the form $\bigoplus_{i=1}^n a_i x_i = 0$ that holds on W, the same group equation but on E_2^n, i.e. $\bigoplus_{i=1}^n a_i y_i = 0$, also has to hold on W. Therefore, the same holds for H and H', so we can choose H_1^* and H_2^* to be defined by the same linear equations in E_1^n and E_2^n correspondingly. Let $\dim H_1^* = \dim H_2^* = r$. As each H_i is a subgroup of H_i^* of finite index, we also have $\dim H_1 = \dim H_2 = r$ and $\dim H = 2r$.

Now let $H^* = H_1^* \times H_2^*$. Then $\dim(H' \oplus \Gamma^n_{\text{corr}}) = \dim(H^* \oplus \Gamma^n_{\text{corr}})$, because $[H^* \oplus \Gamma^n_{\text{corr}} : H' \oplus \Gamma^n_{\text{corr}}] \leqslant [H^* : H]'$ and therefore $H' \oplus \Gamma^n_{\text{corr}}$ is a subgroup of finite rank. Moving to the tangent spaces, we get the Lie algebras LH^* of H^* and $L\Gamma^n_{\text{corr}}$ of Γ^n_{corr}. Then the Lie algebra of $H^* \oplus \Gamma^n_{\text{corr}}$ is $LH^* + L\Gamma^n_{\text{corr}}$ and we can calculate its dimension as $\dim(LH^* + L\Gamma^n_{\text{corr}}) = \dim LH^* + \dim L\Gamma^n_{\text{corr}} - \dim(LH^* \cap L\Gamma^n_{\text{corr}})$. Since LH^* is defined by the same equations on both sides and $L\Gamma^n_{\text{corr}}$ is defined by equations $x_i = y_i$, we have $\dim(LH^* \cap L\Gamma^n_{\text{corr}}) = \dim LH_i^* = r$. Therefore, $\dim(H^* \oplus \Gamma^n_{\text{corr}}) = \dim(LH^* + L\Gamma^n_{\text{corr}}) = 2r + n - r = r + n$ and $\dim W > r$. As $W \subseteq V' \cap H^*$ we have $\dim(V' \cap H^*) > r$ and translating everything back by a, $\dim(V \cap a \oplus H^*) > r$.

Now apply Lemma 3.19 to $J = H_1^*$ and therefore $\widetilde{J} = H^*$. We obtain a Zariski-open dense subset V_J of V, such that if $\gamma \in V_J$, then $\dim(V \cap \gamma \oplus H^*) \leqslant \dim H_1 = r$. Hence, a cannot belong to V_J. We can finally show how to chose a in order to get $\dim A = 0$. Let $V^o = V^{\text{reg}} \cap \bigcap_{J \leqslant E_1^n} V_J$, where V^{reg} is the set of regular points in V, which is Zariski-open in V. Each $V \setminus V_J$ is a Zariski-closed subset of V of lower dimension, and there are only countably many algebraic subgroups J, so V^o is non-empty. Taking any $a \in V^o$ we have that $\dim A = 0$, which completes the proof. □

We conclude by deriving Theorem 1.5 from the results of this chapter together with [BK18, Corollary 11.7]. Thus, this is an analogue of [Kir19, Theorem 7.2]. Here we have to restrict to the elliptic curves defined over a number field due to the use of the Kummer theory of semiabelian varieties over number fields in [BK18].

Theorem (Theorem 1.5)**.** Let E_1, E_2 be two non-isogenous elliptic curves without complex multiplication defined over a number field. Let Λ_1 and Λ_2 be the corresponding lattices on $\mathbb{C}$ and $\exp_1$, $\exp_2$ their corresponding exponential maps. Assume $\Lambda_1 + \Lambda_2$ is dense in $\mathbb{C}$ and consider the subgroup $\Gamma_{\text{corr}} = \{(\exp_1(z), \exp_2(z)) : z \in \mathbb{C}\} \subseteq E_1 \times E_2$. Then $\mathbb{C}_{\text{corr}} = (\mathbb{C}; +, \cdot, \Gamma_{\text{corr}})$ is quasiminimal.

Proof. By [BK18, Corollary 11.7] it suffices to show that $\mathbb{C}_{\text{corr}}$ is Γ-closed and Γcl has the countable closure property. The former one is Theorem 3.20 and the later one is Corollary 3.13. □

4

Generic function

The theory of a generic function was first presented in [Zil02], where Zilber considered an algebraically closed field of characteristic zero together with a function and all its derivatives, satisfying two conditions: a generalized Schanuel property and a form of Existential closedness. Then by [Zil02, Theorem 3.18] this theory is complete and has quantifier elimination in a certain language extension. Moreover, due to [Zil02, Theorem 3.19] it is ω-stable. In the survey [Zil05b] a slightly different language was considered and the derivatives were dropped from the theory. There was still a question of whether there exists a holomorphic function on $\mathbb{C}$ satisfying the theory of a generic function, until Wilkie suggested a construction based on Liouville numbers in [Wil05]. In [Koi03] Koiran finished Wilkie's proof and, moreover, demonstrated that the theory of a generic function coincides with the limit theory of generic polynomials from [Koi05].

We investigate the theory of a generic function further. In Section 4.2 we display a detailed account of the Hrushovski-style amalgamation construction and give full classification of the types together with their Morley ranks in Section 4.3. Thus we also reprove Zilber's result on ω-stability. For a model $\mathcal{M}$ of our theory we define a closure operator $\mathrm{gcl}_{\mathcal{M}}$ in Section 4.4 and in case of a model of the form $\mathbb{C}_g = (\mathbb{C}; +, \cdot, g)$ with an entire, i.e. holomorphic on $\mathbb{C}$, function g, this closure has the countable closure property. Together with the classification of types, the CCP provides quasiminimality of $\mathbb{C}_g$ by elementary means. Thus we obtain Theorem 1.6, restated below.

Theorem (Theorem 1.6)**.** Let $g : \mathbb{C} \to \mathbb{C}$ be an entire generic function. Then $\mathbb{C}_g$ is quasiminimal.

Moreover, in Section 4.5 we show that the models with countable closure property are exactly the prime models over independent sets (in the sense of gcl). Thus, all models of the form $\mathbb{C}_g$ are isomorphic, i.e., we obtain Theorem 1.7. We state it again as well.

Theorem (Theorem 1.7)**.** Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be entire generic functions. Then $\mathbb{C}_{g_1} \cong \mathbb{C}_{g_2}$.

This roughly follows the strategy outlined by Zilber in [Zil05b].

4.1 Theory of a generic function

In this section we present the theory of a generic function T_{gf} and the construction of an entire generic function on $\mathbb{C}$ developed by Wilkie in [Wil05]. First we display the axiomatization of T_{gf} in the spirit of [Zil05b] and [Koi03]. In [Wil05] Wilkie considered a version of T_{gf} with an exception $g(0) = 0$ to the generalized Schanuel property, and Koiran kept this condition in [Koi03]. In general, one can modify the theory to satisfy any fixed finite set of such exceptions and obtain all the same results. We stick to Wilkie's version with $g(0) = 0$ to avoid unnecessary complications to the notation, formulating a more general setting and stating the main results for it in Section 4.6.

In order to write down the axiom corresponding to the Existential closedness property we need to introduce restrictions on the varieties, similar to the freeness and rotundity that we defined for exponential and elliptic cases in Chapters 2 and 3. We keep the same naming for free varieties, although once again it does not coincide with definitions of freeness in other context. Regarding rotundity, we now call the corresponding property *broadness*, as it has already been used under this name to formulate a form of Existential closedness for the j-function (see for example [AK22]).

Definition 4.1. Let F be an algebraically closed field of characteristic 0, $V \subseteq F^{2n}$ an absolutely irreducible algebraic variety and denote the coordinates of F^{2n} by $x_1, \dots, x_n, y_1, \dots, y_n$.

- For an injective map $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ with $m \in \mathbb{N}$, denote by $V^\sigma \subseteq F^{2m}$ the projection $V^\sigma = \{(x_{\sigma(1)}, \dots, x_{\sigma(m)}, y_{\sigma(1)}, \dots, y_{\sigma(m)}) : (\bar{x}, \bar{y}) \in V\}$.
- We call V *free* if it is not contained in a subvariety defined by an equation of the form $x_i = x_j$ for $i \neq j$ or $x_i = c$ for $c \in F$.
- We call V *broad* if for all $m \leqslant n$ and injective $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ we have $\dim V^\sigma \geqslant m$.
- Assume we have fixed a function $g : F \to F$. For a set $A \subseteq F$, denote by $V_A^\dagger$ the set of all n-tuples $\bar{x} \in F^n$ with all elements of $\bar{x}$ non-zero, distinct from each other and elements of A, such that $(\bar{x}, g(\bar{x})) \in V$.

A variety $V \subseteq F^2$ with $\dim V = 1$ is always broad. In Proposition 4.45 we consider a family of varieties $V \subseteq F^{2n}$ defined by $y_1 = x_2, \dots, y_{n-1} = x_n, y_n = c$ for some constant $c \in F$. These provide non-trivial examples of an absolutely irreducible free and broad variety.

Note that when A is finite, the set $V_A^\dagger$ is definable. Moreover, we can axiomatize the property of being an absolutely irreducible free and broad algebraic variety over $\bar{a}$ of dimension n in the following way. Let $\tilde{V} \subseteq K^{2n+m}$ be an algebraic variety over $\mathbb{Q}$. Then for any $\bar{a} \in K^m$ we get a projection of the fibre $V(\bar{a}) = \{(\bar{x}, \bar{y}) \in F^{2n} : (\bar{x}, \bar{y}, \bar{a}) \in \tilde{V}\}$. Note that $V(\bar{a})$ is an algebraic variety over $\bar{a}$. Recall that in ACF_0 Morley rank and Morley degree coincide with the dimension of a variety and the number of irreducible components of maximal dimension. As these are definable (see for example [HC17, Lemma 5.9]), the properties of being absolutely irreducible and having dimension n are also definable. Furthermore, looking at dimension of V^σ we see that being broad is also definable, while defining freeness is straightforward. Thus, there is a first-order formula $\theta_{\tilde{V}}(\bar{z})$ in the language of fields such that $F \models \theta_{\tilde{V}}(\bar{a})$ if and only if $V(\bar{a})$ is an absolutely irreducible free and broad algebraic variety over $\bar{a}$ of dimension n. We use the formula $\theta_{\tilde{V}}$ to axiomatize the theory of a generic function as follows.

Definition 4.2. Let $\mathcal{L}_{\mathrm{gf}}$ be the language of fields with an additional binary relation $g(x) = y$, where we think of g as a function on the field F. We define the first-order theory T_{gf} in the language $\mathcal{L}_{\mathrm{gf}}$ through the following axiomatization.

1. ACF_0.
2. $g : F \to F$ is a function with $g(0) = 0$.
3. Let $V \subseteq F^{2n}$ be an algebraic variety over $\mathbb{Q}$ with $\dim V < n$. Then the set $V^\dagger_\varnothing$ is empty.
4. Let $V \subseteq F^{2n}$ be an absolutely irreducible, free and broad algebraic variety over $\bar{a}$ of dimension n, i.e., $F \models \theta_{\tilde{V}}(\bar{a})$ for the corresponding family $\tilde{V}$. Then the set $V^\dagger_{\bar{a}}$ is non-empty.

Note that we follow [Koi05] and for convenience use g as a binary relation rather than a function symbol, since it simplifies the Hrushovski construction.

First two axioms provide the set up: a function g on an algebraically closed field of characteristic zero with a condition $g(0) = 0$. The third axiom constitutes the generalized Schanuel property; while the standard Schanuel conjecture gives a bound on the transcendental degree of linearly independent numbers and their images under exp, the generalized version only requires the numbers to be non-zero and distinct in order to get the same bound for g. The last axiom is a form of Existential closedness; more specifically it is equivalent to existential closedness with respect to strong extensions which are introduced in Definition 4.10.

In [Koi05] Koiran considers generic polynomials, that is polynomials $f \in \mathbb{C}[x]$ with zero constant term and all other coefficients algebraically independent over $\mathbb{Q}$. Then the limit theory of generic polynomials is the set of all sentences which are true for all generic polynomials of sufficiently high degree. Note that we work here in the same language $\mathcal{L}_{\mathrm{gf}}$. Koiran provides a first-order axiomatization for the limit theory of generic polynomials in [Koi05, Section 3] and shows that this theory is consistent and complete in [Koi05, Section 3.1, Proposition 2]. The limit theory of generic polynomials coincides with the theory of a generic function by [Koi03, Section 2.3, Theorem 2] and therefore T_{gf} is also consistent and complete.

Finally, let us present the construction for a model of T_{gf} developed in [Wil05]. It was inspired by Liouville numbers, which are certain transcendental numbers that are well-approximated by rational numbers, Wilkie named his functions, which are well-approximated by rational polynomials, Liouville functions. Note that despite the name they have nothing to do with the Liouville lambda function. Since Liouville functions are well-approximated by rational polynomials, in [Wil05] Wilkie managed to show that such functions are "very transcendental" in the sense that they satisfy the generalized Schanuel property. Hence, the only condition left to check was the form of Existential closedness, and Wilkie proved it for a variety over $\mathbb{Q}$, while Koiran finished his proof in [Koi03] for an arbitrary variety. Let us now define the Liouville functions and state these results.

Definition 4.3. A holomorphic function $L : \mathbb{C} \to \mathbb{C}$ is called a *Liouville function* if it has Taylor series of the form $L(x) = \sum\limits_{n=1}^{\infty} \frac{x^n}{a_n}$ where a_n are non-zero integers satisfying

$$\forall k \geqslant 1 \; \exists N \; \forall n > N \; |a_{n+1}| > |a_n|^{n^k}.$$

Note that Taylor series of this form converge everywhere on $\mathbb{C}$, so any choice of coefficients with the property above produces a well-defined entire function. Recall that for a function $f : \mathbb{C} \to \mathbb{C}$ we denote by $\mathbb{C}_f$ the structure $(\mathbb{C}; +, \cdot, f)$.

Theorem 4.4. *Let $L : \mathbb{C} \to \mathbb{C}$ be a Liouville function. Then we have $\mathbb{C}_L \models T_{\mathrm{gf}}$.*

Proof. See [Wil05, Theorem 1] and [Koi03, Theorem 2]. □

In this thesis we investigate the theory T_{gf} and do not make use of the Liouville functions per se. However, we point out that they serve as concrete examples of entire generic functions and therefore Theorems 1.6 and 1.7 apply to them as well.

4.2 Amalgamation construction

In this section we use Hrushovski's amalgamation-with-predimension method to construct the Fraïssé limit of the theory T_{gf}. We have already seen some parts of this construction in Section 3.3, where we considered (strong) Γ-fields with a predimension function. Similarly, in this section we are defining gf-fields, strong embeddings and a suitable predimension function. We go further and establish an amalgamation category (as defined in [BK18, Section 5] and called an $\aleph_0$-amalgamation category in [Kir09, Section 2.6]), showing that the obtained limit is indeed a model of T_{gf}. We mostly follow the approach from [BK18], also displayed in [GK24] and start by defining gf-fields, the objects of the future category.

Definition 4.5. A *gf-field* is a field K of characteristic 0 together with a subset $D(K) \subseteq K$, called *the domain* of K, and a map $g : D(K) \to K$ such that $0 \in D(K)$, $g(0) = 0$ and K is generated as a field by $D(K) \cup g(D(K))$. In case of multiple gf-fields, we denote g by g_K to avoid confusion.

We call a gf-field K *full* if $D(K) = K$ and K is algebraically closed and denote by $\mathbb{Q}_g$ the gf-field structure on $\mathbb{Q}$ with $D(\mathbb{Q}_g) = \{0\}$ and $g(0) = 0$.

We also consider extensions of gf-fields.

Definition 4.6. Let L be a gf-field. A *gf-subfield* K of L is a gf-field consisting of a subset $D(K) \subseteq D(L)$, the restriction $g_K = g_L|_{D(K)}$ and the subfield of L generated by $D(K) \cup g_K(D(K))$, which we also denote by K.

Whenever K is a gf-subfield of L, we also call $K \subseteq L$ a *gf-field extension*. Finally, an embedding of fields $f : K \hookrightarrow L$ is an *gf-field embedding* if the image of K is a gf-subfield of L.

The introduced notation might be a bit misleading since we can have gf-subfields $K, K' \subseteq L$ with $D(K) \neq D(K')$, while K and K' are the same subfields of L. Here is a toy example.

Example 4.7. Let K, K' and L all be $\mathbb{Q}(s,t)$ field-wise, where $s, t \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$. Take $D(K) = \{0, s\}$, $D(K') = \{0, t\}$ and $D(L) = \{0, s, t\}$ with $g(s) = t$ and $g(t) = s$. Then K and K' are the same subfields of L, but different gf-subfields of L.

Thus a more useful notion is that of a generated gf-subfield, clarifying that what really determines a gf-subfield is the subset $D(K)$.

Definition 4.8. Let K be a gf-field and $A \subseteq D(K)$. Then the *gf-subfield generated by* A is the gf-subfield $\langle A \rangle$ of K with $D(\langle A \rangle) = A \cup \{0\}$ and $g_{\langle A \rangle} = g_K|_{A \cup \{0\}}$. Thus, field-wise $\langle A \rangle$ is the subfield generated by $A \cup g_K(A)$. Note that in case A is empty, we get the gf-subfield $\mathbb{Q}_g$. When A is a finite tuple $\bar{a}$, we write $\langle \bar{a} \rangle$.

For convenience we switch between notations depending on the context. Now we are ready to define our analogue of Hrushovski's predimension function.

Definition 4.9. Let K be a gf-field, $A \subseteq D(K)$ a subset and $\bar{b} \in D(K)$ a finite tuple. Then $\delta(\bar{b}/A) = \mathrm{td}(\bar{b}, g(\bar{b})/\langle A \rangle) - |\bar{b} \setminus (A \cup \{0\})|$.

Note that adding elements from $A \cup \{0\}$ or $\bar{b}$ itself to $\bar{b}$ does not change $\delta(\bar{b}/A)$. Thus, most of the time we are going to assume that all elements of $\bar{b}$ are non-zero, distinct from each other and elements of A. In this case the formula gets simplified to $\delta(\bar{b}/A) = \mathrm{td}(\bar{b}, g(\bar{b})/\langle A \rangle) - |\bar{b}|$.

Following Hrushovski's method we also define strong embeddings.

Definition 4.10. Let K be a gf-field and $A \subseteq D(K)$. Then A is a *strong* subset if for any finite tuple $\bar{b} \in D(K)$ we have $\delta(\bar{b}/A) \geqslant 0$. In this case, the corresponding gf-field extension $\langle A \rangle \subseteq K$ is called a *strong extension* and we denote it by $\langle A \rangle \lhd K$. If $f : K \hookrightarrow L$ is a gf-field embedding with the image of K being strong in L, we call it a *strong embedding* and denote it by $f : K \overset{\lhd}{\hookrightarrow} L$.

Note that $\delta(\bar{b}/A) < 0$ is a definable property, so a strong subset of a model F of T_{gf} is also strong in an elementary extension of F. The operations of intersection and union on the subsets of the domain of a gf-field give rise to the operations of meet and join on the generated gf-subfields.

Definition 4.11. Let K be a gf-field and $\{A_i \subseteq D(K) : i \in I\}$ a collection of subsets. Then the *meet* of the gf-subfields $\{\langle A_i \rangle : i \in I\}$ is the gf-subfield $\langle \bigcap_{i \in I} A_i \rangle$ and the *join* is the gf-subfield $\langle \bigcup_{i \in I} A_i \rangle$.

Note that field-wise, while the field $\langle \bigcup_{i \in I} A_i \rangle$ coincides with the field generated by $\bigcup_{i \in I} \langle A_i \rangle$, the field $\langle \bigcap_{i \in I} A_i \rangle$ does not always coincide with the field $\bigcap_{i \in I} \langle A_i \rangle$. Here is an example of such behaviour.

Example 4.12. Let $K = \mathbb{Q}(\alpha, \beta, \gamma)$, where $\alpha, \beta, \gamma \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$. Turn K into a gf-field by letting $D(K) = \{0, \alpha, \alpha + 1\}$ and $g(\alpha) = \beta$, $g(\alpha + 1) = \gamma$. Consider two gf-subfields $\langle \alpha \rangle$ and $\langle \alpha + 1 \rangle$ of K. Then their meet is just $\mathbb{Q}_g$, though field-wise their intersection would be $\mathbb{Q}(\alpha, \beta) \cap \mathbb{Q}(\alpha, \gamma) = \mathbb{Q}(\alpha)$.

We prove a useful lemma about properties of the predimension function δ.

Lemma 4.13. 1. *(Addition formula)*

Consider a gf-field extension of the form $\langle A \rangle \subseteq \langle A\bar{b} \rangle \subseteq \langle A\bar{b}\bar{c} \rangle$. Then $\delta(\bar{b}\bar{c}/A) = \delta(\bar{b}/A) + \delta(\bar{c}/A\bar{b})$.

2. *(Submodularity)*

Let K be a gf-field with $A \subseteq B \subseteq D(K)$ and $\bar{c} \in D(K) \setminus B$. Then $\delta(\bar{c}/A) \geqslant \delta(\bar{c}/B)$.

3. *(Finite character)*

Let $\langle A \rangle \subseteq K$ be a gf-field extension. Then for any finite tuple $\bar{b} \in D(K)$ there exists $\bar{a} \in A$ such that for any $A' \subseteq A$ we have $\delta(\bar{b}/A) = \delta(\bar{b}/A'\bar{a})$.

Proof. 1. The equality holds for both the transcendence degree and the cardinality of the tuple.

2. We have $\mathrm{td}(\bar{c}, g(\bar{c})/\langle A \rangle) \geqslant \mathrm{td}(\bar{c}, g(\bar{c})/\langle B \rangle)$, while $|\bar{c} \setminus (A \cup \{0\})| = |\bar{c} \setminus (B \cup \{0\})|$, since $\bar{c} \in D(K) \setminus B$.

3. Since transcendence degree has finite character, there is a subfield $L \subseteq \langle A \rangle$ finitely generated over $\mathbb{Q}$ such that $\mathrm{td}(\bar{b}, g(\bar{b})/\langle A \rangle) = \mathrm{td}(\bar{b}, g(\bar{b})/L)$. Note that then for any intermediate field $L \subseteq L' \subseteq \langle A \rangle$ we have $\mathrm{td}(\bar{b}, g(\bar{b})/\langle A \rangle) = \mathrm{td}(\bar{b}, g(\bar{b})/L')$. Let $\alpha_1, \ldots, \alpha_n \in \langle A \rangle$ generate L. Then there is a finite tuple $\bar{a}_0 \in A$ such that $\bar{a}_0 \cup g(\bar{a}_0)$ generate $\alpha_1, \ldots, \alpha_n$ and thus L. Hence, we get $L \subseteq \langle \bar{a}_0 \rangle$.

   Now let $\bar{a}_1 = \bar{b} \cap A$ and take $\bar{a} \in A$ to be $\bar{a}_0\bar{a}_1$. Then for any $A' \subseteq A$ we get $\mathrm{td}(\bar{b}, g(\bar{b})/\langle A \rangle) = \mathrm{td}(\bar{b}, g(\bar{b})/\langle A'\bar{a} \rangle)$, since $L \subseteq \langle A'\bar{a} \rangle \subseteq \langle A \rangle$, and $|\bar{b} \setminus (A \cup \{0\})| = |\bar{b} \setminus (A'\bar{a} \cup \{0\})|$, since $\bar{b} \cap A \subseteq \bar{a}$. Therefore, $\delta(\bar{b}/A) = \delta(\bar{b}/A'\bar{a})$. □

We use the lemma to prove that the strongness of gf-field embeddings is transitive.

Proposition 4.14. *Let K be a gf-field with subsets $A \subseteq B \subseteq D(K)$ such that $\langle A \rangle \lhd \langle B \rangle$ and $\langle B \rangle \lhd K$. Then $\langle A \rangle \lhd K$.*

Proof. Consider a finite tuple $\bar{c} \in D(K)$. Let $\bar{c}_1 = \bar{c} \cap B$ and write $\bar{c}$ as $\bar{c}_1\bar{c}_2$, where $\bar{c}_2 \in D(K) \setminus B$. First apply addition formula to the gf-field extension $\langle A \rangle \subseteq \langle A\bar{c}_1 \rangle \subseteq \langle A\bar{c} \rangle$ in order to get $\delta(\bar{c}/A) = \delta(\bar{c}_1/A) + \delta(\bar{c}_2/A\bar{c}_1)$. We want to show that both parts of this sum are non-negative. Since $\bar{c}_2 \in D(K) \setminus B$ and $A\bar{c}_1 \subseteq B$, we can apply submodularity to get $\delta(\bar{c}_2/A\bar{c}_1) \geqslant \delta(\bar{c}_2/B)$. Finally, using $\langle A \rangle \lhd \langle B \rangle$ and $\langle B \rangle \lhd K$ as well as $\bar{c}_1 \in B$, we get $\delta(\bar{c}_1/A) \geqslant 0$ and $\delta(\bar{c}_2/B) \geqslant 0$. Hence, $\delta(\bar{c}/A) \geqslant 0$. □

Next we show that the intersection of strong subsets is also strong and therefore the meet of strong gf-subfields is also a strong gf-subfield.

Lemma 4.15. *Let $\{A_i \subseteq D(K) : i \in I\}$ be a collection of strong subsets of the domain of a gf-field K. Then $\bigcap_{i \in I} A_i \subseteq D(K)$ is also strong.*

Proof. First consider two strong subsets A_1 and A_2. Let us prove $\langle A_1 \cap A_2 \rangle \lhd \langle A_1 \rangle$. Suppose $\bar{b} \in A_1$. We can assume $\bar{b} \in A_1 \setminus A_1 \cap A_2$, i.e. $\bar{b} \in A_1 \setminus A_2$. Then by submodularity $\delta(\bar{b}/A_1 \cap A_2) \geqslant \delta(\bar{b}/A_2)$. But A_2 is a strong subset, hence $\delta(\bar{b}/A_2) \geqslant 0$. Therefore, $\delta(\bar{b}/A_1 \cap A_2) \geqslant 0$ and $\langle A_1 \cap A_2 \rangle \lhd \langle A_1 \rangle$. Then by transitivity of strongness $A_1 \cap A_2$ is a strong subset of $D(K)$.

Thus we are done with the case of finite I. For the general case consider a finite tuple $\bar{b} \in D(K)$. We can assume $\bar{b} \in D(K) \setminus \bigcap_{i \in I} A_i$. Hence, for every element b_j of $\bar{b}$ there exists $i_j \in I$ such that $b_j \notin A_{i_j}$. Let $I_0 \subseteq I$ be the finite subset consisting of all such i_j. Then $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I_0} A_i$, while we still have $\bar{b} \in D(K) \setminus \bigcap_{i \in I_0} A_i$. Therefore, by submodularity $\delta(\bar{b}/\bigcap_{i \in I} A_i) \geqslant \delta(\bar{b}/\bigcap_{i \in I_0} A_i)$. But as we already proved the case of I being finite, $\delta(\bar{b}/\bigcap_{i \in I_0} A_i) \geqslant 0$ and $\bigcap_{i \in I} A_i$ is strong. □

This property allows us to introduce the notion of a hull, the smallest strong extension of a set.

Definition 4.16. Let K be a gf-field and $A \subseteq D(K)$ a subset. Then the *hull* of A is defined as

$$\lceil A \rceil = \bigcap \{B \subseteq D(K) : A \subseteq B \text{ and } \langle B \rangle \lhd K\}.$$

Note that since $K \lhd K$, the collection of subsets in the definition is non-empty and we have $A \subseteq \lceil A \rceil$. Moreover, by Lemma 4.15, $\lceil A \rceil$ is strong in K, so $\lceil A \rceil$ is indeed the smallest strong extension of A. Although the hull depends on the gf-field K, whenever we have $A \subseteq K \lhd L$, the hull of A with respect to L coincides with the one with respect to K. The hull also has finite character in the following sense.

Proposition 4.17. *Let K be a gf-field and $A \subseteq D(K)$ a subset. Then* $\lceil A \rceil = \bigcup\limits_{\bar{a} \in A} \lceil \bar{a} \rceil$.

Proof. Let $S = \bigcup\limits_{\bar{a} \in A} \lceil \bar{a} \rceil$. Clearly, $S \subseteq \lceil A \rceil$ and $A \subseteq S$, so it suffices to show that S is strong in K. Consider a tuple $\bar{b} \in D(K)$ with all elements non-zero, distinct from each other and elements of S. Then there exists $\bar{a} \in A$ such that $\operatorname{td}(\bar{b}, g(\bar{b})/S) = \operatorname{td}(\bar{b}, g(\bar{b})/\lceil \bar{a} \rceil)$. By strongness of $\lceil \bar{a} \rceil$, we get $\delta(\bar{b}/S) = \delta(\bar{b}/\lceil \bar{a} \rceil) \geqslant 0$. Therefore, S is strong in K and $\lceil A \rceil = \bigcup\limits_{\bar{a} \in A} \lceil \bar{a} \rceil$. □

Due to the finite character, it suffices to characterize hulls of finite sets.

Proposition 4.18. *Let K be a gf-field, $A \subseteq D(K)$ a strong subset and $\bar{b} \in D(K)$ a finite tuple. Then there exists a finite tuple $\bar{c} \in D(K)$ with all elements non-zero, distinct from each other and elements of $A\bar{b}$, such that $\lceil A\bar{b} \rceil = A\bar{b}\bar{c}$. Moreover, $\bar{c}$ is unique (up to reordering) and is characterized by having $\delta(\bar{c}/A\bar{b})$ of minimal value m among $\delta(\bar{c}'/A\bar{b})$ for finite tuples $\bar{c}' \in D(K)$; and for any $\bar{c}' \in D(K)$ with $\delta(\bar{c}'/A\bar{b}) = m$ we have $\bar{c} \subseteq \bar{c}'$ (as sets).*

Proof. First note that, since A is strong, we have $\delta(\bar{b}\bar{c}'/A) \in \mathbb{Z}_{\geqslant 0}$ for any finite tuple $\bar{c}' \in D(K)$. By the addition formula, $\delta(\bar{c}'/A\bar{b}) = \delta(\bar{b}\bar{c}'/A) - \delta(\bar{b}/A) \geqslant -\delta(\bar{b}/A)$, hence, there exists a minimal value m of $\delta(\bar{c}'/A\bar{b})$. Choose $\bar{c}$ to be a tuple of the smallest length with $\delta(\bar{c}/A\bar{b}) = m$. Then its elements are distinct from each other and elements of $A\bar{b}$, since otherwise we could remove them and have the same $\delta(\bar{c}/A\bar{b})$. Let us show that $\lceil A\bar{b} \rceil = A\bar{b}\bar{c}$. Then the uniqueness follows from the uniqueness of the hull.

Take a finite tuple $\bar{d} \in D(K)$. Then $\delta(\bar{d}/A\bar{b}\bar{c}) = \delta(\bar{b}\bar{c}\bar{d}/A) - \delta(\bar{b}\bar{c}/A) \geqslant 0$. Hence, $A\bar{b}\bar{c}$ is strong in K and $\lceil A\bar{b} \rceil \subseteq A\bar{b}\bar{c}$, implying that there exists a subtuple $\bar{e} \subseteq \bar{c}$ such that $\lceil A\bar{b} \rceil = A\bar{b}\bar{e}$. Note that this works for any $\bar{c}' \in D(K)$ with

$\delta(\bar{c}'/A\bar{b}) = m$. Since $\lceil A\bar{b} \rceil$ is strong in K, it is also strong in $\langle A\bar{b}\bar{c} \rangle$. Hence, $\delta(\bar{b}\bar{c}/A) = \delta(\bar{b}\bar{c}/A\bar{b}\bar{e}) + \delta(\bar{b}\bar{e}/A) \geqslant \delta(\bar{b}\bar{e}/A)$. Then by construction of $\bar{c}$, $\delta(\bar{b}\bar{c}/A) = \delta(\bar{b}\bar{e}/A)$ and since $\bar{c}$ was chosen to be the shortest tuple, $\bar{e} = \bar{c}$. Finally, if we have $\delta(\bar{b}\bar{c}'/A) = m$ for some $\bar{c}' \in D(K)$, then as noted above, $\bar{e} \subseteq \bar{c}'$, i.e., $\bar{c} \subseteq \bar{c}'$. □

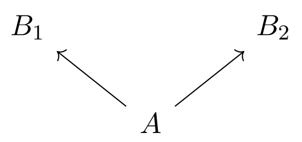
Now we are finally ready to move to the construction of the relevant amalgamation category.

Definition 4.19. Let the category $\mathcal{G}$ consist of the gf-fields K with $\mathbb{Q}_g \lhd K$ as the objects and strong embeddings as the arrows. We also define $\mathcal{G}^{<\aleph_0}$ to be the full subcategory of $\mathcal{G}$ consisting of the gf-fields with finite domain.

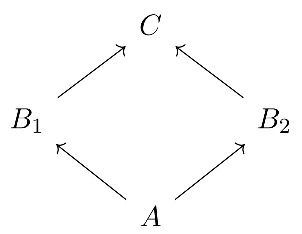
Note that since strongness of embeddings is transitive, $\mathcal{G}$ actually forms a category. Also, any model of T_{gf} is a full gf-field and an object in $\mathcal{G}$.

In order to construct a specific model of T_{gf}, the Fraïssé limit, we use a variant of the renowned Fraïssé's amalgamation theorem (see [Hod93, Chapter 7] for more details) and show that $\mathcal{G}$ is an amalgamation category as defined in [BK18, Definition 5.3] (or equivalently [Kir09, Definition 2.17]). Then by [Kir09, Theorem 2.18] there exists a universal and saturated $U \in \mathcal{G}$. Let us first list the six properties needed to check for $\mathcal{G}$ to be an amalgamation category.

1. Every arrow in $\mathcal{G}$ is a monomorphism.
2. $\mathcal{G}$ has direct limits of ω-chains.
3. $\mathcal{G}^{<\aleph_0}$ has at most countably many objects up to isomorphism.
4. Each object in $\mathcal{G}^{<\aleph_0}$ has at most countably many extensions in $\mathcal{G}^{<\aleph_0}$ up to isomorphism.
5. $\mathcal{G}^{<\aleph_0}$ has the amalgamation property (AP), i.e. any diagram of the form

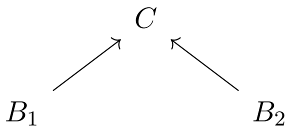


   can be completed to a commuting square



   in $\mathcal{G}^{<\aleph_0}$.

6. $\mathcal{G}^{<\aleph_0}$ has the joint embedding property (JEP), i.e. for every $B_1, B_2 \in \mathcal{G}^{<\aleph_0}$ there is $C \in \mathcal{G}^{<\aleph_0}$ and arrows



in $\mathcal{G}^{<\aleph_0}$.

The only non-trivial condition among these is the amalgamation property, with JEP following from it since $\mathbb{Q}_g$ strongly embeds into every gf-field. Hence we prove the first four properties, and then move to AP. Recall that for a field extension $A \subseteq F$ and a tuple $\bar{a} \in F$, the *locus* $\mathrm{Loc}(\bar{a}/A)$ is the zero set of all polynomials $p(\bar{x}) \in A[\bar{x}]$ such that $p(\bar{a}) = 0$.

Proposition 4.20. *The category $\mathcal{G}$ satisfies properties 1-4 of an amalgamation category.*

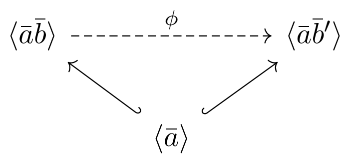
Proof. 1. Strong embeddings are monomorphisms since they are injective.

2. Let $K_0 \lhd K_1 \lhd \ldots$ be an ω-chain in $\mathcal{G}$. Then its limit is the join $K = \bigcup_{i \in \mathbb{N}} K_i$. Note that for every $i \in \mathbb{N}$ we have $K_i \lhd K$. Indeed, let $K_i = \langle A_i \rangle$ and $\bar{b} \in D(K)$. Then $\bar{b}$ is contained in A_j for some $j \geqslant i$ with $K_j = \langle A_j \rangle$ and $\delta(\bar{b}/A_i) \geqslant 0$ since $K_i \lhd K_j$.

   Now suppose every K_i is strongly embedded into some gf-field L by ϕ_i. Then in order to get a commuting diagram, we have to embed K into L by $\phi = \bigcup_{i \in \mathbb{N}} \phi_i$. The embedding ϕ is strong by the finite character of δ from Lemma 4.13, since we can check δ on a gf-subfield of $\phi(K)$ of the form $\langle \bar{a} \rangle$, which has to be contained in one of $\phi_i(K_i)$.

3. As each gf-field is an extension of $\mathbb{Q}_g$, this property follows from the next one.

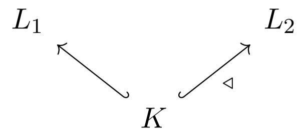
4. Suppose we have a gf-field extension of the form $\langle \bar{a} \rangle \subseteq \langle \bar{a}\bar{b} \rangle$ in $\mathcal{G}^{<\aleph_0}$. Then this extension is defined by $\mathrm{Loc}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle)$, i.e. if we also have $\langle \bar{a} \rangle \subseteq \langle \bar{a}\bar{b}' \rangle$ with $\mathrm{Loc}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle) = \mathrm{Loc}(\bar{b}', g(\bar{b}')/\langle \bar{a} \rangle)$, we get an isomorphism $\phi : \langle \bar{a}\bar{b} \rangle \to \langle \bar{a}\bar{b}' \rangle$ commuting over $\langle \bar{a} \rangle$.



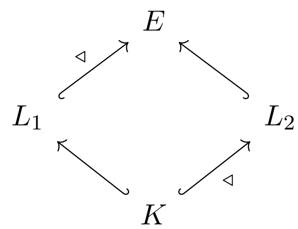
Since $\langle \bar{a} \rangle$ is countable, there are only countably many possibilities for $\operatorname{Loc}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle)$. Hence, there are countably many extensions of $\langle \bar{a} \rangle$ in $\mathcal{G}^{<\aleph_0}$. □

Now our goal is to obtain AP. We start by proving two lemmas, first one being an analogue of [BK18, Proposition 5.7].

Lemma 4.21 (Asymmetric amalgamation property)**.** *Assume we have the following diagram of gf-fields with K full and strong in L_2.*



Then there exists a gf-field E completing the diagram to a commuting square so that L_1 is strong in E.



Moreover, if K is strong in L_1, then L_2 is also strong in E.

Proof. Since K is algebraically closed, we can take E to be the free amalgam of L_1 and L_2 in the sense of fields. That is, as fields L_1 and L_2 are linearly disjoint over K in E and therefore $L_1 \cap L_2 = K$. Let $D(E) = D(L_1) \cup D(L_2)$, i.e. E is the join of L_1 and L_2. Note that since $L_1 \cup L_2$ generates E as a field, this indeed defines a gf-field structure on E. It suffices to show that $L_1 \subseteq E$ is strong.

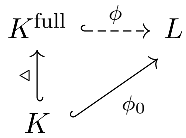
Let $A_1 = D(L_1)$ and $A_2 = D(L_2)$. Take a finite tuple $\bar{b} \in D(E) = A_1 \cup A_2$ and to simplify calculations assume all elements of $\bar{b}$ are non-zero, distinct from each other and elements of A_1. Since $A_1 \cap A_2 = K$, that means $\bar{b} \in A_2 \setminus K$. From K being strong in L_2, we have $\delta(\bar{b}/K) = \operatorname{td}(\bar{b}, g(\bar{b})/K) - |\bar{b}| \geqslant 0$. Hence, it suffices to prove $\operatorname{td}(\bar{b}, g(\bar{b})/L_1) = \operatorname{td}(\bar{b}, g(\bar{b})/K)$, which follows from L_1 and L_2 being algebraically independent over K.

In case we also have $K \lhd L_1$, since the construction of E is symmetric, the same proof works to show $L_1 \lhd E$. □

Next we establish that any gf-field can be strongly extended to a full one in a universal way. Note that this is not exactly universal property in the categorical sense as we do not get uniqueness.

Lemma 4.22. *Let K be a gf-field. Then there exists a full gf-field K^{full} with $K \lhd K^{\text{full}}$ such that for any full gf-field L with a strong embedding $\phi_0 : K \overset{\lhd}{\hookrightarrow} L$*

we can complete the diagram to a commuting triangle by a strong embedding $\phi : K^{\text{full}} \overset{\lhd}{\hookrightarrow} L$, *i.e.* $\phi|_K = \phi_0$.



Proof. We construct K^{full} as a union of a countable chain. Let $K_0 = K$. To construct K_{i+1} from K_i, let $S_i = K_i^{\text{alg}} \setminus D(K_i)$, where K_i^{alg} is the algebraic closure of K_i. Then define $K_{i+1} = K_i^{\text{alg}}(t_\alpha)_{\alpha \in S_i}$, where t_α are algebraically independent over K_i, with $D(K_{i+1}) = K_i^{\text{alg}}$ and $g(\alpha) = t_\alpha$ for each $\alpha \in S_i$. Denote $A_i = D(K_i)$ for all $i \in \mathbb{N}$.

Claim 1. For any $i \in \mathbb{N}$ we have $K_i \lhd K_{i+1}$. Moreover, for any finite tuple $\bar{b} \in A_{i+1}$, we have $\delta(\bar{b}/A_i) = 0$.

Proof. Let $\bar{b} \in A_{i+1}$ and assume all elements of $\bar{b}$ are non-zero and distinct from each other and elements of A_i. Then $\bar{b} \in S_i$ and we can write it as $\bar{b} = \alpha_1, \ldots, \alpha_n$ with $g(\alpha_i) = t_{\alpha_i}$. Therefore, $\text{td}(\bar{b}, g(\bar{b})/A_i) = \text{td}(\alpha_1, \ldots, \alpha_n, t_{\alpha_1}, \ldots, t_{\alpha_n}/A_i) = n$, since $\alpha_1, \ldots, \alpha_n \in A_i^{\text{alg}}$ while $t_{\alpha_1}, \ldots, t_{\alpha_n}$ are algebraically independent over A_i. Thus, we get $\delta(\bar{b}/A_i) = 0$. $\square$

Now let K^{full} be the direct limit of the chain of K_i as constructed in Proposition 4.20. We get $D(K^{\text{full}}) = \bigcup_{i \in \mathbb{N}} D(K_i) = \bigcup_{i \in \mathbb{N}} K_i^{\text{alg}} = K^{\text{full}}$ and $K \lhd K^{\text{full}}$. The field K^{full} is also algebraically closed, since we have $K_i^{\text{alg}} \subseteq K_{i+1}$ for all $i \in \mathbb{N}$. So the only thing left to show is the universal part.

Let $\phi_0 : K \overset{\lhd}{\hookrightarrow} L$ with L full. We define ϕ as a union of coherent $\phi_i : K_i \overset{\lhd}{\hookrightarrow} L$, constructing them by induction on i. Note that we already have ϕ_0. Assume that we have constructed $\phi_i : K_i \overset{\lhd}{\hookrightarrow} L$. Then we define $\phi_{i+1} : K_{i+1} \overset{\lhd}{\hookrightarrow} L$ by first extending ϕ_i to an embedding of K_i^{alg} into L using the algebraic closedness of L (this extension is not unique) and then taking $\phi_{i+1}(t_\alpha) = g_L(\phi_{i+1}(\alpha))$. Note that these embeddings of $K_i^{\text{alg}} \setminus K_i$ into L are the only non-unique part of our construction and therefore ϕ is going to be unique up to a gf-field automorphism of K^{full} over K. Then we extend ϕ_{i+1} to a unique field embedding $K_{i+1} = K_i^{\text{alg}}(t_\alpha)_{\alpha \in S_i} \hookrightarrow K$, which is possible since for any $\alpha_1, \ldots, \alpha_n \in S_i$ all non-zero and distinct, we have

$$\text{td}(g_L(\phi_{i+1}(\alpha_1)), \ldots, g_L(\phi_{i+1}(\alpha_n))/\phi_i(K_i)) = n$$

from ϕ_i being strong and therefore $g_L(\phi_{i+1}(\alpha_1)), \ldots, g_L(\phi_{i+1}(\alpha_n))$ are algebraically independent over $\phi(K_i)$.

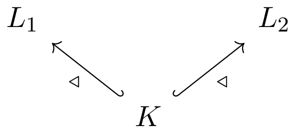
Now let us check that ϕ_{i+1} is strong. Consider a tuple $\bar{b} \in L$. Using finite character of δ from Lemma 4.13, choose a finite tuple $\bar{a} \in A_{i+1}$ such that

$\delta(\bar{b}/\phi_{i+1}(A_{i+1})) = \delta(\bar{b}/\phi_{i+1}(\bar{a}A_i))$. Applying the addition formula from Lemma 4.13 to the gf-field extension $\langle A_i \rangle \subseteq \langle A_i\bar{a} \rangle \hookrightarrow \langle \phi_{i+1}(A_i\bar{a})\bar{b} \rangle$ we get $\delta(\bar{b}/\phi_{i+1}(\bar{a}A_i)) = \delta(\phi_{i+1}(\bar{a})\bar{b}/\phi_i(A_i)) - \delta(\bar{a}/A_i)$. But $\delta(\bar{a}/A_i) = 0$ by Claim 1, so $\delta(\bar{b}/\phi_{i+1}(A_{i+1})) = \delta(\phi_{i+1}(\bar{a})\bar{b}/\phi_i(A_i))$, which is non-negative by strongness of $\phi(A_i)$. Hence, ϕ_{i+1} is strong.

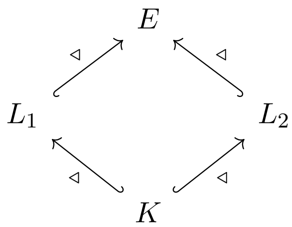
Finally, since the direct limit of ω-chains of strong embeddings is strong, as was shown in Proposition 4.20, we get $\phi : K^{\text{full}} \overset{\triangleleft}{\hookrightarrow} L$. □

Now we are ready to prove the amalgamation property.

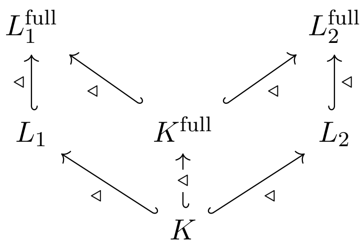
Proposition 4.23. *Suppose K, L_1 and L_2 are gf-fields with finite domains and there are strong embeddings $K \overset{\triangleleft}{\hookrightarrow} L_1$ and $K \overset{\triangleleft}{\hookrightarrow} L_2$.*



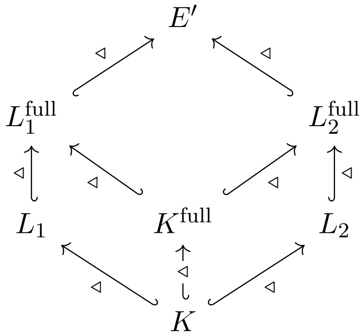
Then there exists a gf-field E with finite domain completing the diagram to a commuting square by strong embeddings.



Proof. Using Lemma 4.22, we construct K^{full}, L_1^{full} and L_2^{full} and then use the universal part of Lemma 4.22 to obtain the following diagram.



Now applying Lemma 4.21 to K^{full}, L_1^{full} and L_2^{full} we get a gf-field E'.



Finally let E be the join of L_1 and L_2 inside of the gf-field of E'. Then E has a finite domain $D(E) = D(L_1) \cup D(L_2)$. The embeddings $K_i \hookrightarrow E$ are also strong since for any $\bar{c} \in D(E)$ we have $\delta(\bar{c}/D(L_i)) \geqslant 0$ by strongness of $K_i \overset{\triangleleft}{\hookrightarrow} E'$. □

Gathering AP with the properties proved before, we see that $\mathcal{G}$ is an amalgamation category.

Corollary 4.24. *The category $\mathcal{G}$ is an amalgamation category.*

Proof. Since for any object K we have a morphism $\mathbb{Q}_H \overset{\triangleleft}{\hookrightarrow} K$, JEP follows from AP. Other properties are shown in Proposition 4.20 and Proposition4.23. □

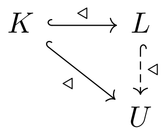
Applying the Amalgamation Theorem from [Kir09, Theorem 2.18], we obtain a gf-field $U \in \mathcal{G}$ called the Fraïssé limit, which satisfies the following properties.

1. (Universality)

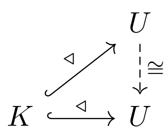
   Any gf-field $K \in \mathcal{G}$ with finite $D(K)$ can be strongly embedded into U.

2. (Saturation)

   For any gf-fields $K, L \in \mathcal{G}$ with finite domains and strong embeddings $K \overset{\triangleleft}{\hookrightarrow} L$ and $K \overset{\triangleleft}{\hookrightarrow} U$, there is a strong embedding of L into U completing the diagram to a commuting triangle.



It also follows from these properties that for any gf-field $K \in \mathcal{G}$ with finite $D(K)$ if there are two ways to strongly embed it into U, we have an isomorphism of U making the diagram commute. We call this property homogeneity.



Now we claim that this gf-field U is actually a model of T_{gf}.

Theorem 4.25. *The Fraïssé limit U of $\mathcal{G}$ satisfies the axioms of T_{gf}.*

Proof. We go one by one through the axioms in Definition 4.2.

1. First we need to show that $U \models \mathrm{ACF}_0$, i.e. that U is algebraically closed. Consider an irreducible polynomial $p(x) \in U[x]$ and let S be the finite set of its coefficients. Then there exists a finite tuple $\bar{a} \in D(U)$ such that $S \subseteq \langle \bar{a} \rangle$. Moreover, by Proposition 4.18 there exists $\bar{b} \in D(U)$ with $\lceil \bar{a} \rceil = \bar{a}\bar{b}$. Let $K = \langle \bar{a}\bar{b} \rangle$, note that $K \triangleleft U$. We want to construct a strong gf-field extension

$K \lhd L$ with L containing a root of $p(x)$ in order to use saturation of U and find a root of $p(x)$ in U.

Field-wise let $L' = K[x]/p(x)$ and $L = L'(t)$, where t is transcendental over L'. We turn $K \subseteq L$ into a gf-field extension by letting $D(L) = \bar{a}\bar{b}x$ and $g(x) = t$. It remains to show that this extension is strong. Consider any finite tuple $\bar{c} \in D(L)$ with all elements non-zero, distinct from each other and elements of $D(K) = \bar{a}\bar{b}$. Then $\bar{c} = x$ and we get $\delta(x/\bar{a}\bar{b}) = \operatorname{td}(x, t/\langle\bar{a}\bar{b}\rangle) - 1 = 0$. Therefore, $K \lhd L$ and $p(x)$ has a root in U.

2. Next we need to demonstrate that g is a total function on U with $g(0) = 0$. It suffices to show that $D(U) = U$. Suppose it is not and let $u \in U \setminus D(U)$. Then the same way as above we can find $\bar{a}\bar{b} \in D(U)$ such that for $K = \langle\bar{a}\bar{b}\rangle$ we have $u \in K$ and $K \lhd U$. Once again we construct a strong gf-field extension $K \lhd L$ with $u \in D(L)$ and deduce through saturation of U that $u \in D(U)$.

   Field-wise let $L = K(t)$, where t is transcendental over K. We turn $K \subseteq L$ into a gf-field extension by letting $D(L) = \bar{a}\bar{b}u$ and $g(u) = t$. Then for any finite tuple $\bar{c} \in D(L)$ with all elements non-zero, distinct from each other and elements of $D(K) = \bar{a}\bar{b}$, we get $\bar{c} = u$ and $\delta(u/\bar{a}\bar{b}) = \operatorname{td}(u, t/\langle\bar{a}\bar{b}\rangle) - 1 = 0$. Therefore, $K \lhd L$ and $u \in D(U)$. Contradiction.

3. In terms of gf-fields, the generalized Schanuel property just says that $\mathbb{Q}_g$ is strong and therefore follows from U being an object in $\mathcal{G}$. Let us write this out in details.

   Let $V \subseteq U^{2n}$ be an algebraic variety over $\mathbb{Q}$ with $\dim V < n$. Consider an n-tuple $\bar{a} \in U$ with distinct non-zero coordinates. Since $\mathbb{Q}_g \lhd U$, we have $\delta(\bar{a}/\mathbb{Q}_g) \geqslant 0$, i.e. $\operatorname{td}(\bar{a}, g(\bar{a})/\mathbb{Q}) \geqslant n$. Hence, $(\bar{a}, g(\bar{a})) \notin V$, as otherwise we would have $\operatorname{td}(\bar{a}, g(\bar{a})/\mathbb{Q}) \leqslant \dim V < n$.

4. Finally, we deduce the last axiom from the saturation of U. Let $\bar{a} \in U$ be a finite tuple and $V \subseteq U^{2n}$ an absolutely irreducible free and broad algebraic variety over $\bar{a}$ of dimension n. We can assume that $\bar{a}$ is strong by taking its hull if necessary. Let $(b_1, \ldots, b_n, c_1, \ldots, c_n)$ generic in V over $K = \langle\bar{a}\rangle$ and consider the field extension $K \subseteq L = K(\bar{b}\bar{c})$. We turn it into a gf-field extension by letting $D(L) = \bar{a}\bar{b}$ and $g(b_i) = c_i$. Note that this is well-defined since V is free. In L the set $V^{\dagger}_{\bar{a}}$ contains the point $(b_1, \ldots, b_n)$, so it remains to embed L into U.

   First we prove $K \lhd L$. It suffices to consider a subtuple $\bar{b}' \subseteq \bar{b}$ and show $\delta(\bar{b}'/\bar{a}) \geqslant 0$. Let $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be an injective map with $\bar{b}' = b_{\sigma(1)}, \ldots, b_{\sigma(m)}$. Since V is broad, $\dim V^{\sigma} \geqslant m$, thus $\operatorname{td}(\bar{b}', g(\bar{b}')/K) \geqslant m$ and $K \lhd L$. Now we can just use saturation property of U and strongly embed L into U over K. Thus, U satisfies the Existential closedness axiom. □

Note that it follows from the property of saturation via the back-and-forth method that U is an ω-saturated model. The construction of the Fraïssé limit helps us to classify types in the next section.

4.3 Classification of types

The goal of this section is to obtain a reasonable classification of types in T_{gf} and establish their Morley ranks. As a consequence, we reprove Zilber's results from [Zil02] on ω-stability of T_{gf}. Furthermore, by looking at Morley ranks we manage to characterize the (model-theoretical) algebraic closure in T_{gf}. Let us start by defining formulas that play the main role in this section. Recall the notation $V_{\bar{a}}^{\dagger}$ from Definition 4.1.

Definition 4.26. Let $F \models T_{\mathrm{gf}}$, $\bar{a} \in F$ a finite tuple, $V \subseteq F^{2n}$ an algebraic variety over $\bar{a}$. Then we denote by $\phi_{V,\bar{a}}(x_1, \ldots, x_n)$ the formula defining $(x_1, \ldots, x_n) \in V_{\bar{a}}^{\dagger}$. For $k \in \{0, \ldots, n\}$ we denote by $\psi_{V,\bar{a}}^k(x_1, \ldots, x_k)$ the formula $\exists x_{k+1}, \ldots, \exists x_n \; \phi_{V,\bar{a}}(x_1, \ldots, x_n)$. In case of $k = 0$ we get a sentence $\psi_{V,\bar{a}}^0$ and in case of $k = n$ we get $\psi_{V,\bar{a}}^n = \phi_{V,\bar{a}}$. Note that $\phi_{V,\bar{a}}$ and $\psi_{V,\bar{a}}^k$ are formulas with parameters in $\bar{a}$.

We connect the formulas $\phi_{V,\bar{a}}$ to the predimension function δ in the following way.

Lemma 4.27. *Let $F \models T_{\mathrm{gf}}$, $A \subseteq F$ a strong subset and $\bar{b} = (b_1, \ldots, b_n) \in F$ an n-tuple with each element b_i non-zero, distinct from each other and elements of A. Then $\delta(\bar{b}/A) = 0$ if and only if there exists $\bar{a} \in A$ and a variety V over $\bar{a}$ of dimension n such that $F \models \phi_{V,\bar{a}}(b_1, \ldots, b_n)$. Moreover, we can take $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle A\rangle)$.*

Proof. First suppose $\delta(\bar{b}/A) = 0$. Let $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle A\rangle)$. Then there exists $\bar{a} \in A$ that V is defined over, such that $F \models \phi_{V,\bar{a}}(\bar{b})$. Moreover, since all elements of $\bar{b}$ are non-zero, distinct from each other and elements of A and $\delta(\bar{b}/A) = 0$, we have $\mathrm{td}(\bar{b}, g(\bar{b})/\langle A\rangle) = n$ and $\dim V = n$.

For the other direction we have $\delta(\bar{b}/A) \geqslant 0$ since A is strong. But as $\dim V = n$ and $(\bar{b}, g(\bar{b})) \in V$, we also have $\mathrm{td}(\bar{b}, g(\bar{b})/\langle A\rangle) \leqslant n$. Therefore, $\delta(\bar{b}/A) = 0$ and $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle A\rangle)$. □

Moreover, whenever A is a full gf-field, V is an absolutely irreducible, free and broad variety.

Lemma 4.28. *Let $F \models T_{\mathrm{gf}}$, $K \lhd F$ a strong full gf-subfield of F and $\bar{b} = (b_1, \ldots, b_n) \in F$ an n-tuple with each element b_i non-zero, distinct from each other and elements of K. Then $\delta(\bar{b}/K) = 0$ if and only if there exists $\bar{a} \in K$ and an absolutely irreducible, free and broad variety V over $\bar{a}$ of dimension n such that $F \models \phi_{V,\bar{a}}(b_1, \ldots, b_n)$. Moreover, we can take $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/K)$.*

Proof. Due to Lemma 4.27, it suffices to show that if $\delta(\bar{b}/K) = 0$, then $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/K)$ is absolutely irreducible, free and broad. Indeed, V is absolutely irreducible since K is algebraically closed. If V is contained in a variety defined by $x_i = c$, then $c \in K$ and there is a contradiction with $b_i \notin K$. Similarly, being contained in a variety defined by $x_i = x_j$ would contradict $b_i \neq b_j$.

For broadness let $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be an injective map. Then V^σ contains a point $(\bar{b}', g(\bar{b}'))$ for a corresponding subtuple $\bar{b}' \subseteq \bar{b}$. Since A is strong, we have $\delta(\bar{b}'/A) \geqslant 0$, so $\mathrm{td}(\bar{b}', g(\bar{b}')/A) \geqslant m$. Thus, $\dim V^\sigma \geqslant m$. □

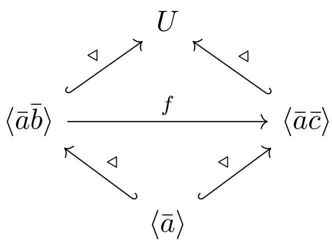
It turns out that the formulas of the form $\psi^k_{V,\bar{a}}$ isolate types over strong finite sets. We prove it by working in the Fraïssé limit U.

Proposition 4.29. *Let $\bar{a} \in U$ be a strong finite tuple. Let $V \subseteq U^{2n}$ be an algebraic variety over $\bar{a}$ of dimension n. Then whenever both $b_1, \ldots, b_k \in U$ and $c_1, \ldots, c_k \in U$ satisfy $\psi^k_{V,\bar{a}}(x_1, \ldots, x_k)$, they have the same type over $\bar{a}$.*

Proof. Since both $b_1, \ldots, b_k \in U$ and $c_1, \ldots, c_k \in U$ satisfy $\psi^k_{V,\bar{a}}(x_1, \ldots, x_k)$, we can extend them to tuples $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ which satisfy $\phi_{V,\bar{a}}$. Denote these tuples by $\bar{b}$ and $\bar{c}$. Then it suffices to show that $\bar{b}$ and $\bar{c}$ have the same type over $\bar{a}$.

By Lemma 4.27, $\mathrm{Loc}(\bar{b}, g(\bar{b})/\langle\bar{a}\rangle) = \mathrm{Loc}(\bar{c}, g(\bar{c})/\langle\bar{a}\rangle)$, so there exists an isomorphism of gf-fields $f : \langle\bar{a}\bar{b}\rangle \to \langle\bar{a}\bar{c}\rangle$, mapping $\bar{b}$ to $\bar{c}$ and identical on $\langle\bar{a}\rangle$.

Moreover, $\delta(\bar{b}/\bar{a}) = \delta(\bar{c}/\bar{a}) = 0$, hence by addition formula and since $\bar{a}$ is strong, $\bar{a}\bar{b}$ and $\bar{a}\bar{c}$ are also strong in U. Therefore, we get the following diagram.



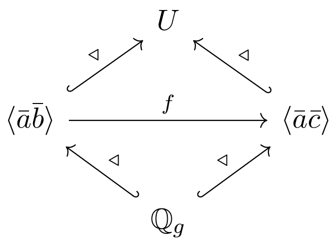
It follows from the homogeneity of U that f can be extended to an automorphism $\hat{f}$ of U. Hence, $\bar{b}$ and $\bar{c}$ have the same type over $\bar{a}$. □

Note that we assumed that the parameters are strong. In order to work with parameters which are not strong, we isolate the type of the hull, using the same method.

Proposition 4.30. *Let $\bar{a} \in U$ and $\lceil\bar{a}\rceil = \bar{a}\bar{b}$. Denote $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle\bar{a}\rangle)$. Then $\phi_{V,\bar{a}}$ isolates $\mathrm{tp}(\bar{b}/\bar{a})$. Moreover, if $\bar{c} \models \phi_{V,\bar{a}}$, then $\bar{c}$ is a permutation of $\bar{b}$.*

Proof. Suppose $\bar{c} \models \phi_{V,\bar{a}}$. First assume $\bar{c} \neq \bar{b}$ as sets. Since $\bar{b}$ and $\bar{c}$ have the same length, by Proposition 4.18 $\delta(\bar{c}/\bar{a}) > \delta(\bar{b}/\bar{a})$, hence $\mathrm{td}(\bar{c}, g(\bar{c})/\langle\bar{a}\rangle) > \mathrm{td}(\bar{b}, g(\bar{b})/\langle\bar{a}\rangle)$. But that contradicts $\bar{c} \in V$, so $\bar{c}$ is a permutation of $\bar{b}$. Note that from the equality of transcendence degrees we also get $V = \mathrm{Loc}(\bar{c}, g(\bar{c})/\langle\bar{a}\rangle)$

We want to show it has the same type over $\bar{a}$. Since $\bar{a}\bar{b}$ is strong, $\bar{a}\bar{c}$ is also strong. Moreover, as $\mathrm{Loc}(\bar{a}\bar{b}, g(\bar{a}\bar{b})/\mathbb{Q}) = \mathrm{Loc}(\bar{a}\bar{c}, g(\bar{a}\bar{c})/\mathbb{Q})$, so similarly to Proposition 4.29, we obtain an isomorphism $f : \langle \bar{a}\bar{b} \rangle \to \langle \bar{a}\bar{c} \rangle$ and the following diagram



Extending f to an automorphism of U we get $\mathrm{tp}(\bar{a}\bar{b}/\varnothing) = \mathrm{tp}(\bar{a}\bar{c}/\varnothing)$. Therefore, $\mathrm{tp}(\bar{b}/\bar{a}) = \mathrm{tp}(\bar{c}/\bar{a})$ and $\phi_{V,\bar{a}}$ isolates $\mathrm{tp}(\bar{b}/\bar{a})$. □

Proposition 4.17 and Proposition 4.30 show that strong full gf-fields are (model-theoretically) algebraically closed, and we find it more convenient to work over them rather than arbitrary parameters. As we will see later in Proposition 4.39, all (model-theoretically) algebraically closed sets are actually strong full gf-fields. We start the classification of types by considering types which are going to correspond to finite non-zero Morley ranks.

Proposition 4.31. *Let $F \models T_{\mathrm{gf}}$, $K \lhd F$ a strong full gf-subfield of F and $V \subseteq F^{2n}$ an absolutely irreducible, free and broad algebraic variety over K of dimension n. Then*

$$p_{V,K}(x_1, \ldots, x_n) = \{\phi_{V,\bar{a}}(x_1, \ldots, x_n) : \bar{a} \in K \textit{ and } V \textit{ is defined over } \bar{a}\}$$

defines a complete n-type.

Proof. First we show that $p_{V,K}$ is satisfiable. Consider a finite subset of $p_{V,K}$. Then in order to realize this finite subset, it suffices to realize $\phi_{V,\bar{a}}$ for some $\bar{a} \in K$ with V defined over $\bar{a}$. Since $F \models T_{\mathrm{gf}}$, the set $V^{\dagger}_{\bar{a}}$ is non-empty. Therefore, there are $b_1, \ldots, b_n \in F$ with $F \models \phi_{V,\bar{a}}(b_1, \ldots, b_n)$ and $p_{V,K}$ is satisfiable.

Now let us prove that $p_{V,K}$ is complete. Let $\alpha(x_1, \ldots, x_n)$ be a formula over K. Then it uses finitely many parameters, denote them by $\bar{a} \in K$. By extending $\bar{a}$ and using Proposition 4.18 we can assume that $\bar{a}$ is strong and V is defined over $\bar{a}$. Since $\phi_{V,\bar{a}}$ isolates the type by Proposition 4.29, we have either $\phi_{V,\bar{a}} \models \alpha$ or $\phi_{V,\bar{a}} \models \neg\alpha$. Therefore, either $p_{V,K} \models \alpha$ or $p_{V,K} \models \neg\alpha$. □

We also denote by $q_{V,K}$ the projection of $p_{V,K}$ onto the first coordinate. In cases, when V and V' have certain symmetries, the types $q_{V,K}$ and $q_{V',K}$ can coincide.

Proposition 4.32. *Let $F \models T_{\text{gf}}$, $K \triangleleft F$ a strong full gf-subfield of F and $V \subseteq F^{2n}$ an absolutely irreducible, free and broad algebraic variety over K of dimension n. Let $m \in \mathbb{Z}_{\geqslant 0}$ be the minimal natural number such that there exists $W \subseteq F^{2m}$ an absolutely irreducible, free and broad algebraic variety over K of dimension m with $q_{V,K} = q_{W,K}$.*

Then any such W is a projection of V, i.e. there is an injective $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $W = V^{\sigma}$. Moreover, if there are two such varieties $W_1, W_2 \subseteq F^{2m}$, then there exists a bijective $\tau : \{1, \ldots, m\} \to \{1, \ldots, m\}$ with $\tau(1) = 1$ and $W_1 = W_2^{\tau}$.

Proof. Let $b_1 \models q_{V,K}$. Then by compactness there is an extension $F \preceq F_1$ and $b_2, \ldots, b_n \in F_1$, such that $\bar{b} \models p_{V,K}$. Then by Lemma 4.28, $\delta(\bar{b}/K) = 0$ and $V = \text{Loc}(\bar{b}, g(\bar{b})/K)$. Consider the hull $\lceil Kb_1 \rceil = Kb_1\bar{c}$. Although we calculate the hull in the extension F_1, since elementary embedding $F \preceq F_1$ is strong, it coincides with the hull calculated with respect to F. Thus, $\bar{c} \in F$. Recall from Proposition 4.18 that $\delta(b_1\bar{c}/K)$ is minimal among tuples $b_1\bar{c}'$, while being non-negative. As $\delta(\bar{b}/K) = 0$, we ought to have $\delta(b_1\bar{c}/K) = 0$ and, moreover, $b_1\bar{c} \subseteq \bar{b}$.

Let $W = \text{Loc}(b_1\bar{c}, g(b_1\bar{c})/K)$. Then W is a projection of V and is absolutely irreducible, free and broad by Lemma 4.28. Let r be the length of $b_1\bar{c}$. Note that all elements of $b_1\bar{c}$ are non-zero, distinct from each other and elements of K by Proposition 4.18 and $b_1 \models q_{V,K}$. Then as $\delta(b_1\bar{c}/K) = 0$, we get $\dim W = r$. Finally, since $b_1 \models q_{W,K}$ and $q_{W,K}$, $q_{V,K}$ are complete types, we have $q_{W,K} = q_{V,K}$.

Suppose there is another variety $W' \subseteq F^{2r'}$ satisfying the conditions with dimension $r' < r$. Then by compactness there is an elementary extension $F \preceq F_2$ and a tuple $\bar{d} \in F$ with $b_1\bar{d} \models q_{W',K}$. Working in F_2, we see that by Lemma 4.28, $\delta(b_1\bar{d}/K) = 0$, which contradicts $\bar{c}$ being the shortest tuple. Note that this still holds in the amalgamation since elementary extensions are strong and $\bar{c}$ is defined by the hull. Therefore, $r' \geqslant r$, so $m = r$ and W satisfies the conditions of the proposition.

Finally, take again another variety $W' \subseteq F^{2m}$ satisfying the conditions. Once again construct an elementary extension $F \preceq F_3$ and $\bar{d} \in F_3$ with $b_1\bar{d} \models q_{W',K}$. Working in F_3 we see that since $\delta(b_1\bar{d}/K) = 0$, the tuples $\bar{c}$ and $\bar{d}$ coincide as sets. Thus, there is a permutation $\tau : \{1, \ldots, m\} \to \{1, \ldots, m\}$ with $\tau(1) = 1$ such that τ applied to the tuple $b_1\bar{c}$ produces the tuple $b_1\bar{d}$. Then since $W = \text{Loc}(b_1\bar{c}, g(b_1\bar{c})/K)$ and $W' = \text{Loc}(b_1\bar{d}, g(b_1\bar{d})/K)$ by Lemma 4.28, we get $W' = W^{\tau}$. □

It turns out that among 1-types over strong full gf-fields, there is only one type besides the types of the form $q_{V,K}$. As we will show later, it is the type of Morley rank ω and thus it is the generic type of T_{gf}.

Proposition 4.33. *Let $F \models T_{\text{gf}}$ and $K \triangleleft F$ a strong full gf-subfield of F. Consider the set $\mathcal{V}$ of all possible absolutely irreducible, free and broad algebraic varieties* *over* K *with* $V \subseteq F^{2n}$ *and* $\dim V = n$. *Then*

$$q_{\text{gen},K} = \{\neg\psi^1_{V,\bar{a}} : V \in \mathcal{V}, \bar{a} \in K \text{ and } V \text{ is defined over } \bar{a}\} \cup \{x \neq a : a \in K\}$$

defines a complete type.

Proof. First we use compactness to show that $q_{\text{gen},K}$ is satisfiable. Consider a strong finite tuple $\bar{a} \in K$. Since U is ω-saturated, there is an elementary embedding of $\bar{a}$ into U and since being strong is an elementary property, we have $\langle\bar{a}\rangle \triangleleft U$. Then it suffices to show that there is $b \in U$ such that $b \models q_{\text{gen},\bar{a}}$. By Lemma 4.27, that is equivalent to there being no $\bar{b}' \in U$ such that $\delta(b\bar{b}'/\bar{a}) = 0$. Note that in this case $\delta(b/\bar{a}) = 1$.

Consider a gf-field extension $\langle\bar{a}\rangle \subseteq \langle\bar{a}b\rangle$ with $\text{td}(b, g(b)/\langle\bar{a}\rangle) = 2$. Note that this extension is unique up to isomorphism and $\delta(b/\bar{a}) = 1$. We also have $\langle\bar{a}\rangle \triangleleft \langle\bar{a}b\rangle$ by construction. Using saturation we can strongly embed $\langle\bar{a}b\rangle$ into U. Then for any $\bar{c} \in U$ we get $\delta(\bar{c}/\bar{a}b) \geqslant 0$, so $\delta(b\bar{c}/\bar{a}) \geqslant 1$ by addition formula. Thus, $q_{\text{gen},K}$ is satisfiable.

In order to show completeness, consider once again a strong finite tuple $\bar{a} \in K$ elementary embedded into U. Then it suffices to show that whenever $b_0, b_1 \in U$ satisfy $q_{\text{gen},\bar{a}}$, they have the same type over $\bar{a}$. Note that since both b_0 and b_1 cannot be extended to a tuple with the predimension over $\bar{a}$ being zero, $\langle b_0\bar{a}\rangle \triangleleft U$ and $\langle b_1\bar{a}\rangle \triangleleft U$. There is an gf-field isomorphism between $\langle b_0\bar{a}\rangle$ and $\langle b_1\bar{a}\rangle$ which is identical on $\langle\bar{a}\rangle$, since such gf-field extension is unique. Then by homogeneity we can extend it to an isomorphism of U and obtain $\text{tp}(b_0/\bar{a}) = \text{tp}(b_1/\bar{a})$. □

Thus we have concluded the classification of types. To be more precise, each 1-type over a strong full gf-field $K \triangleleft F$ for some $F \models T_{\text{gf}}$ has one of the following forms:

- $x = a$ for $a \in K$,
- $q_{V,K}$ for $V \subseteq F^{2n}$ absolutely irreducible, free and broad algebraic variety over K of $\dim = n$,
- $q_{\text{gen},K}$.

Using this classification we can deduce ω-stability as well as rewrite any formula as a boolean combination of the formulas of the form $\psi^1_{V,\bar{a}}$ for certain $\bar{a}$ and V. Alternatively, this result can be obtained by careful consideration of the Quantifier Elimination result in [Zil02, Theorem 3.18].

Theorem 4.34. *Let* $F \models T_{\text{gf}}$, $K \triangleleft F$ *a strong full gf-subfield of* F, $\bar{a} \in K$ *a finite tuple and* $\alpha(x, \bar{a})$ *a formula. Then* $F \models \alpha(x, \bar{a}) \leftrightarrow \psi(x)$, *where* $\psi(x)$ *is a boolean combination of the formulas of the form* $\psi^1_{V,\bar{b}}(x)$ *and* $x = b'$ *with* $\bar{b}, b' \in K$ *and* $V \subseteq F^{2n}$ *an absolutely irreducible, free and broad algebraic variety over* $\bar{b}$ *with* $\dim V = n$. *Note that then* $\psi(x)$ *is a formula with parameters in* K.

Proof. It suffices to show that whenever two elements of F agree on the formulas of the form $\psi^1_{V,\bar{b}}(x)$ and $x = b'$, their types over K coincide. That follows from Proposition 4.31 and Proposition 4.33. □

Before proving ω-stability, let us recall the definition of an ω-stable theory. For a more profound exposition see [TZ12, Section 5.2].

Definition 4.35. Let T be a complete theory with infinite models. Then T is *ω-stable* if in any model of T, over any countable set of parameters and for any n, there are countably many n-types.

It actually suffices to only consider 1-types. The following fact can be found, for example, in [TZ12, Lemma 5.2.2].

Fact 4.36. *A theory T is ω-stable if an only if in any model of T and over any countable set of parameters, there are countably many 1-types.*

Now we can easily deduce that T_{gf} is ω-stable, as also proven in [Zil02, Theorem 3.19]. Recall the construction of a full closure K^{full} of a gf-field K from Lemma 4.22. We use the same notation in the theorem below, where we work inside a model F and take a closure of a set $\lceil A \rceil$ under g and the field-theoretical algebraic closure, obtaining a strong full gf-subfield $\lceil A \rceil^{\mathrm{full}}$. Indeed, the closures are isomorphic and the only difference is that we are now working inside a specific model.

Theorem 4.37. *The theory T_{gf} is ω-stable.*

Proof. Consider a model $F \models T_{\mathrm{gf}}$ and a countable subset $A \subseteq F$ of parameters. Note that the hull $\lceil A \rceil$ is also countable by Propositions 4.17 and 4.18. Moreover, the full closure $\lceil A \rceil^{\mathrm{full}}$ is also countable. Note that $A \subseteq \lceil A \rceil^{\mathrm{full}}$, so the number of types over $\lceil A \rceil^{\mathrm{full}}$ is not less than over A. But $\lceil A \rceil^{\mathrm{full}}$ is a strong full gf-subfield of F by Lemma 4.22, so we can apply the type classification, and since there are countably many algebraic varieties over $\lceil A \rceil^{\mathrm{full}}$, there are also countably many 1-types over $\lceil A \rceil^{\mathrm{full}}$. □

We finalize the section by characterizing the (model-theoretical) algebraic closure and looking at the Morley ranks of types. Let us briefly recall the definition of a Morley rank, using notation from [Mar02, Section 6.2].

Definition 4.38. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\phi(\bar{x})$ a formula with parameters in $\mathcal{M}$. Consider an $\aleph_0$-saturated $\mathcal{N}$ elementary extension of $\mathcal{M}$. Then for an ordinal α, we define $\mathrm{MR}(\phi) \geqslant \alpha$ by

- $\mathrm{MR}(\phi) \geqslant 0$ if $\phi(\mathcal{N}) \neq \varnothing$;

- $\mathrm{MR}(\phi) \geqslant \alpha+1$ if there are formulas $(\psi_i(\bar{x}))_{i<\omega}$ with parameters in $\mathcal{N}$ such that $(\psi_i(\mathcal{N}))_{i<\omega}$ is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{N})$ and $\mathrm{MR}(\psi_i) \geqslant \alpha$ for all $i < \omega$;
- in case α is a limit ordinal, $\mathrm{MR}(\phi) \geqslant \alpha$ if for all $\beta < \alpha$ we have $\mathrm{MR}(\phi) \geqslant \beta$.

We say $\mathrm{MR}(\phi) = \alpha$ if $\mathrm{MR}(\phi) \geqslant \alpha$ but $\mathrm{MR}(\phi) \ngeqslant \alpha + 1$. In case $\mathrm{MR}(\phi) \geqslant \alpha$ for all ordinals α, we say $\mathrm{MR}(\phi) = \infty$. In case $\mathrm{MR}(\phi) \ngeqslant \alpha$, i.e. $\phi(\mathcal{N}) = \varnothing$, we say $\mathrm{MR}(\phi) = -1$.

For a set $S \subseteq \mathcal{M}^n$ definable by a formula $\phi(\bar{x})$, we define $\mathrm{MR}(S) = \mathrm{MR}(\phi)$. For a type $p \in S_n(A)$, we define $\mathrm{MR}(p) = \inf\{\mathrm{MR}(\phi) : \phi \in p\}$. Note that if p is isolated by ϕ, then $\mathrm{MR}(p) = \mathrm{MR}(\phi)$. For $A \subseteq \mathcal{M}$ and $\bar{a} \in \mathcal{M}$ we define $\mathrm{MR}(\bar{a}/A) = \mathrm{MR}(\mathrm{tp}(\bar{a}/A))$.

Note that $\mathrm{MR}(\phi) = 0$ is equivalent to ϕ defining a finite set. Therefore, for $A \subseteq \mathcal{M}$ and $a \in \mathcal{M}$, we have $\mathrm{MR}(a/A) = 0$ if and only if there exists a formula $\phi(x)$ with parameters in A such that the set $\phi(\mathcal{M}) = \{b \in \mathcal{M} : \mathcal{M} \models \phi(b)\}$ is finite and contains a. In this case we say that $a \in \mathcal{M}$ *is algebraic over* A and denote by $\mathrm{acl}(A)$ the set of all elements algebraic over A. We call $\mathrm{acl}(A)$ *the (model-theoretical) algebraic closure of* A and denote by A^{alg} the field-theoretical algebraic closure (just as we did in Lemma 4.22), in order to distinguish between them. Note that acl is a closure operator. We start by showing that the (model-theoretical) algebraic closure coincides with the full closure of the hull of the set, i.e. algebraically closed sets are exactly the strong full gf-fields.

Proposition 4.39. *Let* $F \models T_{\mathrm{gf}}$ *and* $A \subseteq F$ *a subset. Then* $\mathrm{acl}(A) = \lceil A \rceil^{\mathrm{full}}$.

Proof. We start with the inclusion $\lceil A \rceil^{\mathrm{full}} \subseteq \mathrm{acl}(A)$. By Proposition 4.17, $\lceil A \rceil = \bigcup_{\bar{a} \in A} \lceil \bar{a} \rceil$. Hence, to show $\lceil A \rceil \subseteq \mathrm{acl}(A)$ it suffices to show $\lceil \bar{a} \rceil \subseteq \mathrm{acl}(A)$ for each $\bar{a} \in A$. Let $\lceil \bar{a} \rceil = \bar{a}\bar{b}$. Applying Proposition 4.30, we can find a formula only realized by $\bar{b}$ and permutations of $\bar{b}$, therefore $\bar{b} \in \mathrm{acl}(A)$ and $\lceil A \rceil \subseteq \mathrm{acl}(A)$. Finally, $\lceil A \rceil^{\mathrm{full}} \subseteq \mathrm{acl}(A)$ since $\mathrm{acl}(A)$ is closed under g and field-theoretic algebraic closure.

Now we deal with the inclusion $\mathrm{acl}(A) \subseteq \lceil A \rceil^{\mathrm{full}}$. Recall that $\lceil A \rceil^{\mathrm{full}}$ is a strong full gf-subfield by Lemma 4.22 and denote it by K. We show $\mathrm{acl}(K) \subseteq K$ as then $\mathrm{acl}(A) \subseteq \mathrm{acl}(K) \subseteq K$. Let $b_1 \in \mathrm{acl}(K)$, so $\mathrm{tp}(b_1/K)$ is realized by only finitely many elements in F, denote them by $b_1, \dots, b_k$. By our classification of types, it suffices to show that $q_{\mathrm{gen},K}$ and $q_{V,K}$ for an absolutely irreducible, free and broad variety $V \subseteq F^{2n}$ over K of dimension n have to be realized by other elements beside $b_1, \dots, b_k$. Consider the strong full gf-subfield $L = \lceil K\bar{b} \rceil^{\mathrm{full}}$. Then we have a type $q_{\mathrm{gen},L}$ realized by some element $c \in F$. But $c \neq b_i$ for $i = 1, \dots, k$ and $c \models q_{\mathrm{gen},K}$, so $\mathrm{tp}(b_1/K) \neq q_{\mathrm{gen},K}$. Now consider a type $q_{V,L}$ and $c \in F$ realizing it. Then once again $c \neq b_i$ for $i = 1, \dots, k$ and $c \models q_{V,K}$, so $\mathrm{tp}(b_1/K) \neq q_{V,K}$. Therefore, $b_1 \in K$ and we are done. □

Let us prove a small lemma about the algebraic closure which will turn out to be useful later.

Lemma 4.40. *Let $F \models T_{\mathrm{gf}}$ and $A \subseteq F$ a strong subset. Suppose $b \in \mathrm{acl}(A) \setminus A$. Then there exist $b_1, \ldots, b_n \in F \setminus A$ non-zero and distinct from each other such that $b_n = b$ and for each $i \in \{1, \ldots, n\}$ we have $\delta(b_1, \ldots, b_i/A) = 0$.*

Proof. By Proposition 4.39, $\mathrm{acl}(A) = \lceil A \rceil^{\mathrm{full}}$. Since A is already strong, $\mathrm{acl}(A) = A^{\mathrm{full}}$, i.e. the closure of A under the field-theoretic algebraic closure and the function g. Thus, there exist $b_1, \ldots, b_n \in F$ with $b_n = b$ such that each b_{i+1} either belongs to $(A \cup \{b_1, \ldots, b_i\})^{\mathrm{alg}} \setminus A$ or $b_{i+1} = g(t)$ for $t \in A \cup \{b_1, \ldots, b_i\}$ with $b_{i+1} \notin A$. Let us prove $\delta(b_1, \ldots, b_i/A) = 0$ by induction on i.

First observe that $\delta(b_0/A) = \mathrm{td}(b_0, g(b_0)/\langle A \rangle) - 1 \leqslant 0$. But since A is strong, $\delta(b_0/A) = 0$. Now for the step of induction it suffices to show $\delta(b_{i+1}/Ab_0 \ldots b_i) \leqslant 0$ by the addition property of δ and strongness of A. Indeed, $\delta(b_{i+1}/Ab_0 \ldots b_i) = \mathrm{td}(b_{i+1}, g(b_{i+1})/\langle Ab_0 \ldots b_i \rangle) - 1 \leqslant 0$. □

We also provide a construction for a class of strongly minimal sets. Let us first recall the definition of a strongly minimal set, following [Mar02, Definition 6.1.2].

Definition 4.41. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\phi(\bar{x})$ a formula with parameters in $\mathcal{M}$. Then ϕ is *strongly minimal* if it defines an infinite set and for any elementary extension $\mathcal{N}$ of $\mathcal{M}$, all definable subsets of $\phi(\mathcal{N})$ are finite or cofinite. Furthermore, a definable subset $S \subseteq \mathcal{M}^n$ is *strongly minimal* if it is defined by a strongly minimal formula.

Note that for a strongly minimal formula ϕ we have $\mathrm{MR}(\phi) = 1$ and, therefore, for a strongly minimal subset S we also have $\mathrm{MR}(S) = 1$. Furthermore, acl restricted to a strongly minimal set becomes a pregeometry. We provide a class of examples of strongly minimal formulas for the theory T_{gf}. Let us start by characterizing the varieties that correspond to these examples, which we call perfectly broad by analogy with perfectly rotund varieties in the exponential case (see, for example, [Kir09, Definition 2.28]). Recall the notation V^σ from Definition 4.1.

Definition 4.42. Let F be an algebraically closed field of characteristic 0, $V \subseteq F^{2n}$ an absolutely irreducible algebraic variety. Then V is *perfectly broad* if $\dim V = n$ and for all $m < n$ and injective $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$ we have $\dim V^\sigma > m$.

Proposition 4.43. *Let $F \models T_{\mathrm{gf}}$, $\bar{a} \in F$ a strong tuple and $V \subseteq F^{2n}$ an absolutely irreducible, free and perfectly broad variety over $\bar{a}$. Then the formula $\phi_{V,\bar{a}}$ is strongly minimal.*

Proof. First note that whenever $\bar{b} \models \phi_{V,\bar{a}}$, we get $\bar{a}\bar{b}$ strong and for every non-empty proper subtuple $\bar{b}' \subseteq \bar{b}$, the tuple $\bar{a}\bar{b}'$ is not strong. Indeed, by Lemma 4.27 we have $\delta(\bar{b}/\bar{a}) = 0$ and since $\bar{a}$ is strong, $\bar{a}\bar{b}$ is also strong. If $\bar{b}' \subseteq \bar{b}$ is a non-empty proper subtuple, then $\mathrm{Loc}(\bar{b}', g(\bar{b}')/\langle\bar{a}\rangle) = V^\sigma$ for some injective $\sigma : \{1, \dots, m\} \to \{1, \dots, n\}$ with $m = |\bar{b}'| < n$. Thus, $\mathrm{td}(\bar{b}', g(\bar{b}')/\langle\bar{a}\rangle) > m$ by perfect broadness of V and $\delta(\bar{b}'/\bar{a}) > 0$. Then $\bar{a}\bar{b}'$ cannot be strong by $\delta(\bar{b}/\bar{a}) = 0$ and additivity. Note that this does not depend on the model we are working in.

Now take an elementary extension F_0 of F and a subset S_0 of $S = \phi_{V,\bar{a}}(F_0)$ defined by a formula α. We can assume α has strong parameters $\bar{a}\bar{a}'$. Consider $\bar{b} \models \phi_{V,\bar{a}}$. Then $\bar{a}\bar{b}$ strong and $\bar{a}\bar{b} \cap \bar{a}\bar{a}' = \bar{a}(\bar{b} \cap \bar{a}')$ is also strong. But $\bar{b} \cap \bar{a}'$ is a subtuple of $\bar{b}$, so it has to be either empty or the whole $\bar{b}$. Therefore, the set S is a union of a set defined by $\phi_{V,\bar{a}\bar{a}'}$ and finite sets defined by $\bar{x} = \bar{a}'_0$ for some subtuple $\bar{a}'_0 \subseteq \bar{a}'$. Note that this is a finite set of formulas. Since these formulas isolate types over $\bar{a}\bar{a}'$, α has to be equivalent to their disjunction. Thus, it defines a finite or a cofinite subset of S and $\phi_{V,\bar{a}}$ is strongly minimal. □

Zilber's Trichotomy Conjecture provides a classification of every strongly minimal set into three categories, and despite Hrushovski's counterexample in [Hru93], still holds in various settings such as Zariski geometries (see [HZ96]). For the precise statement of the trichotomy and a more detailed exposition, one can have a look at [TZ12, Chapter 10]. We suspect that the strongly minimal sets of T_{gf} should fall into the degenerate case, i.e. for any strongly minimal set S and a subset $A \subseteq S$, we have $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$. However, we leave this question as well as the full classification of strongly minimal sets for future research.

Next we show that, in general, the types of the form $q_{V,A}$ have finite Morley rank.

Proposition 4.44. *Let $F \models T_{\mathrm{gf}}$, $K \lhd F$ a strong full gf-subfield, $V \subseteq F^{2n}$ an absolutely irreducible, free and broad algebraic variety over K of dimension n. Then $\mathrm{MR}(p_{V,K}) \leqslant n$ and therefore $\mathrm{MR}(q_{V,K}) \leqslant n$.*

Proof. Note that it suffices to show $\mathrm{MR}(\phi_{V,\bar{a}}) \leqslant n$ for $\bar{a} \in K$ a strong tuple. We do that by induction on n. In case $n = 1$ the variety V is perfectly broad, so the base case follows from Proposition 4.43 and strongly minimal formulas having $\mathrm{MR} = 1$. Consider the step of the induction, and suppose there exist $(\alpha_i(\bar{x}))_{i<\omega}$ defining pairwise disjoint subsets of $\phi_{V,\bar{a}}(F)$ with $\mathrm{MR}(\alpha_i) \geqslant n$ for all $i < \omega$. Consider α_i for some $i < \omega$ and suppose it has strong parameters $\bar{a}\bar{a}'$.

Let $\bar{b} \models \alpha_i$ and thus $\bar{b} \models \phi_{V,\bar{a}}$. Then by Lemma 4.27 $\delta(\bar{b}/\bar{a}) = 0$. Let $L = \mathrm{acl}(\bar{a}\bar{a}')$ and denote $\bar{c} = \bar{b} \setminus L$ and $\bar{d} = \bar{b} \cap L$. By addition formula $\delta(\bar{b}/\bar{a}) = \delta(\bar{c}/\bar{a}\bar{d}) + \delta(\bar{d}/\bar{a})$ and since $\bar{a}$ is strong and $\delta(\bar{d}/\bar{a}) \geqslant 0$, we have $\delta(\bar{c}/\bar{a}\bar{d}) \leqslant 0$. By submodularity $\delta(\bar{c}/\bar{a}\bar{d}) \geqslant \delta(\bar{c}/L)$, so $\delta(\bar{c}/L) = 0$. Hence, we can apply Lemma

4.28 and obtain an absolutely irreducible, free and broad variety W such that $\bar{c} \models \phi_{W,\bar{a}\bar{a}'}$.

Therefore, $\mathrm{tp}(\bar{b}/\bar{a}\bar{a}')$ is isolated by the formula $\phi_{W,\bar{a}\bar{a}'}(\bar{x}) \wedge \xi(\bar{y})$, where $\xi(\bar{y})$ defines $\bar{y} = \bar{d}$. By induction hypothesis, this formula has MR $< n$ unless $\bar{b}$ does not intersect L and $\bar{c} = \bar{b}$. Note that in this case $W = V$. Thus, in order to have $\mathrm{MR}(\alpha_i) \geqslant n$, we need for $\alpha_i(F)$ to contain $\phi_{V,\bar{a}\bar{a}'}(F)$. But any two such sets $\phi_{V,\bar{a}\bar{a}'}(F)$ and $\phi_{V,\bar{a}\bar{a}''}(F)$ intersect since $\phi_{V,\bar{a}\bar{a}'\bar{a}''}(F)$ has to be non-empty. Hence, we get a contradiction and $\mathrm{MR}(\phi_{V,\bar{a}}) \leqslant n$. □

Although we have not established the Morley ranks of each $q_{V,A}$ precisely, we can construct a type of Morley rank n for each n by considering sets of the form $\{x \in F : g^{(n)}(x) = a\}$, where $g^{(n)}$ is an n-iteration of g.

Proposition 4.45. *Let $F \models T_{\mathrm{gf}}$ and $a_1 \in F$. Let $\bar{a} = \lceil a_1 \rceil$ and for every $n \in \mathbb{N}$ let $V_n \subseteq F^{2n}$ be the n-dimensional variety defined by $y_1 = x_2, \ldots, y_{n-1} = x_n, y_n = a_1$. So $b \models q_{V_n,\bar{a}}$ means exactly $g^{(n)}(b) = a_1$. Note that V_n is an absolutely irreducible free and broad variety of dimension n. Then* $\mathrm{MR}(q_{V_n,\bar{a}}) = n$.

Proof. By Proposition 4.44, we already know $\mathrm{MR}(q_{V_n,\bar{a}}) \leqslant n$. Hence, it suffices to show by induction on n that $\mathrm{MR}(q_{V_n,\bar{a}}) \geqslant n$.

We start with the case $n = 1$. To show that $\mathrm{MR}(q_{V_1,\bar{a}}) \geqslant 1$, we just need to prove that whenever $b \models q_{V_1,\bar{a}}$, i.e. $g(b) = a_1$ and $b \notin \bar{a}$, we have $b \notin \mathrm{acl}(\bar{a})$. Suppose $b \in \mathrm{acl}(\bar{a})$, then by Lemma 4.40 there are $b_1, \ldots, b_m$ non-zero, distinct from each other and $\bar{a}$, such that $b_m = b$ and for each $i \in \{1, \ldots, m\}$ we have $\delta(b_1, \ldots, b_i/\bar{a}) = 0$. Hence, we get $\delta(b_0, \ldots, b_{n-1}/\bar{a}) = 0$. But since $g(b) = a_1 \in \bar{a}$ we have $\mathrm{td}(b, g(b)/\langle \bar{a}, b_0, \ldots, b_{n-1} \rangle) = 0$ and $\delta(b_0, \ldots, b_n/\bar{a}) = -1$, contradicting Lemma 4.40.

Now let us deal with the induction step. Consider $q_{V_{n+1},\bar{a}}$. For every $b_1 \models q_{V_1,\bar{a}}$, i.e. $g(b_1) = a_1$ and $b_1 \notin \bar{a}$, consider the variety $W_{b_1} \subseteq F^{2n}$ defined by $y_1 = x_2, \ldots, y_{n-1} = x_n, y_n = b_1$. Then by induction step, $\mathrm{MR}(q_{W_{b_1},\bar{b}}) = n$ for $\bar{b} = \lceil b_1 \rceil$. Since $\mathrm{MR}(q_{V_1,\bar{a}}) = 1$, there are countably many b_1 and for each of them we get a formula $\psi^1_{W_{b_1},\bar{b}}$ with MR $= n$. They define disjoint subsets of $\psi^1_{V_{n+1},\bar{a}}(F)$. Hence, $\mathrm{MR}(q_{V_{n+1},\bar{a}}) \geqslant n + 1$. □

Finally, we calculate the Morley rank of the generic type to be ω.

Theorem 4.46. *Let $F \models T_{\mathrm{gf}}$ and $A \subseteq F$ a strong subset. Then* $\mathrm{MR}(p_{\mathrm{gen},A}) = \omega$.

Proof. By Proposition 4.45, $\mathrm{MR}(x = x) \geqslant \omega$. Denote by α the ordinal $\mathrm{MR}(x = x)$. By Proposition 4.44, $\mathrm{MR}(\psi^1_{V,\bar{a}}) \leqslant n$ for an absolutely irreducible, free and broad V over $\bar{a}$ of dim $= n$ and since Morley rank of a disjunction is the maximum of Morley ranks, we get $\mathrm{MR}(\neg\psi^1_{V,\bar{a}}) = \alpha$. Similarly, as $\mathrm{MR}(x = a) = 0$, we get $\mathrm{MR}(x \neq a) = \alpha$ and $\mathrm{MR}(p_{\mathrm{gen},A}) = \alpha$. By the same argument, for formulas of the form $\neg\psi^1_{V,\bar{a}}$ and $x \neq a$, any conjuction also has MR $= \alpha$.

By Theorem 4.34, every formula in one variable is a boolean combination of the formulas of the form $\psi^1_{V,\bar{a}}$ and $x = a$. Hence, each of them has either a finite Morley rank or $\mathrm{MR} = \alpha$. But then $\alpha = \omega$. □

4.4 Quasiminimality

The goal of this section is to prove quasiminimality of $\mathbb{C}_g = (\mathbb{C}; +, \cdot, g)$, whenever g is an entire, i.e. holomorphic on $\mathbb{C}$, generic function. We define a pregeometry gcl on models of T_{gf} by letting $a \in \mathrm{gcl}(A)$ if and only if a is not generic over A. In case of $\mathbb{C}_g$ with g entire and generic, we show that gcl has the countable closure property. Then the formulas of the form $\psi^1_{V,\bar{a}}$ have to define countable sets, and therefore we can use the same strategy as for $\mathbb{C}_{\mathrm{field}}$ and apply Theorem 4.34 to see that $\mathbb{C}_g$ is quasiminimal.

Definition 4.47. Let $F \models T_{\mathrm{gf}}$. We define a closure operator gcl_F on F by saying that $b_1 \in \mathrm{gcl}_F(A)$ if there are $b_2, \ldots, b_n \in F$ such that $\delta(b_1, \ldots, b_n / \lceil A \rceil) = 0$.

Note that by Proposition 4.18 if $b_1 \in \mathrm{gcl}_F(A) \setminus \lceil A \rceil$, we can assume that there exist distinct non-zero $b_2, \ldots, b_n \in F \setminus \lceil A \rceil$ such that $\lceil A \rceil \bar{b} = \lceil A b_1 \rceil$ and $\delta(b_1, \ldots, b_n / \lceil A \rceil) = 0$. Moreover, in this case we have $\bar{b} \in \mathrm{gcl}_F(A)$. Hence, $\mathrm{gcl}_F(A)$ is strong. Then applying Lemma 4.40 we see that $\mathrm{gcl}_F(A)$ is algebraically closed.

We can also characterize gcl_F through non-generic types. We first require two small lemmas that we are also going to use later.

Lemma 4.48. *Let $F \models T_{\mathrm{gf}}$ and $A \subseteq F$ a strong subset. Suppose $\bar{a}_1, \ldots, \bar{a}_n \in F$ are tuples such that $\delta(\bar{a}_i / A) = 0$ for each $i \in \{1, \ldots, n\}$. Then $\delta(\bar{a}_1, \ldots, \bar{a}_n / A) = 0$.*

Proof. It suffices to prove the statement for the case $n = 2$. Suppose $\delta(\bar{a}\bar{b}/A) = 0$ and $\delta(\bar{a}\bar{c}/A) = 0$, where $\bar{a} = \bar{a}\bar{b} \cap \bar{a}\bar{c}$ and we can assume that $\bar{a}\bar{b}\bar{c} \cap A = \varnothing$. By addition formula, $\delta(\bar{a}\bar{b}\bar{c}/A) = \delta(\bar{a}\bar{b}/A) + \delta(\bar{c}/A\bar{a}\bar{b}) = \delta(\bar{c}/A\bar{a}\bar{b})$. Now using submodularity and $\bar{c} \cap A\bar{a}\bar{b} = \varnothing$, we get $\delta(\bar{c}/A\bar{a}\bar{b}) \leqslant \delta(\bar{c}/A\bar{a})$. Once again using the addition formula we obtain $\delta(\bar{c}/A\bar{a}) = \delta(\bar{a}\bar{c}/A) - \delta(\bar{a}/A) \leqslant 0$. Therefore, $\delta(\bar{a}\bar{b}\bar{c}/A) = 0$ and we are done. □

Lemma 4.49. *Let $F \models T_{\mathrm{gf}}$, $A \subseteq F$ a strong subset and $b_1 \in F$. Suppose there is an elementary extension $F \preceq F'$ and elements $b_2, \ldots, b_n \in F'$ such that $\delta(b_1, \ldots, b_n / \mathrm{acl}(A)) = 0$. Then there are also $b'_2, \ldots, b'_m \in F$ with $\delta(b_1, b'_2, \ldots, b'_m / A) = 0$.*

Proof. Take the shortest tuple $b_2, \ldots, b_n$ in F' such that $\delta(\bar{b}/\mathrm{acl}(A)) = 0$. As we know that such tuple exists, we can assume that we start with it from the start. Then by Proposition 4.18, $\mathrm{acl}(A)\bar{b} = \lceil \mathrm{acl}(A) b_1 \rceil$. Here hull is calculated with respect to F', but since elementary embedding $F \preceq F'$ is strong, it coincides with the hull calculated with respect to F. Thus, $\bar{b}$ belongs to F.

Using finite character, choose $\bar{a} \in \mathrm{acl}(A)$ such that $\delta(\bar{b}/A'\bar{a}) = 0$ for any $A' \subseteq \mathrm{acl}(A)$. Since $\bar{a} \in \mathrm{acl}(A)$, we can use Lemma 4.40 and extend this tuple so that $\delta(\bar{a}/A) = 0$. Since $A \subseteq \mathrm{acl}(A)$ we get $\delta(\bar{b}/A\bar{a}) = 0$ and by addition formula $\delta(\bar{a}\bar{b}/A) = \delta(\bar{b}/A\bar{a}') + \delta(\bar{a}/A) = 0$. Thus, we can take the tuple $b_1, b'_2, \dots, b'_m$ to be $\bar{a}\bar{b}$. □

Proposition 4.50. *Let $F \models T_{\mathrm{gf}}$, $A \subseteq F$ and $b_1 \in F$. Then $b_1 \in \mathrm{gcl}_F(A)$ if and only if $b_1 \in \mathrm{acl}(A)$ or $\mathrm{tp}(b_1/\mathrm{acl}(A))$ is of the form $q_{V,\mathrm{acl}(A)}$ for an absolutely irreducible, free and broad variety $V \subseteq F^{2n}$ over $\bar{a} \in \mathrm{acl}(A)$ of dimension n, or in other words, $\mathrm{tp}(b_1/\mathrm{acl}(A))$ is not the generic type.*

Proof. First assume $b_1 \in \mathrm{gcl}_F(A)$, i.e. there are $b_2, \dots, b_n \in F$ such that $\delta(b_1, \dots, b_n/\lceil A \rceil) = 0$. Suppose $b_1 \notin \mathrm{acl}(A)$. Similarly to Proposition 4.44, let $\bar{c} = \bar{b} \setminus \mathrm{acl}(A)$ and $\bar{d} = \bar{b} \cap \mathrm{acl}(A)$. Note that $b_1 \in \bar{c}$. Then by addition formula $\delta(\bar{b}/\lceil A \rceil) = \delta(\bar{c}/\lceil A \rceil \bar{d}) + \delta(\bar{d}/\lceil A \rceil)$ while $\delta(\bar{d}/\lceil A \rceil) \geqslant 0$. Hence, $\delta(\bar{c}/\lceil A \rceil \bar{d}) \leqslant 0$ and by submodularity $\delta(\bar{c}/\mathrm{acl}(A)) \leqslant \delta(\bar{c}/\lceil A \rceil \bar{d})$. So we have $\delta(\bar{c}/\mathrm{acl}(A)) = 0$ and by Lemma 4.28 $\bar{c} \models \phi_{V,\bar{a}}$ for an absolutely irreducible, free and broad variety V over $\bar{a} \in \mathrm{acl}(A)$ of dimension $m = |\bar{c}|$. Thus, $\mathrm{tp}(b_1/\mathrm{acl}(A))$ cannot be generic.

For the other direction note that the case of $b_1 \in \lceil A \rceil$ is clear and the case of $b_1 \in \mathrm{acl}(A)$ follows from Lemma 4.40. Hence we can assume $\mathrm{tp}(b_1/\mathrm{acl}(A)) = q_{V,\mathrm{acl}(A)}$ for an absolutely irreducible, free and broad variety $V \subseteq F^{2n}$ over $\bar{a} \in \mathrm{acl}(A)$ of dimension n. Then by compactness there is an elementary extension $F \preceq F'$ such that we can extend b_1 to a tuple $b_1, \dots, b_n \in F'$ with all elements non-zero, distinct from each other and elements of $\mathrm{acl}(A)$ such that $\bar{b} \models \phi_{V,\bar{a}}$. By Lemma 4.28, $\delta(\bar{b}/\mathrm{acl}(A)) = 0$ and by Lemma 4.49 we can choose $b'_2, \dots, b'_m \in F$ with $\delta(b_1, b'_2, \dots, b'_m/\lceil A \rceil) = 0$. Hence, $b_1 \in \mathrm{gcl}_F(A)$. □

We show that in the case of finite strong A, we have $b \in \mathrm{gcl}_F(A)$ if and only if $\mathrm{tp}(b/A)$ is isolated.

Proposition 4.51. *Let $F \models T_{\mathrm{gf}}$, $\bar{a} \in F$ a strong finite tuple and $b_1 \in F$. Then $b_1 \in \mathrm{gcl}_F(\bar{a})$ if and only if $\mathrm{tp}(b_1/\bar{a})$ is isolated.*

Proof. First suppose $b_1 \in \mathrm{gcl}_F(\bar{a})$. Then there are $b_2, \dots, b_n \in F$ with $\delta(b_1, \dots, b_n/\bar{a}) = 0$. In order to use Proposition 4.29, we elementary embed $\bar{a}\bar{b}$ into U. Then we still have $\bar{a}$ strong in U and $\delta(\bar{b}/\bar{a}) = 0$. If $b_1 \in \bar{a}$ or $b_1 = 0$, then $\mathrm{tp}(b_1/\bar{a})$ is clearly isolated. Otherwise we can assume that $b_1, \dots, b_n$ are all non-zero, distinct from each other and elements of $\bar{a}$. Thus, $\dim V = n$ for $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle)$. Hence, Proposition 4.29 is applicable and $\psi^1_{V,\bar{a}}$ isolates the type $\mathrm{tp}(b_1/\bar{a})$.

For the other direction suppose $b_1 \notin \mathrm{gcl}_F(\bar{a})$. Then by Proposition 4.50, $\mathrm{tp}(b_1/\mathrm{acl}(\bar{a}))$ is the generic type $q_{\mathrm{gen},\mathrm{acl}(\bar{a})}$. We claim that $\mathrm{tp}(b_1/\bar{a})$ is defined by formulas of the form $\neg\psi^1_{V,\bar{a}}$ for all $V \subseteq F^{2n}$ over $\bar{a}$ of dimension n together with formulas of the form $\neg\alpha(x,\bar{a})$ for every $\alpha(x,\bar{a})$ defining a finite set (i.e. formulas $\neg\alpha(x,\bar{a})$ all together are equivalent to $x \notin \mathrm{acl}(\bar{a})$). Denote this set of formulas by Ξ.

First let us show that $b_1 \models \Xi$. Since $\mathrm{tp}(b_1/\mathrm{acl}(\bar{a}) = q_{\mathrm{gen},\mathrm{acl}(\bar{a})}$, we have $b_1 \notin \mathrm{acl}(\bar{a})$ and $b_1 \models \neg\alpha(x,\bar{a})$ for $\alpha(x,\bar{a})$ defining a finite set. Suppose $b_1 \models \psi^1_{V,\bar{a}}$ for some $V \subseteq F^{2n}$ over $\bar{a}$ of dimension n. Then there are $b_2,\ldots,b_n \in F$ with $\delta(\bar{b}/\bar{a}) = 0$. Hence, $b_1 \in \mathrm{gcl}_F(\bar{a})$ and we have a contradiction.

Now suppose $b_1' \models \Xi$. Then $b_1' \notin \mathrm{acl}(\bar{a})$. Thus, either $\mathrm{tp}(b_1'/\mathrm{acl}(\bar{a})) = q_{\mathrm{gen},\mathrm{acl}(\bar{a})}$ or $\mathrm{tp}(b_1'/\mathrm{acl}(\bar{a})) = q_{V,\mathrm{acl}(\bar{a})}$ for some irreducible, free and rotund $V \subseteq F^{2n}$ over $\mathrm{acl}(\bar{a})$ of dimension n. In the latter case, by compactness and Lemma 4.28 there is an elementary extension $F \preceq F'$ and $b_2',\ldots,b_n' \in F'$ such that $\delta(\bar{b}'/\mathrm{acl}(\bar{a})) = 0$. Then by Lemma 4.49 there are also $b_2'',\ldots,b_m'' \in F$ all non-zero, distinct from each other and elements of $\bar{a}$ with $\delta(b_1', b_2'',\ldots,b_m''/\bar{a}) = 0$. Taking $V = \mathrm{Loc}(b_1', b_2'',\ldots,b_m'', g(b_1'), g(b_2''),\ldots,g(b_m'')/\langle\bar{a}\rangle)$ we get $\dim V = m$ and $b_1' \models \psi^1_{V,\bar{a}}$. Thus we get a contradiction with $b_1' \models \Xi$. Therefore, $\mathrm{tp}(b_1'/\mathrm{acl}(\bar{a})) = q_{\mathrm{gen},\mathrm{acl}(\bar{a})} = \mathrm{tp}(b/\mathrm{acl}(\bar{a}))$ and $\mathrm{tp}(b_1/\bar{a}) = \mathrm{tp}(b_1'/\bar{a})$.

Thus, Ξ defines $\mathrm{tp}(b_1/\bar{a})$. As formulas of the form $\psi^1_{V,\bar{a}}$ for $V \subseteq F^{2n}$ over $\bar{a}$ of dimension n or $\alpha(x,\bar{a})$ defining a finite set are isolating by Proposition 4.29, all other types over $\bar{a}$ are isolated. Then $\mathrm{tp}(b_1/\bar{a})$ cannot be isolated as well since the type space $S_1(\bar{a})$ is compact and infinite. Thus, $\mathrm{tp}(b_1/\bar{a})$ is not isolated and we are done. □

Let us show that gcl_F is indeed a closure operator and, moreover, a pregeometry.

Proposition 4.52. *Let $F \models T_{\mathrm{gf}}$. Then gcl_F is a pregeometry.*

Proof. Let us start by showing that gcl_F is a closure operator. We get $A \subseteq \mathrm{gcl}_F(A)$ immediately since $\delta(a/\lceil A\rceil) = 0$ for $a \in A$.

To show monotonicity, assume $A \subseteq B \subseteq F$ and $b \in \mathrm{gcl}_F(A)$. Then by Proposition 4.50 either $b \in \mathrm{acl}(A)$ and therefore also $b \in \mathrm{acl}(B)$ or $b \models q_{V,\mathrm{acl}(A)}$ for an absolutely irreducible, free and broad variety $V \subseteq F^{2n}$ over $\bar{a} \in \mathrm{acl}(A)$ of dimension n. In this case $b \models \psi^1_{V,\bar{a}}$ and since $\bar{a} \in \mathrm{acl}(B)$, the type of b over $\mathrm{acl}(B)$ cannot be generic.

Now assume $b_1 \in \mathrm{gcl}_F(\mathrm{gcl}_F(A))$, i.e. there is a tuple $\bar{b}$ with $\delta(\bar{b}/\mathrm{gcl}_F(A)) = 0$. By finite character we can choose $\bar{a} \in \mathrm{gcl}_F(A)$ such that $\delta(\bar{b}/\mathrm{gcl}_F(A)) = \delta(\bar{b}/\bar{a}A')$ for any $A' \subseteq \mathrm{gcl}_F(A)$. Using Lemma 4.48 and $\bar{a} \in \mathrm{gcl}_F(A)$, we can extend the tuple $\bar{a}$ so that $\delta(\bar{a}/\lceil A\rceil) = 0$. Then we get $\delta(\bar{a}\bar{b}/\lceil A\rceil) = \delta(\bar{a}/\lceil A\rceil) + \delta(\bar{b}/\bar{a}\lceil A\rceil) = 0$. Hence, $b_1 \in \mathrm{gcl}_F(A)$.

Finite character of gcl_F follows directly from the finite character of δ and finite character of the hull. Finally, for the exchange principle consider $b \in$

$\mathrm{gcl}_F(Ac) \setminus \mathrm{gcl}_F(A)$. Note that since $b \notin \mathrm{gcl}_F(A)$, we have $\delta(b/\lceil A \rceil) = 1$ and $\lceil A \rceil b$ strong. As $c \notin \mathrm{gcl}_F(A)$, the same holds for c. From $b \in \mathrm{gcl}_F(Ac)$ we get a tuple $\bar{b}' \in F$ such that $\delta(b\bar{b}'/\lceil A \rceil c) = 0$. Hence, we obtain the following equality

$$\begin{aligned}\delta(c\bar{b}'/\lceil A \rceil b) &= \delta(bc\bar{b}'/\lceil A \rceil) - \delta(b/\lceil A \rceil)\\ &= \delta(bc\bar{b}'/\lceil A \rceil) - 1\\ &= \delta(bc\bar{b}'/\lceil A \rceil) - \delta(c/\lceil A \rceil)\\ &= \delta(b\bar{b}'/\lceil A \rceil c) = 0.\end{aligned}$$

Thus, $c \in \mathrm{gcl}_F(Ab)$ and gcl is a pregeometry. □

Recall from Chapter 1 that for a function g on $\mathbb{C}$ we denote $\mathbb{C}_g = (\mathbb{C}; +, \cdot, g)$ and call g generic if $\mathbb{C}_g \models T_{\mathrm{gf}}$. We show that whenever g is entire and generic, the pregeometry $\mathrm{gcl}_{\mathbb{C}}$ has the countable closure property, as defined in Definition 2.4.

Theorem 4.53. *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire generic function. Consider the corresponding pregeometry $\mathrm{gcl}_{\mathbb{C}}$. Then $\mathrm{gcl}_{\mathbb{C}}$ has the countable closure property.*

Proof. Let $\bar{a} \in \mathbb{C}$ be a finite tuple. Recall that it suffices to show that $\mathrm{gcl}_{\mathbb{C}}(\bar{a})$ is countable due to the finite character of $\mathrm{gcl}_{\mathbb{C}}$. Since $\lceil \bar{a} \rceil$ is also finite by Proposition 4.18, we can assume that $\bar{a}$ is strong. Suppose $b_1 \in \mathrm{gcl}_{\mathbb{C}}(\bar{a})$. Then there are $b_2, \ldots, b_n \in \mathbb{C}$ such that $\delta(\bar{b}/\bar{a}) = 0$. We can assume $b_1, \ldots, b_n$ are non-zero, distinct from each other and elements of $\bar{a}$. Thus, $\mathrm{td}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle) = n$. Let $V = \mathrm{Loc}(\bar{b}, g(\bar{b})/\langle \bar{a} \rangle)$, so $\dim V = n$. Denote $\Gamma_g = \{(\bar{x}, g(\bar{x})) : \bar{x} \in \mathbb{C}^n\}$. Then Γ_g is an analytic set of dimension n. Thus, $V \cap \Gamma_g$ is also an analytic set. We claim that $\dim V \cap \Gamma_g = 0$. Then $(\bar{b}, g(\bar{b}))$ is an isolated point of $V \cap \Gamma_g$ and since there are countably many isolated points for each V and countably many algebraic varieties V defined over $\bar{a}$, we are done.

Let $k = \dim V \cap \Gamma_g$ and assume $k \neq 0$. Note that since g is a function, the projection $\pi_1 : V \cap \Gamma_g \to \mathbb{C}^n$ to the first n coordinates is injective. Thus, its image $\pi_1(V \cap \Gamma_g)$ also has dimension k. Take a point $(\bar{c}, g(\bar{c})) \in V \cap \Gamma_g$ with a neighbourhood $U \subseteq V \cap \Gamma_g$ of the maximal dimension k. Then locally there are k variables $x_{i_1}, \ldots, x_{i_k}$ among $x_1, \ldots, x_n$ such that the corresponding projection $\pi : \mathbb{C}^{2n} \to \mathbb{C}^k$ mapping $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (x_{i_1}, \ldots, x_{i_k})$ defines a homeomorphism between a neighbourhood $U_0 \subseteq U$ of $(\bar{c}, g(\bar{c}))$ and a neighbourhood U_1 of $\pi(\bar{c}, g(\bar{c})) = \bar{c}'$. Denote by $\theta = (\theta_1, \ldots, \theta_{2n})$ the map $(\pi \restriction_{U_0})^{-1}$. Since U is of maximal dimension, $\bar{c}$ consists of non-zero elements, distinct from each other and elements of $\bar{a}$, the maps of the form

- θ_i for $i \leqslant n$,
- $\theta_i - \theta_j$ for $i, j \leqslant n$ and $i \neq j$,

- $\theta_i - a_j$ for $i \leqslant n$ and each element a_j in $\bar{a}$

are non-zero in a neighbourhood of $\bar{c}'$. By shrinking U_1 is necessary, we can assume this neighbourhood is U_1. Choose $\bar{r} \in (\mathbb{Q}^{\text{alg}})^k \cap U_1$ with all elements non-zero, distinct from each other and elements of $\bar{a}$. Then $\theta(\bar{r})$ has the first n elements non-zero, distinct from each other and elements of $\bar{a}$. Furthermore, equations of the form $x_{i_j} = r_j$ do not hold on all points of V as θ is defined on a neighbourhood, so $\text{td}(\theta(\bar{r})/\langle\bar{a}\rangle) < \dim V = n$. Hence, denoting by $\bar{s}$ the first n elements of $\theta(\bar{r})$, we get $\delta(\bar{s}/\bar{a}) < 0$ and a contradiction with $\bar{a}$ being strong.

This concludes the proof as we obtain $\dim V \cap \Gamma_g = 0$. □

Note that we have proven CCP without relating $\text{gcl}_{\mathbb{C}}$ to the pregeometries considered in Chapter 2. Taking $\mathcal{S}$ to be $\{g\}$, $\{g, g'\}$ or $G = \{g^{(n)} : n \in \mathbb{Z}_{\geqslant 0}\}$ produces pregeometries of the form $\text{Dcl}_{\mathcal{S}}$ and closure operators of the form $\text{icl}_{\mathcal{S}}$ naturally arising from the function g. Recall that by Theorem 1.4 in the case of G we get $\text{Dcl}_G = \text{icl}_G$. However, as the theory T_{gf} does not provide information about the derivatives of g, the connection between these closures and the pregeometry $\text{gcl}_{\mathbb{C}}$ seems unclear for now and we leave it for future research.

Finally, we can deduce quasiminimality of $\text{gcl}_{\mathbb{C}}$ using the same method as for $\mathbb{C}_{\text{field}}$. Let us state Theorem 1.6 once again before providing its proof.

Theorem (Theorem 1.6)**.** Let $g : \mathbb{C} \to \mathbb{C}$ be an entire generic function. Then $\mathbb{C}_g$ is quasiminimal.

Proof. Let $\alpha(x, \bar{a})$ define a subset. Then by Theorem 4.34, $\alpha(x, \bar{a})$ is equivalent to a boolean combination of the formulas defining types of finite Morley ranks. By Proposition 4.50, b satisfying such a formula is equivalent to $b \in \text{gcl}_{\mathbb{C}}(\bar{a})$, and by Theorem 4.53 $\text{gcl}_{\mathbb{C}}(\bar{a})$ is countable, so this formula has to define a countable subset of it. Then $\alpha(x, \bar{a})$ defines a countable or cocountable set. □

In the next section we are going to show that any two such $\mathbb{C}_g$ are actually isomorphic.

4.5 Prime models

In this section we show that gcl_F has the countable closure property if and only if $F \models T_{\text{gf}}$ is a prime model over a gcl-independent set. As a corollary we see that all models of the form $\mathbb{C}_g$ for an entire $g : \mathbb{C} \to \mathbb{C}$ are isomorphic. The tools required for this section include forking, so we start by introducing it together with the properties that are going to be useful later. We mostly follow [Mar02, Section 6.3].

Definition 4.54. Let T be a complete ω-stable theory, $\mathcal{M} \models T$.

- Let $A \subseteq B \subseteq \mathcal{M}$, $p \in S_n(A)$, $q \in S_n(B)$, and $p \subseteq q$. Note that in this case we have $\mathrm{MR}(q) \leqslant \mathrm{MR}(p)$. If $\mathrm{MR}(q) < \mathrm{MR}(p)$, we say that q *is a forking extension of* p and that q *forks over* A. If $\mathrm{MR}(q) = \mathrm{MR}(p)$, we say that q is a *nonforking extension of* p.

- Let $\bar{a} \in \mathcal{M}$ and $A, B \subseteq \mathcal{M}$. We say that $\bar{a}$ *is independent from* B *over* A if $\mathrm{tp}(\bar{a}/A \cup B)$ does not fork over A. We write $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A B$.

Note that in our case if q is the generic type, then q is always a nonforking extension. We use this by applying the following straightforward lemma.

Lemma 4.55. *Let $\mathcal{M} \models T_{\mathrm{gf}}$, $a \in \mathcal{M}$ and $A \subseteq B \subseteq \mathcal{M}$. Then whenever $a \notin \mathrm{gcl}_{\mathcal{M}}(B)$, we have $a \mathop{\smile\!\!\!\!\!\!|}_A B$.*

Proof. Since $\mathrm{tp}(a/\mathrm{acl}(B))$ is generic, $\mathrm{MR}(\mathrm{tp}(a/B)) = \omega$. But then $\mathrm{MR}(\mathrm{tp}(a/B))$ cannot be smaller than $\mathrm{MR}(\mathrm{tp}(a/A))$. Hence, $a \mathop{\smile\!\!\!\!\!\!|}_A B$. □

Apart from this lemma we essentially only need two facts about the independence relation: a special case of Open Mapping Theorem as proved in [Mar02, Lemma 6.4.6] and symmetry (see, for example, [Mar02, Lemma 6.3.19]). For a deeper exposition on forking and independence relations, one can have a look at the full section [Mar02, Section 6.3] or read [TZ12, Section 7.1].

Fact 4.56 (Symmetry)**.** *Let T be a complete ω-stable theory, $\mathcal{M} \models T$, $\bar{a}, \bar{b} \in \mathcal{M}$, $A \subseteq \mathcal{M}$. Then if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$, we also have $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$.*

Fact 4.57 (Open Mapping Theorem)**.** *Let T be ω-stable, $\mathcal{M} \models T$. If $A \subseteq B \subseteq \mathcal{M}$, $p \in S_n(A)$, $p' \in S_n(B)$, p' is a nonforking extension of p, and p' is isolated, then p is isolated.*

Now let us move to the second ingredient of this section: prime models over gcl-independent sets. We first recall the general notion of a prime model over a set, as well as the notion of a totally indiscernible sequence.

Definition 4.58. Let T be a theory, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. $\mathcal{M}$ is *prime over* A if for any $\mathcal{N} \models T$ and partial elementary $f : A \to \mathcal{N}$, there is an elementary $f^* : \mathcal{M} \to \mathcal{N}$ extending f.

Definition 4.59. Let $(I, <)$ be a linear order, $\mathcal{M}$ a structure, $A \subseteq \mathcal{M}$ a subset and $(a_i)_{i \in I} \in \mathcal{M}$ a sequence of elements. Then $(a_i)_{i \in I}$ is an *indiscernible sequence over* A if for any $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$ in I we have $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/A)$. Moreover, $(a_i)_{i \in I}$ is a *totally indiscernible sequence over* A if for any distinct $i_1, \ldots, i_n$ and distinct $j_1, \ldots, j_n$ in I we have $\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/A)$, i.e. if the ordering does not matter.

Fact 4.60. *Suppose T is ω-stable, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. Then*

1. *there is a prime model $\mathcal{M}_0 \preceq \mathcal{M}$ over A;*

2. *any two prime models $\mathcal{M}_1, \mathcal{M}_2 \preceq \mathcal{M}$ over A are isomorphic over A;*

3. *the following are equivalent for any model $\mathcal{M}_0 \preceq \mathcal{M}$:*

   (a) *$\mathcal{M}_0$ is prime over A;*

   (b) *$\mathcal{M}_0$ is atomic over A (i.e. all realized types over A are isolated) and does not contain an uncountable totally indiscernible sequence over A.*

Recall as well that in ω-stable theories an indiscernible sequence over a set A is also totally indiscernible over A (see for example [TZ12, Lemma 9.11]). Now let us introduce gcl-independent sets.

Definition 4.61. Let $F \models T_{\mathrm{gf}}$ with the corresponding pregeometry gcl_F on it. We call a subset $A \subseteq F$ gcl_F-*independent* if it is independent with respect to gcl_F, i.e. for any $a \in A$ we have $a \notin \mathrm{gcl}_F(A \setminus \{a\})$.

Note that a gcl-independent set forms a totally indiscernible sequence (over $\varnothing$). Moreover, we show that any gcl-independent set is strong.

Proposition 4.62. *Let $F \models T_{\mathrm{gf}}$ with the corresponding pregeometry gcl_F on it and gcl_F-independent subset $A \subseteq F$. Then A is strong.*

Proof. By finite character of δ, it suffices to show that any finite subset of A is strong. We prove that by assuming a finite (possibly empty) tuple $\bar{a} \in A$ is strong, $a' \notin \mathrm{gcl}_F(\bar{a})$ and showing that in this case $a'\bar{a}$ is also strong.

Indeed, since $\bar{a}$ is strong and $a' \notin \mathrm{gcl}_F(\bar{a})$, we have $\delta(a'/\bar{a}) > 0$, so $\delta(a'/\bar{a}) = 1$. Consider any $\bar{b} \in F$. Then for the same reasons $\delta(\bar{b}a'/\bar{a}) \geq 1$. By addition formula we have $\delta(\bar{b}/a'\bar{a}) = \delta(\bar{b}a'/\bar{a}) - \delta(a'/\bar{a}) \geq 0$, thus $a'\bar{a}$ is strong. Hence, by induction any finite subset of A is strong and A is strong as well. $\square$

We are finally ready to show that models with the countable closure property are exactly prime models over gcl-independent sets. We partially follow the strategy from [Hay16, Lemma 3.5.2].

Theorem 4.63. *Let $F \models T_{\mathrm{gf}}$ with the corresponding pregeometry gcl_F. Then gcl_F has the countable closure property if and only if there exists a gcl-independent set $A \subseteq F$ such that F is prime over A.*

Proof. First assume gcl_F has the countable closure property. Let A be a basis for gcl_F. Then A is gcl-independent and $F = \mathrm{gcl}_F(A)$.

We want to show that F is prime over A by proving that F is atomic over A and does not contain uncountable totally indiscernible sequences over A. Let us start with atomicity. Consider $b_1 \in F$.

Then $b_1 \in \mathrm{gcl}_F(A)$ and by Proposition 4.50, we have either $b_1 \in \mathrm{acl}(A)$ or $\mathrm{tp}(b_1/\mathrm{acl}(A)) = q_{V,\mathrm{acl}(A)}$ for some irreducible, free and rotund $V \subseteq F^{2n}$ over $\bar{a} \in \mathrm{acl}(A)$. If $b_1 \in \mathrm{acl}(A)$, then $\mathrm{tp}(b_1/A)$ is clearly isolated. If $\mathrm{tp}(b_1/\mathrm{acl}(A)) = q_{V,\mathrm{acl}(A)}$, by compactness and Lemma 4.28, there is an elementary extension $F \preceq F'$ and $b_2, \ldots, b_n \in F'$ such that $\delta(b_1, b_2, \ldots, b_n/\mathrm{acl}(A)) = 0$. We can use Lemma 4.49 as A is strong by Proposition 4.62, and then finite character of δ, to find $b'_2, \ldots, b'_m \in F$ all non-zero, distinct from each other and elements of A as well as a strong tuple $\bar{a} \in A$ with $\delta(b_1, b'_2, \ldots, b'_m/\bar{a}) = 0$. Then by Lemma 4.27, we have $b_1 \models \psi^1_{V,\bar{a}}$ for $V = \mathrm{Loc}(b_1, b'_2, \ldots, b'_m, g(b_1), g(b'_2), \ldots, g(b'_m)/\langle\bar{a}\rangle)$ with dimension m. By Lemma 4.29 we can elementarily embed $\bar{a}$ and $b_1, b'_2, \ldots, b'_m$ into the Fraïssé limit U to see that $\psi^1_{V,\bar{a}}$ isolates $\mathrm{tp}(b_1/\bar{a})$. We claim that $\psi^1_{V,\bar{a}}$ also isolates $\mathrm{tp}(b_1/\bar{a}\bar{a}')$ for any finite tuple $\bar{a}' \in A$ and thus isolates $\mathrm{tp}(b_1/A)$.

Consider the formula $\psi^1_{V,\bar{a}\bar{a}'}$. Suppose $c_1 \models \psi^1_{V,\bar{a}}$. Then there are $c_2, \ldots, c_m \in F$ with $\bar{c} \models \phi_{V,\bar{a}}$. Each c_i belongs to $\mathrm{gcl}_F(\bar{a})$, while elements of $\bar{a}'$ cannot belong to $\mathrm{gcl}_F(\bar{a})$ as A is gcl_F-independent. Thus, elements of $\bar{c}$ are distinct from elements of $\bar{a}'$ and $c_1 \models \psi^1_{V,\bar{a}\bar{a}'}$. Therefore, $\psi^1_{V,\bar{a}}$ implies $\psi^1_{V,\bar{a}\bar{a}'}$. Applying Lemma 4.29 as above we see that $\psi^1_{V,\bar{a}\bar{a}'}$ isolates $\mathrm{tp}(b_1/\bar{a}\bar{a}')$, so $\psi^1_{V,\bar{a}}$ also isolates $\mathrm{tp}(b_1/\bar{a}\bar{a}')$. Hence, it isolates $\mathrm{tp}(b_1/A)$ and F is atomic over A.

Next we show that there are no uncountable totally indiscernible sequences over A in F. Suppose $(b_\alpha)_{\alpha<\kappa}$, where κ is some cardinal, is a totally indiscernible sequence over A. Then each b_α has the same type over A isolated by the same formula with parameters $\bar{a} \in A$. Since $\mathrm{tp}(b_\alpha/\bar{a})$ is isolated, by Proposition 4.51 we have $b_\alpha \in \mathrm{gcl}_F(\bar{a})$ for all $\alpha < \kappa$. But we assumed gcl_F to have the countable closure property, so $(b_\alpha)_{\alpha<\kappa}$ is countable. Therefore, by Fact 4.60, F is prime over A.

For the other direction, as mentioned above, we use the strategy from [Hay16, Lemma 3.5.2]. Assume F is prime over a gcl-independent set A and in order to prove the countable closure property take $\bar{b} \in F$. We want to show that $\mathrm{gcl}_F(\bar{b})$ is countable. Since F is prime over A, it is also atomic over A by Fact 4.60. Take $c \in F$. Then $\mathrm{tp}(c/A)$ is isolated by some formula $\alpha(x, \bar{a})$ for $\bar{a} \in A$ and therefore $\mathrm{tp}(c/\bar{a})$ is also isolated by $\alpha(x, \bar{a})$. By Proposition 4.51, we have $c \in \mathrm{gcl}_F(\bar{a}) \subseteq \mathrm{gcl}_F(A)$. Thus, $\mathrm{gcl}_F(A) = F$ and $\bar{b} \in \mathrm{gcl}_F(A)$. Then by finite character there exists a finite $\bar{a} \in A$ such that $\bar{b} \in \mathrm{gcl}_F(\bar{a})$. Therefore, $\mathrm{gcl}_F(\bar{b}) \subseteq \mathrm{gcl}_F(\bar{a})$ and it suffices to show that $\mathrm{gcl}_F(\bar{a})$ is countable.

Consider $A \setminus \bar{a}$ and enumerate it as $(a'_\alpha)_{\alpha<\kappa}$ for some cardinal κ. Let $\mathcal{N}_0$ be prime over $\bar{a}$ and construct a model $\mathcal{N}_\alpha$ for every $\alpha < \kappa$ by taking $\mathcal{N}_{\alpha+1}$ to be prime over $\mathcal{N}_\alpha a'_\alpha$ and $\mathcal{N}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{N}_\alpha$ for limit λ. Let $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{N}_\alpha$. Note that for every $\alpha < \kappa$ we have $a'_\alpha \notin \mathrm{gcl}_{\mathcal{N}}(\mathcal{N}_\alpha)$. Indeed, by induction we get that for every $\alpha < \kappa$ we have $\mathcal{N}_\alpha = \mathrm{gcl}_{\mathcal{N}_\alpha}(\bar{a}, (a'_\beta)_{\beta<\alpha})$. But A is gcl-independent, so $a'_\alpha \notin \mathrm{gcl}_{\mathcal{N}}(\mathcal{N}_\alpha)$.

We claim that $\mathrm{gcl}_{\mathcal{N}}(\bar{a}) \subseteq \mathcal{N}_0$. Otherwise there would exist $c \in \mathrm{gcl}_{\mathcal{N}}(\bar{a})$ and $\alpha < \kappa$ such that $c \in \mathcal{N}_{\alpha+1} \setminus \mathcal{N}_\alpha$. Since $\mathcal{N}_{\alpha+1}$ is prime over $\mathcal{N}_\alpha a'_\alpha$, the type $p' = \mathrm{tp}(c/\mathcal{N}_\alpha a'_\alpha)$ is isolated, while the type $p = \mathrm{tp}(c/\mathcal{N}_\alpha)$ is not (since $c \notin \mathcal{N}_\alpha$). Then the by Open Mapping Theorem, p' is a forking extension of p, i.e. $c \not\mathrel{\perp\!\!\!\perp}_{\mathcal{N}_\alpha} a'_\alpha$. By Symmetry, $a'_\alpha \not\mathrel{\perp\!\!\!\perp}_{\mathcal{N}_\alpha} c$, and by Lemma 4.55, $a'_\alpha \in \mathrm{gcl}_{\mathcal{N}}(\mathcal{N}_\alpha c)$. On the other hand, $a'_\alpha \notin \mathrm{gcl}_{\mathcal{N}}(\mathcal{N}_\alpha)$, so $c \notin \mathrm{gcl}_{\mathcal{N}}(\mathcal{N}_\alpha)$. That contradicts $c \in \mathrm{gcl}_{\mathcal{N}}(\bar{a})$.

Finally, since $\mathcal{N}_0$ is countable, $\mathrm{gcl}_{\mathcal{N}}(\bar{a})$ is countable. Then we can embed F into $\mathcal{N}$, since F is prime over A, and see that $\mathrm{gcl}_F(\bar{a})$ has to be countable as well. □

Now it is easy to deduce the categoricity result for the models of the form $\mathbb{C}_g$, where g is entire. Let us state it once again before providing the proof

Theorem (Theorem 1.7)**.** Let g_1, g_2 be entire generic functions. Then $\mathbb{C}_{g_1} \cong \mathbb{C}_{g_2}$.

Proof. By Theorems 4.53 and 4.63, both structures are prime over some gcl-independent sets $A_1, A_2 \subseteq \mathbb{C}$. Then A_1 and A_2 both have cardinality continuum and any bijection between them is a partial elementary function. Hence, by uniqueness of prime models, the structures have to be isomorphic. □

4.6 Generalisation

In this final section we consider a version of the theory of a generic function with arbitrary restrictions instead of $g(0) = 0$. For a function $\alpha : S \subseteq \mathbb{C} \to \mathbb{C}$ with S finite we define a theory T^α_{gf} in the language $\mathcal{L}_{\mathrm{gf}}$ expanded by constants for elements of S and $\mathrm{im}\alpha$. Then T^α_{gf} constitutes an axiomatization for the theory of a generic function with restriction $g \restriction_S = \alpha$. Note that the Liouville functions do not form models of these theories any more although it might be possible to make them into ones. We leave this question out of the scope of the thesis. We expect that with suitable adjustments the results of Sections 4.2, 4.3, 4.4 and 4.5 hold for T^α_{gf}.

Fix a finite subset $S \subseteq \mathbb{C}$ and a function $\alpha : S \subseteq \mathbb{C} \to \mathbb{C}$. The following is an analogue of Definition 4.1.

Definition 4.64. Let F be an algebraically closed field of characteristic 0, $g : F \to F$ a function and $V \subseteq F^{2n}$ an absolutely irreducible algebraic variety. Denote the coordinates of F^{2n} by $x_1, \dots, x_n, y_1, \dots, y_n$.

For a set $A \subseteq F$, denote by $V^\dagger_{A,S}$ the set of all n-tuples $\bar{x} \in F^n$ with all elements of $\bar{x}$ distinct from each other and elements of $A \cup S$, such that $(\bar{x}, g(\bar{x})) \in V$.

Note that in this notation, a set $V^\dagger_A$ as defined in Definition 4.1 would be denoted by $V^\dagger_{A,\{0\}}$. Now we can provide the axiomatization for T^α_{gf}, an analogue of Definition 4.2.

Definition 4.65. Let $\mathcal{L}_{\mathrm{gf}}^{\alpha}$ be the language $\mathcal{L}_{\mathrm{gf}}$ extended by finitely many constants for each element of $S \cup \mathrm{im}\ \alpha$. We define the first-order theory T_{gf}^{α} in the language $\mathcal{L}_{\mathrm{gf}}^{\alpha}$ through the following axiomatization.

1. ACF_0.
2. $g : F \to F$ is a function with $g \restriction_S = \alpha$.
3. Quantifier-free type of $S \cup \mathrm{im}\ \alpha$ in $\mathbb{C}_{\mathrm{field}}$.
4. Let $V \subseteq F^{2n}$ be an algebraic variety over $\mathbb{Q}$ with $\dim V < n$. Then the set $V_{\varnothing,S}^{\dagger}$ is empty.
5. Let $V \subseteq F^{2n}$ be an absolutely irreducible, free and broad algebraic variety over $\bar{a}$ of dimension n. Then the set $V_{\bar{a},S}^{\dagger}$ is non-empty.

Note that T_{gf} coincides with T_{gf}^{α} for $\alpha : \{0\} \to \mathbb{C}$ defined by $\alpha(0) = 0$. Then we can construct the Fraïssé limit the same way as was done in Section 4.2, but taking care of points in S instead of just 0. This would give us consistency and completeness of T_{gf}^{α}, while similarly copying the methods of Section 4.3 provides ω-stability. Finally, following the strategy of Sections 4.4 and 4.5, we expect to see that for any entire function $g : \mathbb{C} \to \mathbb{C}$ with $\mathbb{C}_g^{\alpha} = (\mathbb{C}; +, \cdot, g, S \cup \mathrm{im}\ \alpha) \models T_{\mathrm{gf}}^{\alpha}$, the structure $\mathbb{C}_g^{\alpha}$ is quasiminimal and choosing a different such g produces an isomorphic structure.

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