Higher Homological Algebra and Fractional Calabi-Yau Algebras

Jordan Haden

School of Engineering, Mathematics and Physics
University of East Anglia

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Abstract

In Part I, we present a family of selfinjective algebras of type D, which are Morita equivalent to skew group algebras of the 3-preprojective algebras of type A. One-in-three is itself a 3-preprojective algebra, and the corresponding 2-representation-finite algebras are fractional Calabi-Yau. We show that our work is connected to modular invariants for $\mathrm{SU}(3)$, and give recipes to construct 2-Auslander-Reiten quivers for an arbitrary 2-representation-finite algebra.

In Part II, we reinterpret Thomas' construction of a Bridgeland stability condition on the dg perfect derived category of a classical zigzag algebra of type A, explaining why the family of stable objects takes the form it does. Our attempts to generalise this construction to higher zigzag algebras of type A fail, but we are able to prove a certain group acts by spherical twists in this case. We also study a connection between tilting hearts of t-structures and mutating quivers with potentials.

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Overview

Following the background in Chapter 1, this thesis splits into two parts.

Part I consists of results from my article [33], which concerns Iyama and Oppermann's higher homological algebra. These authors introduced d-representation-finite algebras [43], which are algebras with global dimension at most d whose module categories have so-called d-cluster-tilting subcategories. Herschend and Iyama [37] showed that d-representation finiteness is closely linked to the fractional Calabi-Yau property. If an algebra has finite global dimension then its bounded derived category has a Serre functor, an autoequivalence satisfying a certain duality. If a power of the Serre functor coincides with a shift, the algebra is said to be fractional Calabi-Yau. Given a d-representation-finite algebra Λ, one can construct a selfinjective algebra Π called its $(d+1)$-preprojective algebra. Grant [32] showed that Λ is fractional Calabi-Yau if and only if the Nakayama automorphism of Π has finite order.

We present a family of selfinjective algebras of type D. Using a construction of Giovannini and Pasquali [29], we show they are Morita equivalent to skew group algebras of the well-studied 3-preprojective algebras of type A. We also show they are isomorphic to algebras considered by Evans and Pugh [21, 22] in the context of $\mathrm{SU}(3)$ modular invariants. One-in-three of the selfinjective algebras of type D are 3-preprojective; using Grant's theorem, we show that the corresponding 2-representation-finite algebras are fractional Calabi-Yau. Finally, we give recipes to construct 2-Auslander-Reiten quivers for an arbitrary 2-representation-finite algebra.

Part II concerns stability conditions on triangulated categories, which were introduced by Bridgeland in [12]. There are two equivalent definitions: one uses the heart of a bounded t-structure and one uses a slicing. One of the most remarkable aspects of Bridgeland's definition is that the set of all stability conditions on a triangulated category is a complex manifold, whose geometry is closely related to Happel-Reiten-Smalø tilting. In the article [63], Thomas constructed a stability condition on the dg perfect derived category of a classical zigzag algebra of type A, using the slicing definition. The stable objects of this stability condition appear to be indexed by indecomposable modules over a linearly oriented Dynkin type A quiver. By reinterpreting Thomas' construction using the heart definition, we are able to explain why this is the case.

Our original aim for this part of the thesis was to construct a stability condition on the dg perfect derived category of a higher zigzag algebra of type A, whose family of stable objects admits an explicit classification. Unfortunately, our attempts to generalise Thomas' construction failed in this aim. We detail why, and prove two results which represent small steps towards determining the stability manifold in this case. The first, which we believe to be well-known to the experts, links HRS tilting with Derksen-Weyman-Zelevinsky mutation of quivers with potentials. The second shows a certain group acts on the triangulated category of interest by spherical twists.

Chapter 1

Background

1.1 Preliminaries

In this section we fix our conventions, introduce the categories we will work with, and recall some of their properties. Our convention is that $0 \in \mathbb{N}$.

1.1.1 Module categories

This material is standard and may be found, for example, in [1].

Throughout this thesis, an *algebra* Λ means an associative unital algebra over an algebraically-closed field k, and a Λ-*module* means a finitely-generated right Λ-module: these form an abelian category $\operatorname{mod}\Lambda$. Writing Λ^{op} for the opposite algebra of Λ, we identify $\operatorname{mod}(\Lambda^{\mathrm{op}})$ with the category of left Λ-modules. Given another algebra Γ, we identify $\operatorname{mod}(\Gamma \otimes \Lambda^{\mathrm{op}})$ - where unadorned $\otimes$ always means $\otimes_k$ - with the category of Λ-Γ-bimodules.

Assume Λ is finite dimensional for the rest of this subsection. The regular module decomposes as $\Lambda \cong e_1\Lambda \oplus e_2\Lambda \oplus \cdots \oplus e_n\Lambda$, where $\{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive pairwise-orthogonal idempotents. We call Λ *basic* if $e_i\Lambda \ncong e_j\Lambda$ for $i \neq j$. Recall that, given an object X of an additive category $\mathscr{C}$, $\operatorname{add} X$ denotes the smallest full additive subcategory of $\mathscr{C}$ which contains X and is closed under isomorphisms and direct summands. The projective objects of $\operatorname{mod}\Lambda$ are the objects of $\operatorname{add}\Lambda$.

There are two standard dualities (contravariant equivalences) associated to $\operatorname{mod}\Lambda$. The *$k$-duality* is

$$(-)^* = \operatorname{Hom}_k(-,k)\colon \operatorname{mod}\Lambda \xrightarrow{\sim} \operatorname{mod}(\Lambda^{\mathrm{op}})$$

and the *Λ-duality* is

$$\operatorname{Hom}_\Lambda(-,\Lambda)\colon \operatorname{mod}\Lambda \xrightarrow{\sim} \operatorname{mod}(\Lambda^{\mathrm{op}}).$$

Composing them gives the *Nakayama functor*

$$\nu = \operatorname{Hom}_\Lambda(-,\Lambda)^*\colon \operatorname{mod}\Lambda \xrightarrow{\sim} \operatorname{mod}\Lambda.$$

In fact, ν is naturally isomorphic to $-\otimes_\Lambda \Lambda^*$. From this, it is easy to see that ν maps Λ to $\Lambda^* \cong (\Lambda e_1)^* \oplus (\Lambda e_2)^* \oplus \cdots \oplus (\Lambda e_n)^*$. The injective objects of $\operatorname{mod}\Lambda$ are the objects of $\operatorname{add}(\Lambda^*)$.

Let us introduce two more important functors. Given $M \in \operatorname{mod}\Lambda$, take a minimal projective presentation

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0.$$

The *transpose* $\operatorname{Tr} M$ of M is $\operatorname{Coker}(\operatorname{Hom}_\Lambda(f,\Lambda)) \in \operatorname{mod}(\Lambda^{\mathrm{op}})$; this assignment induces a functor $\operatorname{Tr}\colon \operatorname{mod}\Lambda \to \operatorname{mod}(\Lambda^{\mathrm{op}})$. The *Auslander-Reiten translations* are the endofunctors

$$\tau = (\operatorname{Tr}(-))^*,\ \tau^- = \operatorname{Tr}((-)^*)$$

of $\operatorname{mod}\Lambda$. They induce mutually quasi-inverse equivalences

$$\begin{tikzcd}
\underline{\operatorname{mod}}\Lambda \arrow[r, bend left, "\tau"] & \overline{\operatorname{mod}}\Lambda \arrow[l, bend left, "\tau^-"]
\end{tikzcd}$$

between the stable and costable module categories.

We now introduce three important families of algebras, following [23].

Definition 1.1.1. A finite-dimensional algebra Λ is called

1. *selfinjective* if $\Lambda \in \operatorname{mod}\Lambda$ is injective;
2. *Frobenius* if $\Lambda \cong \Lambda^*$ as (right) Λ-modules;
3. *symmetric* if $\Lambda \cong \Lambda^*$ as Λ-Λ-bimodules.

Clearly, every symmetric algebra is Frobenius.

Let Λ be a Frobenius algebra. Then there exists $\sigma \in \operatorname{Aut}(\Lambda)$ such that ${}_\sigma\Lambda \cong \Lambda^*$ as Λ-Λ-bimodules, where ${}_\sigma\Lambda \in \operatorname{mod}(\Lambda \otimes \Lambda^{\mathrm{op}})$ has underlying vector space Λ, and $a \otimes b \in \Lambda \otimes \Lambda^{\mathrm{op}}$ acts by multiplying on the right by a and on the left by $\sigma(b)$. We call σ the *Nakayama automorphism* of Λ; it is unique up to inner automorphisms [23, II]. If $e \in \Lambda$ is a primitive idempotent then $(\Lambda e)^* \cong \sigma(e)\Lambda$. Hence Λ is selfinjective.

Recall that two algebras Λ and Γ are called *Morita equivalent* if there is an equivalence of categories $\operatorname{mod}\Lambda \simeq \operatorname{mod}\Gamma$. The following says that "selfinjective" and "symmetric" are Morita invariant properties.

Proposition 1.1.2. *[23, III] Let Λ be a selfinjective (resp. symmetric) algebra, and let Γ be an algebra such that $\operatorname{mod}\Lambda \simeq \operatorname{mod}\Gamma$. Then Γ is selfinjective (resp. symmetric).*

Note that "Frobenius" is not a Morita invariant property.

1.1.2 Derived categories

The material in this subsection may be found in [64, §1, 10].

Let $\mathscr{A}$ be an abelian category. A *(cochain) complex* X over $\mathscr{A}$ is a sequence of $\mathscr{A}$-morphisms

$$\cdots \longrightarrow X^{-1} \xrightarrow{d_X^{-1}} X^0 \xrightarrow{d_X^0} X^1 \longrightarrow \cdots$$

such that $d_X^i \circ d_X^{i-1} = 0$ for all $i \in \mathbb{Z}$. This relation implies that $\operatorname{Im} d_X^{i-1} \subseteq \operatorname{Ker} d_X^i$, so for each $i \in \mathbb{Z}$ we have the *i-th cohomology* $H^i(X) = \operatorname{Ker} d_X^i / \operatorname{Im} d_X^{i-1} \in \mathscr{A}$.

Given another complex Y, a *cochain map* $f\colon X \to Y$ is a family of $\mathscr{A}$-morphisms

$f^i\colon X^i \to Y^i$ such that such that $f^{i+1} \circ d_X^i = d_Y^i \circ f^i$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-1} \arrow[r, "d_X^{-1}"] \arrow[d, "f^{-1}"'] & X^0 \arrow[r, "d_X^0"] \arrow[d, "f^0"'] & X^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \\
\cdots \arrow[r] & Y^{-1} \arrow[r, "d_Y^{-1}"'] & Y^0 \arrow[r, "d_Y^0"'] & Y^1 \arrow[r] & \cdots
\end{tikzcd}$$

This relation implies that f^i induces an $\mathscr{A}$-morphism $H^i(X) \to H^i(Y)$. If this is an isomorphism for all $i \in \mathbb{Z}$, then f is called a *quasi-isomorphism*.

There is an abelian category $\mathbf{C}(\mathscr{A})$ of complexes over $\mathscr{A}$ and their cochain maps. For each $i \in \mathbb{Z}$, the assignment $X \mapsto H^i(X)$ induces a functor $H^i\colon \mathbf{C}(\mathscr{A}) \to \mathscr{A}$. An important autoequivalence of $\mathbf{C}(\mathscr{A})$ is the *shift functor* [1], where $X[1]^i = X^{i+1}$, $d_{X[1]}^i = -d_X^{i+1}$ and $f[1]^i = f^{i+1}$. Note that the n-th power of [1] is written $[n]$.

Given a cochain map $f\colon X \to Y$, its *mapping cone* $\operatorname{cone}(f) \in \mathbf{C}(\mathscr{A})$ is the complex with $\operatorname{cone}(f)^i = X^{i+1} \oplus Y^i$ and

$$d_{\operatorname{cone}(f)}^i = \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix} : X^{i+1} \oplus Y^i \to X^{i+2} \oplus Y^{i+1}.$$

Two cochain maps $f, g\colon X \to Y$ are *homotopic*, written $f \sim g$, if there exists a family of $\mathscr{A}$-morphisms $s^i\colon X^i \to Y^{i-1}$ such that $f^i - g^i = s^{i+1} \circ d_X^i + d_Y^{i-1} \circ s^i$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{-1} \arrow[r, "d_X^{-1}"] \arrow[d, shift right, "f^{-1}"'] \arrow[d, shift left, "g^{-1}"] & X^0 \arrow[r, "d_X^0"] \arrow[d, shift right, "f^0"'] \arrow[d, shift left, "g^0"] \arrow[dl, dashed, "s^0"'] & X^1 \arrow[r] \arrow[d, shift right, "f^1"'] \arrow[d, shift left, "g^1"] \arrow[dl, dashed, "s^1"'] & \cdots \\
\cdots \arrow[r] & Y^{-1} \arrow[r, "d_Y^{-1}"'] & Y^0 \arrow[r, "d_Y^0"'] & Y^1 \arrow[r] & \cdots
\end{tikzcd}$$

A cochain map $f\colon X \to Y$ is called a *homotopy equivalence* if there exists some $f'\colon Y \to X$ such that $f' \circ f \sim \mathrm{id}_X$ and $f \circ f' \sim \mathrm{id}_Y$. All homotopy equivalences are quasi-isomorphisms, but the converse is not true.

One verifies that $\sim$ is an equivalence relation on $\operatorname{Hom}_{\mathbf{C}(\mathscr{A})}(X, Y)$ which is compatible with composition. Hence we define the *homotopy category* $\mathbf{K}(\mathscr{A})$ of $\mathscr{A}$, where $\mathrm{Ob}(\mathbf{K}(\mathscr{A})) = \mathrm{Ob}(\mathbf{C}(\mathscr{A}))$ and $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(X, Y) = \mathrm{Hom}_{\mathbf{C}(\mathscr{A})}(X, Y)/\sim$. Note that $\mathbf{K}(\mathscr{A})$ is a triangulated category with suspension functor $[1]$, and the distinguished triangles are of the form

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow X[1].$$

The *derived category* $\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A})[\mathrm{qis}^{-1}]$ is the localisation of $\mathbf{K}(\mathscr{A})$ at the collection of all quasi-isomorphisms. For precisely what this means, we refer to [64, §10.3]. Vaguely, it means we retain the objects and morphisms of $\mathbf{K}(\mathscr{A})$, while formally adjoining inverses for the quasi-isomorphisms. Note that $\mathbf{D}(\mathscr{A})$ inherits triangulated structure from $\mathbf{K}(\mathscr{A})$.

We usually work with the *bounded derived category* $\mathbf{D}^{\mathrm{b}}(\mathscr{A})$, which is the full (triangulated) subcategory of $\mathbf{D}(\mathscr{A})$ on objects X such that $H^i(X) = 0$ for $|i| \gg 0$. If $\mathscr{A} = \mathrm{mod}\,\Lambda$ for some algebra Λ, we write $\mathbf{D}^{\mathrm{b}}(\Lambda)$ in place of $\mathbf{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$.

A functor $F\colon \mathscr{A} \to \mathscr{B}$ between abelian categories can always be extended to functors $\mathbf{C}(\mathscr{A}) \to \mathbf{C}(\mathscr{B})$ and $\mathbf{K}(\mathscr{A}) \to \mathbf{K}(\mathscr{B})$. Sometimes, F can be extended to a *derived functor* $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$. We refer to [64, §10.5] for details.

1.1.3 Dg categories

We follow [46] in our exposition.

Throughout this thesis, *graded* means $\mathbb{Z}$-graded. Given a graded algebra A, there is an abelian category $\mathrm{grmod}\,A$ of graded A-modules; its morphisms are homogeneous A-module morphisms of degree 0. Given $M \in \mathrm{grmod}\,A$ and $n \in \mathbb{Z}$, $M\{n\}$ is the graded A-module with $(A\{n\})^i = A^{i+n}$.

A *differential-graded (dg) algebra* consists of a graded algebra A, together with a homogeneous linear map $d_A\colon A \to A$ of degree 1 such that $d_A^2 = 0$ and

$$d_A(ab) = d_A(a)b + (-1)^{\deg_A(a)} a d_A(b)$$

for all $a, b \in A$. Its *total cohomology* is $H^\bullet(A) = \operatorname{Ker} d_A / \operatorname{Im} d_A$. This is a graded algebra, whose i-th graded part $H^i(A) = (H^\bullet(A))^i$ we call the *i-th cohomology*.

A *dg A-module* consists of a graded A-module M, together with a homogeneous linear map $d_M\colon M \to M$ of degree 1 such that $d_M^2 = 0$ and

$$d_M(xa) = d_M(x)a + (-1)^{\deg_M(x)} d_A(a)$$

for all $a \in A$, $x \in M$. Its *total cohomology* is $H^\bullet(M) = \operatorname{Ker} d_M / \operatorname{Im} d_M \in \operatorname{grmod} A$ and its *i-th cohomology* is $H^i(M) = (H^\bullet(M))^i$.

Let N be another dg A-module. We define $\mathcal{H}om_A(M, N)$ to be the dg A-module with $\mathcal{H}om_A(M, N)^i = \operatorname{Hom}_{\operatorname{grmod} A}(M, N\{i\})$, the space of homogeneous A-module morphisms $M \to N$ of degree i. The differential ∂ is defined by

$$\partial(f) = d_N \circ f - (-1)^{\deg(f)} f \circ d_M.$$

The *dg category of dg A-modules* $\operatorname{dgmod} A$ has objects the dg A-modules, and morphisms given by $\operatorname{Hom}_{\operatorname{dgmod} A}(M, N) = \mathcal{H}om_A(M, N)$. As the name suggests, this is an example of a dg category (see [46, §2.2] for a definition).

The *abelian category of dg A-modules* $\mathbf{C}_{\mathrm{dg}}(A)$ has the same objects as $\operatorname{dgmod} A$, with $\operatorname{Hom}_{\mathbf{C}_{\mathrm{dg}}(A)}(M, N) = \operatorname{Ker}(\partial\colon \mathcal{H}om_A(M, N)^0 \to \mathcal{H}om_A(M, N)^1)$. Interpreting the objects of $\mathbf{C}_{\mathrm{dg}}(A)$ as graded A-modules which are also complexes, the morphisms of $\mathbf{C}_{\mathrm{dg}}(A)$ are homogeneous A-module morphisms of degree 0 which are also cochain maps. In particular, there are notions of quasi-isomorphism, shift functor, and mapping cone defined as in §1.1.2.

The *homotopy category of dg A-modules* $\mathbf{K}_{\mathrm{dg}}(A)$ has the same objects as $\operatorname{dgmod} A$, with $\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(M, N) = H^0(\mathcal{H}om_A(M, N))$. It is a triangulated category with suspension functor $[1]$, and the distinguished triangles are of the form

$$M \xrightarrow{f} N \longrightarrow \mathrm{cone}(f) \longrightarrow M[1].$$

The *derived category of dg A-modules* $\mathbf{D}_{\mathrm{dg}}(A) = \mathbf{K}_{\mathrm{dg}}(A)[\mathrm{qis}^{-1}]$ is the localisation of $\mathbf{K}_{\mathrm{dg}}(A)$ at the collection of all quasi-isomorphisms. It inherits triangulated structure from $\mathbf{K}_{\mathrm{dg}}(A)$.

We usually work with

1. the *finite-dimensional derived category* $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(A)$, which is the full (triangulated) subcategory of $\mathbf{D}_{\mathrm{dg}}(A)$ on objects M such that $\dim_k H^\bullet(M) < \infty$;
2. the *perfect derived category* $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(A)$, which is the smallest full triangulated subcategory of $\mathbf{D}_{\mathrm{dg}}(A)$ which contains A and is closed under direct summands.

Finally, the *cohomology category of dg A-modules* is the graded category $\mathbf{H}_{\mathrm{dg}}(A)$ with the same objects as $\mathrm{dgmod}\, A$ and

$$\mathrm{Hom}^\bullet_{\mathbf{H}_{\mathrm{dg}}(A)}(M, N) := H^\bullet(\mathcal{H}om_A(M, N)) \cong \mathrm{Ext}^\bullet_{\mathbf{K}_{\mathrm{dg}}(A)}(M, N).$$

Clearly, information is lost when passing from $\mathrm{dgmod}\, A$ to $\mathbf{H}_{\mathrm{dg}}(A)$, but this can be recovered. Kadeishvili's theorem[1] states that $\mathbf{H}_{\mathrm{dg}}(A)$ can be equipped with the structure of an A_∞-category, unique up to A_∞-quasi-isomorphism, such that $\mathbf{H}_{\mathrm{dg}}(A)$ is A_∞-quasi-isomorphic to $\mathrm{dgmod}\, A$. Roughly, this means that for any sequence of objects $M_1, \dots, M_n \in \mathbf{H}_{\mathrm{dg}}(A)$, there exists a homogeneous linear map

$$m_n \colon \mathrm{Hom}^\bullet_{\mathbf{H}_{\mathrm{dg}}(A)}(M_{n-1}, M_n) \otimes \cdots \otimes \mathrm{Hom}^\bullet_{\mathbf{H}_{\mathrm{dg}}(A)}(M_1, M_2) \to \mathrm{Hom}^\bullet_{\mathbf{H}_{\mathrm{dg}}(A)}(M_1, M_n)$$

of degree $2 - n$, such that certain coherence laws are satisfied. In this case, $m_1 = 0$ and $m_2 = \circ$, the composition in $\mathbf{H}_{\mathrm{dg}}(A)$. We refer to [52, §1.2,5.1] for details.

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[1] Kadeishvili's original theorem [45] concerns A_∞-algebras; it appears to have been folklore that it can be extended to A_∞-categories. A proof in this generality may be found in [10, §3].

1.2 Preprojective and zigzag algebras

Parts I and II of this thesis concern higher preprojective and higher zigzag algebras respectively. In this section, we introduce the classical versions of these algebras to provide context for our work.

A *quiver* Q consists of a finite set Q_0 of vertices and a finite set Q_1 of arrows, together with functions $s, t \colon Q_1 \to Q_0$, which map arrows to their source and target respectively. A *path* $p = \alpha_1\alpha_2 \cdots \alpha_n$ in Q is a sequence of arrows with $t(\alpha_i) = s(\alpha_{i+1})$. If $t(\alpha_n) = s(\alpha_1)$ then p is called an n-*cycle*. A 1-cycle is called a *loop*. The *path algebra* kQ has basis the set of all paths in Q (plus the vertex idempotents e_i), with multiplication given by concatenation. Any hereditary algebra Λ satisfies $\Lambda \cong kQ$ for some quiver Q. For an introduction to quivers see e.g. [61].

Recall that an algebra Λ is called *representation finite* if $\operatorname{mod} \Lambda$ contains only finitely many indecomposable objects up to isomorphism. Gabriel's theorem tell us that whether kQ has this property is independent of the orientation of Q.

Theorem 1.2.1. *[26] Let Q be a quiver. The path algebra kQ is representation finite if and only if the underlying graph of Q is an ADE Dynkin diagram (see Figure 1.1).*

Another way to understand this classification is via the preprojective algebra, which was introduced by Gel'fand and Ponomarev in [27].

Definition 1.2.2. Let Q be a quiver. Then $\overline{Q}$ is the quiver obtained from Q by adding, for each arrow $i \xrightarrow{\alpha} j$, a reverse arrow $j \xrightarrow{\alpha^*} i$. Define $\rho = \sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha) \in k\overline{Q}$. The *preprojective algebra* of Q is

$$\Pi(Q) = k\overline{Q}/\langle \rho \rangle.$$

Example 1.2.3. Let Q be $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then

$$\Pi(Q) = k\left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\alpha^*"] \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\beta^*"] \end{tikzcd}\right) \Big/ \langle \alpha\alpha^*, \beta\beta^* - \alpha^*\alpha, \beta^*\beta \rangle .$$

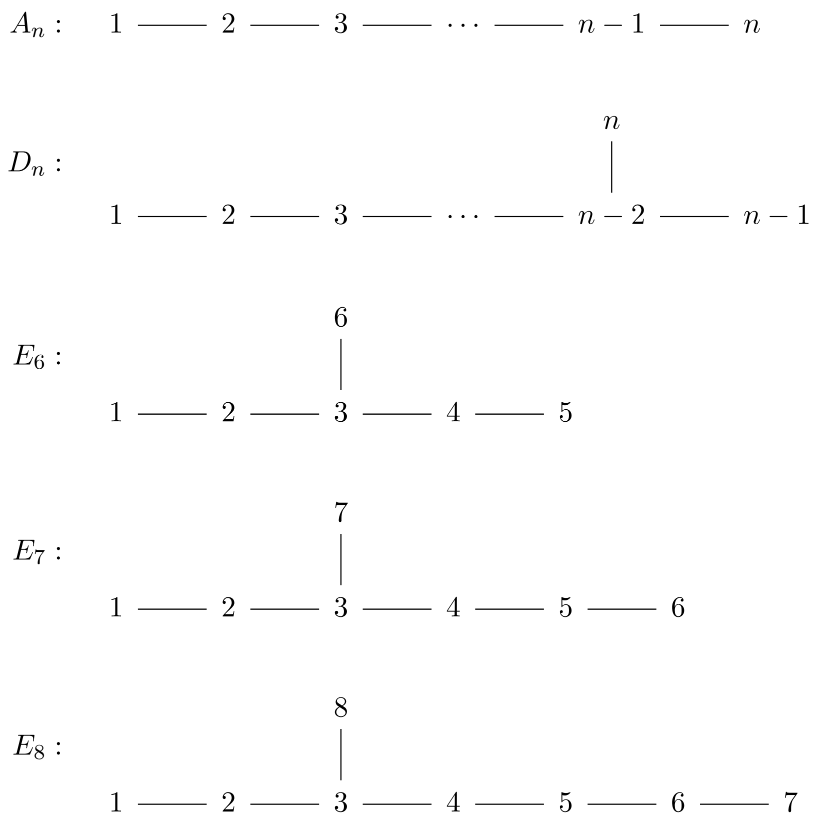


Figure 1.1: The ADE Dynkin diagrams.

Note that $\Pi(Q)$ contains kQ as a subalgebra, and that $\Pi(Q) \cong \bigoplus_{i\geq 0} \tau^{-i}(kQ)$ as kQ-modules: hence the name preprojective algebra.

Baer, Geigle and Lenzing [2] alternatively defined the preprojective algebra of Q as

$$\bigoplus_{i\geq 0} \mathrm{Hom}_{kQ}(kQ, \tau^{-i}(kQ)),$$

where the product of $f\colon kQ \to \tau^{-i}(kQ)$ and $g\colon kQ \to \tau^{-j}(kQ)$ is

$$g \cdot f = \tau^{-i}(g) \circ f\colon kQ \to \tau^{-(i+j)}(kQ)$$

(c.f. Definition 1.4.5). It was shown by Crawley-Boevey and Ringel that the two definitions coincide [16, 58]. From the latter definition, one can deduce that $\Pi(Q)$ is finite dimensional if and only if Q has finite representation type (i.e. Q is ADE Dynkin). In this case, $\Pi(Q)$ is actually Frobenius. This was first stated by Ringel and Schofield [59] and proved by Brenner, Butler and King [11]; see also [31] for a nice alternative proof.

Definition 1.2.4. Let Q be a quiver. The *zigzag algebra* $Z(Q)$ is the path algebra of $\overline{Q}$ modulo the following relations.

1. All paths $i \longrightarrow j \longrightarrow k$ with $i \neq k$ are zero.
2. All 2-cycles at a given vertex are equal.

Example 1.2.5. Let Q be $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then

$$Z(Q) = k\left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\alpha^*"] \arrow[r, bend left, "\beta"] & 3 \arrow[l, bend left, "\beta^*"] \end{tikzcd}\right) \Big/ \langle \alpha\beta, \beta\beta^* - \alpha^*\alpha, \beta^*\alpha^* \rangle,$$

c.f. Example 1.2.3.

The terminology “zigzag algebra” first appeared in [39, §3]. There is a more conceptual definition in terms of trivial extension algebras, but the above will suffice for us. Note that $Z(Q)$ is always finite dimensional and symmetric. If Q is ADE Dynkin then $Z(Q)$ is quadratic dual to the preprojective algebra $\Pi(Q)$.

In this thesis, we equip $Z(Q)$ with the path-length grading and view it as a dg algebra with trivial differential (i.e. $d = 0$).

1.3 Quivers with potentials

Quivers with potentials play a central role in this thesis. For example, they are used to classify 3-preprojective algebras (see Theorem 1.4.8). They were introduced in [17]; we follow the treatment in [8].

Given a quiver Q, let $[kQ, kQ] = \langle xy - yx \mid x, y \in kQ\rangle$ be the linear subspace of kQ spanned by all commutators. We call $(kQ)_{\mathrm{cyc}} := kQ/[kQ, kQ]$ *cycle space*. To see why, consider a path $p = \alpha_1\alpha_2 \cdots \alpha_n$ in Q.

- If p is not a cycle then $p = \alpha_1(\alpha_2 \cdots \alpha_n) - (\alpha_2 \cdots \alpha_n)\alpha_1 \in [kQ, kQ]$, so $p = 0$ in $(kQ)_{\mathrm{cyc}}$.
- If p is a cycle then $\alpha_1(\alpha_2 \cdots \alpha_n) - (\alpha_2 \cdots \alpha_n)\alpha_1 \in [kQ, kQ]$, so all cyclic permutations of p are identified in $(kQ)_{\mathrm{cyc}}$.

An element $W \in (kQ)_{\mathrm{cyc}}$ is called a *potential* on Q, while the pair (Q, W) is called a *quiver with potential (QP)*. By the previous discussion, W is a formal linear combination of cycles in Q, considered up to cyclic equivalence.

For each $\alpha \in Q_1$, define a linear map $\partial_\alpha \colon (kQ)_{\mathrm{cyc}} \to kQ$ such that, for any cycle $c = \alpha_1\alpha_2 \cdots \alpha_n$ in Q,

$$\partial_\alpha(c) = \sum_{i\,:\,\alpha_i = \alpha} \alpha_{i+1} \cdots \alpha_1\alpha_n \cdots \alpha_{i-1}.$$

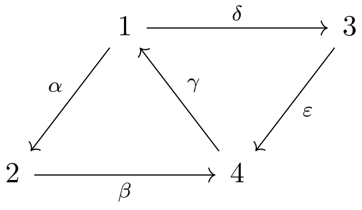
One can think of ∂_α as formal differentiation with respect to α.

Definition 1.3.1. Let (Q, W) be a QP. The associated *Jacobian algebra* is

$$\mathrm{J}(Q, W) = kQ/\langle \partial_\alpha W \mid \alpha \in Q_1\rangle.$$

We call (Q, W) *Jacobi finite* if $\mathrm{J}(Q, W)$ is finite dimensional.

Example 1.3.2. Let Q be



and let $W = \alpha\beta\gamma - \gamma\delta\varepsilon$. Then

$$\partial_\alpha W = \beta\gamma,\ \partial_\beta W = \gamma\alpha,\ \partial_\gamma W = \alpha\beta - \delta\varepsilon,$$
$$\partial_\delta W = -\varepsilon\gamma,\ \partial_\varepsilon W = -\gamma\delta.$$

Thus $\mathrm{J}(Q, W)$ has basis $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \varepsilon, \alpha\beta = \delta\varepsilon\}$.

We are particularly interested in the following class of QPs.

Definition 1.3.3. [38, Definition 3.6] A QP (Q, W) is called *selfinjective* if $\mathrm{J}(Q, W)$ is a selfinjective algebra.

Definition 1.3.4. [38, Definition 3.1] Let (Q, W) be a QP. To each subset $C \subseteq Q_1$, we associate a non-negative grading on kQ, given on arrows by

$$\deg_C(\alpha) = \begin{cases} 1 & \text{if } \alpha \in C, \\ 0 & \text{else.} \end{cases}$$

We call C a *cut* if W is homogeneous of degree 1 with respect to this grading. We say (Q, W) has *enough cuts* if, for every arrow $\alpha \in Q_1$, there exists a cut C containing α.

If C is a cut then there is an induced grading on $\mathrm{J}(Q, W)$. Its degree 0 part is denoted $\mathrm{J}(Q, W)_C$. Clearly, $\mathrm{J}(Q, W)_C \cong \mathrm{J}(Q, W)/\langle C\rangle$.

Example 1.3.5. Let (Q, W) be as in Example 1.3.2. Then $\{\alpha, \delta\}$ and $\{\gamma\}$ are both cuts, but $\{\alpha, \beta\}$ is not. Choose $C = \{\gamma\}$. Then $\mathrm{J}(Q, W)_C$ is the path algebra of

$$\begin{tikzcd}
 & 1 \arrow[r, "\delta"] \arrow[dl, "\alpha"'] & 3 \arrow[dl, "\varepsilon"] \\
2 \arrow[r, "\beta"'] & 4 &
\end{tikzcd}$$

modulo the relation $\alpha\beta = \delta\varepsilon$.

1.3.1 Ginzburg dg algebras

Ginzburg dg algebras were introduced in [28]. They provide a way to construct Calabi-Yau triangulated categories (see §1.5). We consider two types: one is attached to a quiver and one is attached to a QP.

Definition 1.3.6. Let Q be a quiver and let $N \in \mathbb{N}$, $N \geq 2$. The *N-Ginzburg quiver* of Q is the graded quiver Q^N with vertices $Q_0^N = Q_0$ and

1. the arrows of Q in degree 0;
2. for each arrow $i \xrightarrow{\alpha} j$ in Q, a reverse arrow $j \xrightarrow{\alpha^*} i$ in degree $2 - N$;
3. for each $i \in Q_0^N$, a loop $i \xrightarrow{t_i} i$ in degree $1 - N$.

Definition 1.3.7. Let Q be a quiver. The *Ginzburg dg algebra* $\Gamma(Q)$ of Q is the graded path algebra of Q^2 with differential d satisfying

1. $d(\alpha) = d(\alpha^*) = 0$ for all $\alpha \in Q_1$;
2. $d(t_i) = \sum_{\alpha \in Q_1} e_i(\alpha\alpha^* - \alpha^*\alpha)$ for all $i \in Q_0$.

Example 1.3.8. Let Q be $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then $\Gamma(Q)$ is the graded path algebra of

$$\begin{tikzcd}
1 \arrow[loop above, "t_1"] \arrow[r, bend left, "\alpha"] & 2 \arrow[loop above, "t_2"] \arrow[l, bend left, "\alpha^*"] \arrow[r, bend left, "\beta"] & 3 \arrow[loop above, "t_3"] \arrow[l, bend left, "\beta^*"]
\end{tikzcd}$$

where $\deg(\alpha) = \deg(\beta) = 0 = \deg(\alpha^*) = \deg(\beta^*)$, $d(t_i) = -1$, and the differential satisfies

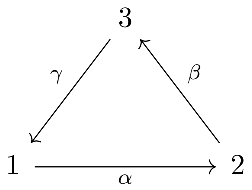
$$d(\alpha) = d(\alpha^*) = d(\beta) = d(\beta^*) = 0;$$
$$d(t_1) = \alpha\alpha^*,\ d(t_2) = \beta\beta^* - \alpha^*\alpha,\ d(t_3) = -\beta^*\beta.$$

For any quiver Q, it is easy to see that $H^0(\Gamma(Q)) \cong \Pi(Q)$.

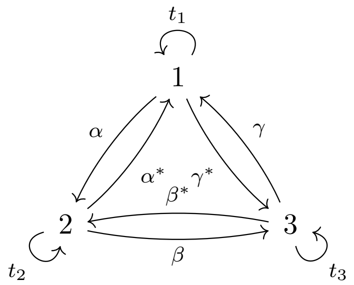
Definition 1.3.9. Let (Q, W) be a QP. The *Ginzburg dg algebra* $\Gamma(Q, W)$ of (Q, W) is the graded path algebra of Q^3 with differential d satisfying

1. $d(\alpha) = 0$ for all $\alpha \in Q_1$;
2. $d(\alpha^*) = \partial_\alpha W$ for all $\alpha \in W$;
3. $d(t_i) = \sum_{\alpha \in Q_1} e_i(\alpha\alpha^* - \alpha^*\alpha)$.

Example 1.3.10. Let Q be



and let $W = \alpha\beta\gamma$. Then $\Gamma(Q, W)$ is the graded path algebra of



where $\deg(\alpha) = \deg(\beta) = \deg(\gamma) = 0$, $\deg(\alpha^*) = \deg(\beta^*) = \deg(\gamma^*) = -1$,

$\deg(t_i) = -2$, and the differential satisfies

$$\begin{aligned} d(\alpha) &= d(\beta) = d(\gamma) = 0; \\ d(\alpha^*) &= \beta\gamma,\ d(\beta^*) = \gamma\alpha,\ d(\gamma^*) = \alpha\beta; \\ d(t_1) &= \alpha\alpha^* - \gamma^*\gamma,\ d(t_2) = \beta\beta^* - \alpha^*\alpha,\ d(t_3) = \gamma\gamma^* - \beta^*\beta. \end{aligned}$$

For any QP (Q, W), it is easy to see that $H^0(\Gamma(Q, W)) \cong \mathrm{J}(Q, W)$.

1.3.2 Mutation

One of the key features of [17] is the introduction of QP mutation, which generalises Fomin-Zelevinsky's quiver mutation (see [24]). First, we need a certain splitting theorem.

Definition 1.3.11. [17, §4] Let (Q, W) and (Q', W') be two QPs on the same vertex set Q_0.

1. We say (Q, W) and (Q', W') are *right-equivalent* if there exists an algebra isomorphism $f\colon kQ \to kQ'$ such that $f(W) = W'$ in $(kQ')_{\mathrm{cyc}}$.
2. The *direct sum* $(Q, W) \oplus (Q', W')$ is the QP with vertices Q_0, arrows $Q_1 \sqcup Q_1'$ and potential $W + W'$.

Theorem 1.3.12. *[17, Theorem 4.6] Let (Q, W) be a QP without loops. Then (Q, W) is right-equivalent to the direct sum of a QP $(Q_{\mathrm{red}}, W_{\mathrm{red}})$ such that all cycles in W_{red} have length at least 3, and a QP $(Q_{\mathrm{triv}}, W_{\mathrm{triv}})$ such that $\mathrm{J}(Q_{\mathrm{triv}}, W_{\mathrm{triv}}) \cong k$.*

We are now ready to define mutation of QPs. We say a QP (Q, W) is *mutable* at $k \in Q_0$ if Q does not contain any loops, nor any 2-cycles starting at k.

Definition 1.3.13. [17, §5] Let (Q, W) be a QP mutable at $k \in Q_0$. Let $(\widetilde{Q}, \widetilde{W})$ be the QP obtained from (Q, W) using the following algorithm.

1. Replace each arrow α incident to k with an arrow α^* in the opposite direction.

2. For each path $j \xrightarrow{\alpha^*} k \xrightarrow{\beta^*} i$ in the new quiver,
   (a) add an arrow $i \xrightarrow{[\alpha\beta]} j$,
   (b) replace $\alpha\beta$ with $[\alpha\beta]$ in the potential,
   (c) add a term $[\alpha\beta]\beta^*\alpha^*$ to the potential.

The *mutation* $\mu_k(Q, W)$ of (Q, W) at k is defined to be $(\widetilde{Q}_{\mathrm{red}}, \widetilde{W}_{\mathrm{red}})$.

Example 1.3.14. Let (Q, W) be as in Example 1.3.10 and set $k = 2$. Then $\widetilde{Q}$ is

$$\begin{tikzcd}
& 3 \arrow[dl, bend right, "\gamma"'] \arrow[dr, "\beta^*"] & \\
1 \arrow[ur, bend right, "{[\alpha\beta]}"'] & & 2 \arrow[ll, "\alpha^*"]
\end{tikzcd}$$

and $\widetilde{W} = [\alpha\beta]\gamma + [\alpha\beta]\beta^*\alpha^*$. In this case, $\widetilde{Q}_{\mathrm{red}}$ and $\widetilde{Q}_{\mathrm{triv}}$ are

$$\begin{tikzcd}
3 \arrow[dr, "\beta^*"] & \\
1 & 2 \arrow[l, "\alpha^*"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 3 \arrow[dl, bend right, "\gamma"'] & \\
1 \arrow[ur, bend right, "{[\alpha\beta]}"'] & & 2
\end{tikzcd}$$

respectively, while $\widetilde{W}_{\mathrm{red}} = 0$ and $\widetilde{W}_{\mathrm{triv}} = [\alpha\beta]\gamma$.

Theorem 1.3.15. *[48, Theorem 3.2] Let (Q, W) be a QP mutable at $k \in Q_0$. Let Γ and Γ' be the Ginzburg dg algebras of (Q, W) and $\mu_k(Q, W)$, respectively. There is an equivalence of categories $\mathbf{D}_{\mathrm{dg}}(\Gamma) \simeq \mathbf{D}_{\mathrm{dg}}(\Gamma')$, descending to an equivalence of categories $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma) \simeq \mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma')$.*

1.4 Higher homological algebra

This section and §1.5 comprise the core of the background, especially for Part I.

Definition 1.4.1. [43, Definitions 2.1-2] Let Λ be a finite-dimensional algebra and let $d \in \mathbb{Z}^+$. An object $M \in \operatorname{mod}\Lambda$ is called *d-cluster tilting* if

$$\operatorname{add} M = \{N \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(M, N) = 0,\ 1 \leq i \leq d-1\} \text{ and}$$
$$\operatorname{add} M = \{N \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(N, M) = 0,\ 1 \leq i \leq d-1\}.$$

We call Λ *d-representation finite* if $\operatorname{gl.dim}\Lambda \leq d$ and $\operatorname{mod}\Lambda$ contains a d-cluster-tilting object.

Definition 1.4.2. [41, §1.4.1] Let Λ be a finite-dimensional algebra, $d \in \mathbb{Z}^+$. The *d-Auslander-Reiten translations* are the endofunctors $\tau_d = \tau\Omega^{d-1}$ and $\tau_d^- = \tau^-\Omega^{1-d}$ of $\operatorname{mod}\Lambda$, where Ω and Ω^- are the syzygy and cosyzygy functors respectively.

Given a d-representation-finite algebra Λ, we define the subcategory

$$\mathscr{M}_\Lambda = \operatorname{add}\{\tau_d^{-i}\Lambda \mid i \geq 0\} \subset \operatorname{mod}\Lambda.$$

Proposition 1.4.3. *Let Λ be a d-representation-finite algebra.*

1. *[42, Theorem 1.6] For any d-cluster-tilting object $M \in \operatorname{mod}\Lambda$, we have that $\operatorname{add} M = \mathscr{M}_\Lambda$.*
2. *[41, Theorem 2.3] The d-Auslander-Reiten translations induce mutually quasi-inverse equivalences*
$$\begin{tikzcd} \underline{\mathscr{M}_\Lambda} \arrow[r, bend left, "\tau_d"] & \overline{\mathscr{M}_\Lambda} \arrow[l, bend left, "\tau_d^-"] \end{tikzcd}$$
between the stable and costable categories of $\mathscr{M}_\Lambda$.

We call $\mathscr{M}_\Lambda$ the *d-cluster-tilting subcategory* of $\operatorname{mod}\Lambda$ (c.f. [42, Definition 1.1]).

Example 1.4.4. An algebra Λ is 1-representation finite if and only if it is representation finite and hereditary (i.e. $\Lambda \cong kQ$ for an ADE Dynkin quiver Q). In this case, we recover the classical Auslander-Reiten translations, and $\mathscr{M}_\Lambda = \operatorname{mod}\Lambda$.

Definition 1.4.5. [37, §2] Let Λ be a d-representation-finite algebra. The associated

$(d+1)$-*preprojective algebra* is

$$\Pi(\Lambda) = \bigoplus_{i \geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau_d^{-i}\Lambda),$$

where the product of $f\colon \Lambda \to \tau_d^{-i}\Lambda$ and $g\colon \Lambda \to \tau_d^{-j}\Lambda$ is

$$g \cdot f = \tau_d^{-i}(g) \circ f\colon \Lambda \to \tau_d^{-(i+j)}\Lambda.$$

Hence $\Pi(\Lambda)$ is non-negatively graded, with degree i part $\mathrm{Hom}_\Lambda(\Lambda, \tau_d^{-i}\Lambda)$. This grading is called the *tensor grading*.

Example 1.4.6. If $\Lambda = kQ$ for an ADE quiver Q, we recover the classical preprojective algebra of Q.

Proposition 1.4.7. *[44, Corollary 3.4] Let Λ be a d-representation-finite algebra. Then $\Pi(\Lambda)$ is selfinjective. If Λ is also basic then $\Pi(\Lambda)$ is Frobenius.*

We understand 2-representation-finite algebras especially well, thanks to the following classification of Herschend and Iyama.

Theorem 1.4.8. *[38, Theorem 3.11] Let (Q, W) be a selfinjective QP, and let $C \subseteq Q_1$ be a cut.*

1. *$\mathrm{J}(Q, W)_C$ is 2-representation finite, and its 3-preprojective algebra is $\mathrm{J}(Q, W)$.*
2. *Every basic 2-representation-finite algebra arises this way.*

In particular, the grading on $\mathrm{J}(Q, W)$ induced by C coincides with the tensor grading.

We now present an important family of examples.

1.4.1 3-Preprojective algebras of type A

Definition 1.4.9. Define $\omega\colon \mathbb{N}^3 \to \mathbb{N}^3$ by $\omega(x_0, x_1, x_2) = (x_1, x_2, x_0)$.

Given $x, y \in \mathbb{N}^3$, we write $x \sim y$ if $x = \omega^j(y)$ for some $j \in \{0, 1, 2\}$.

Definition 1.4.10. [43, Definition 5.1] Let $n \in \mathbb{Z}$, $n \geq 2$. Let $\mathcal{A}^n$ be the quiver with vertices

$$\mathcal{A}_0^n = \{x \in \mathbb{N}^3 \mid x_0 + x_1 + x_2 = n - 1\}$$

and arrows

$$\mathcal{A}_1^n = \bigcup_{i=0}^{2} \{x \xrightarrow{\alpha_i} x + v_i \mid x, x + v_i \in \mathcal{A}_0^n\},$$

where $v_0 = (-1, 1, 0)$, $v_1 = (0, -1, 1)$ and $v_2 = (1, 0, -1)$.

Example 1.4.11. $\mathcal{A}^4$ is the following quiver. Note we often write $x_0x_1x_2$ in place of (x_0, x_1, x_2) and $x \xrightarrow{i} y$ in place of $x \xrightarrow{\alpha_i} y$.

$$\begin{tikzcd}
 & & & 003 \arrow[dl, "2"'] & & & \\
 & & 102 \arrow[rr, "0"'] \arrow[dl, "2"'] & & 012 \arrow[ul, "1"'] \arrow[dl, "2"'] & & \\
 & 201 \arrow[rr, "0"'] \arrow[dl, "2"'] & & 111 \arrow[ul, "1"'] \arrow[rr, "0"'] \arrow[dl, "2"'] & & 021 \arrow[ul, "1"'] \arrow[dl, "2"'] & \\
300 \arrow[rr, "0"'] & & 210 \arrow[ul, "1"'] \arrow[rr, "0"'] & & 120 \arrow[ul, "1"'] \arrow[rr, "0"'] & & 030 \arrow[ul, "1"']
\end{tikzcd}$$

Definition 1.4.12. Let $n \geq 2$. Define a Jacobian algebra $\Pi_{\mathcal{A}}^n = \mathrm{J}(\mathcal{A}^n, W_{\mathcal{A}}^n)$ over $\mathbb{C}$ using the potential $W_{\mathcal{A}}^n = \sum_c \lambda_{\mathcal{A}}^n(c)c$, where the sum is taken over all 3-cycles $c = e_x\alpha_{i_0}\alpha_{i_1}\alpha_{i_2}$ in $\mathcal{A}^n$, and

$$\lambda_{\mathcal{A}}^n(c) = \begin{cases} 1 & \text{if } (i_0, i_1, i_2) \sim (0, 1, 2), \\ -1 & \text{if } (i_0, i_1, i_2) \sim (0, 2, 1). \end{cases}$$

Informally, this is the sum of all anti-clockwise 3-cycles minus the sum of all clockwise 3-cycles.

Remark 1.4.13. If $\alpha_i e_x$ lies on an edge of $\mathcal{A}^n$ then $\partial_{\alpha_i e_x} W_{\mathcal{A}}^n = e_x\alpha_{i+1}\alpha_{i-1}$, while if $\alpha_i e_x$ is an internal arrow then $\partial_{\alpha_i e_x} W_{\mathcal{A}}^n = e_x(\alpha_{i+1}\alpha_{i-1} - \alpha_{i-1}\alpha_{i+1})$. Hence $\Pi_{\mathcal{A}}^n$ is the path algebra of $\mathcal{A}^n$ modulo the following relations.

1. Each length-two path which starts on and ends on the same edge of $\mathcal{A}^n$, whose midpoint is not on that edge, is zero.
2. Each rhombus in $\mathcal{A}^n$ commutes.

Thus $\Pi^n_{\mathcal{A}}$ is precisely the algebra called $\widehat{\Lambda}^{(2,n)}$ in [43, Definition 5.1].

Note that ω permutes $\mathcal{A}^n_0$, inducing an automorphism of $\mathcal{A}^n$ such that

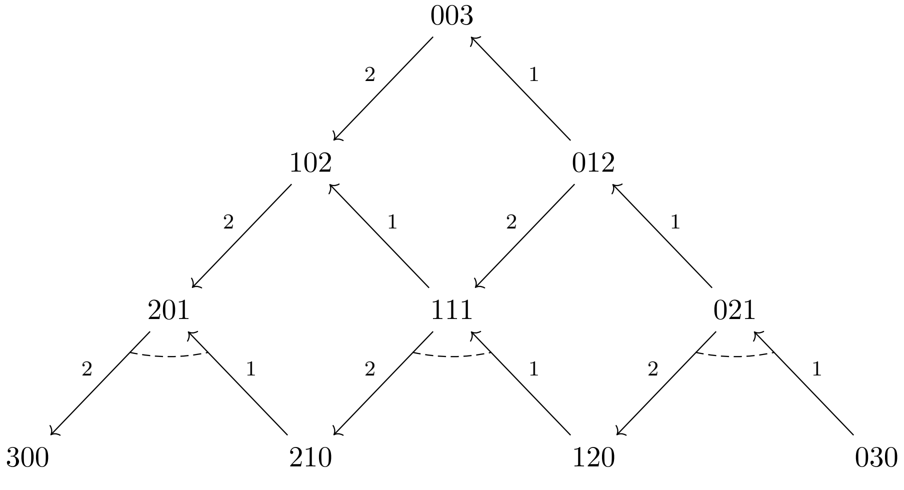
$$(x \xrightarrow{i} x + v_i) \mapsto (\omega(x) \xrightarrow{i-1} \omega(x) + v_{i-1}),$$

where we used that $\omega(x + v_i) = \omega(x) + v_{i-1}$. In particular, this automorphism preserves $W^n_{\mathcal{A}}$, so we get an induced automorphism of $\Pi^n_{\mathcal{A}}$ which we also call ω by abuse of notation.

Proposition 1.4.14. *[37, Theorem 3.5]. For all $n \geq 2$, the algebra $\Pi^n_{\mathcal{A}}$ is Frobenius, and its Nakayama automorphism is ω.*

Theorem 1.4.8 then implies that, for any $n \geq 2$ and cut $C \subset \mathcal{A}^n_1$, $\Pi^n_{\mathcal{A}}/\langle C \rangle$ is 2-representation-finite, and its 3-preprojective algebra is $\Pi^n_{\mathcal{A}}$. This first appeared as [43, Proposition 5.48]. Hence $\Pi^n_{\mathcal{A}}$ is called a 3-*preprojective algebra of type A*.

Example 1.4.15. Consider $\Pi^4_{\mathcal{A}}$ (see Example 1.4.11). If $C = \{e_x\alpha_0 \mid x \in \mathcal{A}^n_0\} \subset \mathcal{A}^n_1$, then $\Pi^n_{\mathcal{A}}/\langle C \rangle$ is the path algebra of



where the marked compositions are zero and the rhombi commute. This is isomorphic to the *Auslander algebra* (endomorphism algebra of the direct sum of indecomposable modules) $\mathrm{Aus}(kQ)$ of a linearly oriented Dynkin type A_4 quiver Q. Hence the terminology "type A".

1.4.2 2-Zigzag algebras of type A

Higher zigzag algebras were introduced by Grant in [30]. They have a conceptual definition in terms of twisted trivial extension algebras, but we only need a specific family, which we will define explicitly.

Definition 1.4.16. [30, Example 3.2] Let $n \in \mathbb{N}$, $n \geq 2$. The *2-zigzag algebra of type A_n*, denoted Z_2^n, is the path algebra of $\mathcal{A}^n$ modulo the following relations.

1. For all $x \in \mathcal{A}_0^n$ and $i \in \{0, 1, 2\}$, $e_x \alpha_i^2 = 0$.
2. Each rhombus

$$\begin{tikzcd}
 & y \arrow[dr, "j"] & \\
x \arrow[ur, "i"] \arrow[dr, "j"'] & & z \\
 & y' \arrow[ur, "i"'] &
\end{tikzcd}$$

   in $\mathcal{A}^n$ commutes.

Note Z_2^n is finite dimensional, symmetric, and quadratic dual to the 3-preprojective algebra $\Pi_{\mathcal{A}}^n$ of type A_n [30].

In this thesis, we equip Z_2^n with the path-length grading and consider it as a dg algebra with trivial differential (i.e. $d = 0$).

1.5 Fractional Calabi-Yau algebras

Throughout, we assume triangulated categories are k-linear and Hom-finite.

Definition 1.5.1. [9, Definition 3.1] Let $\mathcal{T}$ be a triangulated category. A *Serre functor* is an autoequivalence $\mathbb{S}$ of $\mathcal{T}$ such that

$$\mathrm{Hom}_{\mathcal{T}}(E, F) \cong \mathrm{Hom}_{\mathcal{T}}(F, \mathbb{S}\, E)^*$$

naturally in $E, F \in \mathcal{T}$.

Example 1.5.2. [34, §4.6] Let Λ be a finite-dimensional algebra with $\mathrm{gl.\,dim}\,\Lambda < \infty$. Then the *derived Nakayama functor*

$$- \otimes^{\mathbf{L}}_{\Lambda} \Lambda^* \colon \mathbf{D}^{\mathrm{b}}(\Lambda) \to \mathbf{D}^{\mathrm{b}}(\Lambda)$$

(see [64, §10.6]) is a Serre functor.

The Serre functor $\mathbb{S}$ on $\mathbf{D}^{\mathrm{b}}(\Lambda)$ gives rise to a derived higher Auslander-Reiten theory.

Definition 1.5.3. [42, §1.2] Let Λ be a finite-dimensional algebra, $d \in \mathbb{Z}^+$. The *derived d-Auslander-Reiten translations* are the autoequivalences $\mathbb{S}_d = \mathbb{S}[-d]$ and $\mathbb{S}_d^- = \mathbb{S}^-[d]$ of $\mathbf{D}^{\mathrm{b}}(\Lambda)$.

Given a d-representation-finite algebra Λ, we define the subcategory

$$\mathcal{U}_\Lambda = \mathrm{add}\{\mathbb{S}_d^{-}\,\Lambda \mid i \in \mathbb{Z}\} \subset \mathbf{D}^{\mathrm{b}}(\Lambda).$$

Proposition 1.5.4. *[42, §1.2] Let Λ be a d-representation-finite algebra.*

1. *For any $M \in \mathrm{mod}\,\Lambda$, $H^0(\mathbb{S}_d^-(M)) \cong \tau_d^-(M)$. Moreover, $\mathcal{U}_\Lambda = \mathcal{M}_\Lambda[d\mathbb{Z}]$.*
2. *We have $\mathbb{S}_d(\mathcal{U}_\Lambda) = \mathbb{S}_d^-(\mathcal{U}_\Lambda) = \mathcal{U}_\Lambda$.*

Note that $\mathcal{U}_\Lambda$ is a d-cluster-tilting subcategory of $\mathbf{D}^{\mathrm{b}}(\Lambda)$ in the sense of [42, §1.2].

The following definition is due to Kontsevich [50].

Definition 1.5.5. Let $\mathcal{T}$ be a triangulated category with Serre functor $\mathbb{S}$ and suspension functor Σ. Let $N, m \in \mathbb{Z}$, $m \neq 0$. We say $\mathcal{T}$ is

1. *N-Calabi-Yau* if there is a natural isomorphism $\mathbb{S} \cong \Sigma^N$;
2. *(N, m)-fractional Calabi-Yau* if there is a natural isomorphism $\mathbb{S}^m \cong \Sigma^N$.

If Λ is a finite-dimensional algebra with $\operatorname{gl.dim} \Lambda < \infty$, we say Λ is (N, m)-fractional Calabi-Yau if this property holds for $\mathbf{D}^{\mathrm{b}}(\Lambda)$.

Remark 1.5.6. Many authors display the Calabi-Yau dimension as N/m, but we prefer (N, m) to reflect the fact one cannot "cancel common factors" in general.

Example 1.5.7.

1. For any quiver Q, $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma(Q))$ is 2-Calabi-Yau [28, Theorem 3.6.4].
2. For any QP (Q, W), $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma(Q, W))$ is 3-Calabi-Yau [28, Theorem 3.6.4].
3. If Q is ADE Dynkin with Coxeter number h then kQ is $(h-2, h)$-fractional Calabi-Yau [55, Theorem 3.8(2)].
4. For any cut $C \subset \mathcal{A}^n_1$, $\Pi^n_{\mathcal{A}}/\langle C \rangle$ is $(2(n-1), n+2)$-fractional Calabi-Yau [19, Remark 2.29].

The fractional Calabi-Yau property is closely related to d-representation finiteness. Consider the following definition.

Definition 1.5.8. [37, Definition 0.3] Let Λ be a finite-dimensional algebra with $\operatorname{gl.dim} \Lambda < \infty$. Given $\phi \in \operatorname{Aut}(\Lambda)$, define

$$\phi^* = - \otimes^{\mathbf{L}}_{\Lambda} ({}_{\phi}\Lambda^*) \colon \mathbf{D}^{\mathrm{b}}(\Lambda) \to \mathbf{D}^{\mathrm{b}}(\Lambda)$$

(see [64, §10.6]). If there is a natural isomorphism

$$\mathbb{S}^m \cong [N] \circ \phi^*$$

for some $\phi \in \operatorname{Aut}(\Lambda)$ and $N, m \in \mathbb{Z}$, we say Λ is *(N, m)-twisted fractional Calabi-Yau*.

Theorem 1.5.9. *[37, Theorem 1.1] Every d-representation-finite algebra is twisted fractional Calabi-Yau.*

By [14, Proposition 2.1], a twisted fractional Calabi-Yau algebra Λ is fractional Calabi-Yau if and only if the twist ϕ has finite order in the group of outer automorphisms of Λ. In the case Λ is d-representation-finite, Grant found necessary and sufficient conditions for this to happen.

Theorem 1.5.10. *[32, Theorem 6.14] Let Λ be basic d-representation-finite, and let σ be the Nakayama automorphism of $\Pi(\Lambda)$. Then Λ is fractional Calabi-Yau if and only if σ has finite order.*

Remark 1.5.11. We have presented simplified versions of the previous two theorems: they also explain how to compute the Calabi-Yau dimension.

1.6 Bridgeland stability conditions

The following material will be used in Part II. As well as the original paper of Bridgeland [12], we highly recommend the survey of Barbieri [3].

Recall the following concept, due to Beilinson, Bernstein and Deligne [7].

Definition 1.6.1. A pair $(\mathscr{T}^{\leq 0}, \mathscr{T}^{\geq 0})$ of full subcategories of a triangulated category $\mathscr{T}$ is called a *t-structure* if

1. $\Sigma\mathscr{T}^{\leq 0} \subset \mathscr{T}^{\leq 0}$ and $\Sigma^{-}\mathscr{T}^{\geq 0} \subset \mathscr{T}^{\geq 0}$;
2. $\mathrm{Hom}_{\mathscr{T}}(\mathscr{T}^{\leq 0}, \Sigma^{-}\mathscr{T}^{\geq 0}) = 0$;
3. for each $E \in \mathscr{T}$, there exists a distinguished triangle

$$E^{\leq 0} \longrightarrow E \longrightarrow E^{\geq 1} \longrightarrow \Sigma E^{\leq 0}$$

   with $E^{\leq 0} \in \mathscr{T}^{\leq 0}$ and $E^{\geq 1} \in \Sigma^{-}\mathscr{T}^{\geq 0}$.

The *heart* of the t-structure is the abelian subcategory $\mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0} \subset \mathscr{T}$. The t-structure is called *bounded* if

$$\mathscr{T} = \left(\bigcup_{i\in\mathbb{N}} \Sigma^{-i}\mathscr{T}^{\leq 0}\right) \cap \left(\bigcup_{i\in\mathbb{N}} \Sigma^{i}\mathscr{T}^{\geq 0}\right).$$

In particular, bounded t-structures are determined by their hearts.

Example 1.6.2.

1. Let $\mathscr{A}$ be an abelian category. There is a bounded t-structure on $\mathbf{D}(\mathscr{A})$ whose heart is $\mathscr{A}$.

2. Let A be a dg algebra. There is a bounded t-structure on $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(A)$ whose heart is $\mathrm{mod}(H^0(A))$ [47, Example 7.1].

3. Suppose Z is either $Z(Q)$ for some quiver Q, or Z_2^n for some $n \in \mathbb{N}_{>0}$. By [47, Theorem 8.1], there is a bounded t-structure on $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z)$ whose heart is the smallest full subcategory which contains Z and is closed under extensions and direct summands.

Given an abelian or triangulated category $\mathscr{C}$, we denote by $\mathrm{K}(\mathscr{C})$ its Grothendieck group. Fix a finite-rank free lattice L and an epimorphism $\ell\colon \mathrm{K}(\mathscr{C}) \to L$. We say a group homomorphism $\mathscr{Z}\colon \mathrm{K}(\mathscr{C}) \to \mathbb{C}$ has the *support property* with respect to a subset $X \subset \mathrm{Ob}(\mathscr{C})$ if $\mathscr{Z}$ factors through ℓ, and there exists a norm $||\cdot||$ on $L \otimes \mathbb{R}$ and a constant $c \in \mathbb{R}_{>0}$ such that, for all $E \in X$, $|\mathscr{Z}(E)| \geq c||\ell(E)||$.

Definition 1.6.3. [60] Let $\mathscr{A}$ be an abelian category, and let $\mathscr{Z}\colon \mathrm{K}(\mathscr{A}) \to \mathbb{C}$ be a group homomorphism such that, for all $0 \neq E \in \mathscr{A}$,

$$\mathscr{Z}(E) \in \overline{\mathbb{H}} := \{r\exp(i\pi\phi) \mid r > 0,\ 0 < \phi \leq 1\}.$$

The *phase* of $0 \neq E \in \mathscr{A}$ is $\phi(E) = \arg\mathscr{Z}(E)/\pi \in (0,1]$. We call E *$\mathscr{Z}$-semistable* (resp. *$\mathscr{Z}$-stable*) if every proper subobject $0 \neq F \subset E$ satisfies $\phi(F) \leq \phi(E)$ (resp. $\phi(F) < \phi(E)$). We say $\mathscr{Z}$ is a *stability function* if it has the *Harder-Narasimhan* *property*: every $0 \neq E \in \mathscr{A}$ has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E,$$

where each subquotient $F_j := E_j / E_{j-1}$ is $\mathscr{Z}$-semistable with phases

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

Definition 1.6.4. [12] A *stability condition* $(\mathscr{H}, \mathscr{Z})$ on a triangulated category $\mathscr{T}$ consists of the heart $\mathscr{H}$ of a bounded t-structure, together with a stability function $\mathscr{Z} \colon \mathrm{K}(\mathscr{H}) \to \mathbb{C}$ which has the support property with respect to $\mathscr{Z}$-semistable objects.

The support property did not appear in the original definition of Bridgeland, but was introduced in [51, §1.2].

Definition 1.6.5. [12, Definition 3.3] A slicing $\mathscr{P}$ of a triangulated category $\mathscr{T}$ consists of a full additive subcategory $\mathscr{P}(\phi) \subset \mathscr{T}$ for each $\phi \in \mathbb{R}$, such that the following axioms are satisfied.

1. For all $\phi \in \mathbb{R}$, $\mathscr{P}(\phi + 1) = \mathscr{P}(\phi)[1]$.
2. If $\phi > \psi$, $E \in \mathscr{P}(\phi)$ and $F \in \mathscr{P}(\psi)$, then $\mathrm{Hom}_{\mathscr{T}}(E, F) = 0$.
3. For each $0 \neq E \in \mathscr{T}$, there exist real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and a collection of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r, dash] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

such that $A_j \in \mathscr{P}(\phi_j)$ for all $1 \leq j \leq n$.

Proposition 1.6.6. *[12, Proposition 5.3] A stability condition on $\mathscr{T}$ is equivalent to a pair $\sigma = (\mathscr{P}, \mathscr{Z})$ consisting of a slicing $\mathscr{P}$, together with a group homomorphism $\mathscr{Z} \colon \mathrm{K}(\mathscr{T}) \to \mathbb{C}$ which has the support property with respect to non-zero objects of $\bigcup_{\phi \in \mathbb{R}} \mathscr{P}(\phi)$, and such that $\mathscr{Z}(E) \in \mathbb{R}_{>0} \exp(i\pi\phi)$ for all $0 \neq E \in \mathscr{P}(\phi)$.*

Sketch. Let $(\mathcal{H}, \mathcal{Z})$ be a stability condition on $\mathcal{T}$ in the first sense. For each $0 < \phi \leq 1$, $i \in \mathbb{Z}$, set $\mathcal{P}(\phi{+}i) = \mathrm{add}\{E[i] \mid E \text{ is } \mathcal{Z}\text{-semistable of phase } \phi\}$. This defines a slicing. Pairing with $\mathrm{K}(\mathcal{T}) \xrightarrow{\sim} \mathrm{K}(\mathcal{H}) \xrightarrow{\mathcal{Z}} \mathbb{C}$ gives a stability condition in the second sense.

Conversely, if $(\mathcal{P}, \mathcal{Z})$ is a stability condition on $\mathcal{T}$ in the second sense, then the abelian category $\mathcal{P}(0,1]$, defined as the extension-closure of $\bigcup_{\phi \in (0,1]} \mathcal{P}(\phi)$, is the heart of a bounded t-structure. Pairing with $\mathrm{K}(\mathcal{P}(0,1]) \xrightarrow{\sim} \mathrm{K}(\mathcal{T}) \xrightarrow{\mathcal{Z}} \mathbb{C}$ gives a stability condition in the first sense.

The non-zero objects (resp. simple objects) of $\mathcal{P}(\phi)$ are called *σ-semistable* (resp *σ-stable*) of *phase* ϕ. We call $\mathcal{Z}$ a *central charge*.

The following is one of the most remarkable aspects of Bridgeland's definition.

Theorem 1.6.7. *[12, Theorem 1.2], [6, Appendix A], [5] Denote by* $\mathrm{Stab}(\mathcal{T})$ *the set of all stability conditions on a triangulated category* $\mathcal{T}$ *(for a fixed choice of* $\mathrm{K}(\mathcal{T}) \overset{\ell}{\twoheadrightarrow} L$*). If* $\mathrm{Stab}(\mathcal{T})$ *is non-empty then it is a complex manifold. Moreover, it is locally isomorphic to the space* $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathcal{T}), \mathbb{C})$ *via the forgetful map* $(\mathcal{H}, \mathcal{Z}) \mapsto \mathcal{Z}$*.*

Bridgeland's original theorem concerns the set of *locally-finite* stability conditions, but local finiteness is implied by the support property. The structure of the stability manifold is closely linked to Happel-Reiten-Smalø tilting.

Definition 1.6.8. [35] Let $\mathcal{H}_1, \mathcal{H}_2$ be hearts of bounded t-structures on a triangulated category $\mathcal{T}$. Then $(\mathcal{H}_1, \mathcal{H}_2)$ is called a *tilting pair* if $\mathcal{H}_2$ lies in the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{H}_1$, $\mathcal{H}_1[-1]$ and closed under extensions.

Proposition 1.6.9. *[66, Corollary 5.2] Let* $\sigma = (\mathcal{H}_\sigma, \mathcal{Z}_\sigma)$ *and* $\tau = (\mathcal{H}_\tau, \mathcal{Z}_\tau)$ *be stability conditions on a triangulated category* $\mathcal{T}$*. Then* σ, τ *lie in the same connected component of* $\mathrm{Stab}(\mathcal{T})$ *if and only if* $\mathcal{H}_\sigma$*,* $\mathcal{H}_\tau$ *are related by a finite sequence of tilting pairs.*

We can provide a more concrete description under stricter assumptions about the hearts involved. Call a heart *finite* if it is finite length and has finitely many simple objects up to isomorphism. Denote by $\operatorname{Stab}(\mathcal{H}) \subset \operatorname{Stab}(\mathcal{T})$ the space of stability conditions supported on a given heart $\mathcal{H}$ of $\mathcal{T}$.

Proposition 1.6.10. *[60, §1] Let $\mathcal{H}$ be the finite heart of a bounded t-structure on a triangulated category $\mathcal{T}$. Then any group homomorphism $\mathcal{Z}\colon \mathrm{K}(\mathcal{H}) \to \mathbb{C}$ with $\mathcal{Z}(S) \in \overline{\mathbb{H}}$ for all simple objects $S \in \mathcal{H}$ is automatically a stability function with the support property. In particular, $\operatorname{Stab}(\mathcal{H}) \cong \overline{\mathbb{H}}^n$, where n is the rank of $\mathrm{K}(\mathcal{H})$.*

Definition 1.6.11. [35] Let $\mathcal{H}$ be the finite heart of a bounded t-structure on a triangulated category $\mathcal{T}$, and let $S \in \mathcal{H}$ be a simple object. The *backward tilt* of $\mathcal{H}^\flat_S$ of $\mathcal{H}$ at S is the smallest full extension-closed subcategory of $\mathcal{T}$ containing $S[-1]$ and

$$S^\perp := \{E \in \mathcal{H} \mid \operatorname{Hom}_{\mathcal{H}}(S, E) = 0\}.$$

The *forward tilt* $\mathcal{H}^\sharp_S$ is the smallest full extension-closed subcategory containing $S[1]$ and

$${}^\perp S := \{E \in \mathcal{H} \mid \operatorname{Hom}_{\mathcal{H}}(E, S) = 0\}.$$

We have tilting pairs $(\mathcal{H}, \mathcal{H}^\flat_S)$ and $(\mathcal{H}^\sharp_S, \mathcal{H})$. Moreover, $(\mathcal{H}^\flat_S)^\sharp_{S[-1]} \simeq \mathcal{H} \simeq (\mathcal{H}^\sharp_S)^\flat_{S[1]}$.

Proposition 1.6.12. *[12], [65]. Let $\mathcal{H}_1, \mathcal{H}_2$ be hearts of bounded t-structures on a triangulated category $\mathcal{T}$, such that $\mathcal{H}_1$ is finite. Take a simple object $S \in \mathcal{H}_1$. Suppose $\varnothing \neq \mathcal{W}_S \subset \operatorname{Stab}(\mathcal{H}_1)$ is the real-codimension 1 locus on which S has phase 1, and all other simple objects have phase less than 1, then*

$$\mathcal{H}_1 = (\mathcal{H}_2)^\flat_S \iff \operatorname{Stab}(\mathcal{H}_1) \cap \overline{\operatorname{Stab}(\mathcal{H}_2)} = \mathcal{W}_S.$$

Part I

Higher preprojective algebras

Introduction

Many mathematical objects admit a classification by Dynkin diagrams, perhaps most famously the semisimple Lie algebras over $\mathbb{C}$. Gabriel's theorem says that the path algebra of a quiver is representation finite if and only if its underlying graph is an ADE Dynkin diagram, which happens precisely when the preprojective algebra of the quiver is finite dimensional. The group $\mathbb{Z}_2$ acts on the type A Dynkin diagrams by rotating them through π. One obtains the type D diagrams by taking $\mathbb{Z}_2$-orbifolds, i.e. quotienting by this action while duplicating the fixed vertex, whenever one exists (see Table A).

A related classification appears in [21, 22], where Evans and Pugh study Jacobian algebras over so-called $\mathcal{ADE}$ graphs, introduced by Di Francesco and Zuber in work on $\mathrm{SU}(3)$ modular invariants [18]. The group $\mathbb{Z}_3$ acts on the type $\mathcal{A}$ graphs by rotating them through $2\pi/3$. One obtains the type $\mathcal{D}$ graphs by taking $\mathbb{Z}_3$-orbifolds, i.e. quotienting by this action while triplicating the fixed vertex, whenever one exists. The type $\mathcal{A}$ algebras are well-studied. Indeed, we show they are isomorphic to the 3-preprojective algebras of type A. In this part I present work from my article [33], which started as a study of the type $\mathcal{D}$ algebras using higher homological algebra.

We present a family of selfinjective algebras, which are Morita equivalent to skew group algebras of the 3-preprojective algebras of type A by a construction of Giovannini and Pasquali [29]. Our definition is standalone, in the sense that it makes no reference to type A. This unlocks the possibility of locally reconstructing the algebra

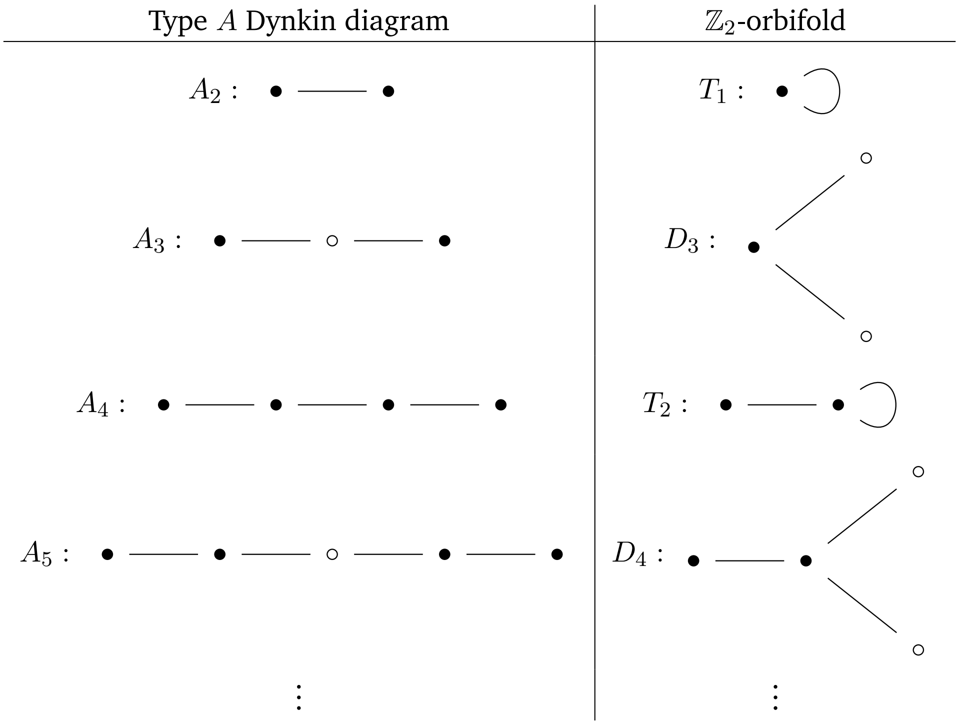


Table A: Type D Dynkin diagrams (and tadpole diagrams) arising as $\mathbb{Z}_2$-orbifolds of type A diagrams.

around a given vertex, which we utilise in the proof of Lemma 3.1.4. In §2.3 we show these algebras are isomorphic to the type $\mathcal{D}$ algebras of Evans and Pugh.

Using the classification of Herschend and Iyama [38], we show that one in three of the selfinjective algebras are 3-preprojective. We call these the 3-preprojective algebras of type D by analogy with Dynkin diagrams. By considering their Nakayama automorphisms and applying Grant's theorem [30], we show that the corresponding 2-representation-finite algebras are fractional Calabi-Yau. Finally, we give a recipe to construct 2-Auslander-Reiten quivers for an arbitrary basic 2-representation-finite algebra.

Remark. Following the convention in [21, 22], we denote by $\mathcal{D}^n$ the quiver obtained as an orbifold of $\mathcal{A}^n$ (and label the corresponding algebras accordingly). However, only every third quiver is the quiver of a 3-preprojective algebra. We could have chosen to only label those with a $\mathcal{D}$. However, it is not obvious how one should index in this case (see Table B).

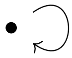
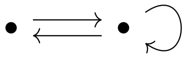
Quiver	Our notation	3-preprojective algebra?	Alternative notation
	$\mathcal{D}^2$	✗	$\mathcal{T}^?$
	$\mathcal{D}^3$	✗	$\mathcal{T}^?$
-	$\mathcal{D}^4$	✓	$\mathcal{D}^?$
-	$\mathcal{D}^5$	✗	$\mathcal{T}^?$
-	$\mathcal{D}^6$	✗	$\mathcal{T}^?$
-	$\mathcal{D}^7$	✓	$\mathcal{D}^?$
⋮	⋮	⋮	⋮

Table B: Two options for notation. See Figure 2.1 for the missing quivers.

Chapter 2

3-Preprojective algebras of type D

Denote by $\succ$ the strict lexicographic order on $\mathbb{N}^3$. Throughout, $\equiv$ denotes congruence modulo 3. Given $x, y \in \mathbb{N}^3$, write $x \sim_{\equiv} y$ if x is component-wise congruent modulo 3 to $\omega^j(y)$ for some $j \in \{0, 1, 2\}$ (see Definition 1.4.9).

Definition 2.0.1. Let $n \in \mathbb{N}$, $n \geq 2$. Define

$$\begin{aligned} Q_0^n &= \{x \in \mathbb{N}^3 \mid x_0 + x_1 + x_2 = n - 1,\ x \succ \omega(x),\ x \succ \omega^2(x)\}, \\ Q_1^n &= \bigcup_{i,j=0}^{2} \{x \xrightarrow{\alpha_{i,j}} \omega^j(x) + v_i \mid x, \omega^j(x) + v_i \in Q_0^n\}, \end{aligned}$$

where $v_0 = (-1, 1, 0)$, $v_1 = (0, -1, 1)$ and $v_2 = (1, 0, -1)$.

If $n \not\equiv 1$, let $\mathcal{D}^n$ be the quiver with vertices $\mathcal{D}_0^n = Q_0^n$ and arrows $\mathcal{D}_1^n = Q_1^n$.

If $n = 3m + 1$ for some $m \in \mathbb{N}_{>0}$, write $X = (m, m, m)$, and let $\mathcal{D}^n$ be the quiver with vertices $\mathcal{D}_0^n = Q_0^n \cup \{X_0, X_1, X_2\}$ (i.e. take three copies of X) and arrows

$$\mathcal{D}_1^n = Q_1^n \cup \{X - v_0 \xrightarrow{\beta_k} X_k,\ X_k \xrightarrow{\gamma_k} X + v_2 \mid k = 0, 1, 2\}.$$

Some examples are given in Figure 2.1. Note we often write $x_0x_1x_2$ in place of (x_0, x_1, x_2), $x \xrightarrow{(i,j)} y$ in place of $x \xrightarrow{\alpha_{i,j}} y$, and $x \xrightarrow{i} y$ in place of $x \xrightarrow{\alpha_{i,0}} y$.

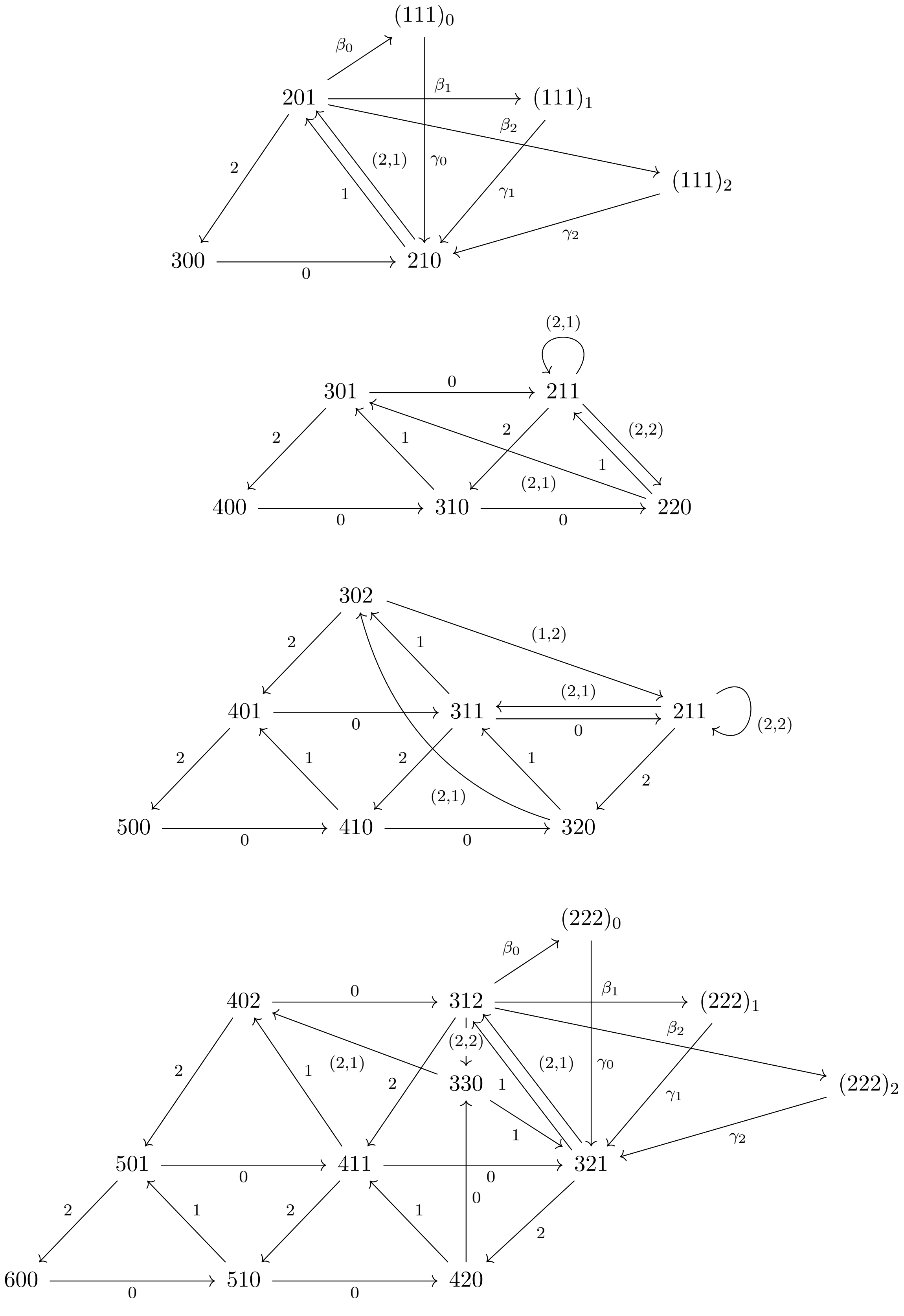


Figure 2.1: $\mathcal{D}^4, \mathcal{D}^5, \mathcal{D}^6, \mathcal{D}^7$.

Definition 2.0.2. Let $n \geq 2$. Define a Jacobian algebra $\Pi^n_{\mathcal{D}} = \mathrm{J}(\mathcal{D}^n, W^n_{\mathcal{D}})$ over $\mathbb{C}$ using the potential $W^n_{\mathcal{D}} = \sum_c \lambda^n_{\mathcal{D}}(c)c$, where the sum is taken over all 3-cycles in $\mathcal{D}^n$, and $\lambda^n_{\mathcal{D}}$ is defined as follows.

1. Suppose $c = e_x\alpha_{i_0,j_0}\alpha_{i_1,j_1}\alpha_{i_2,j_2}$ contains three distinct arrows. If $j_0+j_1+j_2 \equiv 0$ then
$$\lambda^n_{\mathcal{D}}(c) = \begin{cases} 1 & \text{if } (i_0 + j_0, i_1 + j_0 + j_1, i_2) \sim_{\equiv} (0, 1, 2), \\ -1 & \text{if } (i_0 + j_0, i_1 + j_0 + j_1, i_2) \sim_{\equiv} (0, 2, 1). \end{cases}$$
Otherwise, $\lambda^n_{\mathcal{D}}(c) = 0$.

2. If $c = e_x\alpha^3_{i,j}$ for a loop $e_x\alpha_{i,j}$ then
$$\lambda^n_{\mathcal{D}}(c) = \begin{cases} \frac{1}{3} & \text{if } (i + j, i + 2j, j) \sim_{\equiv} (0, 1, 2), \\ -\frac{1}{3} & \text{if } (i + j, i + 2j, i) \sim_{\equiv} (0, 2, 1). \end{cases}$$

3. If $n \equiv 1$ then let $\zeta = e^{2\pi i/3}$. For each $k \in \{0, 1, 2\}$, $\lambda^n_{\mathcal{D}}(\alpha_1\beta_k\gamma_k) = -1$ and $\lambda^n_{\mathcal{D}}(\alpha_{2,1}\beta_k\gamma_k) = \zeta^k$.

Example 2.0.3. The potential on $\mathcal{D}^4$ is

$$W^4_{\mathcal{D}} = \alpha_0\alpha_1\alpha_2 + \cancel{0\alpha_0\alpha_{2,1}\alpha_2} + \sum_{k=0}^{2}(\zeta^k\alpha_{2,1}\beta_k\gamma_k - \alpha_1\beta_k\gamma_k).$$

Note we often write α_i for $\alpha_{i,0}$. Hence $\Pi^4_{\mathcal{D}}$ is $\mathbb{C}\mathcal{D}^4$ modulo the ideal generated by

$$\begin{array}{rl} \alpha_0\alpha_1, & \alpha_1\alpha_2, \\ \displaystyle\sum_{k=0}^{2}\zeta^k\beta_k\gamma_k, & \displaystyle\alpha_2\alpha_0 - \sum_{k=0}^{2}\beta_k\gamma_k, \\ \zeta^k\alpha_{2,1}\beta_k - \alpha_1\beta_k, & \zeta^k\gamma_k\alpha_{2,1} - \gamma_k\alpha_{2,1}. \end{array}$$

Example 2.0.4. The potential on $\mathcal{D}^5$ is

$$\begin{aligned}W^5_{\mathcal{D}} = e_{400}\alpha_0\alpha_1\alpha_2 - e_{301}\alpha_0\alpha_2\alpha_1 + e_{310}\alpha_0\alpha_1\alpha_2 \\ + e_{301}\alpha_0\alpha_{2,2}\alpha_{2,1} - e_{220}\alpha_1\alpha_{2,1}\alpha_{2,2} + \frac{e_{211}\alpha^3_{2,1}}{3}.\end{aligned}$$

2.1 Morita equivalence to skew group algebras

Definition 2.1.1. Let G be a finite group acting on an algebra Λ by automorphisms. The *skew group algebra* $\Lambda \# G$ has underlying vector space $\Lambda \otimes kG$, and multiplication satisfying $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$ for all $a, b \in \Lambda$, $g, h \in G$.

In [29], Giovannini and Pasquali study the case $\Lambda = \mathrm{J}(Q, W)$ is a Jacobian algebra and G is a finite cyclic group. Under certain assumptions, they give a recipe to construct a QP $(\widetilde{Q}, \widetilde{W})$ such that $\mathrm{J}(\widetilde{Q}, \widetilde{W})$ is Morita equivalent to $\Lambda \# G$, building on [57]. In particular, their assumptions are satisfied when (Q, W) is *strongly planar* and G *acts by rotations* (definitions to follow) [29, Lemma 6.5].

Definition 2.1.2. [38, Definition 8.1] Let (Q, W) be a QP. Its *canvas* $X_{(Q,W)}$ is the following 2-dimensional CW complex (see e.g. [36, §0]).

1. The 0-cells are the vertices of Q, i.e. $X^0_{(Q,W)} = Q_0$.
2. The 1-cells are indexed by the arrows of Q. For each $\alpha \in Q_1$, the attaching map $\phi^1_\alpha \colon \{0, 1\} \to X^0_{(Q,W)}$ satisfies $\phi^1_\alpha(0) = s(\alpha)$, $\phi^1_\alpha(1) = t(\alpha)$.
3. The 2-cells are indexed by the cycles appearing in W. For each $\alpha \in Q_1$, fix a characteristic map $\varepsilon^1_\alpha \colon [0, 1] \to X_{(Q,W)}$ extending ϕ^1_α. For each $c = \alpha_0 \cdots \alpha_{n-1}$, the attaching map $\phi^2_c \colon S^1 \to X^1_{(Q,W)}$ satisfies

$$\phi^2_c\left(\cos\left(\frac{2\pi}{n}(i+t)\right), \sin\left(\frac{2\pi}{n}(i+t)\right)\right) = \varepsilon^1_{\alpha_i}(t)$$

for integers $0 \le i < n$ and real numbers $0 \le t < 1$.

Informally, the 1-skeleton is the underlying graph of Q, and to obtain the canvas we glue a 2-cell along each cycle appearing in W.

Definition 2.1.3. [29, Definition 6.3] A QP (Q, W) is called *strongly planar* if there is an embedding $X_{(Q,W)} \hookrightarrow \mathbb{R}^2$ whose image is homeomorphic to a disc.

Lemma 2.1.4. *For each $n \geq 2$, $(\mathcal{A}^n, W_{\mathcal{A}}^n)$ is strongly planar.*

Proof. Identify $\mathbb{R}^2$ with the plane $P = \{x \in \mathbb{R}^3 \mid x_0 + x_1 + x_2 = n - 1\}$. Embed the vertices of $\mathcal{A}^n$ in the obvious way. For each $e_x\alpha_i \in \mathcal{A}_1^n$, embed the 1-cell $D^1_{e_x\alpha_i}$ as the line segment joining x and $x + v_i$. The induced embedding $X_{(\mathcal{A}^n, W_{\mathcal{A}}^n)} \hookrightarrow P$ has image $\{x \in P \mid x_0, x_1, x_2 \geq 0\}$, which is a closed triangle lying in the plane and therefore homeomorphic to a disc. □

Definition 2.1.5. [29, Definition 6.4] Let (Q, W) be a strongly planar QP, and let G be a finite cyclic group acting on Q. Then *G acts on (Q, W) by rotations* if

1. there is an embedding $X_{(Q,W)} \hookrightarrow \mathbb{R}^2$ such that the action of a generator of G is induced by a rotation of the plane;
2. the action of G is faithful;
3. every cycle c appearing in W is one of the following types:
   (a) c contains no vertices fixed by G;
   (b) c contains precisely one vertex fixed by G (counted with multiplicity[1]);
   (c) c contains precisely one vertex not fixed by G (counted with multiplicity);
   (d) c only contains vertices fixed by G.

Lemma 2.1.6. *For each $n \geq 2$, $\mathbb{Z}_3$ acts on $(\mathcal{A}^n, W_{\mathcal{A}}^n)$ by rotations.*

Proof. Since $\omega^3 = \mathrm{id}$, there is a group action

$$\begin{aligned} \mathbb{Z}_3 &\to \mathrm{Aut}(\mathcal{A}^n), \\ j &\mapsto \omega^j. \end{aligned}$$

---

[1] Hence if c contains two arrows whose targets are fixed (even if these happen to be the same vertex), it fails condition (b). Analogously for condition (c).

1. Consider the embedding $X_{(\mathcal{A}^n, W^n_{\mathcal{A}})} \hookrightarrow P$ from the proof of Lemma 2.1.4. Then ω is induced by rotating P clockwise through $\frac{2\pi}{3}$ about $(\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-1}{3})$.

2. Since $n \geq 2$, $(n-1, 0, 0)$ is a vertex of $\mathcal{A}^n$ not fixed by ω or ω^2, so the action is faithful.

3. Note $x \in \mathcal{A}^n_0$ is fixed by $\mathbb{Z}_3$ if and only if $x_0 = x_1 = x_2$. But $x_0 + x_1 + x_2 = n-1$, so this occurs precisely when $n = 3m+1$ for some $m \in \mathbb{N}_{>0}$. In this case, there is a unique fixed vertex (m, m, m). Let c be a cycle appearing in $W^n_{\mathcal{A}}$. Then c passes through any given vertex of $\mathcal{A}^n$ at most once. Since there is at most one vertex fixed by $\mathbb{Z}_3$, c is either type (a) or (b). □

Theorem 2.1.7. *[29, Theorem 3.20] Let G be a finite cyclic group acting by rotations on a strongly planar QP (Q, W). Then G acts on $\mathrm{J}(Q, W)$. Furthermore, there is a construction giving a QP $(\widetilde{Q}, \widetilde{W})$ and an idempotent $\eta \in \mathrm{J}(Q, W)\#G$ such that $\mathrm{J}(\widetilde{Q}, \widetilde{W}) \cong \eta(\mathrm{J}(Q, W)\#G)\eta$, and this isomorphism induces a Morita equivalence between $\mathrm{J}(\widetilde{Q}, \widetilde{W})$ and $\mathrm{J}(Q, W)\#G$.*

We apply their construction to $(\mathcal{A}^n, W^n_{\mathcal{A}})$, and conclude that $(\widetilde{\mathcal{A}^n}, \widetilde{W^n_{\mathcal{A}}}) = (\mathcal{D}^n, W^n_{\mathcal{D}})$.

Vertices

Let $V_1 = \{x \in \mathcal{A}^n_0 \mid x \succ \omega(x),\ x \succ \omega^2(x)\}$. Writing $X = (\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-1}{3})$, let $V_2 = \{X\}$ if $n \equiv 1$, and $V_2 = \varnothing$ otherwise. Note that $V_1 \sqcup V_2$ is a complete set of representatives of the $\mathbb{Z}_3$-orbits of vertices of $\mathcal{A}^n$, and that V_2 contains precisely the fixed vertices. Now $\widetilde{\mathcal{A}^n_0} = V_1$ if $n \not\equiv 1$, while $\widetilde{\mathcal{A}^n_0} = V_1 \cup \{X_0, X_1, X_2\}$ if $n \equiv 1$ [29, Notation 3.9-11]. Note the idempotent is $\eta = (\sum_{x \in V_1 \cup V_2} e_x) \otimes 0 \in \Pi^n_{\mathcal{A}} \# \mathbb{Z}_3$, where we mean the group element $0 \in \mathbb{Z}_3$.

Arrows

For each arrow in $\mathcal{A}^n_1$, we fix a representative of its $\mathbb{Z}_3$-orbit, and define arrows in $\widetilde{\mathcal{A}^n}$ corresponding to these representatives [29, Notation 3.13].

1. Consider an arrow in $\mathcal{A}^n_1$ between two vertices not fixed by $\mathbb{Z}_3$. There is a unique arrow in its orbit whose target is in $\widetilde{\mathcal{A}^n_0}$, and it must be of the form $\omega^j(x) \xrightarrow{\alpha_i} \omega^j(x) + v_i$ for some $x \in \widetilde{\mathcal{A}^n_0}$, $i, j \in \{0, 1, 2\}$. We define a corresponding arrow $x \xrightarrow{\alpha_{i,j}} \omega^j(x) + v_i$ in $\widetilde{\mathcal{A}^n_1}$.

2. Suppose $n \equiv 1$. There are three arrows in $\mathcal{A}^n_1$ whose target is X, all in the same $\mathbb{Z}_3$-orbit. Only one of them has its source in $\widetilde{\mathcal{A}^n_0}$, namely $X - v_0 \xrightarrow{0} X$. We define corresponding arrows $\{X - v_0 \xrightarrow{\beta_k} X_k \mid k = 0, 1, 2\}$ in $\widetilde{\mathcal{A}^n_1}$. There are three arrows in $\mathcal{A}^n_1$ whose source is X, all in the same $\mathbb{Z}_3$-orbit. Only one of them has its target in $\widetilde{\mathcal{A}^n_0}$, namely $X \xrightarrow{2} X + v_2$. We define corresponding arrows $\{X_k \xrightarrow{\gamma_k} X + v_2 \mid k = 0, 1, 2\}$ in $\widetilde{\mathcal{A}^n_1}$.

Potential

We fix a complete set $C_1 \sqcup C_2$ of representatives of the $\mathbb{Z}_3$-orbits of 3-cycles in $\mathcal{A}^n$, where C_1 contains the cycles that do not pass through the fixed vertex X, and C_2 contains the cycles that do. In particular, if $n \not\equiv 1$ then $C_2 = \varnothing$. To each $c \in C_1$ we associate a cycle $\widetilde{c}$ in $\widetilde{\mathcal{A}^n}$, while to each $c \in C_2$ we associate three cycles $\widetilde{c_0}, \widetilde{c_1}, \widetilde{c_2}$ in $\widetilde{\mathcal{A}^n}$ [29, Notation 3.17].

1. Consider a 3-cycle in $\mathcal{A}^n$ that does not contain X. Choose a representative c of its $\mathbb{Z}_3$-orbit that contains at least one vertex from $\widetilde{\mathcal{A}^n_0}$. Then c is of the form

$$x \xrightarrow{i'_0} y' \xrightarrow{i'_1} z' \xrightarrow{i_2} x,$$

where $x \in \widetilde{\mathcal{A}^n_0}$, $\{i'_0, i'_1, i_2\} = \{0, 1, 2\}$, $y' = x + v_{i'_0}$ and $z' = x + v_{i'_0} + v_{i'_1}$. Then $y := \omega^{j_0}(y') \in \widetilde{\mathcal{A}^n_0}$ and $z := \omega^{j_0 + j_1}(z') \in \widetilde{\mathcal{A}^n_0}$ for unique $j_0, j_1 \in \{0, 1, 2\}$. Let $i_0, i_1, j_2 \in \{0, 1, 2\}$ satisfy $i_0 \equiv i'_0 - j_0$, $i_1 \equiv i'_1 - j_0 - j_1$ and $j_2 \equiv -j_0 - j_1$. Define $\widetilde{c}$ to be the cycle

$$x \xrightarrow{(i_0, j_0)} y \xrightarrow{(i_1, j_1)} z \xrightarrow{(i_2, j_2)} x$$

in $\widetilde{\mathcal{A}^n}$. See Remark 2.1.8 for a proof $\widetilde{c}$ exists.

2. Suppose $n \equiv 1$. There are six cycles in $W_{\mathcal{A}}^n$ containing X, in two disjoint $\mathbb{Z}_3$-orbits. As representatives of these orbits we choose

$$\begin{aligned} c^- :& \quad X + v_2 \xrightarrow{1} X - v_0 \xrightarrow{0} X \xrightarrow{2} X + v_2, \\ c^+ :& \quad X + v_2 \xrightarrow{0} X - v_1 \xrightarrow{1} X \xrightarrow{2} X + v_2. \end{aligned}$$

For each $k \in \{0, 1, 2\}$, we define the cycles

$$\begin{aligned} \widetilde{c_k^-} :& \quad X + v_2 \xrightarrow{1} X - v_0 \xrightarrow{\beta_k} X_k \xrightarrow{\gamma_k} X + v_2, \\ \widetilde{c_k^+} :& \quad X + v_2 \xrightarrow{(2,1)} X - v_0 \xrightarrow{\beta_k} X_k \xrightarrow{\gamma_k} X + v_2 \end{aligned}$$

in $\widetilde{\mathcal{A}^n}$. We also define $p(c^-) = 0$, $p(c^+) = -1$. Informally, this will adjust for the fact that $X - v_1 \notin \widetilde{\mathcal{A}_0^n}$ but $\omega(X - v_1) \in \widetilde{\mathcal{A}_0^n}$.

By [29, Notation 3.18], the potential on $\widetilde{\mathcal{A}^n}$ is

$$\widetilde{W_{\mathcal{A}}^n} = \sum_{c \in C_1} \frac{|\mathbb{Z}_3 \cdot c| \lambda_{\mathcal{A}}(c)}{3} \widetilde{c} + \sum_{c \in C_2} \lambda_{\mathcal{A}}(c) \sum_{k=0}^{2} \zeta^{-p(c)k} \widetilde{c_k}.$$

Remark 2.1.8. We prove that the cycle $\widetilde{c}$ defined above exists in $\widetilde{\mathcal{A}^n}$. Recall c is the cycle

$$x \xrightarrow{i_0'} \omega^{-j_0}(y) \xrightarrow{i_1'} \omega^{-j_0-j_1}(z) \xrightarrow{i_2} x$$

in $\mathcal{A}^n$, where $x, y, z \in \widetilde{\mathcal{A}_0^n}$, $\{i_0', i_1', i_2\} = \{0, 1, 2\}$ and $j_0, j_1 \in \{0, 1, 2\}$.

1. Consider $x \xrightarrow{i_0'} \omega^{-j_0}(y)$. Applying ω^{j_0} shows that the arrow in its $\mathbb{Z}_3$ orbit with target in $\widetilde{\mathcal{A}_0^n}$ is $\omega^{j_0}(x) \xrightarrow{i_0} y$, where $i_0 \in \{0, 1, 2\}$ satisfies $i_0 \equiv i_0' - j_0$. Hence there is an arrow $x \xrightarrow{(i_0,j_0)} y$ in $\widetilde{\mathcal{A}_1^n}$.

2. Consider $\omega^{-j_0}(y) \xrightarrow{i_1'} \omega^{-j_0-j_1}(z)$. Applying $\omega^{j_0+j_1}$ shows that the arrow in its $\mathbb{Z}_3$-orbit with target in $\widetilde{\mathcal{A}_0^n}$ is $\omega^{j_1}(y) \xrightarrow{i_1} z$, where $i_1 \in \{0, 1, 2\}$ satisfies $i_1 \equiv i_1' - j_0 - j_1$. Hence there is an arrow $y \xrightarrow{(i_1,j_1)} z$ in $\widetilde{\mathcal{A}_0^n}$.

3. Note that $\omega^{-j_0-j_1}(z) \xrightarrow{i_2} x$ already has target in $\widetilde{\mathcal{A}^n_0}$, so there is an arrow $z \xrightarrow{(i_2,j_2)} x$ in $\widetilde{\mathcal{A}^s_1}$, where $j_2 \in \{0,1,2\}$ satisfies $j_2 \equiv -j_0 - j_1$.

Therefore the cycle $\widetilde{c}$ given by $x \xrightarrow{(i_0,j_0)} y \xrightarrow{(i_1,j_1)} z \xrightarrow{(i_2,j_2)} x$ exists as claimed.

Theorem 2.1.9. *For each* $n \geq 2$, $(\widetilde{\mathcal{A}^n}, \widetilde{W^n_{\mathcal{A}}}) = (\mathcal{D}^n, W^n_{\mathcal{D}})$, *so* $\Pi^n_{\mathcal{D}} \cong \eta(\Pi^n_{\mathcal{A}} \# \mathbb{Z}_3)\eta$. *In particular,* $\Pi^n_{\mathcal{D}}$ *is Morita equivalent to* $\Pi^n_{\mathcal{A}} \# \mathbb{Z}_3$.

Proof. By inspection, $\widetilde{\mathcal{A}^n_0} = \mathcal{D}^n_0$, $\widetilde{\mathcal{A}^n_1} \subseteq \mathcal{D}^n_1$. To see that $\mathcal{D}^n_1 \subseteq \widetilde{\mathcal{A}^n_1}$, take $e_x\alpha_{i,j} \in \mathcal{D}^n_1$. Then $x, \omega^j(x) + v_i \in \widetilde{\mathcal{A}^n_0}$, so $e_{\omega^j(x)}\alpha_i$ is an arrow in $\mathcal{A}^n_1$ between two vertices not fixed by $\mathbb{Z}_3$, whose target is in $\widetilde{\mathcal{A}^n_0}$. Hence $e_x\alpha_{i,j} \in \widetilde{\mathcal{A}^n_1}$ by construction, and we conclude that $\widetilde{\mathcal{A}^n} = \mathcal{D}^n$.

Note that $\widetilde{W^n_{\mathcal{A}}}$ and $W^n_{\mathcal{D}}$ are both linear combinations of 3-cycles in $\mathcal{D}^n$. Hence to show they are equal it is enough to check, for every 3-cycle c in $\mathcal{D}^n$, that its coefficient $\widetilde{\lambda^n_{\mathcal{A}}}(c)$ in $\widetilde{W^n_{\mathcal{A}}}$ is equal to its coefficient $\lambda^n_{\mathcal{D}}(c)$ in $W^n_{\mathcal{D}}$.

1. Suppose $c = e_x\alpha_{i_0,j_0}\alpha_{i_1,j_1}\alpha_{i_2,j_2}$ contains three distinct arrows. In the case $j_0+j_1+j_2 \not\equiv 0$, c does not appear in $\widetilde{W^n_{\mathcal{A}}}$ by construction, so $\widetilde{\lambda^n_{\mathcal{A}}}(c) = 0 = \lambda^n_{\mathcal{D}}(c)$. Otherwise, let $i'_0, i'_1 \in \{0,1,2\}$ satisfy $i'_0 \equiv i_0 + j_0$ and $i'_1 \equiv i_1 + j_0 + j_1$. Then $c = \widetilde{d}$, where d is the cycle $e_x\alpha_{i'_0}\alpha_{i'_1}\alpha_{i_2}$ in $\mathcal{A}^n$. Hence

$$\begin{aligned}
\widetilde{\lambda^n_{\mathcal{A}}}(c) &= \frac{|\mathbb{Z}_3 \cdot d|\lambda^n_{\mathcal{A}}(d)}{3} \\
&= \lambda^n_{\mathcal{A}}(d) \\
&= \begin{cases} 1 & \text{if } (i_0 + j_0, i_1 + j_0 + j_1, i_2) \sim_{\equiv} (0,1,2) \\ -1 & \text{if } (i_0 + j_0, i_1 + j_0 + j_1, i_2) \sim_{\equiv} (0,2,1) \end{cases} \\
&= \lambda_{\mathcal{D}}(c).
\end{aligned}$$

2. Suppose $c = e_x\alpha^3_{i,j}$ for a loop $e_x\alpha_{i,j}$. Let $i', i'' \in \{0,1,2\}$ satisfy $i' \equiv i + j$ and

$i'' \equiv i + 2j$. Then $c = \widetilde{d}$, where d is the cycle $e_x\alpha_{i'}\alpha_{i''}\alpha_i$ in $\mathcal{A}^n$. Hence

$$\begin{aligned}
\widetilde{\lambda^n_{\mathcal{A}}}(c) &= \frac{|\mathbb{Z}_3 \cdot d|\lambda^n_{\mathcal{A}}(d)}{3} \\
&= \frac{\lambda^n_{\mathcal{A}}(d)}{3} \\
&= \begin{cases} \frac{1}{3} & \text{if } (i+j, i+2j, i) \sim_{\equiv} (0,1,2), \\ \\ -\frac{1}{3} & \text{if } (i+j, i+2j, i) \sim_{\equiv} (0,2,1) \end{cases} \\
&= \lambda^n_{\mathcal{D}}(c).
\end{aligned}$$

3. Suppose $n \equiv 1$ and suppose c is a 3-cycle in $\mathcal{D}^n$ containing X_k for some k. Then either $c = \alpha_1\beta_k\gamma_k = \widetilde{c_k^-}$ or $c = \alpha_{2,1}\beta_k\gamma_k = \widetilde{c_k^+}$. But

$$\begin{aligned}
\widetilde{\lambda^n_{\mathcal{A}}}(\alpha_1\beta_k\gamma_k) &= \lambda^n_{\mathcal{A}}(c^-)\zeta^{-p(c^-)k} = -1 = \lambda^n_{\mathcal{D}}(\alpha_1\beta_k\gamma_k), \\
\widetilde{\lambda^n_{\mathcal{A}}}(\alpha_{2,1}\beta_k\gamma_k) &= \lambda^n_{\mathcal{A}}(c^+)\zeta^{-p(c^+)k} = \zeta^k = \lambda^n_{\mathcal{D}}(\alpha_{2,1}\beta_k\gamma_k).
\end{aligned}$$

Therefore $(\widetilde{\mathcal{A}^n}, \widetilde{W^n_{\mathcal{A}}}) = (\mathcal{D}^n, W^n_{\mathcal{D}})$, and Theorem 2.1.7 completes the claim. □

2.2 Taking cuts

Lemma 2.2.1. *Suppose $n \not\equiv 1$. There does not exist a cut of $(\mathcal{D}^n, W^n_{\mathcal{D}})$.*

Proof. If $n = 3m$ for some $m \in \mathbb{N}_{>0}$ then there is a loop $e_x\alpha_{2,2} \in \mathcal{D}^n_1$, where $x = (m, m, m-1)$. The 3-cycle $e_x\alpha^3_{2,2}$ appears in $W^n_{\mathcal{D}}$ with non-zero coefficient. For any subset $C \subseteq \mathcal{D}^n_1$, $\deg_C(e_x\alpha^3_{2,2}) \in \{0, 3\}$, so C is not a cut.

Similarly, if $n = 3m + 2$ for some $m \in \mathbb{N}$ then there is a loop $e_x\alpha_{2,1} \in \mathcal{D}^n_1$, where $x = (m+1, m, m)$. The 3-cycle $e_x\alpha^3_{2,1}$ appears in $W^n_{\mathcal{D}}$ with non-zero coefficient. For any subset $C \subseteq \mathcal{D}^n_1$, $\deg_C(e_x\alpha^3_{2,1}) \in \{0, 3\}$, so C is not a cut. □

Definition 2.2.2. [29, §7] Let G be a finite cyclic group acting by rotations on a strongly planar QP (Q, W). A cut C of (Q, W) is *G-invariant* if $g(\alpha) \in C$ for all $g \in G$, $\alpha \in C$.

Lemma 2.2.3. *Let $n = 3m + 1$ for some $m \in \mathbb{N}_{>0}$. Then $(\mathcal{D}^n, W^n_{\mathcal{D}})$ has enough cuts.*

Proof. As usual, put $X = (m, m, m)$. Take an arrow $\delta \in \mathcal{D}^n_1$. There are three possibilities.

1. Suppose $\delta = e_x\alpha_{i,j}$ for some $x \in \mathcal{D}^n_0$, $i, j \in \{0, 1, 2\}$. In this case, define $\delta' = e_{\omega^j(x)}\alpha_i \in \mathcal{A}^n_1$.
2. Suppose $\delta = \beta_k$ for some $k \in \{0, 1, 2\}$. In this case, define $\delta' = \alpha_0 e_X \in \mathcal{A}^n_1$.
3. Suppose $\delta = \gamma_k$ for some $k \in \{0, 1, 2\}$. In this case, define $\delta' = e_X\alpha_2 \in \mathcal{A}^n_1$.

By [29, Proposition 8.2], there must exist a $\mathbb{Z}_3$-invariant cut C' of $(\mathcal{A}^n, W^n_{\mathcal{A}})$ containing δ'. Hence by [29, Proposition 7.3],

$$\begin{aligned} C = \{e_x\alpha_{i,j} \mid x \in Q^n_0,\ e_{\omega^j(x)}\alpha_i \in C\} \cup \{\beta_0, \beta_1, \beta_2 \mid \alpha_0 e_X \in C'\} \\ \cup \{\gamma_0, \gamma_1, \gamma_2 \mid e_X\alpha_2 \in C'\} \end{aligned} \tag{2.1}$$

is a cut of $(\mathcal{D}^n, W^n_{\mathcal{D}})$. Since $\delta' \in C'$, we have $\delta \in C$. Therefore $(\mathcal{D}^n, W^n_{\mathcal{D}})$ has enough cuts. □

Lemma 2.2.4. *[29, Corollary 2.6] Let G be a finite group acting on a Frobenius algebra Λ by automorphisms. If the Nakayama automorphism of Λ generates the subgroup $\mathrm{Im}(G) \subseteq \mathrm{Aut}(\Lambda)$, then $\Lambda\#G$ is symmetric.*

Proposition 2.2.5. *For each $n \geq 2$, $\Pi^n_{\mathcal{D}}$ is symmetric. If $n \equiv 1$ then for any cut C, $\Pi^n_{\mathcal{D}}/\langle C\rangle$ is 2-representation-finite, and its 3-preprojective algebra is $\Pi^n_{\mathcal{D}}$.*

Proof. By Proposition 1.4.14, $\Pi^n_{\mathcal{A}}$ is Frobenius and its Nakayama automorphism is ω, which generates $\mathrm{Im}(\mathbb{Z}_3) \subseteq \mathrm{Aut}(\Pi^n_{\mathcal{A}})$. Hence Lemma 2.2.4 shows that $\Pi^n_{\mathcal{A}}\#\mathbb{Z}_3$ is symmetric. Since being symmetric is Morita invariant, the same is true of $\Pi^n_{\mathcal{D}}$. In particular, $(\mathcal{D}^n, W^n_{\mathcal{D}})$ is a selfinjective QP, so Theorem 1.4.8 yields the second statement. □

If $n \equiv 1$, we call $\Pi^n_{\mathcal{D}}$ a 3-*preprojective algebra of type D*. Given a cut C, we call $\Pi^n_{\mathcal{D}}/\langle C\rangle$ a 2-*representation-finite algebra of type D*. The following, which is immediate from [29, Theorem 7.9], shows that the various 2-representation-finite algebras associated to $\Pi^n_{\mathcal{D}}$ are closely related.

Corollary 2.2.6. *Let $n \equiv 1$. Then $\Pi^n_{\mathcal{D}}/\langle C\rangle$ is derived equivalent to $\Pi^n_{\mathcal{D}}/\langle C'\rangle$ for any pair of cuts C, C'.*

2.3 Connection to operator algebras

In [21, 22], Evans and Pugh study Jacobian algebras of the $\mathcal{A}$ and $\mathcal{D}$ quivers (interpreted as the $\mathrm{SU}(3)$ ADE graphs of Di Francesco and Zuber [18], see also [15]) with respect to different potentials. We show that their algebras are isomorphic to the ones considered here. We use the letter V to denote the Evans-Pugh potentials, in contrast with the letter W for ours.

Fix $n \geq 2$ and write $q = e^{i\pi/(n+2)}$. For each $m \in \mathbb{Z}$ there is a *quantum number* $[m] = (q^m - q^{-m})/(q - q^{-1})$.

2.3.1 Type A

Definition 2.3.1. [21, Theorem 5.1] Define a potential on $\mathcal{A}^n$ by $V^n_{\mathcal{A}} = \sum_c \mu^n_{\mathcal{A}}(c)c$, where the sum is taken over all 3-cycles in $\mathcal{A}^n$ and, for each $x = (x_0, x_1, x_2) \in \mathcal{A}^n_0$,

$$\mu^n_{\mathcal{A}}(e_x\alpha_0\alpha_1\alpha_2) = \frac{\sqrt{[x_1+1][x_1+2][x_2+1][x_2+2][x_1+x_2+2][x_1+x_2+3]}}{[2]},$$
$$\mu^n_{\mathcal{A}}(e_x\alpha_0\alpha_2\alpha_1) = \frac{\sqrt{[x_1+1][x_1+2][x_2][x_2+1][x_1+x_2+2][x_1+x_2+3]}}{[2]}.$$

Remark 2.3.2. We believe there is a small typo in [21, Theorem 5.1]. Namely, the formula (14) should have $[k+m+3][k+m+4]$ in place of $[k+m+2][k+m+3]$. Translating into our notation gives the above definition.

Lemma 2.3.3. *[17, Proposition 3.7] Let (Q, W) be a QP and Q' be a quiver. An* *isomorphism* $f\colon kQ \to kQ'$ *induces an isomorphism* $\mathrm{J}(Q, W) \cong \mathrm{J}(Q', f(W))$.

We first replace the commutativity relations in $\Pi^n_{\mathcal{A}}$ with anti-commutativity relations, adapting the method in [30, §3.3].

Lemma 2.3.4. *Let* $|W^n_{\mathcal{A}}|$ *be the potential on* $\mathcal{A}^n$ *given by the sum of all 3-cycles, each with coefficient* 1. *Then* $\Pi^n_{\mathcal{A}} \cong \mathrm{J}(\mathcal{A}^n, |W^n_{\mathcal{A}}|)$.

Proof. Define $\mathrm{par}_i(x) = (-1)^{n-x_{i+1}}$ for all $x = (x_0, x_1, x_2) \in \mathcal{A}^n_0$ and $i \in \{0,1,2\}$, where $x_3 := x_0$. Let $f\colon \mathbb{C}\mathcal{A}^n \to \mathbb{C}\mathcal{A}^n$ be the unique algebra automorphism such that $f(e_x\alpha_i) = \mathrm{par}_i(x)e_x\alpha_i$ for all $e_x\alpha_i \in \mathcal{A}^n_1$. Consider a cycle c in $W^n_{\mathcal{A}}$. If $c = e_x\alpha_0\alpha_1\alpha_2$ for some $x \in \mathcal{A}^n_0$ then $\lambda^n_{\mathcal{A}}(c) = 1$. Hence

$$\begin{aligned} f(\lambda^n_{\mathcal{A}}(c)c) &= \mathrm{par}_0(x)\,\mathrm{par}_1(x+v_0)\,\mathrm{par}_2(x+v_0+v_1)c \\ &= (-1)^{n-x_1}(-1)^{n-x_2}(-1)^{n-(x_0+1)}c \\ &= (-1)^{2n+2}c \\ &= c, \end{aligned}$$

where we used that $x_0+x_1+x_2 = n-1$. Similarly, if $c = e_x\alpha_0\alpha_2\alpha_1$ for some $x \in \mathcal{A}^n_0$ then $\lambda^n_{\mathcal{A}}(c) = -1$. Hence

$$\begin{aligned} f(\lambda^n_{\mathcal{A}}(c)c) &= -\,\mathrm{par}_0(x)\,\mathrm{par}_2(x+v_0)\,\mathrm{par}_1(x+v_0+v_2)c \\ &= -(-1)^{n-x_1}(-1)^{n-(x_0-1)}(-1)^{n-(x_2-1)}c \\ &= -(-1)^{2n+3}c \\ &= c. \end{aligned}$$

Therefore $f(W^n_{\mathcal{A}}) = |W^n_{\mathcal{A}}|$, meaning $\Pi^n_{\mathcal{A}} \cong \mathrm{J}(\mathcal{A}^n, |W^n_{\mathcal{A}}|)$ by Lemma 2.3.3. □

Proposition 2.3.5. *For all* $n \geq 2$, $\Pi^n_{\mathcal{A}} \cong \mathrm{J}(\mathcal{A}^n, V^n_{\mathcal{A}})$.

Proof. By Lemma 2.3.4, it is enough to show that $\mathrm{J}(\mathcal{A}^n, |W^n_{\mathcal{A}}|) \cong \mathrm{J}(\mathcal{A}^n, V^n_{\mathcal{A}})$. For each $x = (x_0, x_1, x_2) \in \mathcal{A}_0^n$, let

$$\begin{aligned}
\operatorname{coef}_0(x) &= \frac{\sqrt[4]{[x_1+1][x_1+2][x_1+x_2+2][x_1+x_2+3]}}{\sqrt[3]{[2]}},\\
\operatorname{coef}_1(x) &= \frac{\sqrt[4]{[x_1][x_1+1][x_2+1][x_2+2]}}{\sqrt[3]{[2]}},\\
\operatorname{coef}_2(x) &= \frac{\sqrt[4]{[x_2][x_2+1][x_1+x_2+1][x_1+x_2+2]}}{\sqrt[3]{[2]}}.
\end{aligned}$$

Let $g\colon \mathbb{C}\mathcal{A}^n \to \mathbb{C}\mathcal{A}^n$ the the unique automorphism such that $g(e_x\alpha_i) = \operatorname{coef}_i(x)e_i\alpha_i$ for all $e_x\alpha_i \in \mathcal{A}_1^n$. Consider a cycle c in $|W_{\mathcal{A}}^n|$. If $c = e_x\alpha_0\alpha_1\alpha_2$ for some $x \in \mathcal{A}_0^n$, then

$$g(c) = \operatorname{coef}_0(x)\operatorname{coef}_1(x+v_0)\operatorname{coef}_2(x+v_0+v_1)c.$$

Note that

$$\begin{aligned}
\operatorname{coef}_0(x) &= \frac{\sqrt[4]{[x_1+1][x_1+2][x_1+x_2+2][x_1+x_2+3]}}{\sqrt[3]{[2]}},\\
\operatorname{coef}_1(x+v_0) &= \frac{\sqrt[4]{[x_1+1][x_1+2][x_2+1][x_2+2]}}{\sqrt[3]{[2]}},\\
\operatorname{coef}_2(x+v_0+v_1) &= \frac{\sqrt[4]{[x_2+1][x_2+2][x_1+x_2+2][x_1+x_2+3]}}{\sqrt[3]{[2]}}.
\end{aligned}$$

Hence

$$g(c) = \frac{\sqrt{[x_1+1][x_1+2][x_2+1][x_2+2][x_1+x_2+2][x_1+x_2+3]}}{[2]}c = \mu_{\mathcal{A}}^n(c)c.$$

Similarly, if $c = e_x\alpha_0\alpha_2\alpha_1$ for some $x \in \mathcal{A}_0^n$, then

$$g(c) = \operatorname{coef}_0(x)\operatorname{coef}_2(x+v_0)\operatorname{coef}_1(x+v_0+v_2)c.$$

Note that

$$\begin{aligned}
\operatorname{coef}_2(x+v_0) &= \frac{\sqrt[4]{[x_2][x_2+1][x_1+x_2+2][x_1+x_2+3]}}{\sqrt[3]{[2]}},\\
\operatorname{coef}_1(x+v_0+v_2) &= \frac{\sqrt[4]{[x_1+1][x_1+2][x_2][x_2+1]}}{\sqrt[3]{[2]}}.
\end{aligned}$$

Hence

$$g(c) = \frac{\sqrt{[x_1+1][x_1+2][x_2][x_2+1][x_1+x_2+2][x_1+x_2+3]}}{[2]}c = \mu_{\mathcal{A}}^n(c)c.$$

Therefore $g(|W_{\mathcal{A}}^n|) = V_{\mathcal{A}}^n$, meaning $\mathrm{J}(\mathcal{A}^n, |W_{\mathcal{A}}^n|) \cong \mathrm{J}(\mathcal{A}^n, V_{\mathcal{A}}^n)$ by Lemma 2.3.3. □

2.3.2 Type D

The only way $(\mathcal{A}^n, V_{\mathcal{A}}^n)$ differs from $(\mathcal{A}^n, W_{\mathcal{A}}^n)$ is that the coefficient of each cycle in the potential has been multiplied by a non-zero scalar. In particular, they have the same canvas. Hence $(\mathcal{A}^n, V_{\mathcal{A}}^n)$ is a strongly planar QP on which $\mathbb{Z}_3$ acts by rotations, so we may apply Theorem 2.1.7. Rescaling the coefficients does not affect the construction of the quiver, nor of the idempotent, nor does it change which cycles appear in the potential. Hence we obtain a potential

$$\widetilde{V_{\mathcal{A}}^n} = \sum_{c \in C_1} \frac{|\mathbb{Z}_3 \cdot c|\mu_{\mathcal{A}}^n(c)}{3}\widetilde{c} + \sum_{c \in C_2} \mu_{\mathcal{A}}^n(c) \sum_{k=0}^{2} \zeta^{-p(c)k}\widetilde{c_k}$$

such that $\mathrm{J}(\mathcal{D}^n, \widetilde{V_{\mathcal{A}}^n}) \cong \eta(\mathrm{J}(\mathcal{A}^n, V_{\mathcal{A}}^n)\#\mathbb{Z}_3)\eta$, where we slightly abuse notation and consider η as an element of $\mathrm{J}(\mathcal{A}^n, V_{\mathcal{A}}^n)\#\mathbb{Z}_3$.

Definition 2.3.6. [21, Theorem 6.1-2] Define a potential on $\mathcal{D}^n$ by $V_{\mathcal{D}}^n = \sum_c \mu_{\mathcal{D}}^n(c)c$, where the sum is taken over all 3-cycles in $\mathcal{D}^n$, and $\mu_{\mathcal{D}}^n$ is defined as follows.

If $n = 3m+1$ for some $m \in \mathbb{N}_{>0}$ then, for each $k \in \{0,1,2\}$,

$$\mu_{\mathcal{D}}^n(\alpha_1\beta_k\gamma_k) = \zeta^k \frac{[m]\sqrt{[m+1]^3[m+2]}}{\sqrt{3}[2]},$$
$$\mu_{\mathcal{D}}^n(\alpha_{2,1}\beta_k\gamma_k) = \zeta^{-k} \frac{[m+2]\sqrt{[m][m+1]^3}}{\sqrt{3}[2]}.$$

For all other cycles c (including when $n \not\equiv 1$), $\mu_{\mathcal{D}}^n(c) = \widetilde{\mu_{\mathcal{A}}^n}(c)$.

Remark 2.3.7. In [21, Theorem 6.1-2], the authors explicitly give the coefficients of six cycles, and say that the rest are given by the coefficients of the corresponding cycles in $V_{\mathcal{A}}^n$. If we assume that $\mu_{\mathcal{D}}^n(c) = \widetilde{\mu_{\mathcal{A}}^n}(c)$ for all c, we already get the desired coefficients for four of the distinguished cycles, so we omit them from the definition.

Lemma 2.3.8. *For all* $n \geq 2$, $\mathrm{J}(\mathcal{D}^n, \widetilde{V^n_{\mathcal{A}}}) \cong \mathrm{J}(\mathcal{D}^n, V^n_{\mathcal{D}})$.

Proof. If $n \not\equiv 1$ then $\widetilde{V^n_{\mathcal{A}}} = V^n_{\mathcal{D}}$, so assume $n = 3m+1$ for some $m \in \mathbb{N}_{>0}$. Let $h \colon \mathbb{C}\mathcal{D}^n \to \mathbb{C}\mathcal{D}^n$ be the unique automorphism such that $h(\beta_k) = \frac{\zeta^k}{\sqrt{3}}\beta_k$ for each $k \in \{0,1,2\}$, and such that h is the identity on all other arrows.

If $c = \alpha_1\beta_k\gamma_k$ then $c = \widetilde{c^-_k}$, where $c^- = e_{(m+1,m-1,m)}\alpha_0\alpha_2\alpha_1$. Hence

$$\begin{aligned} h(\widetilde{\mu^n_{\mathcal{A}}}(c)) &= \zeta^{k-p(c^-)k}\frac{\mu^n_{\mathcal{A}}(c^-)}{\sqrt{3}}c \\ &= \zeta^k\frac{[m][m+1]\sqrt{[2m+1][2m+2]}}{\sqrt{3}[2]}c \\ &= \zeta^k\frac{[m]\sqrt{[m+1]^3[m+2]}}{\sqrt{3}[2]}c \\ &= \mu^n_{\mathcal{D}}(c)c, \end{aligned}$$

where we used that $p(c^-) = 0$, and that $[l] = [3m+3-l]$ for all $1 \leq l \leq 2m+2$ [21, Lemma 4.3]. Similarly, if $c = \alpha_{2,1}\beta_k\gamma_k$ then $c = \widetilde{c^+_k}$, where $c^+ = e_{(t+1,t,t-1)}\alpha_0\alpha_1\alpha_2$. Recalling $p(c^+) = -1$, we have

$$\begin{aligned} h(\widetilde{\mu^n_{\mathcal{A}}}(c)c) &= \zeta^{k-p(c^+)k}\frac{\mu^n_{\mathcal{A}}(c^+)}{\sqrt{3}}c \\ &= \zeta^{2k}\frac{[m+1]\sqrt{[m][m+2][2m+1][2m+2]}}{\sqrt{3}[2]}c \\ &= \zeta^{-k}\frac{[m+2]\sqrt{[m][m+1]^3}}{\sqrt{3}[2]}c \\ &= \mu^n_{\mathcal{D}}(c). \end{aligned}$$

Therefore $h(\widetilde{V^n_{\mathcal{A}}}) = V^n_{\mathcal{D}}$, meaning $\mathrm{J}(\mathcal{D}^n, \widetilde{V^n_{\mathcal{A}}}) \cong \mathrm{J}(\mathcal{D}^n, V^n_{\mathcal{D}})$ by Lemma 2.3.3. □

Proposition 2.3.9. *For all* $n \geq 2$, $\Pi^n_{\mathcal{D}} \cong \mathrm{J}(\mathcal{D}^n, V^n_{\mathcal{D}})$.

Proof. Recall the maps f and g from the proofs of Lemma 2.3.4 and Proposition 2.3.5. They both act as the identity on length-zero idempotents, so we have an isomorphism $gf \otimes \mathrm{id} \colon \Pi^n_{\mathcal{A}}\#\mathbb{Z}_3 \to \mathrm{J}(\mathcal{A}^n, V^n_{\mathcal{A}})\#\mathbb{Z}_3$ which preserves η. Thus we have a chain of isomorphisms

$$\Pi^n_{\mathcal{D}} \cong \eta(\Pi^n_{\mathcal{A}}\#\mathbb{Z}_3)\eta \overset{gf\otimes\mathrm{id}}{\to} \eta(\mathrm{J}(\mathcal{A}^n, V^n_{\mathcal{A}})\#\mathbb{Z}_3)\eta \cong \mathrm{J}(\mathcal{D}^n, \widetilde{V^n_{\mathcal{A}}}) \to \mathrm{J}(\mathcal{D}^n, V^n_{\mathcal{D}}),$$

where the last map is h if $n \equiv 1$, and id otherwise (see the proof of Lemma 2.3.8).

□

Remark 2.3.10. In [22], the authors determine the Nakayama automorphisms of $\mathrm{J}(\mathcal{A}^n, V^n_{\mathcal{A}})$ and $\mathrm{J}(\mathcal{D}^n, V^n_{\mathcal{D}})$. Thus, the above result provides an alternative way to prove Proposition 2.2.5.

Chapter 3

2-Representation-finite algebras of type D

3.1 The fractional Calabi-Yau property

Let $\Lambda = \mathrm{J}(Q, W)_C$ be a basic 2-representation-finite algebra, and let σ be the Nakayama automorphism of its 3-preprojective algebra $\Pi = \mathrm{J}(Q, W)$. Then for all $i \in Q_0$, $(\Pi e_i)^* \cong \sigma(e_i)\Pi$ as Π-modules. By [42, Proposition 1.3], there exists a function $l \colon Q_0 \to \mathbb{Z}$ such that $(\Lambda e_i)^* \cong \tau_2^{-l(i)}(\sigma(e_i)\Lambda)$ for all $i \in Q_0$. Equipping Π with the tensor grading, [31, Proposition 3.2, §4] tells us that for all $i \in Q_0$, $(\Pi e_i)^* \cong \sigma(e_i)\Pi\{l(i)\}$ as graded Π-modules.

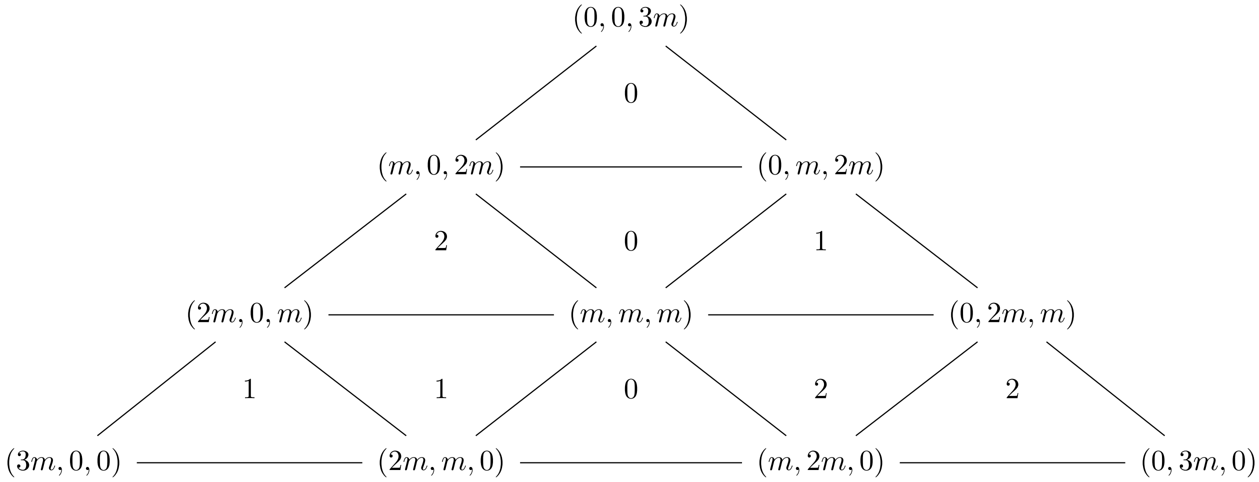
The following is an amalgamation of results due to Grant.

Proposition 3.1.1. *Maintaining the above notation, suppose there exists $k \in \mathbb{N}_{>0}$ such that $\sigma^k = \mathrm{id}$. Then*

1. *there exists $N \in \mathbb{Z}$ such that for all $i \in Q_0$, $\sum_{j=1}^{k} l(\sigma^j(i)) = N$;*
2. *Λ is $(2N, k+N)$-fractional Calabi-Yau.*

Proof. In [31, §3.4], it is demonstrated that if $f \in \mathrm{Hom}_\Lambda(e_i\Lambda, \tau_2^{-r}(e_j\Lambda))$, then we have $\sigma(f) \in \operatorname{Hom}_\Lambda(\sigma(e_i)\Lambda, \tau_2^{l(i)-l(j)-r}(\sigma(e_j)\Lambda))$, meaning (σ, l) is a *degree-adjusted automorphism* in the sense of [32, Definition 4.19]. Hence [32, Lemma 6.15] yields the first statement. In fact, (σ, l) is a tr-*graded Nakayama automorphism* in the sense of [32, Definition 4.21], so [32, Theorem 6.14] completes the result. □

Definition 3.1.2. Let $n = 3m+1$. Let K' be the $\mathbb{Z}_3$-invariant cut of $(\mathcal{A}^n, W^n_{\mathcal{A}})$ defined by the following diagram.



Here, the label i means that all arrows α_i in that region (including those on the edge) should be cut. Let K be the induced cut of $(\mathcal{D}^n, W^n_{\mathcal{D}})$.

See Figure 3.1 for K' and K in the case $m = 2$.

Remark 3.1.3. Note that all arrows in K' appear in the leftmost set of (2.1), see the proof of Lemma 2.2.3. Hence $e_x\alpha_{i,j} \in K \iff e_{\omega^j(x)}\alpha_i \in K'$.

Recall that the *socle* $\operatorname{Soc}(M)$ of a module M is its maximal semisimple submodule.

Lemma 3.1.4. *Let* $n = 3m+1$ *and let* $x = (3m, 0, 0)$. *Then* $\operatorname{Soc}(e_x\Pi^n_{\mathcal{D}}) = \langle p \rangle$, *where*

$$p = \begin{cases} e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2^{(3m-1)/2} & \text{if } m \text{ is odd,} \\ e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^{(3m-2)/2} & \text{if } m \text{ is even.} \end{cases}$$

In particular, $\deg_K(p) = m$.

The proof proceeds by recursively computing the space $e_x\Pi^n_{\mathcal{D}}(k)$ of paths starting at

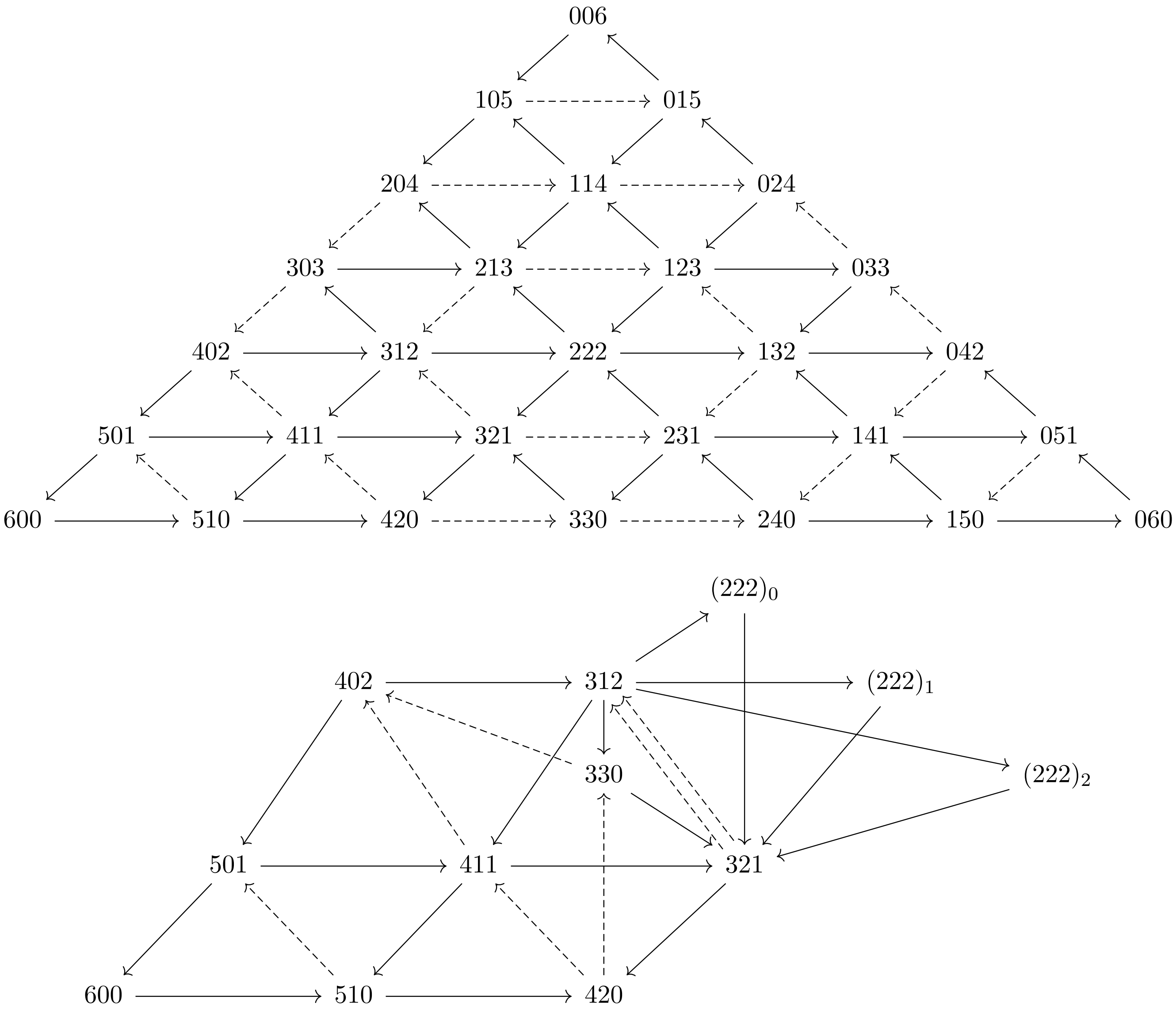


Figure 3.1: The $\mathbb{Z}_3$-invariant cut K' of $\mathcal{A}^7$ and the induced cut K of $\mathcal{D}^7$, indicated by the dashed arrows.

x of length k. The $m = 1$ case is slightly different, so we treat it separately.

Example 3.1.5. Let $m = 1$, so the distinguished cut is $K = \{\alpha_1, \alpha_{2,1}\}$. Clearly, $e_x\Pi^4_{\mathcal{D}}(0) = \langle e_x\rangle$ and $e_x\Pi^4_{\mathcal{D}}(1) = \langle \alpha_0\rangle$. One can extend this to a path of length 2 by following α_1 or $\alpha_{2,1}$. Note that $\alpha_0\alpha_1 = 0$, while there is no relation reducing $\alpha_0\alpha_{2,1}$. Hence $e_x\Pi^4_{\mathcal{D}}(2) = \langle \alpha_0\alpha_{2,1}\rangle$. One can extend this to a path of length 3 by following α_2 or β_k, for some $k \in \{0, 1, 2\}$. Note that $\alpha_0\alpha_{2,1}\beta_k = \frac{1}{\zeta^k}\alpha_0\alpha_1\beta_k = 0$, while there is no relation reducing $\alpha_0\alpha_{2,1}\alpha_2$. Hence $e_x\Pi^4_{\mathcal{D}}(3) = \langle \alpha_0\alpha_{2,1}\alpha_2\rangle$. The only way to extend this path is by following α_0, but

$$\alpha_0\alpha_{2,1}\alpha_1\alpha_0 = \alpha_0\alpha_{2,1}\sum_{k=0}^{2}\beta_k\gamma_k = \sum_{k=0}^{2}\frac{1}{\zeta_k}\alpha_0\alpha_1\beta_k = 0.$$

Therefore $\mathrm{Soc}(e_x\Pi^4_{\mathcal{D}})$ is generated by $\alpha_0\alpha_{2,1}\alpha_2$, which has degree 1 as claimed.

Proof of Lemma 3.1.4. Let $m \geq 2$. To clean up the notation, write $\Pi = \mathrm{J}(\mathcal{D}, W)$ in place of $\Pi^n_{\mathcal{D}} = \mathrm{J}(\mathcal{D}^n, W^n_{\mathcal{D}})$. Clearly, $e_x\Pi(0) = \langle e_x\rangle$ and $e_x\Pi(1) = \langle e_x\alpha_0\rangle$. There are two ways to extend this to a path of length 2.

$$\begin{tikzcd}
 & (3m-1,0,1) \arrow[dl, "2"'] \arrow[rr, "0"] & & (3m-2,1,1) \arrow[dl, "2"] & \\
x = (3m,0,0) \arrow[rr, "0"'] & & (3m-1,1,0) \arrow[ul, dashed, "1"'] \arrow[rr, dashed, "0"'] & & (3m-2,2,0)
\end{tikzcd}$$

Note that $\partial_{\alpha_2 e_x}W = e_x\alpha_0\alpha_1$, so $e_x\alpha_0\alpha_1 = 0$. There is no relation reducing $e_x\alpha_0^2$, since the two arrows lie in 3-cycles which do not share a common edge. So certainly $e_x\alpha_0^2 \neq 0$, and this path generates $e_x\Pi(2)$.

Case 1: m is even.

Let $2 \leq k < \frac{3m}{2}$ and assume that $e_x\Pi(k) = \langle e_x\alpha_0^k\rangle$. There are two ways to extend this to a path of length $k + 1$.

$$\begin{tikzcd}
y+f_1 \arrow[rr, "0"] & & y-f_2 \arrow[dl, "2"'] & & \\
& y=(3m-k+1,k-1,0) \arrow[ul, "1"] \arrow[rr, "0"'] & & y+f_0 \arrow[ul, dashed, "1"'] \arrow[r, dashed, "0"'] & y+2f_0
\end{tikzcd}$$

Note that $\partial_{\alpha_2 e_y} W = e_y(\alpha_0\alpha_1 - \alpha_1\alpha_0)$, so $e_x\alpha_0^k\alpha_1 = e_x\alpha_0^{k-1}\alpha_1\alpha_0$, which is 0 by assumption. As before, there is no relation reducing $e_x\alpha_0^{k+1}$, so this path generates $e_x\Pi(k+1)$. By induction, $e_x\Pi(k) = \langle e_x\alpha_0^k\rangle$ for all $k \leq \frac{3m}{2}$.

There are two ways to extend $e_x{\alpha_0}^{3m/2}$ to a path of length $\frac{3m+2}{2}$.

$$\begin{tikzcd}
(\frac{3m+2}{2},0,\frac{3m-2}{2}) & & \\
& (\frac{3m}{2},\frac{3m}{2},0) \arrow[ul, dashed, "(2,1)"'] \arrow[dr, dashed, "1"] & \\
(\frac{3m+2}{2},\frac{3m-4}{2},1) \arrow[rr, "0"] & & (\frac{3m}{2},\frac{3m-2}{2},1) \arrow[dl, "2"] \\
& z=(\frac{3m+2}{2},\frac{3m-2}{2},0) \arrow[ul, "1"] \arrow[uu, "0"] &
\end{tikzcd}$$

Note that $\partial_{\alpha_2 e_z} W = e_z(\alpha_0\alpha_1 - \alpha_1\alpha_0)$, so $e_x{\alpha_0}^{3m/2}\alpha_1 = e_x{\alpha_0}^{(3m-2)/2}\alpha_1\alpha_0 = 0$. There is no relation reducing $e_x{\alpha_0}^{3m/2}\alpha_{2,1}$, so this path generates $e_x\Pi(\frac{3m+2}{2})$.

There are two ways to extend $e_x{\alpha_0}^{3m/2}\alpha_{2,1}$ to a path of length $\frac{3m+4}{2}$.

$$\begin{tikzcd}
& (\frac{3m+2}{2},0,\frac{3m-2}{2}) \arrow[rr, dashed, "0"] \arrow[dl, dashed, "2"'] & & (\frac{3m}{2},1,\frac{3m-2}{2}) \arrow[dl, "(2,2)"'] & \\
(\frac{3m+4}{2},0,\frac{3m-4}{2}) & & w=(\frac{3m}{2},\frac{3m}{2},0) \arrow[ul, "(2,1)"] \arrow[rr, "1"'] & & (\frac{3m}{2},\frac{3m-2}{2},1) \arrow[ul, shift left, "(2,1)"] \arrow[ul, shift right, "1"']
\end{tikzcd}$$

Note that $\partial_{\alpha_{2,2}} W = e_y(\alpha_{2,1}\alpha_0 - \alpha_1\alpha_{2,1})$, so $e_x{\alpha_0}^{3m/2}\alpha_{2,1}\alpha_0 = e_x{\alpha_0}^{3m/2}\alpha_1\alpha_{2,1} = 0$. There is no relation reducing $e_x{\alpha_0}^{3m/2}\alpha_{2,1}\alpha_2$, so this path generates $e_x\Pi(\frac{3m+4}{2})$.

Let $1 \leq k < \frac{3m-2}{2}$ and assume that $e_x\Pi(\frac{3m+2}{2}+k) = \langle e_x{\alpha_0}^{3m/2}\alpha_{2,1}\alpha_2^k\rangle$. There are two ways to extend this to a path of length $\frac{3m+2}{2}+k+1$.

$$\begin{tikzcd}
 & & v = (\frac{3m+2}{2}+k-1, 0, \frac{3m-2}{2}-k+1) \arrow[rr, "0"] \arrow[dl, "2"'] & & v+f_0 \arrow[dl, "2"] \\
 & v+f_2 \arrow[rr, dashed, "0"'] \arrow[dl, dashed, "2"'] & & v-f_1 \arrow[ul, "1"'] & \\
v+2f_2 & & & &
\end{tikzcd}$$

Note that $\partial_{\alpha_1 e_v} W = e_v(\alpha_2\alpha_0 - \alpha_0\alpha_2)$, so $e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^k\alpha_0 = e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^{k-1}\alpha_0\alpha_2$, which is 0 by assumption. There is no relation reducing $e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^{k+1}$, so this path generates $e_x\Pi(\frac{3m+2}{2}+k+1)$. By induction, $e_x\Pi(\frac{3m+2}{2}+k) = \langle e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^k\rangle$ for all $k \leq \frac{3m-2}{2}$.

Now, $t(e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^{(3m-2)/2}) = x$. There is only one arrow with source x, namely $e_x\alpha_0$ (see start of proof). But since $\partial_{\alpha_1 e_{(3m-1,0,1)}} W = e_{(3m-1,0,1)}(\alpha_0\alpha_2 - \alpha_2\alpha_0)$, we have that $e_x\alpha_0^{3m/2}\alpha_{2,1}\alpha_2^{(3m-2)/2}\alpha_0 = 0$. Hence $e_x\Pi(k) = 0$ for all $k > 3m$, and $\mathrm{Soc}(e_x\Pi)$ is generated by the path p as claimed. By Remark 3.1.3, we have that $\deg_K(e_x\alpha_{i,j}) = \deg_{K'}(e_{\omega^j(x)}\alpha_i)$. Hence

$$\deg_K(p) = \deg_{K'}(e_x\alpha_0^{3m/2}) + \deg_{K'}(\alpha_2^{3m/2}e_x).$$

By definition of K',

$$\deg_{K'}(e_x\alpha_0^{3m/2}) = \deg_{K'}(\alpha_2^{3m/2}e_x) = \frac{m}{2}.$$

Therefore we indeed have $\deg_K(p) = m$.

Case 2: m is odd.

Arguing as in Case 1, one finds that $e_x\Pi(\frac{3m-1}{2}) = \langle e_x\alpha_0^{(3m-1)/2}\rangle$. There are two ways to extend this to a path of length $\frac{3m+1}{2}$.

$$\begin{tikzcd}
 & & (\frac{3m+1}{2}, 0, \frac{3m-1}{2}) & \\
(\frac{3m+3}{2}, \frac{3m-5}{2}, 1) \arrow[rr, "0"] & & (\frac{3m+1}{2}, \frac{3m-3}{2}, 1) \arrow[dl, "2"] & \\
 & y = (\frac{3m+3}{2}, \frac{3m-3}{2}, 0) \arrow[ul, "1"] \arrow[rr, "0"'] & & (\frac{3m+1}{2}, \frac{3m-1}{2}, 0) \arrow[ul, dashed, "1"'] \arrow[uul, dashed, "{(2,1)}"']
\end{tikzcd}$$

Note that $\partial_{\alpha_2 e_y} W^n_{\mathfrak{D}} = e_y(\alpha_0\alpha_1 - \alpha_1\alpha_0)$, so $e_x\alpha_0^{(3m-1)/2}\alpha_1 = e_x\alpha_0^{(3m-3)/2}\alpha_1\alpha_0 = 0$. There is no relation reducing $e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}$, so this path generates $e_x\Pi(\frac{3m+1}{2})$.

There are two ways to extend $e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}$ to a path of length $\frac{3m+3}{2}$.

$$\begin{tikzcd}
 & (\frac{3m+1}{2}, 0, \frac{3m-1}{2}) \arrow[dl, dashed, "2"'] \arrow[ddrr, dashed, "{(1,2)}"] & & \\
(\frac{3m+3}{2}, 0, \frac{3m-3}{2}) & & & \\
 & (\frac{3m+1}{2}, \frac{3m-3}{2}, 1) \arrow[rr, "0"] & & (\frac{3m-1}{2}, \frac{3m-1}{2}, 1) \arrow[dl, "2"] \\
 & & z = (\frac{3m+1}{2}, \frac{3m-1}{2}, 0) \arrow[ul, "1"] \arrow[uuul, "{(2,1)}"] &
\end{tikzcd}$$

Note that $\partial_{\alpha_2 e_z} W^n_{\mathfrak{D}} = e_z(\alpha_{2,1}\alpha_{1,2} - \alpha_1\alpha_0)$, so

$$e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_{1,2} = e_x\alpha_0^{(3m-1)/2}2\alpha_1\alpha_0 = 0.$$

There is no relation reducing $e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2$, so this path generates $e_x\Pi(\frac{3m+3}{2})$.

There are two ways to extend $e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2$ to a path of length $\frac{3m+5}{2}$.

$$\begin{tikzcd}
 & & w = (\frac{3m+1}{2}, 0, \frac{3m-1}{2}) \arrow[dl, "2"'] \arrow[drr, bend left, "{(1,2)}"] & & \\
 & (\frac{3m+3}{2}, 0, \frac{3m-3}{2}) \arrow[rr, dashed, "0"'] \arrow[dl, dashed, "2"'] & & (\frac{3m+1}{2}, 1, \frac{3m-3}{2}) \arrow[ul, "1"'] & (\frac{3m-1}{2}, \frac{3m-1}{2}, 1) \arrow[l, bend left, "{(2,1)}"] \\
(\frac{3m+5}{2}, 0, \frac{3m-5}{2}) & & & &
\end{tikzcd}$$

Note that $\partial_{\alpha_1 e_w} W_{\mathcal{D}}^n = e_w(\alpha_2\alpha_0 - \alpha_{1,2}\alpha_{2,1})$, so

$$e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2\alpha_0 = e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_{1,2}\alpha_{2,1} = 0.$$

There is no relation reducing $e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2^2$, so this path generates $e_x\Pi(\frac{3m+5}{2})$.

Arguing as in Case 1, one finds that $e_x\Pi(3m) = \langle e_x\alpha_0^{(3m-1)/2}\alpha_{2,1}\alpha_2^{(3m-1)/2}\rangle$, while $e_x\Pi(k) = 0$ for all $k > 3m$. Hence $\operatorname{Soc}(e_x\Pi)$ is generated by the path p as claimed. Using Remark 3.1.3,

$$\deg_K(p) = \deg_{K'}(e_x\alpha_0^{(3m-1)/2}) + \deg_{K'}(\alpha_2^{(3m+1)/2}e_x).$$

By definition of K',

$$\deg_{K'}(e_x\alpha_0^{(3m-1)/2}) = \frac{m-1}{2},\ \deg_{K'}(\alpha_2^{(3m+1)/2}e_x) = \frac{m+1}{2}.$$

Therefore we indeed have $\deg_K(p) = m$. ☐

Theorem 3.1.6. *Let $n = 3m+1$. For any cut C, $\Pi_{\mathcal{D}}^n/\langle C\rangle$ is $(2m, m+1)$-fractional Calabi-Yau.*

Proof. By Corollary 2.2.6, it is enough to prove the statement for $C = K$. Proposition 3.1.1(1) with $\sigma = \mathrm{id}$ and $k = 1$ implies the function $l\colon \mathcal{D}_0^n \to \mathbb{Z}$ takes some constant value N. Again writing $x = (3m, 0, 0)$, we have that $(\Pi_{\mathcal{D}}^n e_x)^* \cong e_x\Pi_{\mathcal{D}}^n\{N\}$ as graded $\Pi_{\mathcal{D}}^n$-modules. In Lemma 3.1.4 we show that $\operatorname{Soc}(e_x\Pi_{\mathcal{D}}^n)$ is generated by a path p of degree m, so $e_x\Pi_{\mathcal{D}}^n$ exists in degrees 0 through m. From this we deduce that $(\Pi_{\mathcal{D}}^n e_x)^*$ exists in degrees $-m$ through 0, and that $N = m$. Therefore $\Pi_{\mathcal{D}}^n/\langle K\rangle$ is $(2m, m+1)$-fractional Calabi-Yau by Proposition 3.1.1(2). ☐

Recall the discussion at the start of this section. If the function $l\colon Q_0 \to \mathbb{Z}$ takes some constant value N, then Λ is called $(N+1)$-*homogeneous* [37, Definition 1.2]. The following is immediate from the proof of Theorem 3.1.6.

Corollary 3.1.7. *Let $n = 3m+1$. For any cut C, $\Pi_{\mathcal{D}}^n/\langle C\rangle$ is $(m+1)$-homogeneous.*

3.2 2-Auslander-Reiten quivers

We give recipes to construct 2-Auslander-Reiten quivers for a basic 2-representation-finite algebra $\Lambda = \mathrm{J}(Q,W)_C$. Recall the function $l\colon Q_0 \to \mathbb{Z}$ from §3.1.

Proposition 3.2.1. *The* 2*-Auslander-Reiten quiver of* $\mathcal{U}_\Lambda \subset \mathbf{D}^{\mathrm{b}}(\Lambda)$ *is isomorphic to the quiver* Γ *with vertices*

$$\Gamma_0 = \{(x,i) \mid x \in Q_0,\ i \in \mathbb{Z}\}$$

and arrows

$$\begin{aligned}\Gamma_1 =& \{(x,i) \longrightarrow (y,i) \mid (y \longrightarrow x) \in Q_1 \backslash C,\ i \in \mathbb{Z}\}\\ &\cup \{(x,i) \longrightarrow (y,i+1) \mid (y \longrightarrow x) \in C,\ i \in \mathbb{Z}\}.\end{aligned}$$

The 2*-Auslander-Reiten quiver of* $\mathcal{M}_\Lambda \subset \operatorname{mod}\Lambda$ *is isomorphic to the full subquiver* Γ' *of* Γ *with vertices*

$$\Gamma'_0 = \{(x,i) \mid x \in Q_0,\ 0 \leq i \leq l(x)\}.$$

Proof. It follows from Proposition 1.5.4(1), and the fact $\mathbb{S}^-((\Lambda e_x)^*) \cong e_x\Lambda$ for all $x \in Q_0$, that $\mathbb{S}_2^{-i}(e_x\Lambda) \cong \tau_2^{-i}(e_x\Lambda)$ if $0 \leq i \leq l(x)$, while $\mathbb{S}_2^{-i}(e_x\Lambda)$ is concentrated in a single non-zero degree otherwise. From this, using that Λ is basic, one deduces that the isoclasses of indecomposable objects of $\mathcal{U}_\Lambda = \operatorname{add}\{\mathbb{S}_2^{-i}(e_x\Lambda) \mid x \in Q_0,\ i \in \mathbb{Z}\}$ (i.e. the vertices of its 2-AR quiver) are in bijection with the elements of Γ_0.

Note that $\Pi(\Lambda) \cong \bigoplus_{r\geq 0} \operatorname{Hom}_{\mathcal{U}_\Lambda}(\Lambda, \mathbb{S}_2^{-r}\,\Lambda)$ [37, §2]. Hence for all $x, y \in Q_0$, $i, j \geq 0$,

$$\operatorname{Hom}_{\mathcal{U}_\Lambda}(\mathbb{S}_2^{-i}(e_x\Lambda), \mathbb{S}_2^{-j}(e_y\Lambda)) \cong \operatorname{Hom}_{\mathcal{U}_\Lambda}(e_x\Lambda, \mathbb{S}_2^{i-j}(e_y\Lambda)) \cong e_y\Pi^{j-i}e_x.$$

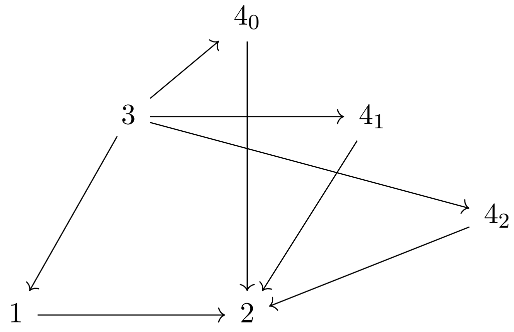
Suppose $j = i$. Then $\operatorname{Hom}_{\mathcal{U}_\Lambda}(\mathbb{S}_2^{-i}(e_x\Lambda), \mathbb{S}_2^{-i}(e_y\Lambda)) \cong e_y\Pi^0 e_x \cong e_y\Lambda e_x$ has basis consisting of paths from y to x in Λ. In particular, the irreducible morphisms from $\mathbb{S}_2^{-i}(e_x\Lambda)$ to $\mathbb{S}_2^{-i}(e_y\Lambda)$ in $\mathcal{U}_\Lambda$ (i.e. arrows $(x,i) \longrightarrow (y,i)$ in the 2-AR quiver) are in bijection with arrows $y \longrightarrow x$ in $Q_1 \backslash C$.

Suppose $j = i + 1$. Then $\mathrm{Hom}_{\mathscr{U}_\Lambda}(\mathbb{S}_2^{-i}(e_x\Lambda), \mathbb{S}_2^{-(i+1)}(e_y\Lambda)) \cong e_y\Pi^1 e_x$ has basis consisting of degree 1 paths from y to x in Π. In particular, the irreducible morphisms from $\mathbb{S}_2^{-i}(e_x\Lambda)$ to $\mathbb{S}_2^{-(i+1)}(e_y\Lambda)$ in $\mathscr{U}_\Lambda$ (i.e. arrows $(x, i) \longrightarrow (y, i+1)$ in the 2-AR quiver) are in bijection with arrows $y \longrightarrow x$ in C.

There are no more arrows in Γ_1. Indeed, if $j < i$ then $\Pi^{j-i} = 0$ since Π is non-negatively graded. If $j > i + 1$ then no morphism in $\mathrm{Hom}_{\mathscr{U}_\Lambda}(\mathbb{S}_2^{-i}\Lambda, \mathbb{S}_2^{-j}\Lambda)$ is irreducible: it corresponds to a path in Π of degree $j - 1 \geq 2$, which can always be factored into two paths of smaller degree since Π is generated in degree 1.

Finally, since $\tau_2^{-i}(e_x\Lambda) = 0$ for all $i > l(x)$, the isoclasses of indecomposable objects of $\mathscr{M}_\Lambda = \mathrm{add}\{\tau_2^{-i}\Lambda \mid i \geq 0\}$ (i.e. the vertices of its 2-AR quiver) are in bijection with the elements of Γ_0'. □

Example 3.2.2. Let $\Lambda = \Pi_{\mathbb{D}}^4/\langle C\rangle$, where $C = \{\alpha_1, \alpha_{2,1}\}$ is the pair of parallel arrows. To simplify notation, relabel as follows.



Using the definition $\tau_2^- = \tau^-\Omega^-$, one computes that

$$e_1\Lambda = \begin{smallmatrix}1\\2\end{smallmatrix},\ e_2\Lambda = 2,\ e_3\Lambda = \begin{smallmatrix}3\\14_04_14_2\\22\end{smallmatrix},\ e_{4_k}\Lambda = \begin{smallmatrix}4_k\\2\end{smallmatrix},$$

$$\tau_2^-(e_1\Lambda) = \begin{smallmatrix}3\\1\end{smallmatrix},\ \tau_2^-(e_2\Lambda) = \begin{smallmatrix}33\\14_04_14_2\\2\end{smallmatrix},\ \tau_2^-(e_3\Lambda) = 3,\ \tau_2^-(e_{4_k}) = \begin{smallmatrix}3\\4_k\end{smallmatrix}.$$

The 2-AR quivers of $\mathscr{U}_\Lambda \subset \mathbf{D}^{\mathrm{b}}(\Lambda)$ and $\mathscr{M}_\Lambda \subset \mathrm{mod}\,\Lambda$ are given in Figure 3.2.

The derived 2-Auslander-Reiten translate $\mathbb{S}_2^-$ acts by moving one step to the right,

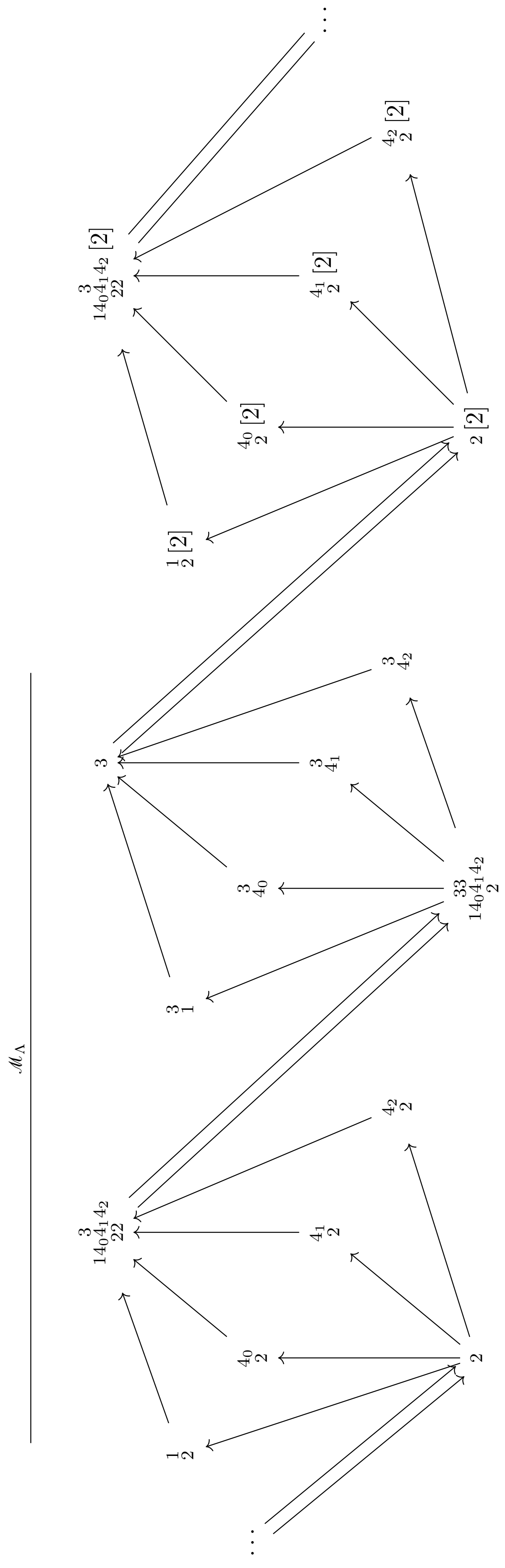


Figure 3.2: The 2-AR quiver of $\mathscr{U}_\Lambda \subset \mathbf{D}^{\mathrm{b}}(\Lambda)$ from Example 3.2.2. Restricting to the indicated region gives the 2-AR quiver of $\mathscr{M}_\Lambda \subset \operatorname{mod} \Lambda$.

so has orbits as follows (c.f. Λ is $(2,2)$-fractional Calabi-Yau).

$$\begin{gathered}
\cdots \mapsto 2 \mapsto \begin{smallmatrix} 33 \\ 14_0 4_1 4_2 \\ 2 \end{smallmatrix} \mapsto 2\,[2] \mapsto \cdots \\
\cdots \mapsto \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \mapsto \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \mapsto \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}[2] \mapsto \cdots \\
\cdots \mapsto \begin{smallmatrix} 4_k \\ 2 \end{smallmatrix} \mapsto \begin{smallmatrix} 3 \\ 4_k \end{smallmatrix} \mapsto \begin{smallmatrix} 4_k \\ 2 \end{smallmatrix}[2] \mapsto \cdots \\
\cdots \mapsto \begin{smallmatrix} 3 \\ 14_0 4_1 4_2 \\ 22 \end{smallmatrix} \mapsto 3 \mapsto \begin{smallmatrix} 3 \\ 14_0 4_1 4_2 \\ 22 \end{smallmatrix}[2] \mapsto \cdots
\end{gathered}$$

In general, if $\Lambda = \Pi^n_{\mathcal{D}}/\langle C \rangle$ for some $n = 3m+1$ and cut C, $\mathbb{S}_2^-$ has an orbit

$$\cdots \mapsto e_x\Lambda \mapsto \tau_2^-(e_x\Lambda) \mapsto \cdots \mapsto \tau_2^{1-m}(e_x\Lambda) \mapsto (\Lambda e_x)^* \mapsto e_x\Lambda[2] \mapsto \cdots$$

for each $x \in \mathcal{D}^n_0$ (c.f. Λ is $(2m, m+1)$-fractional Calabi-Yau). These orbits are disjoint. Indeed, $\tau_2^{-i}(e_x\Lambda) \cong \tau_2^{-j}(e_y\Lambda)$ implies $\tau_2^{j-i}(e_x\Lambda) \cong e_y\Lambda$. Since $e_y\Lambda$ is projective, we must have $i = j$, whence $x = y$ since Λ is basic.

In [42, Example 6.13(b)], Iyama presents a different family that could be called 2-representation-finite algebras of type D. Namely, the Auslander algebras of Dynkin type D quivers.

Corollary 3.2.3. *Let $n = 3m+1$, let $C \subset \mathcal{D}^n_1$ be a cut, and let Q be a quiver of Dynkin type D. Then $\Pi^n_{\mathcal{D}}/\langle C \rangle$ is not Morita equivalent to $\operatorname{Aus}(kQ)$.*

Proof. Write $\Lambda = \Pi^n_{\mathcal{D}}/\langle C \rangle$, $\Gamma = \operatorname{Aus}(kQ)$. Assume for a contradiction there exists an equivalence of categories $F\colon \operatorname{mod}\Lambda \xrightarrow{\sim} \operatorname{mod}\Gamma$. If M is a d-cluster-tilting object of $\operatorname{mod}\Lambda$, then $F(M)$ is a d-cluster-tilting object of $\operatorname{mod}\Gamma$. By Proposition 1.4.3,

$$\mathscr{M}_\Lambda = \operatorname{add} M \simeq \operatorname{add} F(M) = \mathscr{M}_\Gamma.$$

In particular, the 2-Auslander-Reiten quivers of $\mathscr{M}_\Lambda$ and $\mathscr{M}_\Gamma$ coincide. Now, note that if C contains one element of $\{e_{(m+1,m,m-1)}\alpha_1, e_{(m+1,m,m-1)}\alpha_{2,1}\}$ then it also contains the other, since C must contain precisely one arrow from the cycles $\alpha_1\beta_0\gamma_0$

and $\alpha_{2,1}\beta_0\gamma_0$.

$$\begin{tikzcd}
(m+1,m-1,m) \arrow[rr, "\beta_0"] & & (m,m,m)_0 \arrow[dl, "\gamma_0"] \\
& (m+1,m,m-1) \arrow[ul, shift right, "(2,1)"'] \arrow[ul, shift left, "1"] &
\end{tikzcd}$$

Hence, either C or Λ contains a pair of parallel arrows, so the same is true of the 2-Auslander-Reiten quiver of $\mathscr{M}_\Lambda$ by Proposition 3.2.1. But by [42, Theorem 6.12], the 2-Auslander-Reiten quiver of $\mathscr{M}_A$ does not contain a pair of parallel arrows. This contradiction shows that F cannot exist. $\square$

Remark 3.2.4. It is an open question whether, given $n = 3m + 1$ and a cut $C \subset \mathcal{D}_1^n$, there exists a quiver Q of Dynkin type D such that $\Lambda = \Pi_{\mathcal{D}}^n / \langle C \rangle$ is derived equivalent to $\Gamma = \mathrm{Aus}(kQ)$. We cannot argue as above. An equivalence $\mathbf{D}^{\mathrm{b}}(\Lambda) \to \mathbf{D}^{\mathrm{b}}(\Gamma)$ must map $\mathscr{U}_\Lambda$ to a d-cluster-tilting subcategory of $\mathbf{D}^{\mathrm{b}}(\Gamma)$, but not necessarily $\mathscr{U}_\Gamma$, since d-cluster-tilting subcategories of derived categories are not unique.

Future work

1. **Bases.** Finding a basis for $\Pi^n_{\mathfrak{D}}$ is difficult, even for small n.

   Example. Let $n = 4$ (see Figure 2.1 and Example 2.0.3). Then

   $$\begin{aligned} e_{300}\Pi^4_{\mathfrak{D}} &= \langle e_{300}, \alpha_0, \alpha_0\alpha_{2,1}, \alpha_0\alpha_{2,1}\alpha_2\rangle, \\ e_{210}\Pi^4_{\mathfrak{D}} &= \langle e_{210}, \alpha_1, \alpha_{2,1}, \alpha_1\beta_0, \alpha_1\beta_1, \alpha_1\beta_2, \alpha_{2,1}\alpha_2, \alpha_{2,1}\alpha_2\alpha_0\rangle, \\ e_{201}\Pi^4_{\mathfrak{D}} &= \langle e_{201}, \alpha_2, \beta_0, \beta_1, \beta_2, \alpha_2\alpha_0, \beta_0\gamma_0, \alpha_2\alpha_0\alpha_{2,1}\rangle, \\ e_{(111)_k}\Pi^4_{\mathfrak{D}} &= \langle e_{(111)_k}, \gamma_k, \gamma_k\alpha_1, \gamma_k\alpha_1\beta_k\rangle \text{ for all } k \in \{0,1,2\}. \end{aligned}$$

   In particular, $\dim_{\mathbb{C}} \Pi^4_{\mathfrak{D}} = 32$.

   One calculates as in Lemma 3.1.4, recursively computing the space of paths starting at each vertex with a given length. To make these calculations we had to derive the relations

   (a) $\alpha_1\beta_0\gamma_0 = \zeta^2\alpha_1\beta_1\gamma_1 = \zeta\alpha_1\beta_2\gamma_2$ and $\alpha_{2,1}\beta_0\gamma_0 = \alpha_{2,1}\beta_1\gamma_1 = \alpha_{2,1}\beta_2\gamma_2$;

   (b) $\beta_0\gamma_0\alpha_1 = \zeta^2\beta_1\gamma_1\alpha_1 = \zeta\beta_2\gamma_2\alpha_1$ and $\beta_0\gamma_0\alpha_{2,1} = \beta_1\gamma_1\alpha_{2,1} = \beta_2\gamma_2\alpha_{2,1}$;

   (c) if $i \neq j$ then $\gamma_i\alpha_1\beta_j = \gamma_i\alpha_{2,1}\beta_j = 0$.

   To see (a), note that

   $$\alpha_{2,1}\beta_2\gamma_2 = -\zeta\alpha_{2,1}\beta_0\gamma_0 - \zeta^2\alpha_{2,1}\beta_1\gamma_1 = -\zeta\alpha_1\beta_0\gamma_0 - \zeta\alpha_1\beta_1\gamma_1, \text{ but also}$$
   $$\alpha_{2,1}\beta_2\gamma_2 = \zeta\alpha_1\beta_2\gamma_2 = -\zeta^2\alpha_1\beta_0\gamma_0 - \alpha_1\beta_1\gamma_1,$$

   so $\zeta\alpha_1\beta_0\gamma_0 + \zeta\alpha_1\beta_1\gamma_1 = \zeta^2\alpha_1\beta_0\gamma_0 + \alpha_1\beta_1\gamma_1$, meaning $\alpha_1\beta_0\gamma_0 = \zeta^2\alpha_1\beta_1\gamma_1$. One similarly shows $\alpha_1\beta_0\gamma_0 = \zeta\alpha_1\beta_2\gamma_2$, and the fact $\alpha_1\beta_k\gamma_k = \zeta^k\alpha_{2,1}\beta_k\gamma_k$ for all $k \in \{0,1,2\}$ implies the second statement.

Part (b) is demonstrated analogously. To see (c), assume $i \neq j$. Then we have $\gamma_i\alpha_1\beta_j = \zeta^i\gamma_i\alpha_{2,1}\beta^j$, but also $\gamma_i\alpha_1\beta_j = \zeta^j\gamma_i\alpha_{2,1}\beta_j$. This is a contradiction unless $\gamma_i\alpha_1\beta_j = \gamma_i\alpha_{2,1}\beta_j = 0$.

2. **Greater d.** The $(d+1)$-preprojective algebras of type A are defined for all $d \in \mathbb{Z}_{>0}$ ($d = 1$ is the classical preprojective algebra of a Dynkin type A quiver). The corresponding d-representation-finite algebras are always obtained through quotienting by the ideal generated by a cut [43, §5]. Let $\{\Pi^n_{\mathcal{A},d} \mid n \geq 2\}$ be the family of $(d+1)$-preprojective algebras of type A, for some $d > 2$. There is an obvious $\mathbb{Z}_{d+1}$-action on each $\Pi^n_{\mathcal{A},d}$, so some goals are as follows.

   (a) Generalise Definitions 2.0.1 and 2.0.2 to give a family $\{\Pi^n_{\mathcal{D},d} \mid n \geq 2\}$ of algebras such that $\Pi^n_{\mathcal{D},d}$ is Morita equivalent to $\Pi^n_{\mathcal{A},d}\#\mathbb{Z}_{d+1}$.

   (b) Prove $\Pi^n_{\mathcal{D},d}$ is a $(d+1)$-preprojective algebra precisely when d is 1 modulo $d+1$.

   (c) Prove that the d-representation-finite algebras associated to each $\Pi^n_{\mathcal{D},d}$ are fractional Calabi-Yau.

   The methods employed would almost certainly have to be different to those in this thesis. Perhaps most significantly, when $d > 2$ one no longer has the classification of Herschend and Iyama (c.f. Theorem 1.4.8).

3. **Postnikov and orbifold diagrams.** A *Postnikov diagram* $\mathcal{P}$ is a collection of oriented curves in a disc, which intersect each other transversally in an alternating way. Hence they naturally give rise to a quiver with potential $(Q_{\mathcal{P}}, W_{\mathcal{P}})$. Pasquali [56] showed that $\Lambda_{\mathcal{P}} := \mathrm{J}(Q_{\mathcal{P}}, W_{\mathcal{P}})$ is selfinjective if and only if $\mathcal{P}$ is k-symmetric, i.e. invariant under rotation through $2\pi/k$. In this case Baur, Pasquali and Velasco [4] defined a corresponding *orbifold diagram* $\mathcal{O}$, which also has an associated Jacobian algebra $\Lambda_{\mathcal{O}} := \mathrm{J}(Q_{\mathcal{O}}, W_{\mathcal{O}})$. They showed

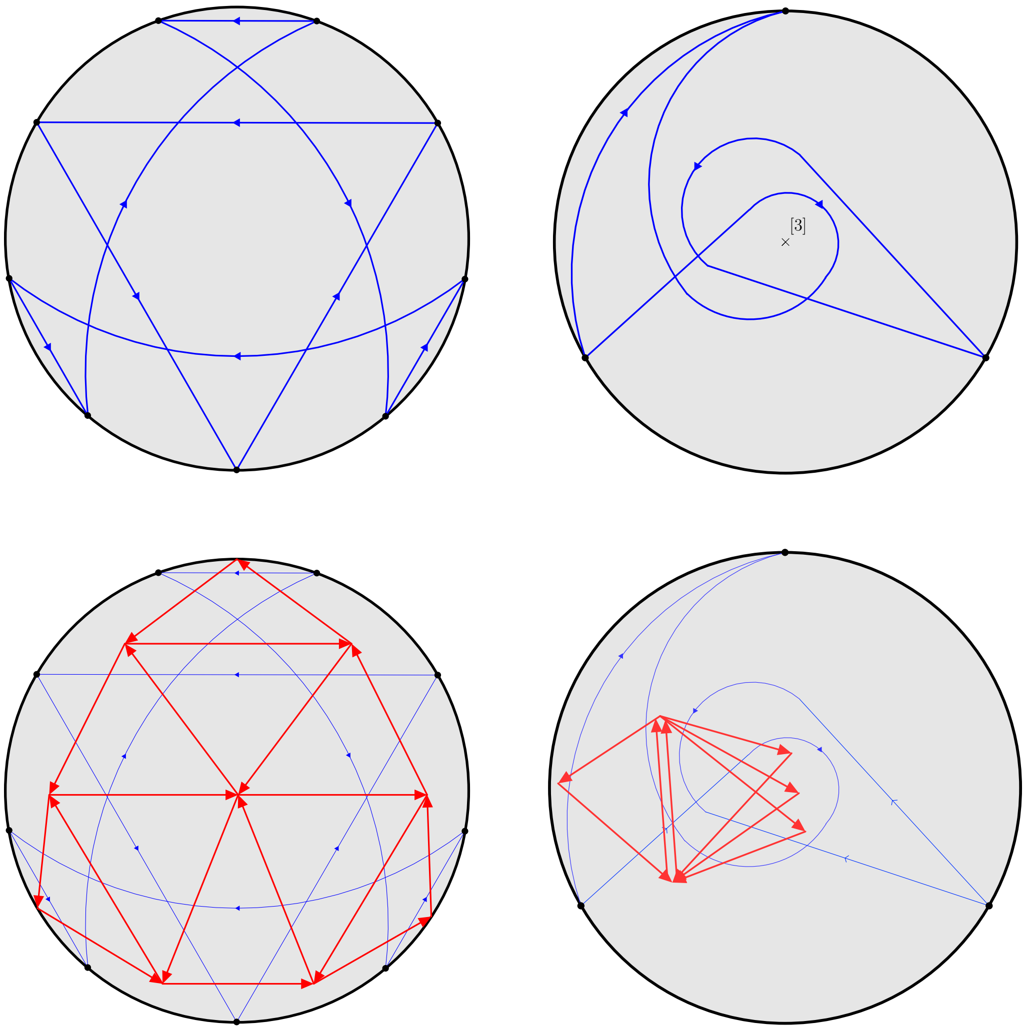


Figure C: Top: A 3-symmetric Postnikov diagram $\mathcal{P}$ and the corresponding orbifold diagram $\mathcal{O}$. Bottom: The quivers $Q_{\mathcal{P}}$ and $Q_{\mathcal{O}}$.

$(Q_{\mathcal{O}}, W_{\mathcal{O}})$ is precisely the QP obtained by applying the construction of [29] to $(Q_{\mathcal{P}}, W_{\mathcal{P}})$, and hence deduced that $\Lambda_{\mathcal{O}}$ is symmetric.

Example. Figure C shows a 3-symmetric Postnikov diagram $\mathcal{P}$ on the left, and its corresponding orbifold diagram $\mathcal{O}$ on the right. In fact, we have $\Lambda_{\mathcal{P}} \cong \Pi^4_{\mathcal{A}}$ and $\Lambda_{\mathcal{O}} \cong \Pi^4_{\mathcal{D}}$. See [4, 56] for details on how to construct the QPs from the diagrams.

Grant's theorem (Theorem 1.5.10) implies that taking cuts of $\Lambda_{\mathcal{P}}$ and $\Lambda_{\mathcal{O}}$ yields fractional Calabi-Yau algebras. The former appeared as [32, Theorem 6.22], while the latter is a novel observation. Even knowing this, calculating the

Calabi-Yau dimensions of these algebras remains difficult, and requires explicit quiver computations as in the style of Lemma 3.1.4. Therefore, a potential project is to find an algorithm which computes the Calabi-Yau dimension directly from the Postnikov or orbifold diagram. One of the benefits of this is that it would provide a way to assign novel numerical invariants to Postnikov and orbifold diagrams.

Part II

Higher zigzag algebras

Introduction

Stability conditions on triangulated categories were introduced by Bridgeland [12]. There are two equivalent definitions: one uses the heart of a bounded t-structure and one uses a slicing. In the article [63], Thomas constructs a stability condition $\sigma = (\mathscr{P}, \mathscr{Z})$ on the dg perfect derived category $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ of the zigzag algebra of type A_n, using the slicing definition. The σ-stable objects appear to be indexed, up to shift, by the indecomposable modules over the linearly oriented Dynkin type A_n quiver Q_n.

In §4.1, we reinterpret Thomas' construction using the heart definition. The heart in question is the smallest full subcategory $\mathscr{H} \subset \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ which contains Z_n and is closed under extensions and direct summands. This interpretation gives a way to explain why the stable objects are indexed by the indecomposable kQ_n-modules. Writing Π_n for $\Pi(Q_n)$, there is a duality of categories $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Pi_n) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ descending to a duality of hearts $\mathrm{mod}(\Pi_n) \xrightarrow{\sim} \mathscr{H}$. We show this induces a map $\{\text{indecomposable } kQ_n\text{-modules}\} \to \{\mathscr{Z}\text{-stable objects}\}$, and argue that it must be a bijection.

Our original hope for this project was to construct a stability condition on the dg perfect derived category $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$ of the 2-zigzag algebra of type A_n, whose family of stable objects admits an explicit classification. We note that finding a stability condition is fairly easy, given that we can exhibit a finite heart, but that it is not clear how to find one whose stable objects have a "nice" description. In §4.2 we detail our attempts to generalise Thomas' construction, and how they failed in this aim. We therefore turn our attention away from individual stability conditions and towards the stability manifold. A long-term goal is to explicitly determine (a connected component of) $\mathrm{Stab}(\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z^n_2))$. In Chapter 5, we take two small steps in this direction.

The $\mathrm{Ext}^\bullet$-spaces of $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z^n_2)$ (or rather the equivalent category $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z^n_2)$, see below) coincide with the corresponding $\mathrm{Hom}^\bullet$-spaces of $\mathbf{H}_{\mathrm{dg}}(Z^n_2)$. By Kadeishvili's theorem, the latter can be given the structure of an A_∞-category. If A is a dg algebra such that the A_∞-structure on $\mathbf{H}_{\mathrm{dg}}(A)$ is 3-*cyclic*, Kontsevich and Soibelman [51] give a recipe to associate an End^1-quiver with potential to any set of objects $\mathcal{C}$. If this set is *cluster*, they define a mutation operation, and claim it induces Derksen-Weyman-Zelevinsky mutation on the level of End^1-QPs. The Koszul dual result is proved by Keller and Yang in [48] (Theorem 1.3.15 in this thesis), but we have been unable to find a proof on this side of Koszul duality in the literature. We write down a proof that KS mutation induces DWZ mutation of the End^1-quiver. The relevance to stability conditions comes via tilting: given a heart of $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(A)$ whose set of stable objects is a cluster collection in $\mathbf{H}_{\mathrm{dg}}(A)$, the set of stable objects of a tilted heart is a mutated cluster collection.

Finally, we prove that a certain group acts on $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z^n_2)$ by spherical twists. This generalises the classical case, where there is a braid group action on $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ by spherical twists, and transfers a result of Grant to our setting. We argue that the action preserves a certain connected component of the stability manifold. Understanding the group of autoequivalences which preserve a given connected component can play an important role in explicitly determining that component, since it allows one to restrict their attention to a fundamental domain (see e.g. [13, 40, 54]).

Chapter 4

Stability conditions on classical zigzag algebras

4.1 Reinterpreting Thomas' construction

Denote by Q_n the linearly oriented Dynkin type A_n quiver, and let $Z_n = Z(Q_n)$ be its zigzag algebra. Equip Z_n with the path-length grading, and consider it as a dg algebra with trivial differential. In [63], Thomas constructs a stability condition on $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ using the slicing definition (see Proposition 1.6.6).

Definition 4.1.1. [63, Definition 5.3] Fix positive reals $m_1, m_2, \ldots, m_n \in \mathbb{R}_{>0}$ and $0 < \phi_1 < \phi_2 < \cdots < \phi_n \leq 1$. For each $1 \leq j < \ell \leq n$, let $m_{j,\ell} \in \mathbb{R}_{>0}$ and $\phi_{j,\ell} \in (0,1]$ be the unique numbers satisfying $m_{j,\ell} \exp(i\pi\phi_{j,\ell}) = \sum_{j \leq k \leq \ell} m_k \exp(i\pi\phi_k)$. For each $1 \leq j \leq n-1$, let $E_{j,j+1} = \operatorname{cone}(e_{j+1}Z_n[-1] \to e_j Z_n)$. For each $1 \leq j < \ell \leq n$, inductively define $E_{j,\ell} = \operatorname{cone}(e_\ell Z_n[-1] \to E_{j,\ell-1})$. By definition, $E_{j,j} = e_j Z_n$.

Define a slicing $\mathscr{P}$ and central charge $\mathscr{Z}$ as follows. For each $\phi \in (0,1]$ and $k \in \mathbb{Z}$, let

$$\mathscr{P}(\phi + k) = \operatorname{add}\{E_{j,\ell}[k] \mid \phi_{j,\ell} = \phi\} \subset \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n),$$

where our convention is that $\operatorname{add}\varnothing = \{0\}$. Let $\mathscr{Z}\colon \mathrm{K}(\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)) \to \mathbb{C}$ be the linear map satisfying $\mathscr{Z}(e_jZ_n) = m_j \exp(i\pi\phi_j)$ for all $1 \leq j \leq n$. Put $\sigma = (\mathscr{P}, \mathscr{Z})$.

By explicitly verifying that the axioms are satisfied, Thomas proves the following.

Theorem 4.1.2. *[63, Theorem 5.5] In Definition 4.1.1, σ is a stability condition on $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$.*

Example 4.1.3. Let $n = 3$, retaining the notation of Example 1.2.5. Assume that $m_1 = m_2 = m_3$, and set $\phi_1 = \theta$, $\phi_2 = 1 - \theta$, $\phi_3 = 1$ for some $\theta \in (0, \frac{1}{2})$. There are six σ-stable objects up to shift, namely the projectives e_1Z_3, e_2Z_3, e_3Z_3 and three extensions

$$E_{12} = \mathrm{cone}(e_2Z_3[-1] \to e_1Z_3) = \left(e_1Z_3 \oplus e_2Z_3, \begin{pmatrix} 0 & m_\alpha \\ 0 & 0 \end{pmatrix} \right),$$

$$E_{23} = \mathrm{cone}(e_3Z_3[-1] \to e_2Z_3) = \left(e_2Z_3 \oplus e_3Z_3, \begin{pmatrix} 0 & m_\beta \\ 0 & 0 \end{pmatrix} \right),$$

$$E_{13} = \mathrm{cone}(e_3Z_3[-1] \to E_{12}) = \left(e_1Z_3 \oplus e_2Z_3 \oplus e_3Z_3, \begin{pmatrix} 0 & m_\alpha & 0 \\ 0 & 0 & m_\beta \\ 0 & 0 & 0 \end{pmatrix} \right),$$

where m_α and m_β denote left multiplication by α and β respectively. The charges of these objects are displayed on an Argand diagram in Figure 4.1.

For each $n \in \mathbb{Z}_+$ the σ-stable objects, up to shift, appear to be indexed by indecomposable modules over kQ_n. Indeed, given $1 \leq j \leq \ell \leq n$,

$$M_{j,\ell} = \langle \text{paths } p \text{ in } Q_n \mid j \leq s(p) \leq t(p) \leq \ell \rangle$$

is an indecomposable kQ_n-module, and every indecomposable kQ_n-module is of this form. The aim of this section is to explain why this is the case, using the heart definition of a stability condition (see Definition 1.6.4). It is easy to check that the heart $\mathscr{P}(0, 1] \subset \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ corresponding to σ is the smallest full subcategory which contains Z_n and is closed under extensions and direct summands (but not shifts). In future, we will denote this heart by $\mathscr{H}$.

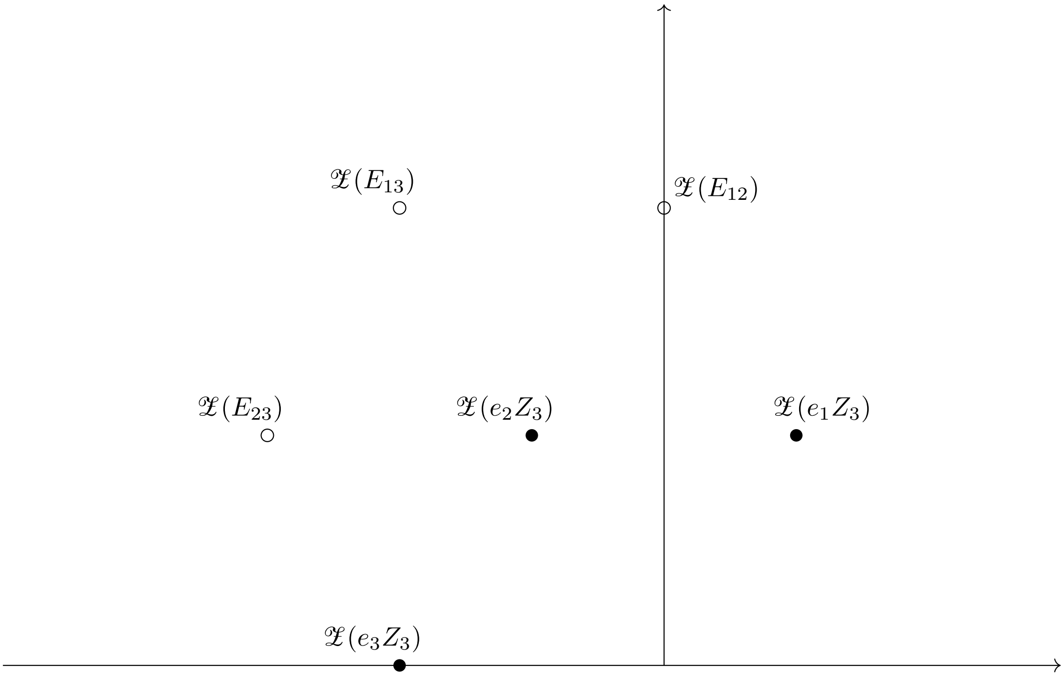


Figure 4.1: Argand diagram showing the charges of the σ-stable objects, up to shift, from Example 4.1.3

Our main tool is the following Koszul duality between Ginzburg dg algebras and zigzag algebras.

Lemma 4.1.4. *[25, Theorem 4.4] Given an ADE Dynkin quiver Q, let $\Gamma = \Gamma(Q)$ and define $\Gamma^! = \mathbf{R}\,\mathcal{H}om_\Gamma(kQ_0, kQ_0)^{\mathrm{op}}$. The contravariant functor*[1]

$$\mathbf{R}\,\mathcal{H}om_\Gamma(-, kQ_0)^{\mathrm{op}} \colon\ \mathbf{D}_{\mathrm{dg}}(\Gamma) \to \mathbf{D}_{\mathrm{dg}}(\Gamma^!)$$

induces a duality of categories $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(\Gamma^!)$.

We refer to $\Gamma^!$ as the *Koszul dual* of Γ.

Lemma 4.1.5. *[20, Corollary 25] Let Q be an ADE Dynkin quiver. There is an isomorphism of dg algebras $\Gamma(Q)^! \cong Z(Q)$.*

Let $\Gamma_n = \Gamma(Q_n)$ and $\Pi_n = \Pi(Q_n)$. We have a duality of categories

$$\Psi := \mathbf{R}\,\mathcal{H}om_{\Gamma_n}(-, k(Q_n)_0)^{\mathrm{op}} \colon\ \mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma_n) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n).$$

---

[1] Here $\mathbf{R}\,\mathcal{H}om_A(M, N) := \mathcal{H}om_A(C, N)$, where C is a cofibrant replacement of M (see [48, §2.12]).

Recall that $\operatorname{mod}(\Pi_n)$ is the heart of a bounded t-structure on $\mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma_n)$. The former coincides with the smallest full subcategory of the latter which contains $k(Q_n)_0$ and is closed under extensions and direct summands (not shifts). Since $\Psi(k(Q_n)_0) \cong Z_n$, Ψ induces a duality of hearts $\operatorname{mod}(\Pi_n) \xrightarrow{\sim} \mathscr{H}$.

Proposition 4.1.6. *In the above notation,* Ψ *induces a function*

$$\{\textit{indecomposable } kQ_n\textit{-modules}\} \to \{\mathscr{Z}\textit{-stable objects}\}.$$

Proof. Let $M \in \operatorname{mod}(\Pi_n)$ be an indecomposable kQ_n-module (i.e. let M be annihilated by the ideal $\langle \alpha^* \mid \alpha \in (Q_n)_1 \rangle$). We show that $E := \Psi(M)$ is $\mathscr{Z}$-stable. Now, $M \cong M_{j,\ell} = \langle \text{paths } p \text{ in } Q_n \mid j \leq s(p) \leq t(p) \leq \ell \rangle$ for some $1 \leq j \leq \ell \leq n$, so there is a sequence

$$M_{j,\ell} \twoheadrightarrow M_{j,\ell-1} \twoheadrightarrow \cdots \twoheadrightarrow M_{j,j+1} \twoheadrightarrow M_{j,j} \tag{4.1}$$

of quotients in $\operatorname{mod}(\Pi_n)$, corresponding to a composition series

$$M_{j,j} \subset M_{j,j+1} \subset \cdots \subset M_{j,\ell-1} \subset M_{j,\ell}$$

in its opposite category with composition factors $k\{e_j\}, k\{e_{j+1}\}, \ldots, k\{e_\ell\}$. Hence the sequence (4.1) is mapped by the duality Ψ to a composition series

$$e_j Z_n \subset \Psi(M_{j,j+1}) \subset \cdots \subset \Psi(M_{j,\ell-1}) \subset E \tag{4.2}$$

in $\mathscr{H}$ with composition factors $e_j Z_n, e_{j+1} Z_n, \ldots, e_\ell Z_n$. Let $0 \neq F \subset E$ be a proper subobject. Then $F \cong \Psi(N)$ for some $M \twoheadrightarrow N$ in $\operatorname{mod}(\Pi_n)$. Hence $N \cong M_{j,r}$ for some $j \leq r < \ell$, and arguing as above we conclude that F has composition factors $e_j Z_n, e_{j+1} Z_n, \ldots, e_r Z_n$. Therefore

$$\begin{aligned} \mathscr{Z}(E) &= \sum_{j \leq k \leq \ell} \mathscr{Z}(e_k Z_n) \\ &= \underbrace{\sum_{j \leq k \leq r} \mathscr{Z}(e_k Z_n)}_{\mathscr{Z}(F)} + \sum_{r+1 \leq k \leq \ell} \mathscr{Z}(e_k Z_n). \end{aligned} \tag{4.3}$$

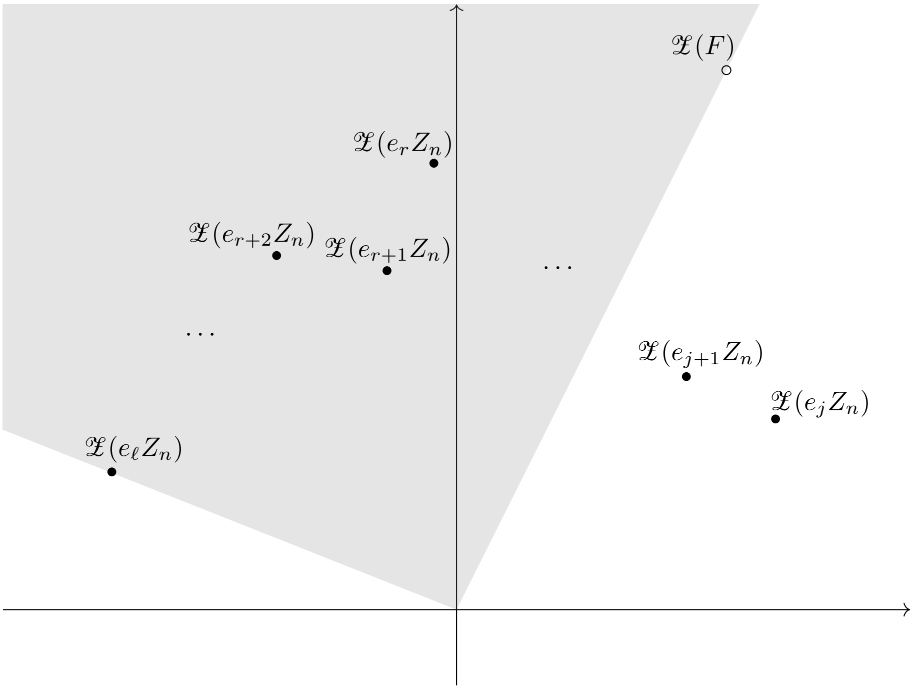


Figure 4.2: Argand diagram showing the charges of various objects from the proof of Proposition 4.1.6. Note that $\mathscr{Z}(E)$ must lie in the shaded region (boundary excluded).

Since $m_k > 0$ and $\phi_k > \max\{\phi_1, \dots, \phi_r\}$ for all $r+1 \leq k \leq \ell$, we certainly have $\arg(\mathscr{Z}(E)) > \arg(\mathscr{Z}(F))$ (see Figure 4.2). Hence E is $\mathscr{Z}$-stable. □

Consider the composition series (4.2). Since $\Psi(M_{j,j+1})$ is a non-split extension of $e_{j+1}Z_n$ by e_jZ_n, we must have $\Psi(M_{j,j+1}) \cong E_{j,j+1}$. Arguing recursively, we deduce that $\Psi(M_{j,\ell}) \cong E_{j,\ell}$ for all $1 \leq j \leq \ell \leq n$. Since the $\mathscr{Z}$-stable objects of $\mathscr{H}$ are precisely the σ-stable objects of $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ with phase in $(0,1]$, and since these are precisely the $E_{j,\ell}$, we deduce that the function in Proposition 4.1.6 is a bijection.

4.2 The failure of generalisations

Note that, in Thomas' construction, we choose the phases of the projective modules to be ordered linearly ($\phi_1 < \phi_2 < \dots < \phi_n$), and that the resulting stability condition has stable objects indexed by indecomposable modules over the linear Dynkin type A_n quiver ($1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$).

Let $m_1, m_2, \dots, m_n \in \mathbb{R}_{>0}$, $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in (0, 1]^n$. Recall that the smallest full subcategory $\mathscr{H} \subset \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ containing Z_n and closed under extensions and direct summands is the heart of a bounded t-structure. Let $\mathscr{Z} \colon \mathrm{K}(\mathscr{H}) \to \mathbb{C}$ be the unique linear map with $\mathscr{Z}(e_j Z_n) = m_j \exp(i\pi\phi_j)$ for all $1 \leq j \leq n$. Then $\mathscr{Z}(e_j Z_n) \in \overline{\mathbb{H}}$ for all $1 \leq j \leq n$, so $(\mathscr{H}, \mathscr{Z})$ is a stability condition by Proposition 1.6.10.

Let Q_ϕ be the quiver with vertex set $\{1, 2, \dots, n\}$, such that there exists a single arrow $i \longrightarrow j$ if and only if $j \in \{i-1, i+1\}$ and $\phi_i < \phi_j$. Provided $\phi_i \neq \phi_{i+1}$ for all $1 \leq i < n$, there is a duality of categories $\Psi \colon \mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma(Q_\phi)) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_n)$ descending to a duality of hearts $\mathrm{mod}(\Pi(Q_\phi)) \xrightarrow{\sim} \mathscr{H}$, as in §4.1.

Question 4.2.1. Does Ψ induce a bijection

$$\{\text{indecomposable } kQ_\phi\text{-modules}\} \to \{\mathscr{Z}\text{-stable objects}\}?$$

We have seen that the answer is affirmative in the case $\phi_1 < \phi_2 < \cdots < \phi_n$. Unfortunately, this appears to be an anomaly.

Example 4.2.2. Let $n = 3$. Assume that $m_1 = m_2 = m_3$, and set $\phi_1 = \theta$, $\phi_2 = 1$, $\phi_3 = 1 - \theta$ for some $\theta \in (0, \frac{1}{3})$. Then Q_ϕ is $1 \longrightarrow 2 \longleftarrow 3$. Note that

$$\mathrm{mod}(\Pi(Q_\phi)) = \mathrm{add}\left\{\boxed{1}, \boxed{2}, \boxed{3}, \boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \boxed{\begin{smallmatrix}3\\2\end{smallmatrix}}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}, \boxed{\begin{smallmatrix}13\\2\end{smallmatrix}}, \begin{smallmatrix}2\\13\end{smallmatrix}, \begin{smallmatrix}2\\13\\2\end{smallmatrix}\right\},$$

where the boxed modules are precisely the indecomposable kQ_ϕ-modules. We exhibit an indecomposable kQ_ϕ-module M such that $\Psi(M)$ is not $\mathscr{Z}$-stable, and a $\mathscr{Z}$-stable object whose preimage N under Ψ is not an indecomposable kQ_ϕ-module.

Let $M = \begin{smallmatrix}13\\2\end{smallmatrix}$. Then M has proper quotient 3 in $\mathrm{mod}(\Pi(Q_\phi))$, so $\Psi(3) = e_3 Z_3$ is a

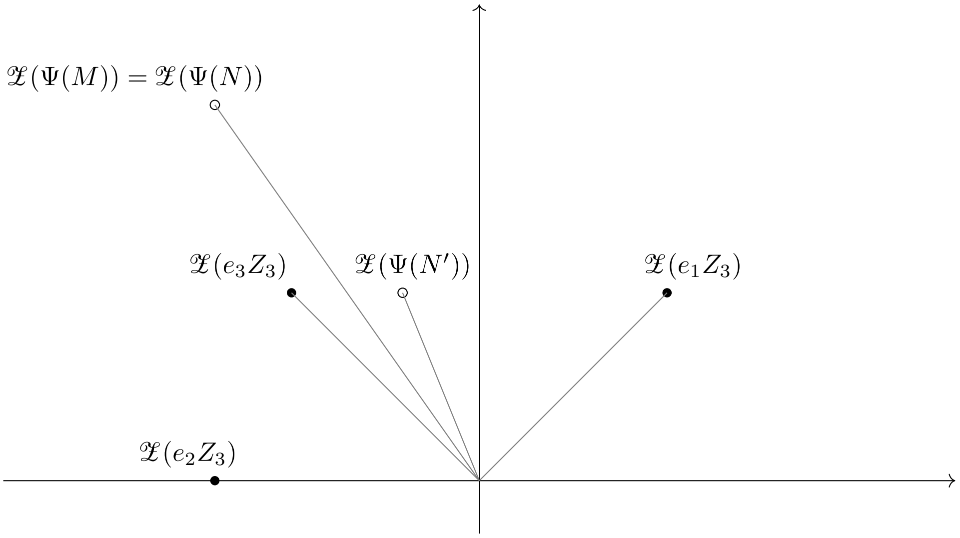


Figure 4.3: Argand diagram showing the charges of various objects from Example 4.2.2.

proper subobject of $\Psi(M)$ in $\mathscr{H}$. But

$$\begin{aligned}\arg \mathscr{Z}(\Psi(M))/\pi &= \arg(\mathscr{Z}(e_1Z_3) + \mathscr{Z}(e_2Z_3) + \mathscr{Z}(e_3Z_3))/\pi \\ &= 1 - \arctan(2\sin(\pi\theta))/\pi \\ &< 1 - \theta \\ &= \arg \mathscr{Z}(e_3Z_3)/\pi.\end{aligned}$$

Hence M is an indecomposable kQ_ϕ-module such that $\Psi(M)$ is not $\mathscr{Z}$-stable.

Let $N = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$. Then N has two proper quotients in $\mathrm{mod}(\Pi(Q_\phi))$, namely $N' = \begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\end{smallmatrix}$. Hence $\Psi(N)$ has two proper subobjects in $\mathscr{H}$, namely $\Psi(N')$ and $\Psi(\begin{smallmatrix}1\end{smallmatrix}) = e_1Z_3$. Note that

$$\arg \mathscr{Z}(\Psi(N'))/\pi = 1 - \arctan\left(\frac{\sin(\pi\theta)}{1-\cos(\pi\theta)}\right) \Big/ \pi,$$

while $\arg \mathscr{Z}(e_1Z_3)/\pi = \theta$. Both of these charges are strictly smaller than

$$\arg \mathscr{Z}(\Psi(N))/\pi = 1 - \arctan(2\sin(\pi\theta))/\pi,$$

so $\Psi(N)$ is a $\mathscr{Z}$-stable object whose preimage N is not an indecomposable kQ_ϕ-module.

Our original hope for this project was to construct, for all $n \in \mathbb{Z}^+$, a stability condition on the dg perfect derived category $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$ of the 2-zigzag algebra of type A_n, whose family of stable objects admits an explicit classification. We reiterate that finding a stability condition is fairly easy, in view of Proposition 1.6.10, but that it is not clear how to find one whose stable objects have a “nice” description.

Had the answer to Question 4.2.1 been affirmative in general, we hoped to use similar methods for $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$. Indeed, we expect (but have not proved) there is an isomorphism of dg algebras $\Gamma(\mathcal{A}^n, W^n_{\mathcal{A}})^! \cong Z_2^n$. As in the classical case, this would give us a duality

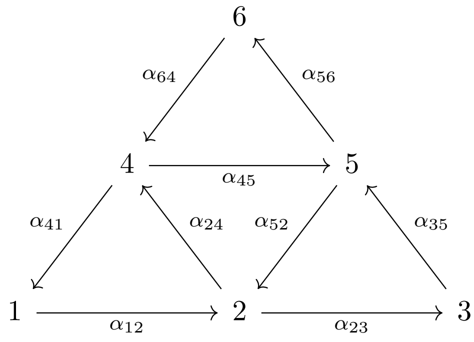
$$\Psi_2 \colon \mathbf{D}^{\mathrm{fd}}_{\mathrm{dg}}(\Gamma(\mathcal{A}^n, W^n_{\mathcal{A}})) \xrightarrow{\sim} \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$$

by [25, Theorem 4.4], descending to a duality of hearts $\mathrm{mod}(\Pi^n_{\mathcal{A}}) \xrightarrow{\sim} \mathscr{H}_2$, where the latter is the smallest full extension-closed subcategory of $\mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$ containing Z_2^n and closed under direct summands. Given $\mathscr{Z} \colon \mathrm{K}(\mathscr{H}_2) \to \mathbb{C}$ with $\mathscr{Z}(e_x Z_2^n) \in \overline{\mathbb{H}}$ for all $x \in \mathcal{A}^n_0$, $(\mathscr{H}_2, \mathscr{Z})$ is a stability condition by Proposition 1.6.10. Define the set $C_{\mathscr{Z}} = \{(x \xrightarrow{i} y) \in \mathcal{A}^n_1 \mid \phi_x \geq \phi_y\}$. For certain choices of $\mathscr{Z}$, $C_{\mathscr{Z}} \subset \mathcal{A}^n_1$ is a cut. In this case, we hoped Ψ_2 would induce a bijection

$$\{\text{indecomposable } \Pi^n_{\mathcal{A}}/\langle C_{\mathscr{Z}}\rangle\text{-modules}\} \xrightarrow{\sim} \{\mathscr{Z}\text{-stable objects}\}.$$

While we still expect the conjectured isomorphism $\Gamma(\mathcal{A}^n, W^n_{\mathcal{A}})^! \cong Z_2^n$ to hold, it has become clear this method will not work.

Example 4.2.3. Let $n = 3$. For convenience, relabel $\mathcal{A}^3$ as follows.



Let $\mathscr{Z}$ be the stability function defined in Figure 4.4. Then $C_{\mathscr{Z}} = \{\alpha_{23}, \alpha_{41}, \alpha_{45}\}$ is a

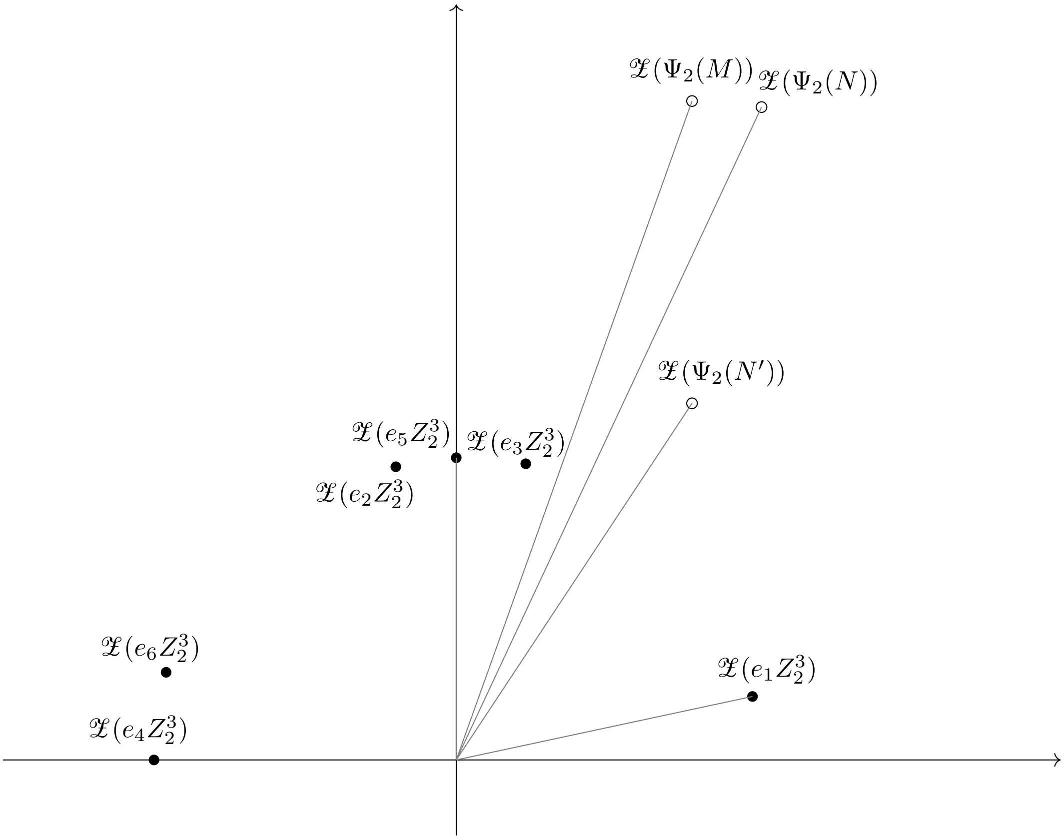


Figure 4.4: Argand diagram showing the choices of $\mathscr{Z}(e_x Z_2^3)$ which define the stability function $\mathscr{Z}$ from Example 4.2.3, as well as the charges of various other objects.

cut.

Let $M = \begin{smallmatrix} 1 & & 5 \\ & 2 & \end{smallmatrix}$. This is an indecomposable $\Pi_{\mathcal{A}}^n/\langle C_{\mathscr{Z}}\rangle$-module such that $\Psi_2(M)$ is not $\mathscr{Z}$-stable. Indeed, the proper quotient $\scriptstyle 5$ in $\operatorname{mod}(\Pi_{\mathcal{A}}^n/\langle C_{\mathscr{Z}}\rangle)$ yields a proper subobject $\Psi_2(\,{\scriptstyle 5}\,) = e_5 Z_2^3$ in $\mathscr{H}_2$ which has greater phase.

Let $N = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$. Then $\Psi_2(N)$ is a $\mathscr{Z}$-stable object whose preimage N is not an indecomposable $\Pi_{\mathcal{A}}^n/\langle C_{\mathscr{Z}}\rangle$-module. Indeed, the proper quotients $N' = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and $\scriptstyle 1$ in $\operatorname{mod}(\Pi_{\mathcal{A}}^n/\langle C_{\mathscr{Z}}\rangle)$ yield proper subobjects $\Psi_2(N')$ and $\Psi_2(\,{\scriptstyle 1}\,) = e_1 Z_2^3$ in $\mathscr{H}_2$, both of which have smaller phase.

Chapter 5

Results on the higher case

In this chapter we work in $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z_2^n)$, the smallest full triangulated subcategory of $\mathbf{K}_{\mathrm{dg}}(Z_2^n)$ which contains Z_2^n and is closed under direct summands. This does not change the setting in an essential way: for any dg algebra A, $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(A) \simeq \mathbf{D}^{\mathrm{per}}_{\mathrm{dg}}(A)$ [62, §3].

5.1 Tilting and mutation

Throughout this section, let A be a dg algebra. Using Kadeishvili's theorem, view $\mathbf{H}_{\mathrm{dg}}(A)$ as an A_∞-category.

Definition 5.1.1. [51, §3.3] The A_∞-structure $(m_1 = 0, m_2 = \circ, m_3, \dots)$ on $\mathbf{H}_{\mathrm{dg}}(A)$ is called 3-*cyclic* if, for each pair of objects $M, N \in \mathbf{H}_{\mathrm{dg}}(A)$, there exists a non-degenerate symmetric pairing

$$\langle -, - \rangle \colon \operatorname{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(N, M) \otimes \operatorname{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(M, N) \to k[-3]$$

which is cyclicly invariant, in the sense that

$$\langle f_n, m_{n-1}(f_{n-1} \otimes \cdots \otimes f_1) \rangle = \langle f_1, m_{n-1}(f_n \otimes \cdots \otimes f_2) \rangle$$

for all $M_1, \dots, M_n \in \mathbf{H}_{\mathrm{dg}}(A)$ and $f_i \in \mathrm{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(M_i, M_{i+1})$, where $M_{n+1} := M_1$.

For the rest of this section, assume the A_∞-structure on $\mathbf{H}_{\mathrm{dg}}(A)$ is 3-cyclic. Note that the pairing induces an isomorphism $\mathrm{Hom}^n_{\mathbf{H}_{\mathrm{dg}}(A)}(M, N) \cong \mathrm{Hom}^{3-n}_{\mathbf{H}_{\mathrm{dg}}(A)}(N, M)^*$ for each $n \in \mathbb{Z}$, $M, N \in \mathbf{H}_{\mathrm{dg}}(A)$.

Definition 5.1.2. [51, §8.1] Let $\mathcal{C}$ be a set of objects of $\mathbf{H}_{\mathrm{dg}}(A)$. The associated Hom^1-*quiver with potential* $(Q_{\mathcal{C}}, W_{\mathcal{C}})$ has

1. vertices the objects of $\mathcal{C}$;
2. arrows $M \longrightarrow N$ in bijection with a basis $B(M, N)$ of $\mathrm{Hom}^1_{\mathbf{H}_{\mathrm{dg}}(A)}(M, N)$;
3. potential $W_{\mathcal{C}} = \sum_{n \geq 3} W^n_{\mathcal{C}}$, where

$$W^n_{\mathcal{C}} = \frac{1}{n} \sum \langle f_n, m_{n-1}(f_{n-1} \otimes \cdots \otimes f_1) \rangle f_1 \cdots f_n.$$

   Here the sum runs over all $M_1, \dots, M_n \in \mathcal{C}$ and $f_i \in B(M_i, M_{i+1})$.

Note that one can define the Hom^1-quiver of a set of objects in any graded category: the 3-cyclic A_∞-structure is only needed to define the potential.

Definition 5.1.3. [51, Definition 20] Let $S \in \mathcal{C}$, a set of objects of $\mathbf{H}_{\mathrm{dg}}(A)$. We say $\mathcal{C}$ is *cluster* at S if

1. each $M \in \mathcal{C}$ is spherical, i.e. $\mathrm{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(M, M) \cong k \oplus k[-3]$;
2. for all $M \neq S$ in $\mathcal{C}$,

$$\mathrm{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M) \cong \mathrm{Hom}^1_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M) \oplus \mathrm{Hom}^2_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M),$$

   where at most one of the summands is non-zero.

Condition 1 ensures $Q_{\mathcal{C}}$ is free from loops. Since $\mathbf{H}_{\mathrm{dg}}(A)$ is 3-cyclic, Condition 2 can be rephrased as

$$\mathrm{Hom}^{\bullet}_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M) \cong \mathrm{Hom}^1_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M) \oplus \mathrm{Hom}^1_{\mathbf{H}_{\mathrm{dg}}(A)}(M, S)^*,$$

where at most one of the summands is non-zero. Therefore $Q_{\mathcal{C}}$ is free from 2-cycles at S. In particular, $(Q_{\mathcal{C}}, W_{\mathcal{C}})$ is mutable at S.

Definition 5.1.4. [51, §8.1] Let $\mathcal{C}$ be a set of objects in $\mathbf{H}_{\mathrm{dg}}(A)$ which is cluster at S. The *mutation* of $\mathcal{C}$ at S is

$$\upsilon_S \mathcal{C} = \{S[-1]\} \cup \{\mathrm{cone}(S[-1] \otimes \mathrm{Hom}^1_{\mathbf{H}_{\mathrm{dg}}(A)}(S, M) \overset{\mathrm{ev}}{\to} M) \mid S \neq M \in \mathcal{C}\}.$$

Just like $\mu_k(Q, W)$ is not necessarily mutable at k, $\upsilon_S \mathcal{C}$ is not necessarily cluster at $S[-1]$. The relevance to stability conditions is explained by the following result, which follows immediately from [49, Proposition 5.4].

Corollary 5.1.5. *Let $\mathcal{H}$ be the finite heart of a bounded t-structure on $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(A)$ whose set of simple objects $\mathcal{C}$ is cluster at S in $\mathbf{H}_{\mathrm{dg}}(A)$. Then $\upsilon_S \mathcal{C}$ is the set of simple objects of $\mathcal{H}^{\flat}_S$.*

We now connect Kontsevich-Soibelman mutation of cluster sets to Derksen-Weyman-Zelevinsky mutation of QPs.

Proposition 5.1.6. *Let $\mathcal{C}$ be a set of objects in $\mathbf{H}_{\mathrm{dg}}(A)$ which is cluster at S. Then $Q_{\upsilon_S \mathcal{C}} \cong \overline{Q_{\mathcal{C}}}$, where $(\overline{Q_{\mathcal{C}}}, \overline{W_{\mathcal{C}}}) = \mu_S(Q_{\mathcal{C}}, W_{\mathcal{C}})$.*

Remark 5.1.7. It is our understanding that actually $(Q_{\upsilon_S \mathcal{C}}, W_{\upsilon_S \mathcal{C}}) \cong (\overline{Q_{\mathcal{C}}}, \overline{W_{\mathcal{C}}})$, and that this is well-known to the experts. Kontsevich and Soibelman make this claim [51, §8.1], while Keller and Yang prove the Koszul dual result in [48] (Theorem 1.3.15 in this thesis). However, we were unable to find a proof written down on this side of Koszul duality, so we feel this work is worth including.

Proof of Proposition 5.1.6. To simplify notation, we write Hom to mean $\mathrm{Hom}_{\mathbf{H}_{\mathrm{dg}}(A)}$ throughout. Partition $\mathcal{C} \backslash \{S\}$ into

$$\mathcal{X} = \{X \in \mathcal{C} \mid \mathrm{Hom}^1(S, X) \neq 0\},$$

$$\mathcal{Y} = \{Y \in \mathcal{C} \mid \mathrm{Hom}^2(S, Y) \neq 0\},$$

$$\mathcal{Z} = \{Z \in \mathcal{C} \mid \mathrm{Hom}^{\bullet}(S, Z) = 0\}.$$

Note that

$$\begin{aligned} \upsilon_S\mathcal{C} &= \{S[-1]\} \cup \{\widehat{X} \mid X \in \mathcal{X}\} \cup \mathcal{Y} \cup \mathcal{Z}, \text{ where} \\ \widehat{X} &= \operatorname{cone}(S[-1] \otimes \operatorname{Hom}^1(S,X) \xrightarrow{\text{ev}} X) \\ &\cong \operatorname{cone}(S^{\oplus n}[-1] \xrightarrow{(f_1 \cdots f_n)} X) \end{aligned}$$

for our chosen basis $B(S,X) = \{f_1, \dots, f_n\}$ of $\operatorname{Hom}^1(S,X)$. For each pair of objects $M, N \in \upsilon_S\mathcal{C}$, we compute a basis for $\operatorname{Hom}^\bullet(M,N)$, which tells us the arrows between M and N in $Q_{\upsilon_S\mathcal{C}}$ (thanks to the duality, there is no need to also compute $\operatorname{Hom}^\bullet(N,M)$). For each pair, these coincide with the arrows in $\overline{Q_{\mathcal{C}}}$.

Firstly, note that $\operatorname{Hom}^\bullet(S[-1],S[-1]) = \operatorname{Hom}^{0,3}(S,S)$. Hence there are no loops at $S[-1]$ in $Q_{\upsilon_S\mathcal{C}}$.

Let $Y \in \mathcal{Y}$. Then $\operatorname{Hom}^\bullet(S[-1],Y) = \operatorname{Hom}^2(S,Y)[1] \cong \operatorname{Hom}^1(Y,S)^*[1]$. Hence each arrow $Y \xrightarrow{g} S$ in $Q_{\mathcal{C}}$ is replaced by an arrow $Y \xleftarrow{g^*} S[-1]$ in $Q_{\upsilon_S\mathcal{C}}$.

Let $Z \in \mathcal{Z}$. Then $\operatorname{Hom}^\bullet(S[-1],Z) = 0$, so $S[-1]$ and Z are not adjacent in $Q_{\upsilon_S\mathcal{C}}$.

From now on, fix $X \in \mathcal{X}$ and $B(S,X) = \{f_1, \dots, f_n\}$. There is a triangle

$$S^{\oplus n}[-1] \xrightarrow{(f_1 \cdots f_n)} X \longrightarrow \widehat{X} \longrightarrow S^{\oplus n} \tag{5.1}$$

in $\mathbf{K}_{\mathrm{dg}}(A)$. Note that $\operatorname{Hom}^\bullet(X,S[-1]) = \operatorname{Hom}^2(X,S)[-1] \cong \operatorname{Hom}^1(S,X)^*[-1]$, $\operatorname{Hom}^\bullet(S^{\oplus n}[-1],S[-1]) = \operatorname{Hom}^{0,3}(S^{\oplus n},S)$. Since $\operatorname{Ext}^\bullet_{\mathbf{K}_{\mathrm{dg}}(A)}(M,N) \cong \operatorname{Hom}^\bullet(M,N)$, applying $\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(-,S[-1])$ to (5.1) yields exact sequences

$$0 \longrightarrow \operatorname{Hom}^0(S^{\oplus n},S) \longrightarrow \operatorname{Hom}^1(\widehat{X},S[-1]) \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Hom}^3(\widehat{X},S[-1]) \longrightarrow \operatorname{Hom}^1(S,X)^*[-1] \xrightarrow{-\circ(f_1 \cdots f_n)} \operatorname{Hom}^3(S^{\oplus n},S) \longrightarrow \operatorname{Hom}^4(\widehat{X},S[-1]) \longrightarrow 0.$$

A basis for $\operatorname{Hom}^1(S,X)^*[-1]$ is $\{f_1^*, \dots, f_n^*\}[-1]$, while a basis for $\operatorname{Hom}^3(S^{\oplus n},S)$ is $\{\mathrm{id}_S^* \circ p_1, \dots, \mathrm{id}_S^* \circ p_n\}$, where p_j denotes projection onto the j-th copy of S. Note that $f_j^* \circ (f_1 \cdots f_n) = \mathrm{id}_S^* \circ p_j$, so these spaces are isomorphic and we deduce that

$\mathrm{Hom}^\bullet(\widehat{X}, S[-1]) \cong \mathrm{Hom}^0(S^{\oplus n}, S)[-1]$. The latter has a basis $\{p_1, \dots, p_n\}[-1]$, so each arrow $S \xrightarrow{f_j} X$ in $Q_{\mathcal{C}}$ is replaced by an arrow $S[-1] \xleftarrow{f_j^\vee} \widehat{X}$ in $Q_{v_S\mathcal{C}}$, where $f_j^\vee$ is the image of p_j in $\mathrm{Hom}^1(\widehat{X}, S[-1])$.

Let $Y \in \mathcal{Y}$, $B(Y, S) = \{g_1, \dots, g_m\}$. Then $\mathrm{Hom}^\bullet(Y, S^{\oplus n}[-1]) = \mathrm{Hom}^1(Y, S^{\oplus n})[-1]$.

1. Assume $\mathrm{Hom}^\bullet(Y, X) = \mathrm{Hom}^1(Y, X)$. This corresponds to an arrangement

$$\begin{tikzcd}
S \arrow[r, "f_1,\dots,f_n"] & X \\
Y \arrow[u, "g_1,\dots,g_m"] \arrow[ur, "h_1,\dots,h_\ell"'] &
\end{tikzcd}$$

in $Q_{\mathcal{C}}$, where $B(Y, X) = \{h_1, \dots, h_\ell\}$. Applying $\mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(Y, -)$ to (5.1) yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}^1(Y, X) \longrightarrow \mathrm{Hom}^1(Y, \widehat{X}) \longrightarrow \mathrm{Hom}^1(Y, S^{\oplus n})[-1] \longrightarrow 0,$$

so $\mathrm{Hom}^\bullet(Y, \widehat{X}) \cong \mathrm{Hom}^1(Y, X) \oplus \mathrm{Hom}^1(Y, S^{\oplus n})$. A basis for $\mathrm{Hom}^1(Y, S^{\oplus n})[-1]$ is $\{i_k \circ g_j \mid 1 \le k \le n,\ 1 \le j \le m\}[-1]$, where i_k denotes inclusion into the k-th copy of S. Thus $Q_{v_S\mathcal{C}}$ contains mn additional arrows $Y \longrightarrow \widehat{X}$, giving the following arrangement.

$$\begin{tikzcd}
S[-1] \arrow[d, "g_1^*,\dots,g_m^*"'] & \widehat{X} \arrow[l, "f_1^\vee,\dots,f_n^\vee"'] \\
Y \arrow[ur, "h_1,\dots,h_\ell;\ (g_j f_k)"'] &
\end{tikzcd}$$

Here $(g_j f_k)$ is the preimage of $i_k \circ g_j$ in $\mathrm{Hom}^1(Y, \widehat{X})$, and we identify h_i with its image in that space.

2. Assume $\mathrm{Hom}^\bullet(Y, X) = \mathrm{Hom}^2(Y, X) \cong \mathrm{Hom}^1(X, Y)^*$. Performing a change of arrows if necessary [17, Definition 2.5], assume $B(X, Y) = \{h_1, \dots, h_\ell\}$ such that, for all $1 \le j \le m$ and $1 \le k \le n$, $f_k \circ g_j$ is either 0 or h_i^* for some

$1 \leq i \leq \ell$. This corresponds to an arrangement

$$\begin{tikzcd}
S \arrow[r, "f_1,\dots,f_n"] & X \arrow[dl, "h_1,\dots,h_\ell"] \\
Y \arrow[u, "g_1,\dots,g_m"] &
\end{tikzcd}$$

in $Q_{\mathcal{C}}$. Applying $\operatorname{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(Y,-)$ to (5.1) yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}^1(Y,\widehat{X}) \longrightarrow \operatorname{Hom}^1(Y,S^{\oplus n})[-1] \xrightarrow{(f_1\cdots f_n)\circ -} \operatorname{Hom}^1(X,Y)^* \longrightarrow \operatorname{Hom}^2(Y,\widehat{X}) \longrightarrow 0,$$

so $\operatorname{Hom}^\bullet(Y,\widehat{X}) \cong \operatorname{Ker}((f_1\dots f_n)\circ -)[1] \oplus \operatorname{Coker}((f_1\dots f_n)\circ -)$. Note that a basis for $\operatorname{Hom}^1(Y,S^{\oplus n})[-1] \oplus \operatorname{Hom}^1(X,Y)^*$ is

$$\{i_k \circ g_j \mid 1 \leq k \leq n,\ 1 \leq j \leq m\}[-1] \cup \{h_1^*,\dots,h_\ell^*\},$$

which is in bijection with the arrows between Y and $\widehat{X}$ in the intermediate quiver $\widetilde{Q_{\mathcal{C}}}$ from DWZ mutation. Write (g_jf_k) for the preimage of $i_k \circ g_j$ in $\operatorname{Hom}^1(Y,\widehat{X})$, and identify h_i^* with its image in $\operatorname{Hom}^2(Y,\widehat{X})$. Using that $(f_1\cdots f_n)\circ(i_k\circ g_j) = f_k \circ g_j$, a basis for $\operatorname{Hom}^\bullet(Y,\widehat{X})$ is

$$(\{(g_jf_k) \mid 1 \leq k \leq n,\ 1 \leq j \leq m\} \cup \{h_1^*,\dots,h_\ell^*\})\backslash\{(g_jf_k),h_i^* \mid f_k \circ g_j = h_i^*\}.$$

But $f_k \circ g_j = h_i^*$ if and only if the cycle $h_ig_jf_k$ appears in $W_{\mathcal{C}}$, if and only if the cycle $h_j[g_jf_k]$ appears in $\widetilde{W_{\mathcal{C}}}$, if and only if this pair is deleted when moving from $\widetilde{Q_{\mathcal{C}}}$ to $\overline{Q_{\mathcal{C}}}$. Hence the arrows between Y and $\widehat{X}$ in $Q_{v_S\mathcal{C}}$ coincide with those in $\overline{Q_{\mathcal{C}}}$.

Let $Z \in \mathcal{Z}$. Then $\operatorname{Hom}^\bullet(S^{\oplus n}[-1],Z) = 0$ so $\operatorname{Hom}^\bullet(\widehat{X},Z) \cong \operatorname{Hom}^\bullet(X,Z)$. Hence the arrows between $\widehat{X}$ and Z are unchanged in $Q_{v_S\mathcal{C}}$.

Let $X' \in \mathcal{X}$ with $B(S,X') = \{f_1',\dots,f_{n'}'\}$. There is a triangle

$$S^{\oplus n'}[-1] \xrightarrow{(f_1'\cdots f_{n'}')} X' \longrightarrow \widehat{X'} \longrightarrow S^{\oplus n'} \tag{5.2}$$

in $\mathbf{K}_{\mathrm{dg}}(A)$. Note that $\mathrm{Hom}^{\bullet}(S^{\oplus n}[-1], X') = \mathrm{Hom}^1(S^{\oplus n}, X')[1]$. Arguing as above, we see that $\mathrm{Hom}^{\bullet}(\widehat{X}, S^{\oplus n'}[-1]) \cong \mathrm{Hom}^0(S^{\oplus n}, S^{\oplus n'})[-1]$.

1. If $X \ncong X'$ then applying $\mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(-, X')$ to (5.1) yields exact sequences

$$0 \longrightarrow \mathrm{Hom}^1(S^{\oplus n}, X')[1] \longrightarrow \mathrm{Hom}^1(\widehat{X}, X') \longrightarrow \mathrm{Hom}^1(X, X') \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}^2(\widehat{X}, X') \longrightarrow \mathrm{Hom}^2(X, X') \longrightarrow 0,$$

where at most one of $\mathrm{Hom}^1(X, X')$, $\mathrm{Hom}^2(X, X')$ is non-zero. In particular, $\mathrm{Hom}^1(\widehat{X}, X') \cong \mathrm{Hom}^1(S^{\oplus n}, X') \oplus \mathrm{Hom}^1(X, X')$, so applying $\mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(\widehat{X}, -)$ to (5.2) yields exact sequences

$$\mathrm{Hom}^0(S^{\oplus n}, S^{\oplus n'})[-1] \overset{(f_1' \cdots f_n') \circ -}{\longrightarrow} \mathrm{Hom}^1(S^{\oplus n}, X') \oplus \mathrm{Hom}^1(X, X') \longrightarrow \mathrm{Hom}^1(\widehat{X}, \widehat{X'}) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}^2(X, X') \longrightarrow \mathrm{Hom}^2(\widehat{X}, \widehat{X'}) \longrightarrow 0.$$

A basis for $\mathrm{Hom}^0(S^{\oplus n}, S^{\oplus n'})[-1]$ is $\{i_k \circ p_j \mid 1 \leq j \leq n,\ 1 \leq k \leq n'\}[-1]$, while a basis for $\mathrm{Hom}^1(S^{\oplus n}, X')$ is $\{f_k' \circ p_j \mid 1 \leq j \leq n,\ 1 \leq k \leq n'\}$. Now $(f_1' \cdots f_n') \circ (i_k \circ p_j) = f_k' \circ p_j$, so these spaces are isomorphic and we deduce that $\mathrm{Hom}^{\bullet}(\widehat{X}, \widehat{X'}) \cong \mathrm{Hom}^{\bullet}(X, X')$. Hence the arrows between $\widehat{X}$ and $\widehat{X'}$ are unchanged in $Q_{v_S \mathcal{C}}$.

2. If $X \cong X'$ then, since X is spherical, applying $\mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(-, X)$ to (5.1) yields exact sequences

$$0 \longrightarrow \mathrm{Hom}^0(\widehat{X}, X) \longrightarrow \mathrm{Hom}^0(X, X) \overset{- \circ (f_1 \cdots f_n)}{\longrightarrow} \mathrm{Hom}^1(S^{\oplus n}, X)[1] \longrightarrow \mathrm{Hom}^1(\widehat{X}, X) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}^3(\widehat{X}, X) \longrightarrow \mathrm{Hom}^3(X, X) \longrightarrow 0.$$

Since $\mathrm{Hom}^0(X, X)$ is generated by id_X,

$$\mathrm{Hom}^{\bullet}(\widehat{X}, X) \cong \frac{\mathrm{Hom}^1(S^{\oplus n}, X)}{\langle (f_1 \cdots f_n) \rangle} \oplus \mathrm{Hom}^3(X, X).$$

Hence applying $\mathrm{Hom}_{\mathbf{K}_{\mathrm{dg}}(A)}(\widehat{X},-)$ to (5.1) yields exact sequences

$$0 \longrightarrow \mathrm{Hom}^0(\widehat{X},\widehat{X}) \longrightarrow \mathrm{Hom}^0(S^{\oplus n},S^{\oplus n})[-1] \xrightarrow{(f_1\cdots f_n)\circ-} \frac{\mathrm{Hom}^1(S^{\oplus n},X)}{\langle (f_1\cdots f_n)\rangle} \longrightarrow \mathrm{Hom}^1(\widehat{X},\widehat{X}) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}^3(X,X) \longrightarrow \mathrm{Hom}^3(\widehat{X},\widehat{X}) \longrightarrow 0.$$

Now $\mathrm{Hom}^0(S^{\oplus n},S^{\oplus n})[-1]$ surjects onto $\frac{\mathrm{Hom}^1(S^{\oplus n},X)}{\langle (f_1\cdots f_n)\rangle}$ via $(f_1\cdots f_n)\circ -$, with kernel generated by $\sum_{j=1}^n i_j \circ p_j$. In particular, $\widehat{X}$ is spherical, meaning there are no loops at $\widehat{X}$ in $Q_{v_S\mathcal{C}}$.

Finally, let $E, F \in \mathcal{Y}\cup\mathcal{Z}$. These objects, and therefore the morphisms between them, are unchanged moving from $\mathcal{C}$ to $v_S\mathcal{C}$. Hence the arrows between them are also unchanged moving from $Q_\mathcal{C}$ to $Q_{v_S\mathcal{C}}$.

We conclude that $Q_{v_S\mathcal{C}} \cong \overline{Q_\mathcal{C}}$. □

Example 5.1.8. Consider $Z = Z_2^3$, retaining the notation of Example 4.2.3. Let $\mathcal{H}_2$ be the smallest full extension-closed subcategory of $\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z)$ containing Z and closed under direct summands. Then $\mathcal{H}_2$ is the heart of a bounded t-structure, whose set of simple objects $\mathcal{P}$ consists of the projective Z-modules. These have trivial differentials, so the same is true of the $\mathcal{H}om$-complexes between them. Hence the A_∞-structure on $\mathrm{End}^\bullet_{\mathbf{H}_{\mathrm{dg}}(Z)}(\mathcal{P})$ has $m_n = 0$ for all $n \neq 2$. In fact, it is 3-cyclic and $\mathcal{P}$ is cluster at all its objects, so we can construct $(Q_\mathcal{P}, W_\mathcal{P})$ and test Proposition 5.1.6. Indeed, $Q_\mathcal{P}$ is

$$\begin{tikzcd}
& & e_6Z \arrow[rd, "f_{56}"] & & \\
& e_4Z \arrow[ru, "f_{64}"] \arrow[rd, "f_{24}"] & & e_5Z \arrow[ll, "f_{45}"] \arrow[rd, "f_{35}"] & \\
e_1Z \arrow[ru, "f_{41}"] & & e_2Z \arrow[ll, "f_{12}"] \arrow[ru, "f_{52}"] & & e_3Z \arrow[ll, "f_{23}"]
\end{tikzcd}$$

where $f_{ij} \colon e_jZ \to e_iZ$ is left multiplication by α_{ij}, and

$$W_{\mathcal{P}} = f_{12}f_{41}f_{24} + f_{23}f_{52}f_{35} + f_{45}f_{64}f_{56} - f_{45}f_{24}f_{52}.$$

The term $f_{12}f_{41}f_{24}$, for example, encodes the relations

$$f^*_{24} = f_{41} \circ f_{12},\ f^*_{41} = f_{12} \circ f_{24},\ f^*_{12} = f_{24} \circ f_{41}.$$

We now perform KS mutation at e_2Z. By direct computation, $Q_{\upsilon_2\mathcal{P}}$ is as follows.

$$\begin{tikzcd}
 & e_6Z \arrow[dr, "f_{56}"] & \\
e_4Z \arrow[ur, "f_{64}"] & & \widehat{e_5Z} \arrow[dl, "f^\vee_{52}"'] \\
 & e_2Z[-1] \arrow[ul, "f^*_{24}"'] \arrow[dr, "f^*_{23}"] & \\
\widehat{e_1Z} \arrow[ur, "f^\vee_{12}"] & & e_3Z \arrow[ll, "(f_{23}f_{12})"]
\end{tikzcd}$$

For each $i \in \{1, 5\}$,

$$\widehat{e_iZ} = \operatorname{cone}(e_2Z[-1] \xrightarrow{f_{i2}} e_iZ) = \left(e_iZ \oplus e_2Z, \begin{pmatrix} 0 & f_{i2} \\ 0 & 0 \end{pmatrix} \right),$$

while

$$f^\vee_{i2} = \begin{pmatrix} 0 & \mathrm{id}_2 \end{pmatrix} \colon e_iZ \oplus e_2Z \to e_2Z[-1],$$

$$(f_{23}f_{12}) = \begin{pmatrix} 0 \\ f_{23} \end{pmatrix} \colon e_3Z \to e_1Z \oplus e_2Z,$$

and we identify f_{56} with

$$\begin{pmatrix} f_{56} \\ 0 \end{pmatrix} : e_6Z \to e_5Z \oplus e_2Z$$

as in the proof of Proposition 5.1.6.

Performing DWZ mutation on $(Q_{\mathcal{P}}, W_{\mathcal{P}})$, one finds that $\widetilde{Q_{\mathcal{P}}}$ is

$$\begin{tikzcd}
& & e_6Z \arrow[dr, "f_{56}"] & & \\
& e_4Z \arrow[ur, "f_{64}"] \arrow[rr, shift left, "{[f_{24}f_{52}]}"] \arrow[dl, shift left, "{[f_{24}f_{12}]}"] & & e_5Z \arrow[ll, shift left, "f_{45}"] \arrow[dl, "f_{52}^\star"'] \arrow[dr, shift left, "f_{35}"] & \\
e_1Z \arrow[ur, shift left, "f_{41}"] \arrow[rr, "f_{12}^\star"'] & & e_2Z \arrow[ul, "f_{24}^\star"'] \arrow[rr, "f_{23}^\star"'] & & e_3Z \arrow[ul, shift left, "{[f_{23}f_{52}]}"] \arrow[llll, bend left, "{[f_{23}f_{12}]}"]
\end{tikzcd}$$

and

$$\begin{aligned}\widetilde{W_{\mathcal{P}}} = [f_{24}f_{12}]f_{41} + [f_{23}f_{52}]f_{35} + f_{45}f_{64}f_{56} - f_{45}[f_{24}f_{52}] \\ + [f_{24}f_{12}]f_{12}^\star f_{24}^\star + [f_{23}f_{12}]f_{12}^\star f_{23}^\star + [f_{23}f_{52}]f_{52}^\star f_{23}^\star + [f_{24}f_{52}]f_{52}^\star f_{24}^\star.\end{aligned}$$

Splitting $(\widetilde{Q_{\mathcal{P}}}, \widetilde{W_{\mathcal{P}}})$ into its reduced and trivial part, one verifies that $\overline{Q_{\mathcal{P}}} \cong Q_{v_2\mathcal{P}}$.

We expect (but have not proved) that the A_∞-structure on $\mathbf{H}_{\mathrm{dg}}(Z_2^n)$ is 3-cyclic for all $n \geq 2$.

5.2 A group action

Definition 5.2.1. [30, Example 4.12] Given $n \geq 2$, let G_2^n be the group with a generator s_x for each $x \in \mathcal{A}_0^n$, and the following relations.

Commutativity: $s_x s_y = s_y s_x$ for all x, y not adjacent in $\mathcal{A}^n$.

Reidemeister 3: $s_x s_y s_x = s_y s_x s_y$ for all $(x \longrightarrow y) \in \mathcal{A}_1^n$.

Cyclic: $s_x s_y s_z s_x = s_y s_z s_x s_y = s_z s_x s_y s_z$ for each 3-cycle $x \longrightarrow y \longrightarrow z \longrightarrow x$ in $\mathcal{A}^n$.

Grant showed there is an action of G_2^n on $\mathbf{D}^{\mathrm{b}}(Z_2^n)$ (with Z_2^n considered as an ordinary algebra) such that s_x acts as the *spherical twist* Tw_x around $e_x Z_2^n$ [30, Theorem 4.24]. In other words, such that

$$s_x(E) = \mathrm{cone}\left(\mathrm{Hom}_{Z_2^n}(e_x Z_2^n, E) \otimes e_x Z_2^n \xrightarrow{\mathrm{ev}} E\right).$$

We prove that the analogous result holds in our setting. The method we use is essentially the same as in [30]. First, we need a technical Lemma.

Lemma 5.2.2. *Let A, B be dg algebras, and let (M, d) be a dg A-B-bimodule such that*

1. $M \cong N \oplus L$ *in* $\mathrm{grmod}(B \otimes A^{\mathrm{op}})$ *for some* N, L;
2. $d(N) \subseteq N$.

Each $s \in \mathrm{Hom}_{\mathrm{grmod}(B \otimes A^{\mathrm{op}})}(L, N)$ *induces a homotopy equivalence*

$$\mathrm{id} + s \colon (M, d - sd + ds) \to (M, d)$$

of dg A-B-bimodules.

Proof. Since $d(N) \subseteq N$, there exist differentials d_N, d_L on N, L and a map of dg bimodules $f \colon (L[-1], -d_L) \to (N, d_N)$ such that $(M, d) \cong \mathrm{cone}(f)$. Given a map $s \colon L \to N$ of graded bimodules, it is standard that f is homotopic to $f - sd_L + d_N s$, leading to a homotopy equivalence $\mathrm{id} + s \colon \mathrm{cone}(f - sd_L + d_N s) \to \mathrm{cone}(f)$. The domain is isomorphic to $(M, d - sd + ds)$. □

Proposition 5.2.3. *There is an action of G_2^n on $\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z_2^n)$ such that s_x acts as the spherical twist Tw_x around $e_x Z_2^n$. In other words, such that*

$$s_x(E) = \mathrm{cone}\left(\mathcal{H}om_{Z_2^n}(e_x Z_2^n, E) \otimes e_x Z_2^n \xrightarrow{\mathrm{ev}} E\right).$$

Proof. To simplify notation, write $Z = Z_2^n$. We need to check that the spherical twists obey the relations of Definition 5.2.1. Note that Tw_x is naturally isomorphic to $- \otimes_Z M_x$, where

$$M_x \coloneqq \operatorname{cone}(Ze_x \otimes e_x Z \overset{m}{\to} Z) = \left(Z \oplus (Ze_x \otimes e_x Z)[1], \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right),$$

m being the multiplication in Z. Given dg Z-bimodules M and N, we write MN for $M \otimes_Z N$.

Commutativity: Let $x, y \in \mathcal{A}_0^n$. Note that

$$M_x(Ze_y \otimes e_y Z) \cong \left((Ze_y \otimes e_y Z) \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Z)[1], \begin{pmatrix} 0 & m \otimes \mathrm{id} \\ 0 & 0 \end{pmatrix} \right).$$

Therefore $M_x M_y \cong \operatorname{cone}(M_x(Ze_y \otimes e_y Z) \overset{\mathrm{id}\, m}{\to} M_x Z)$ is isomorphic to

$$Z \oplus (Ze_x \otimes e_x Z)[1] \oplus (Ze_y \otimes e_y Z)[1] \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Z)[2], \text{ with differential}$$

$$\begin{pmatrix} 0 & m & m & 0 \\ 0 & 0 & 0 & \mathrm{id} \otimes m \\ 0 & 0 & 0 & -m \otimes \mathrm{id} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If x, y are not adjacent in $\mathcal{A}^n$ then $e_x Z e_y = 0$, so the final term vanishes. Reversing the roles of x and y shows $M_x M_y \cong M_y M_x$. Thus there is a natural isomorphism $\mathrm{Tw}_y \circ \mathrm{Tw}_x \cong \mathrm{Tw}_x \circ \mathrm{Tw}_y$ in $\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z)$.

Reidemeister 3: Let $(x \longrightarrow y) \in \mathcal{A}_1^n$. Then $M_x M_y (Ze_x \otimes e_x Z)$ is isomorphic to

$$\begin{aligned} &(Ze_x \otimes e_x Z) \oplus (Ze_x \otimes e_x Ze_x \otimes e_x Z)[1] \oplus (Ze_y \otimes e_y Ze_x \otimes e_x Z)[1] \\ &\qquad \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Ze_x \otimes e_x Z)[2], \text{ with differential} \end{aligned}$$

$$\begin{pmatrix} 0 & m\otimes \mathrm{id} & m\otimes \mathrm{id} & 0 \\ 0 & 0 & 0 & \mathrm{id}\otimes m\otimes \mathrm{id} \\ 0 & 0 & 0 & -m\otimes \mathrm{id}^{\otimes 2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\begin{aligned} Ze_x\otimes e_xZe_x\otimes e_xZ &\cong (Ze_x\otimes \langle e_x\rangle \otimes e_xZ)\oplus (Ze_x\otimes \langle e_x^*\rangle\otimes e_xZ) \\ &\cong (Ze_x\otimes e_xZ)\oplus (Ze_x\otimes e_xZe_y\otimes e_yZe_x\otimes e_xZ). \end{aligned} \tag{5.3}$$

Hence $M_xM_y(Ze_x\otimes e_xZ)$ is isomorphic to

$$\begin{aligned} &(Ze_x\otimes e_xZ)\oplus (Ze_x\otimes e_xZ)[1]\oplus (Ze_y\otimes e_yZe_x\otimes e_xZ)[1] \\ &\qquad \oplus (Ze_x\otimes e_xZe_y\otimes e_yZe_x\otimes e_xZ)[1]\oplus (Ze_x\otimes e_xZe_y\otimes e_yZe_x\otimes e_xZ)[2], \end{aligned}$$

with differential

$$\left(\begin{array}{cc|ccc} 0 & \mathrm{id}^{\otimes 2} & m\otimes \mathrm{id} & m_3\otimes \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -m\otimes \mathrm{id}^{\otimes 2} \\ 0 & 0 & 0 & 0 & \mathrm{id}^{\otimes 4} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Here m_3 denotes the multiplication of three elements in Z. Applying Lemma 5.2.2 with

$$\begin{aligned} N &= (Ze_x\otimes e_xZ)\oplus (Ze_x\otimes e_xZ)[1], \\ L &= (Ze_y\otimes e_yZe_x\otimes e_xZ)[1]\oplus (Ze_x\otimes e_xZe_y\otimes e_yZe_x\otimes e_xZ)[1] \\ &\qquad \oplus (Ze_x\otimes e_xZe_y\otimes e_yZe_x\otimes e_xZ)[2], \\ s &= \begin{pmatrix} 0 & 0 & 0 \\ -m\otimes \mathrm{id} & -m_3\otimes \mathrm{id} & 0 \end{pmatrix} \end{aligned}$$

shows that $M_xM_y(Ze_x \otimes e_xZ)$ is homotopy equivalent to

$$\cancel{\left(N, \begin{pmatrix} 0 & \mathrm{id}^{\otimes 2} \\ 0 & 0 \end{pmatrix}\right)} \oplus \left(L, \left(\begin{array}{c|cc} 0 & 0 & -m \otimes \mathrm{id}^{\otimes 2} \\ \hline 0 & 0 & \mathrm{id}^{\otimes 4} \\ 0 & 0 & 0 \end{array}\right)\right),$$

where we delete the contractible first term. Another application of Lemma 5.2.2 with

$$\begin{aligned} N' &= (Ze_y \otimes e_yZe_x \otimes e_xZ)[1] \\ L' &= (Ze_x \otimes e_xZe_y \otimes e_yZe_x \otimes e_xZ)[1] \oplus (Ze_x \otimes e_xZe_y \otimes e_yZe_x \otimes e_xZ)[2], \\ s' &= \begin{pmatrix} -m \otimes \mathrm{id}^{\otimes 2} & 0 \end{pmatrix} \end{aligned}$$

shows that $M_xM_y(Ze_x \otimes e_xZ)$ is homotopy equivalent to

$$(N', 0) \oplus \cancel{\left(L', \begin{pmatrix} 0 & \mathrm{id}^{\otimes 4} \\ 0 & 0 \end{pmatrix}\right)},$$

where we delete the contractible second term.

Therefore $M_xM_yM_x \cong \mathrm{cone}(M_xM_y(Ze_x \otimes e_xZ) \overset{\mathrm{id}\, m}{\to} M_xM_yZ)$ is homotopy equivalent to

$$\begin{aligned} &Z \oplus (Ze_x \otimes e_xZ)[1] \oplus (Ze_y \otimes e_yZ)[1] \oplus (Ze_x \otimes e_xZe_y \otimes e_yZ)[2] \\ &\quad \oplus (Ze_y \otimes e_yZe_x \otimes e_xZ)[2], \text{ with differential} \end{aligned}$$

$$\begin{pmatrix} 0 & m & m & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} \otimes m & -m \otimes \mathrm{id} \\ 0 & 0 & 0 & -m \otimes \mathrm{id} & \mathrm{id} \otimes m \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Reversing the roles of x and y shows $M_xM_yM_x$ is homotopy equivalent to $M_yM_xM_y$, so there is a natural isomorphism $\mathrm{Tw}_x \circ \mathrm{Tw}_y \circ \mathrm{Tw}_x \cong \mathrm{Tw}_y \circ \mathrm{Tw}_x \circ \mathrm{Tw}_y$ in $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z)$.

Cyclic: Let $x \longrightarrow y \longrightarrow z \longrightarrow x$ be a cycle in $\mathcal{A}^n$. Then $M_x M_y(Ze_z \otimes e_z Z)$ is isomorphic to

$$\begin{aligned}(Ze_z \otimes e_z Z) \oplus (Ze_y \otimes e_y Ze_z \otimes e_z Z)[1] \oplus (Ze_x \otimes e_x Ze_z \otimes e_z Z)[1]\\ \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z)[2], \text{ with differential}\end{aligned}$$

$$\begin{pmatrix} 0 & m \otimes \mathrm{id} & m \otimes \mathrm{id} & 0 \\ 0 & 0 & 0 & -m \otimes \mathrm{id}^{\otimes 2} \\ 0 & 0 & 0 & \mathrm{id} \otimes m \otimes \mathrm{id} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $Ze_x \otimes e_x Ze_z \otimes e_z Z \cong Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z$. Hence $M_x M_y(Ze_z \otimes e_z Z)$ is isomorphic to

$$\begin{aligned}(Ze_z \otimes e_z Z) \oplus (Ze_y \otimes e_y Ze_z \otimes e_z Z)[1] \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z)[1]\\ \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z)[2], \text{ with differential}\end{aligned}$$

$$\left(\begin{array}{cc|cc} 0 & m \otimes \mathrm{id} & m_3 \otimes \mathrm{id} & 0 \\ 0 & 0 & 0 & -m \otimes \mathrm{id}^{\otimes 2} \\ \hline 0 & 0 & 0 & \mathrm{id}^{\otimes 4} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Applying Lemma 5.2.2 with

$$\begin{aligned} N &= (Ze_z \otimes e_z Z) \oplus (Ze_y \otimes e_y Ze_z \otimes e_z Z)[1], \\ L &= (Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z)[1] \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Ze_z \otimes e_z Z)[2], \\ s &= \begin{pmatrix} 0 & 0 \\ -m \otimes \mathrm{id}^{\otimes 2} & 0 \end{pmatrix} \end{aligned}$$

shows that $M_x M_y(Ze_x \otimes e_x Z)$ is homotopy equivalent to

$$\left(N, \begin{pmatrix} 0 & m \otimes \mathrm{id} \\ 0 & 0 \end{pmatrix}\right) \oplus \cancel{\left(L, \begin{pmatrix} 0 & \mathrm{id}^{\otimes 4} \\ 0 & 0 \end{pmatrix}\right)},$$

where we delete the contractible second term.

Hence $M_xM_yM_z \cong \mathrm{cone}(M_xM_y(Ze_z \otimes e_zZ) \overset{\mathrm{id}\, m}{\to} M_xM_yZ)$ is homotopy equivalent to

$$\begin{aligned} &Z \oplus (Ze_x \otimes e_xZ)[1] \oplus (Ze_y \otimes e_yZ)[1] \oplus (Ze_z \otimes e_zZ)[1] \oplus (Ze_x \otimes e_xZ \otimes e_xZ)[2] \\ &\quad \oplus (Ze_y \otimes e_yZe_z \otimes e_zZ)[2], \text{ with differential} \end{aligned}$$

$$\begin{pmatrix} 0 & m & m & m & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}\otimes m & 0 \\ 0 & 0 & 0 & 0 & -m\otimes \mathrm{id} & \mathrm{id}\otimes m \\ 0 & 0 & 0 & 0 & 0 & -m\otimes \mathrm{id} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now, $M_xM_yM_z(Ze_x \otimes e_xZ)$ is isomorphic to

$$\begin{aligned} &(Ze_x \otimes e_xZ) \oplus (Ze_x \otimes e_xZe_x \otimes e_xZ)[1] \oplus (Ze_y \otimes e_yZe_x \otimes e_xZ)[1] \\ &\qquad \oplus (Ze_z \otimes e_zZe_x \otimes e_xZ)[1] \oplus (Ze_x \otimes e_xZe_y \otimes e_yZe_x \otimes e_xZ)[2] \\ &\qquad \oplus (Ze_y \otimes e_yZe_z \otimes e_zZe_x \otimes e_xZ)[2], \text{ with differential} \end{aligned}$$

$$\begin{pmatrix} 0 & m\otimes \mathrm{id} & m\otimes \mathrm{id} & m\otimes \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}\otimes m\otimes \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 & -m\otimes \mathrm{id}^{\otimes 2} & \mathrm{id}\otimes m\otimes \mathrm{id} \\ 0 & 0 & 0 & 0 & 0 & -m\otimes \mathrm{id}^{\otimes 2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Recalling (5.3), by twice applying Lemma 5.2.2 and deleting contractible terms as we did for *Reidemeister 3*, we find that $M_xM_yM_z(Ze_x \otimes e_xZ)$ is homotopy equivalent to

$$\begin{aligned}(Ze_y \otimes e_y Ze_x \otimes e_x Z)[1] \oplus (Ze_z \otimes e_z Ze_x \otimes e_x Z)[1]& \\ \oplus (Ze_y \otimes e_y Ze_z \otimes e_z Ze_x \otimes e_x Z)[2],&\ \text{with differential}\end{aligned}$$

$$\begin{pmatrix} 0 & 0 & \mathrm{id} \otimes m \otimes \mathrm{id} \\ 0 & 0 & -m \otimes \mathrm{id}^{\otimes 2} \end{pmatrix}.$$

Noting $Ze_y \otimes e_y Ze_x \otimes e_x Z \cong Ze_y \otimes e_y Ze_z \otimes e_z Ze_x \otimes e_x Z$, by applying Lemma 5.2.2 and deleting a contractible term as above, we find that $M_x M_y M_z (Ze_x \otimes e_x Z)$ is homotopy equivalent to $((Ze_z \otimes e_z Ze_x \otimes e_x Z)[1], 0)$.

Therefore $M_x M_y M_z M_x \cong \mathrm{cone}(M_x M_y M_z (Ze_x \otimes e_x Z) \overset{\mathrm{id}\, m}{\to} M_x M_y M_z Z)$ is homotopy equivalent to

$$\begin{aligned}Z \oplus (Ze_x \otimes e_x Z)[1] \oplus (Ze_y \otimes e_y Z)[1] \oplus (Ze_z \otimes e_z Z)[1] \oplus (Ze_x \otimes e_x Ze_y \otimes e_y Z)[2]& \\ \oplus (Ze_y \otimes e_y Ze_z \otimes e_z Z)[2] \oplus (Ze_z \otimes e_z Ze_x \otimes e_x Z)[2],\ \text{with differential}&\end{aligned}$$

$$\begin{pmatrix} 0 & m & m & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \otimes m & 0 & -m \otimes \mathrm{id} \\ 0 & 0 & 0 & 0 & -m \otimes \mathrm{id} & \mathrm{id} \otimes m & 0 \\ 0 & 0 & 0 & 0 & 0 & -m \otimes \mathrm{id} & \mathrm{id} \otimes m \end{pmatrix}.$$

Cyclicly permuting the roles of x, y, z shows that $M_x M_y M_z M_x$, $M_y M_z M_x M_y$ and $M_z M_x M_y M_z$ are all homotopy equivalent. Hence there are natural isomorphisms

$$\mathrm{Tw}_x \circ \mathrm{Tw}_z \circ \mathrm{Tw}_y \circ \mathrm{Tw}_x \cong \mathrm{Tw}_y \circ \mathrm{Tw}_x \circ \mathrm{Tw}_z \circ \mathrm{Tw}_y \cong \mathrm{Tw}_z \circ \mathrm{Tw}_y \circ \mathrm{Tw}_x \circ \mathrm{Tw}_z$$

in $\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z)$. □

Let $\mathscr{H} \subset \mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z)$ be the smallest full extension-closed subcategory containing Z and closed under direct summands. Denote by $\mathrm{Stab}^\circ(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z))$ the connected component of $\mathrm{Stab}(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z))$ containing stability conditions supported on $\mathscr{H}$. Given [49, Proposition 5.4], it is easy to check that $\mathrm{Tw}_x \mathscr{H} \simeq (\mathscr{H}^\flat_{e_x Z})^\flat_{e_x Z[-1]}$ for all $x \in \mathcal{A}_0^n$.

Since autoequivalences commute with tilting, $\mathrm{Tw}_x(\mathscr{H}^\flat_{e_y Z}) \simeq (\mathrm{Tw}_x \mathscr{H})^\flat_{\mathrm{Tw}_x(e_y Z)}$ for all $x, y \in \mathcal{A}^n_0$. The analogous statements connecting $(-)^\sharp$ and Tw^- also hold, so the action of G^n_2 on $\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z)$ preserves the connected component $\mathrm{Stab}^\circ(\mathbf{K}^{\mathrm{per}}_{\mathrm{dg}}(Z))$.

Future work

1. **Koszul duality.** For all $n \geq 2$, we expect that $\Gamma(\mathcal{A}^n, W^n_{\mathcal{A}})^! \cong Z^n_2$ as dg algebras.

   Example. Let $n = 2$, retaining the notation of Example 1.3.10. We show that $\Gamma(\mathcal{A}^2, W^2_{\mathcal{A}})^! \cong Z^2_2$ as cochain complexes. Following [48, §2.14], we have a cofibrant resolution

   $$0 \longrightarrow \operatorname{Ker}(\pi_i) \longrightarrow e_i\Gamma \xrightarrow{\pi_i} \langle e_i \rangle \longrightarrow 0$$

   for each $i \in \{1,2,3\}$. Hence $C_i \coloneqq \operatorname{cone}(\operatorname{Ker}(\pi_i) \to e_i\Gamma)$ is a cofibrant replacement for $\langle e_i \rangle$. As graded Γ-modules,

   $$\begin{aligned} C_1 &\cong e_1\Gamma \oplus (t_1\Gamma \oplus \alpha\Gamma \oplus \gamma^*\Gamma)[1] \cong e_1\Gamma \oplus e_1\Gamma[3] \oplus e_2\Gamma[1] \oplus e_3\Gamma[2], \\ C_2 &\cong e_2\Gamma \oplus (\alpha^*\Gamma \oplus t_2\Gamma \oplus \beta\Gamma)[1] \cong e_1\Gamma[2] \oplus e_2\Gamma \oplus e_2\Gamma[3] \oplus e_3\Gamma[1], \\ C_3 &\cong e_3\Gamma \oplus (\gamma\Gamma \oplus \beta^*\Gamma \oplus t_3\Gamma)[1] \cong e_1\Gamma[1] \oplus e_2\Gamma[2] \oplus e_3\Gamma \oplus e_3\Gamma[3]. \end{aligned}$$

   Note that

   $$\operatorname{Hom}_{\operatorname{grmod}\Gamma}(e_i\Gamma, \langle e_j \rangle\{n\}) = \begin{cases} \langle \pi_i \rangle & \text{if } i = j,\ n = 0, \\ 0 & \text{else.} \end{cases}$$

   Therefore $\mathcal{H}om_{\Gamma}(C_i, \langle e_j \rangle)$ has trivial differential for all $i, j \in \{1,2,3\}$ (so in particular, $\Gamma^!$ has trivial differential). Hence

   $$\mathbf{R}\,\mathcal{H}om_{\Gamma}(\langle e_i \rangle, \langle e_j \rangle) = \mathcal{H}om_{\Gamma}(C_i, \langle e_j \rangle) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{grmod}\Gamma}(C_i, \langle e_j \rangle\{n\}). \tag{D}$$

Hence for each $i \in \{1,2,3\}$, one computes that

$$\begin{aligned}\mathbf{R}\,\mathcal{H}om_{\Gamma}(\langle e_i\rangle, \langle e_i\rangle) &\cong k \oplus k[-3],\\ \mathbf{R}\,\mathcal{H}om_{\Gamma}(\langle e_i\rangle, \langle e_{i+1}\rangle) &\cong k[-1],\\ \mathbf{R}\,\mathcal{H}om_{\Gamma}(\langle e_i\rangle, \langle e_{i-1}\rangle) &\cong k[-2],\end{aligned}$$

where the subscripts should be taken modulo 3. Hence $\Gamma(\mathcal{A}^2, W^2_{\mathcal{A}})^! \cong Z^2_2$ as cochain complexes.

Note that this argument may be readily generalised to $n > 2$.

2. **Tilting and mutation.** Recall Proposition 5.1.6. We were able to show that $Q_{\upsilon_S \mathcal{C}} \cong \overline{Q_{\mathcal{C}}}$, but unable to prove that the potentials coincide. Again, we believe this is well-known, but haven't seen a proof written down on this side of Koszul duality.

   Example. Recall Example 5.1.8. After relabelling to match the notation of $Q_{\upsilon_2 \mathcal{P}}$, the reduced potential from the Derksen-Weyman-Zelevinsky mutation is

   $$\overline{W_{\mathcal{P}}} = (f_{23}f_{12})f^{\vee}_{12}f^{*}_{23} - f_{64}f_{56}f^{\vee}_{52}f^{*}_{24}.$$

   This indeed coincides with $W_{\upsilon_2 \mathcal{P}}$. The term $(f_{23}f_{12})f^{\vee}_{12}f^{*}_{23}$ encodes the relations

   $$f_{23} = f^{\vee}_{12} \circ (f_{23}f_{12}),\ (f^{\vee}_{12})^{*} = (f_{23}f_{12}) \circ f^{*}_{23},\ (f_{23}f_{12})^{*} = f^{*}_{23} \circ f^{\vee}_{12}.$$

   The appearance of a 4-cycle reflects the fact that the objects $\widehat{e_1 Z}$ and $\widehat{e_5 Z}$ have non-trivial differentials. This means that certain $\mathcal{H}om$-complexes involving them also have non-trivial differentials, so information is lost when passing to cohomology which must be recorded using higher compositions. Specifically, the term $f_{64}f_{56}f_{52}^{\vee}f_{24}^{*}$ encodes the relations

$$f_{24} = m_3(f_{52}^{\vee} \otimes f_{56} \otimes f_{64}),\ (f_{52}^{\vee})^{*} = m_3(f_{56} \otimes f_{64} \otimes f_{24}^{*}),$$
$$f_{56}^{*} = m_3(f_{64} \otimes f_{24}^{*} \otimes f_{52}^{\vee}),\ f_{64}^{*} = m_3(f_{24}^{*} \otimes f_{52}^{\vee} \otimes f_{56}).$$

At the risk of digressing, this says that the Massey product of three consecutive morphisms in the 4-cycle contains the dual of the remaining morphism (see [53, §3]). The fact $W_{\upsilon_2\mathcal{P}}$ contains no other cycles encodes the fact that all other compositions in $\mathrm{End}^1_{\mathbf{H}_{\mathrm{dg}}(Z)}(\upsilon_2\mathcal{P})$, normal and higher, are zero.

It also remains to be proved that the A_∞-structure on $\mathbf{H}_{\mathrm{dg}}(Z_2^n)$ is 3-cyclic.

3. **Reachable autoequivalences.** In §5.2 we exhibited a group homomorphism $G_2^n \to \mathrm{Aut}^\circ(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z_2^n))$, where the latter is the subgroup of $\mathrm{Aut}(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z_2^n))$ containing autoequivalences which preserve the distinguished connected component $\mathrm{Stab}^\circ(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z_2^n))$ of the stability manifold. Such autoequivalences are called *reachable*. When trying to explicitly determine a connected component of the stability manifold, it is often useful to find the group of reachable autoequivalences, since this allows one to restrict their attention to a fundamental domain (see e.g. [13, 40, 54]). An obvious goal is to determine whether the map $G_2^n \to \mathrm{Aut}^\circ(\mathbf{K}_{\mathrm{dg}}^{\mathrm{per}}(Z_2^n))$ is injective (i.e. the action of G_2^n is faithful) or surjective (i.e. the action of G_2^n is full).

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