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Forcing Axioms and the Continuum Problem: Hilbert's First Problem Revisited

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Abstract. The continuum problem has recently seen a revival, brought about by advances involving strong forcing axioms and other natural principles extending the standard axiomatization for set theory. In this article we survey these developments. This is a slightly edited version of the slides of our ICM talk.

Key words. the continuum problem, axiom (*), forcing axioms, natural axioms

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1 Cantor's paradise. Set theory is the study of sets (of mathematical objects) under the membership relation $\in$. It provides a particularly simple language to model (all?) mathematical notions.

By way of example, following Kuratowski we can make sense of the notion of an ordered pair using the language of set theory by defining

$$(a,b) = \{\{a\},\{a,b\}\}.$$

It is then easy to check that for all a, a', b, b',

$$(a,b) = (a',b') \Longleftrightarrow a = a' \quad \text{and} \quad b = b.'$$

Once we have ordered pairs, we can define relations as sets of ordered pairs, functions as relations f such that (a,b), $(a,b') \in f \Longrightarrow b = b'$, and so on.

Following von Neumann, we can define $0 = \emptyset$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$, and, generally, we can define every specific natural number n to be the set of its predecessor; we can also define the general notion of a natural number.[1]

We can then construct $\mathbb{Z}$ from $\mathbb{N}$ in the standard way, as well as $\mathbb{Q}$ from $\mathbb{Z}$, and $\mathbb{R}$ from $\mathbb{Q}$ (as Dedekind cuts of rationals or classes of Cauchy sequences of rationals), and so on.

Given sets X and Y, we write

- $|X| \le |Y|$ if there is an injective function $f : X \to Y$,
- $|X| = |Y|$ if $|X| \le |Y|$ and $|Y| \le |X|$,[2] and
- $|X| < |Y|$ if $|X| \le |Y|$ but $|Y| \not\le |X|$.

In 1873, Georg Cantor famously proved that although $\mathbb{N}$ and $\mathbb{R}$ are both infinite, $|\mathbb{N}| < |\mathbb{R}|$. Here is a more general version of this theorem.

THEOREM 1.1 (Cantor's theorem). *For every set X, $|X| < |\mathcal{P}(X)|$, where $\mathcal{P}(X) = \{Y : Y \subseteq X\}$.*

Proof. Clearly $|X| \le |\mathcal{P}(X)|$ (take for this the function sending $y \in X$ to $\{y\} \in \mathcal{P}(X)$).

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[1]We can define a natural number as an ordinal which is either the empty set or some successor ordinal and such that each of its members is either the empty set or a successor ordinal. Both the notions of ordinal and of successor ordinal are purely set-theoretic.

[2]By a (nontrivial) result of Cantor, Schröder, and Bernstein, this is equivalent to having a bijection from X onto Y.

To see that $|\mathcal{P}(X)| \not\leq |X|$, suppose that there were an injection from $\mathcal{P}(X)$ into X. There would then be a surjection $f: X \to \mathcal{P}(X)$. Let

$$A = \{y \in X : y \notin f(y)\}.$$

If $A = f(y)$, then

$$y \in f(y) \Longleftrightarrow y \notin f(y),$$

which is absurd. Hence $A \notin \text{range}(f)$ and f cannot be a surjection after all. $\square$

It is not difficult to see that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$. By Cantor's theorem we then have that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \ldots.$$

Thus we have a hierarchy of bigger and bigger infinities.

Ordinals form a natural extension of the natural number sequence. They come with the following natural order:

$$0, 1, 2, \ldots \quad \omega = \aleph_0 = \mathbb{N}, \omega + 1, \ldots \qquad \omega_1 = \aleph_1, \quad \ldots \qquad \ldots.$$

Some ordinals are cardinals. ω, also known as $\aleph_0$, is the first infinite *cardinal.*

$\aleph_1$ is the first ordinal above $\aleph_0$ which is not in bijection with $\aleph_0$; it is the second infinite cardinal. Similarly, $\aleph_2$ is the first ordinal above $\aleph_1$ which is not in bijection with $\aleph_1$; it is the third infinite cardinal.

All these cardinals can be proved to exist. In fact, we can prove that for every ordinal α, the αth infinite cardinal exists. This cardinal is denoted by $\aleph_\alpha$.

If we assume the Axiom of Choice (which was soon to be regarded as one of the core truths about sets and is an indispensable tool in mathematics), then every set is either finite or in bijection with some $\aleph_\alpha$. In this case, the unique $\aleph_\alpha$ such that $|X| = \aleph_\alpha$ is the *cardinality of X.*

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\aleph_0)|$ is also known as $2^{\aleph_0}$. The following is now a natural question.

The Continuum Problem. $2^{\aleph_0} = \aleph_1$? $2^{\aleph_0} = \aleph_2$? In general, which is the exact cardinality of $\mathbb{R}$? In other words, exactly how many reals are there?

$\aleph_0 < 2^{\aleph_0}$ by Cantor's theorem, and Cantor conjectured that $2^{\aleph_0}$ is the least possible value compatible with this inequality.

Cantor's Continuum Hypothesis (CH) is the statement $2^{\aleph_0} = \aleph_1$.

Deciding the truth value of the Continuum Hypothesis, and in general solving the Continuum Problem, was $\sharp 1$ on Hilbert's famous list of open problems for the ICM meeting in 1900; see [7].

A, related, fundamental question is the following.

QUESTION 1.2. *What counts as a solution of the Continuum Problem?*

Early work in set theory actually gave rise to paradoxes (!).

Russell's Paradox. Let $\mathcal{R}$ be the collection of all sets x such that $x \notin x$. If $\mathcal{R}$ is a set, then

$$\mathcal{R} \in \mathcal{R} \Longleftrightarrow \mathcal{R} \notin \mathcal{R}.$$

Hence, working under the assumption that every well-defined collection of objects forms a set leads to contradiction. That is, the early naive set theory was inconsistent.

On the other hand, the intuitions behind set theory looked *natural,* and they seemed to have the potential to serve as a foundation for mathematics. A natural next move was to retreat to a more modest, hopefully *consistent,* but still powerful, "true" theory of sets.

The standard way to make sense of this is to work in the framework of first order logic. That is, we want to put together an $\mathcal{L}_\in$-theory T for set theory which is

(1) hopefully consistent,
(2) powerful, and
(3) hopefully "true."

$\mathcal{L}_\in$ is the first order language with a symbol for $\in$. T is meant to be a list (maybe infinite) of truths about sets.

Some examples of such truths are the following.

- The Axiom of Extensionality: $(\forall x)(\forall y)(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$.
  [Two sets are equal iff they have the same elements.]
- The Axiom of Unordered Pairs: $(\forall x)(\forall y)(\exists z)(\forall w)(w \in z \leftrightarrow w = x \vee w = y)$.
  [For any two sets x and y, the set $\{x, y\}$ exists.]
- The Power Set Axiom: $(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall w)(w \in z \rightarrow w \in x))$.
  [If X is a set, then the power set of X, $\mathcal{P}(X)$, is also a set.]

The standard theory T is known as ZFC (Zermelo–Fraenkel set theory with the Axiom of Choice).

ZFC proves that the (set-theoretic) universe, denoted by V, can be naturally stratified as

$$V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha.$$

Here, $(V_\alpha : \alpha \in \mathrm{Ord})$ is defined by the following:

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. If α is a limit ordinal, then $V_\alpha = \bigcup_{\beta<\alpha} V_\beta = \{x : x \in V_\beta \text{ for some } \beta < \alpha\}$.

It is easy to see that $V_\alpha \subseteq V_\beta$ for all ordinals $\alpha \leq \beta$. The statement $V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ tells us that sets are orderly generated in stages indexed by the ordinals. Sets appearing at a stage α are formed of sets having appeared at stages before α. This is a nice incarnation of Cantor's intuition that sets are "collections of already existing objects."

The hope now was to show that ZFC proves that $2^{\aleph_0} = \aleph_1$ or to show that ZFC proves that $2^{\aleph_0} > \aleph_1$. In this second case, the hope would of course have been that ZFC decides $2^{\aleph_0} = \aleph_\alpha$ for some specific value $\alpha > 1$. This would have clearly constituted a solution of the Continuum Problem.

We say "would have" because

- in 1938, Kurt Gödel proved that if ZFC is consistent, then ZFC $+ 2^{\aleph_0} = \aleph_1$ is also consistent [5];
- in 1963, Paul Cohen proved that if ZFC is consistent, then ZFC $+ 2^{\aleph_0} = \aleph_2$ is also consistent [2].

To prove the consistency of ZFC $+ 2^{\aleph_0} = \aleph_1$ (assuming the consistency of ZFC), Gödel defined, in fact working in the theory ZF (i.e., ZFC without the Axiom of Choice), a minimal subuniverse L of V, called the *constructible universe,* and showed that L satisfies ZFC $+ 2^{\aleph_0} = \aleph_1$.

L can be stratified as $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$, where

(1) $L_0 = \emptyset$;
(2) for every ordinal α, $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$, where $\mathrm{Def}(L_\alpha)$ is the set of all subsets of L_α which are definable over L_α by some formula (possibly with parameters);
(3) if α is a limit ordinal, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

To prove the consistency of ZFC $+ 2^{\aleph_0} = \aleph_2$, Cohen devised the method of *forcing.* This is a very general method which, given a model M of (enough axioms of) ZFC, enables us to build an outer universe $M[G] \supseteq M$ of M satisfying (enough axioms of) ZFC. $M[G]$ is called a *generic extension of* M.

If we choose the *generic object* G carefully, we may be able to arrange that $M[G]$ satisfies some interesting statement, like $2^{\aleph_0} = \aleph_2$. Cohen won the Fields medal for this.

Using the method of forcing one can show, given the consistency of ZFC, that each of the following are consistent:

- ZFC $+ 2^{\aleph_0} = \aleph_2$.
- ZFC $+ 2^{\aleph_0} = \aleph_3$.
- ZFC $+ 2^{\aleph_0} = \aleph_{273453453667889}$.
- ZFC $+ 2^{\aleph_0} = \aleph_{\omega+1}$.
- ZFC $+ 2^{\aleph_0} = \aleph_{\omega_1}$.[3]
- ...

Is this the end of the story? Do these consistency results show that the Continuum Problem is in fact a pseudo-problem? A formalist, who takes our official theory ZFC as the only source of set-theoretic "truth," would in fact answer Yes.

[3] On the other hand, ZFC $+ 2^{\aleph_0} = \aleph_\omega$ is inconsistent.

But this is not the only reasonable position. In fact, the formalist position faces very serious problems. We will now present a way to see this.

2 Trouble with the formalist position. Taking a question like "How many real numbers are there?" to have no answer commits the formalist to denying that there is any such thing as "the set-theoretic universe $\bigcup_{\alpha\in\mathrm{Ord}} V_\alpha$."

For the formalist, all there is to say is things like

$$\text{“}T \text{ proves } \varphi\text{”}$$

or

$$\text{“If } T \text{ is consistent, then it does not prove } \varphi\text{”},$$

where $T = \mathrm{ZFC}$ or T is some given extension of ZFC, and where φ is a sentence.

These proof-theoretic statements are ultimately just arithmetical statements (via standard coding of the relevant syntactical notions into arithmetic). So the formalist seems to be at least committed to the existence of $V_\omega = \bigcup_{n\in\omega} V_n$.[4]

But arithmetic is in fact dependent on the higher V_α's. For example, whether or not a simple (and rather natural) purely arithmetical combinatorial statement known as the Paris–Harrington theorem holds depends on whether or not there are infinite sets in a way which makes it hard to justify the truth of this statement and extensions thereof without making reference to objects which are beyond V_ω.

So $V_{\omega+1}$ seems indispensable.

But the properties of $V_{\omega+1}$ depend crucially on what happens at V_α for higher α's. For example, Borel Determinacy holding depends on the existence of V_α for all $\alpha < \omega_1$ [4].

So the universe should in fact contain V_α for all $\alpha < \omega_1$. But the statement "There is no $X \subseteq \mathbb{R}$ with $\aleph_0 < |X| < |\mathbb{R}|$" (which is equivalent to CH in ZFC) is decided already in $V_{\omega+2}$, so it should be decided!

The question now is: Where should we stop and why?

3 The realist position: Natural axioms. If the set-theoretic universe V is real, then the Continuum Problem has a definite solution. The fact that ZFC does not solve the Continuum Problem means that we need to supplement ZFC with *natural* axioms solving this problem.

The search for natural axioms supplementing ZFC is also known as *Gödel's program.* Gödel indeed made the above point [6]. Although he had proved the consistency of CH, he suspected the CH to be false *in reality* and was hoping that natural axioms would eventually settle the issue.

Adopting $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0} = \aleph_{27}$, as a new axiom with no further justification seems surely ad hoc, and therefore unnatural. How do we decide if an axiom is natural?

We need general criteria to assess the naturalness of axioms. They should address the question: What virtues do we want our theory of sets to have?

3.1 Strong axioms of infinity. Large cardinal (LC) axioms are an (open-ended) hierarchy of axioms asserting the existence of very high cardinals with strong properties. They realize the idea that "the universe is large."

They form a hierarchy of stronger and stronger theories: Given an LC axiom A, $\mathrm{ZFC}+A$ proves the consistency of ZFC, and in fact of $\mathrm{ZFC}+A'$ for any weaker large cardinal axiom A'.

The stronger A is, the more daring $\mathrm{ZFC}+A$ is (i.e., more likely to be inconsistent). We can prove the consistency of LC axioms only by working in a strictly stronger LC theory.

Arguably, the weakest large cardinal axiom is "ω exists." The next natural large cardinal axiom is "There is an inaccessible cardinal." The following are some classical large cardinal axioms.

$\omega <$ inaccessible $<$ weakly compact $<$ measurable $<$ strong $<$ Woodin $<$ superstrong $<$ supercompact $<$ huge $<$ 2-huge $< \ldots$.

An empirical fact is that every natural mathematical theory ever considered can be interpreted relative to $\mathrm{ZFC}+A$ for some LC axiom A. We may take this to mean that large cardinals encapsulate all the expressive power attainable in set theory. In fact, for theories T naturally occurring in mathematics, we can often show

[4] V_ω is in fact mutually interpretable, in a precise sense, with the structure of arithmetic.

that T is *equiconsistent* with ZFC$+A$ for some LC axiom A (i.e., we can prove that T is consistent iff ZFC$+A$ is consistent). This is true at least for theories of relatively low consistency strength. Working in these theories we can often construct a "canonical" inner model with a certain large cardinal configuration LC and show, in turn, that the presence of LC enables us to build a model of our theory. For stronger theories this is typically not possible at present due to the fact that, at present, inner model theory does not reach the large cardinals needed in that case. Still, and as already mentioned, the consistency of every natural theory seems to follow from some sufficiently strong large cardinal assumption.[5]

A natural axiom should therefore be compatible with all consistent large cardinal axioms.

3.2 Invariance with respect to forcing. Forcing is our prime method for proving the independence of some given statement φ from some base set theory; in other words, to prove that if T is consistent, then φ and its negation $\neg\varphi$ are both consistent with T. Therefore, if we want our set theory T to be strong, we had better have that T neutralizes the effects of forcing as much as possible, in the sense of proving, for as many sentences φ as possible, that the truth value of φ cannot be changed by forcing.

The axiom $V = L$ neutralizes the effects of forcing for silly reasons: nontrivial generic extension does not satisfy $V = L$.

However, $V = L$ is *not* compatible with most LC axioms. In fact, if $V = L$, then there are no measurable cardinals.[6]

Quite remarkably, large cardinals in the region of Woodin cardinals also neutralize the effects of forcing to a suitable extent (and are of course compatible with all LC axioms).

THEOREM 3.1 (Woodin, mid 1980s). *Suppose there are arbitrarily large Woodin cardinals. Then the following are equivalent for every sentence φ in the language of set theory.*

1. *φ is true in $L(\mathbb{R})$.*
2. *It can be forced that φ is true in $L(\mathbb{R})$.*

$L(\mathbb{R})$ is the $\subseteq$-minimal subuniverse of ZF (= ZFC without the Axiom of Choice) containing all the reals and all the ordinals. $L(\mathbb{R})$ is where all of classical analysis takes place.

Hence, if there are arbitrarily large Woodin cardinals, classical analysis is immune to the forcing method.

3.3 Maximality with respect to forcing: Forcing axioms. $\mathbb{P} = (P, \leq_{\mathbb{P}})$ is a *partial order* if P is a set and $\leq_{\mathbb{P}}$ is a relation on P which is transitive, antisymmetric, and reflexive on P.

Given a partial order $\mathbb{P} = (P, \leq_{\mathbb{P}})$ (a.k.a. *forcing notion*), $D \subseteq P$ is a *dense* subset of $\mathbb{P}$ if for every $p \in P$ there is some $q \in D$ such that $q \leq_{\mathbb{P}} p$.

$G \subseteq P$ is a *filter* if

- for every $q \in G$ and $p \in P$, if $q \leq_{\mathbb{P}} p$, then $p \in G$;
- for all q_1, $q_2 \in G$ there is some $q \in G$ such that $q \leq_{\mathbb{P}} q_1$ and $q \leq_{\mathbb{P}} q_2$.

If M is some model such that $\mathbb{P} \in M$, we say that G is $\mathbb{P}$-*generic over* M if $G \subseteq \mathbb{P}$ is a filter and $G \cap D \neq \emptyset$ for every dense subset D of $\mathbb{P}$ such that $D \in M$.

THEOREM 3.2 (Cohen). *Suppose M is a transitive model of (enough of)* ZFC. *Let $\mathbb{P} = (P, \leq_{\mathbb{P}}) \in M$ be a forcing notion and let $G \subseteq P$ be $\mathbb{P}$-generic over M. Then the following hold:*

1. *If $\mathbb{P}$ is* nonatomic, *meaning that for every $p \in \mathbb{P}$ there are $q_0 \leq_{\mathbb{P}} p$ and $q_1 \leq_{\mathbb{P}} p$ such that q_0 and q_1 are $\mathbb{P}$-incompatible (i.e., there is no $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} q_0$ and $q \leq_{\mathbb{P}} q_1$), then $G \notin M$.*
2. *There is a $\subseteq$-minimal model $M[G]$ of (enough of)* ZFC *such that*
   (a) *$M \subseteq M[G]$ and*
   (b) *$G \in M[G]$.*

Example. Let $\mathbb{P}$ be the set of finite tuples $(x_0, x_1, \ldots, x_{n-1})$ of 0's and 1's. Given such tuples $\vec{\sigma}_1$, $\vec{\sigma}_2$, let us set $\vec{\sigma}_2 \leq_{\mathbb{P}} \vec{\sigma}_1$ iff $\vec{\sigma}_1$ in an initial segment of $\vec{\sigma}_2$.

Let G be a filter of $\mathbb{P}$. If G meets $D_n = \{\vec{\sigma} \in \mathbb{P} : \text{length}(\vec{\sigma}) > n\}$ for every $n \in \mathbb{N}$, then $c = \bigcup G$ is an infinite sequence $(x_n : n \geq 0)$ of 0's and 1's.

[5]For all we know at present, it could very well be that there are no inner models resembling L to a sufficient extent and which accommodate large cardinals past some critical threshold. If that were the case, then there might be no reason to expect that the consistency strengths of sufficiently strong natural theories fit in the large cardinal hierarchy. The structure of these consistency strengths could then be, conceivably, very complex.

[6]The existence of a measurable cardinal sits very low in the large cardinal hierarchy.

If G meets $\{\vec{\sigma} \in \mathbb{P} : \text{there are } 2n, 2n+2 < \text{length}(\vec{\sigma}), \vec{\sigma}(2n) \neq \vec{\sigma}(2n+2)\}$, then it is not the case that all consecutive even entries of c are equal.

If G meets $\{\vec{\sigma} \in \mathbb{P} : \text{there are } 2n+1, 2n+3 < \text{length}(\vec{\sigma}), \vec{\sigma}(2n+1) \neq \vec{\sigma}(2n+3)\}$, then it is not the case that all consecutive odd entries of c are equal.

If G meets $\{\vec{\sigma} \in \mathbb{P} : \text{there is } n \text{ with } n+10000 < \text{length}(\vec{\sigma}) \text{ such that } \vec{\sigma}(n) = \vec{\sigma}(n+1) = \ldots = \vec{\sigma}(n+10000) = 0\}$, then c has somewhere 10001 consecutive entries all taking value 0.

...

If G is sufficiently generic, then c is a very random sequence. In fact, if G is generic over some model M, then c avoids every regularity pattern expressible within M.

$\mathbb{P}$ in the example above is the simplest possible nonatomic forcing. It is called *Cohen forcing.* And if G is $\mathbb{P}$-generic over a model M, c is a *Cohen real over M.*

By (1) in Cohen's theorem (Theorem 3.2), there are no $\mathbb{P}$-generic filters over V (whenever $\mathbb{P}$ is nonatomic).

On the other hand, we have the following.

Fact 3.3 (ZFC). *If $|M| = \aleph_0$, then for every forcing notion $\mathbb{P} \in M$ there is a $\mathbb{P}$-generic filter G over M.*

Proof. Let $(D_n : n \in \mathbb{N})$ enumerate all dense subsets of $\mathbb{P}$ in M. Let $q_0 \in D_0$. Since D_1 is dense, we may find $q_1 \in D_1$ such that $q_1 \leq_{\mathbb{P}} q_0$. In general, since D_{n+1} is dense, we may find $q_{n+1} \in D_{n+1}$ such that $q_{n+1} \leq_{\mathbb{P}} q_n$. Then

$$G = \{p \in P : q_n \leq_{\mathbb{P}} p \text{ for some } n\}$$

is a $\mathbb{P}$-generic filter over M. □

Fact 3.3 can be seen as a generalization of the Baire Category Theorem.

We can now naturally ask: Is it possible to find strengthenings of this fact applying to larger collections of dense sets? Or, what amounts to the same thing: To what extent can the Baire Category Theorem be generalized?

It turns out that it is consistent with ZFC to have the answer be Yes in some cases—for example, if we restrict the class of forcings in some suitable way and if, for example, "larger" is interpreted as "of size $\aleph_1$." These assertions are known as *forcing axioms.* Their consistency was traditionally shown by forcing "many times" with forcing notions in the relevant class.

Let us now introduce a standard piece of notation for forcing axioms.

Definition 3.4. *Given an infinite cardinal κ and a class Γ of forcing notions,* $\mathrm{FA}_\kappa(\Gamma)$ *is the following assertion:*

Suppose $\mathbb{P}$ is a forcing notion in Γ. If $\mathcal{D}$ is a collection of at most κ-many dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

The first forcing axiom ever considered (by Martin and Solovay in 1970 [8] and by Solovay and Tennenbaum in 1971 [9]) was Martin's Axiom at ω_1, MA_{ω_1}. MA_{ω_1} is

$$\mathrm{FA}_{\aleph_1}(\{\mathbb{P} : \mathbb{P} \text{ has the countable chain condition}\}).$$

Here, $\mathbb{P}$ has the countable chain condition iff every subset of $\mathbb{P}$ consisting of pairwise incompatible conditions is at most countable.

The strongest possible forcing axiom of this sort (i.e., applying to the widest possible class of forcing notions), known as Martin's Maximum (MM), was isolated and proved consistent by Foreman, Magidor, and Shelah in 1984 assuming the consistency of ZFC + "there is a supercompact cardinal" [3].

Definition 3.5. *Martin's Maximum,* MM*, is*

$$\mathrm{FA}_{\aleph_1}(\{\mathbb{P} : \textit{forcing with } \mathbb{P} \textit{ preserves all stationary subsets of } \omega_1\}).$$

Here, a subset S of ω_1 is stationary iff $S \cap C \neq \emptyset$ for every $C \subset \omega_1$ which is unbounded in ω_1 and closed below ω_1 in the order topology on the ordinals.

The existence of "partially generic" filters given by forcing axioms ensures that many of the facts that would hold in the corresponding generic extensions already hold in V. Thus, forcing axioms realize the following "maximality idea":

Any statement (of the right syntactical form) that could possibly hold (by forcing with a forcing notion in the relevant class) is actually true.

Also, the wider the class Γ is, the more "democratic" $\mathrm{FA}_\kappa(\Gamma)$ is (in the sense of not discriminating between possible generic extensions).

Given a set X, $\mathrm{TC}(X) = X \cup \bigcup X \cup \bigcup\bigcup X \cup \ldots = X \cup \{a : a \in b \in X \text{ for some } b\} \cup \{a : a \in b \in c \in X \text{ for some } b, c\} \cup \ldots$.

Let

$$H(\omega_2) = \{X : |\mathrm{TC}(X)| < \aleph_2\}.$$

Thus, $H(\omega_2)$ is the collection of all sets which are "small relative to $\aleph_2$."

$H(\omega_2)$ is a set which is a model of a convenient fragment of ZFC.

Many natural statements, like CH and $\neg$CH, are decided in $H(\omega_2)$.

Forcing axioms have many consequences at the level of $H(\omega_2)$. In particular, MM implies all of the following.

1. (Suslin's hypothesis) If $(L, \leq)$ is a Dedekind-complete dense linear order such that $|\mathcal{I}| \leq \aleph_0$ whenever $\mathcal{I}$ is a collection of pairwise disjoint intervals of L, then $(L, \leq)$ is order-isomorphic to the real line with the usual order.
2. $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.
3. (Moore) There is a set of reals X and a Countryman line[7] C such that whenever $(L, \leq)$ is a linear order such that $\aleph_0 < |L|$, L contains some suborder order-isomorphic to one of the following:
   - X;
   - ω_1;
   - ω_1^* (the reverse of ω_1);
   - C;
   - C^* (the reverse of C).

The consistency proof of MM shows in fact that the following enhanced form MM^{++} of this axiom is consistent. Its formulation makes use of the notion of $\mathcal{P}$-names[8] for objects in the forcing extension as well as of the forcing relation $\Vdash_{\mathcal{P}}$.[9]

Definition 3.6. MM^{++} *is the following statement: For every forcing notion $\mathcal{P}$ preserving stationary subsets of ω_1, every collection $\mathcal{D}$ of $\aleph_1$-many dense subsets of $\mathcal{P}$, and every collection $\{\dot{S}_i : i < \omega_1\}$ of $\mathcal{P}$-names for stationary subsets of ω_1, there is a filter $G \subseteq \mathcal{P}$ such that*

- *$G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$ and*
- *for each i, the interpretation of $\dot{S}_i$ by G, i.e.,*

$$(\dot{S}_i)_G = \{\nu < \omega_1 : (\exists p \in G) p \Vdash_{\mathcal{P}} \nu \in \dot{S}_i\},$$

is stationary.

MM^{++} in fact seemed to decide *all* questions about $H(\omega_2)$ modulo forcing.

3.4 The Axiom of Determinacy. Let A be a set of sequences $(x_i : i \geq 0)$ of natural numbers. Consider the following game G_A between two players, I and II, who alternate picking natural numbers x_i:

$$\begin{array}{c|cccccc} \mathrm{I} & x_0 & & x_2 & & \ldots & \\ \hline \mathrm{II} & & x_1 & & x_3 & & \ldots \end{array}$$

Thus, player I starts the run of the game by picking some $x_0 \in \mathbb{N}$. Then player II responds by picking some $x_1 \in \mathbb{N}$. Then player I picks some $x_2 \in \mathbb{N}$, and so on.

[7]A linear order $(C, \leq)$ is a Countryman line if its square $C \times C$ is a union of countably many chains. In other words, there is a decomposition $C \times C = \bigcup_{n<\omega} X_n$ of the Cartesian product $C \times C$ into countably many pieces X_n such that for each n and for all (x_0, y_0), $(x_1, y_1) \in X_n$, $x_0 \leq x_1$ iff $y_0 \leq y_1$.

[8]Any object in a generic extension $V[G]$ of V by a forcing $\mathcal{P} \in V$ is the interpretation, by the generic filter G, of some $\mathcal{P}$-name $\dot{x} \in V$.

[9]The forcing relation $\Vdash_{\mathcal{P}}$ is a relation, definable in V from $\mathcal{P}$, which captures forcibility in the following sense: for every $p \in \mathcal{P}$, every formula $\varphi(x_0, \ldots, x_n)$, and any $\mathcal{P}$-names $\dot{a}_0, \ldots, a_n$ in V, $p \Vdash_{\mathcal{P}} \varphi(\dot{a}_0, \ldots, \dot{a}_n)$ iff $V[G] \models \varphi((\dot{a}_0)_G, \ldots, (\dot{a}_n)_G)$ for every $\mathcal{P}$-generic filter G over V such that $p \in G$, where for each i, $(\dot{a}_i)_G$ is the interpretation of $\dot{a}_i$ by G.

- Player I wins this run of the game if $(x_i : i \geq 0) \in A$.
- Player II wins if $(x_i : i \geq 0) \notin A$.

We say that A *is determined* if either player I or player II has a winning strategy in G_A (i.e., a strategy ensuring a win for that player no matter how the other player makes their moves).

The *Axiom of Determinacy* (AD) is the statement: "Every set A of sequences of natural numbers is determined."

AD contradicts the Axiom of Choice.

On the other hand, ZF+AD gives a remarkably rich theory. In particular, it provides a very fine analysis of $L(\mathbb{R})$. Also, ZF+AD arguably gives the correct structure theory for the definable sets of reals.[10]

THEOREM 3.7 (Martin and Steel, Woodin, mid 1980s). *Suppose there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then* AD *holds in* $L(\mathbb{R})$.

We will now see a simple application of AD (due to Martin).

A sequence $(b_n)_{n\geq 0}$ in $\mathbb{N}$ *computes* another sequence $(a_n)_{n\geq 0}$ in $\mathbb{N}$ (this is written $(a_n)_{n\geq 0} \leq_{\mathrm{Tu}} (b_n)_{n\geq 0}$) iff there is a computer program (technically, a Turing machine) which outputs $(a_n)_{n\geq 0}$ upon input of $(b_n)_{n\geq 0}$.

$(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are *Turing equivalent* (written $(a_n)_{n\geq 0} \equiv_{\mathrm{Tu}} (b_n)_{n\geq 0}$) if
- $(a_n)_{n\geq 0} \leq_{\mathrm{Tu}} (b_n)_{n\geq 0}$ and
- $(b_n)_{n\geq 0} \leq_{\mathrm{Tu}} (a_n)_{n\geq 0}$.

C is a *Turing cone* if there is some $(a_n)_{n\geq 0}$ such that

$$C = \{(b_n)_n : (a_n)_n \leq_{\mathrm{Tu}} (b_n)_n\}.$$

$(a_n)_{n\geq 0}$ is the *base of* C.

Now, let A be a *Turing invariant* set of sequences (i.e., if $(a_n)_{n\geq 0} \in A$ and $(b_n)_{n\geq 0} \equiv_{\mathrm{Tu}} (a_n)_{n\geq 0}$, then also $(b_n)_{n\geq 0} \in A$).
- If player I has a winning strategy σ in G_A and $\vec{a} = (a_n)_{n\geq 0}$ codes σ, then $C \subseteq A$, where C is the Turing cone with base $\vec{a}$. ($\vec{a} \leq_{\mathrm{Tu}} \vec{b} \Longrightarrow \vec{b} \equiv_{\mathrm{Tu}} \sigma * \vec{b} \in A \Longrightarrow \vec{b} \in A$.)
- If player II has a winning strategy τ in G_A and $\vec{a} = (a_n)_{n\geq 0}$ codes τ, then $C \cap A = \emptyset$, where C is the Turing cone with base $\vec{a}$. (Argue as above.)

$\sigma * \vec{b}$ is the play of G_A resulting from player I moving according to the strategy σ and player II playing the members of $\vec{b}$ in increasing order. (And we define $\vec{b} * \tau$ similarly if τ is a strategy for player II.)

Hence, if AD holds, then every Turing invariant A is either large (i.e., it contains a Turing cone) or small (i.e., it is disjoint from a Turing cone).

Note: If C_1 is the Turing cone with base $(a_n)_{n\geq 0}$ and C_2 is the Turing cone with base $(b_n)_{n\geq 0}$, then $C \subseteq C_1 \cap C_2$, where C is the Turing cone with base $(a_0, b_0, a_1, b_1, a_2, b_2, \ldots)$.

This means that for any finitely many Turing invariant properties $P_1, \ldots, P_n$ of sequences of natural numbers there are choices $Q_i \in \{P_i, \neg P_i\}$, for $i \leq n$, such that the set of sequences satisfying Q_i for all i is large (in particular nonempty).

In fact this is true for any countable collection P_i (for $i \in \mathbb{N}$) of properties.

4 $(*)$. In the 1990s, Woodin defined and studied the following axiom $(*)$ [10].

DEFINITION 4.1. $(*)$ *is the conjunction of* (1) *and* (2).
1. AD *holds in* $L(\mathbb{R})$. *[This follows from large cardinals by Theorem* 3.7.*]*
2. *There is a* $\mathbb{P}_{\max}$*-generic filter* G *over* $L(\mathbb{R})$ *such that* $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$.

Here, $\mathbb{P}_{\max}$ is a certain *homogeneous* forcing notion in $L(\mathbb{R})$ (i.e., with the property that $L(\mathbb{R})[G_1]$ and $L(\mathbb{R})[G_2]$ satisfy the same sentences whenever G_1 and G_2 are $\mathbb{P}_{\max}$-generic filters over $L(\mathbb{R})$), definable over $L(\mathbb{R})$ without parameters.

Assuming the theory of $L(\mathbb{R})$ is frozen under forcing (which follows from large cardinals), it follows that the $L(\mathcal{P}(\omega_1))$'s of any two models of $(*)$ obtained by forcing satisfy exactly the same sentences.

Also, $(*)$ implies the following very strong maximality principle (relative to all generic extensions).

[10] Definable in the sense of being *projective* (equivalently, being definable in the structure $H(\omega_1) = \{x : |\mathrm{TC}(x)| < \aleph_1\}$ from parameters). This is a natural restricted notion of definability which has in particular the feature that all sets of reals falling under it are in $L(\mathbb{R})$. Being in $L(\mathbb{R})$ can itself be rendered as a natural notion of definability.

THEOREM 4.2 (Woodin). *Suppose $(*)$ holds and there are arbitrarily large Woodin cardinals. For every parameter-free Π_2 sentence φ $(= (\forall x)(\exists y)\psi(x, y)$, where all quantifiers in $\psi(x, y)$ are restricted), the following are then equivalent.*

1. *φ is true in $H(\omega_2)$.*
2. *In some forcing extension it holds that φ is true in $H(\omega_2)$.*

Remark 4.3. In 2, the forcing extension need not preserve stationary subsets of ω_1.

In addition, $(*)$ implies that $L(\mathcal{P}(\omega_1))$ is obtained from $L(\mathbb{R})$ by adding to it any subset of ω_1 not in $L(\mathbb{R})$. Given that AD makes $L(\mathbb{R})$ into a "canonical" subuniverse, this means that $(*)$ makes $L(\mathcal{P}(\omega_1))$ into a larger "canonical" subuniverse.

The above facts rendered $(*)$ a very appealing axiom. However, in order for $(*)$ to be truly natural, it would have to be compatible with all consistent LC axioms. The main question was therefore as follows.

QUESTION 4.4. *Is $(*)$ compatible with all consistent LC axioms? Is $(*)$ even forcible over V (assuming enough large cardinals)?*

By Theorem 4.2, $(*)$ is a maximality principle, in a similar fashion as forcing axioms are. This led to the following question, in a similar spirit as Question 4.4.

QUESTION 4.5. *Is* MM *compatible with* $(*)$?

Both questions above had been open since the 1990s, when $\mathbb{P}_{\max}$ forcing was introduced.

4.1 $\mathrm{MM}^{++} \Longrightarrow (*)$. In 2018, we answered Questions 4.4 and 4.5.

THEOREM 4.6 (Asperó and Schindler [1]). MM^{++} *implies* $(*)$.

In particular, if there is a supercompact cardinal κ, then there is a forcing notion of cardinality κ forcing MM^{++}, and therefore forcing $(*)$. All large cardinals there might be above κ are then preserved in the generic extension. Hence, Question 4.4 gets a positive answer, and of course so does Question 4.5 since, as already mentioned, the standard consistency proof of MM actually produces a model of MM^{++}.

This theorem renders $(*)$ a truly natural axiom. And it makes the observed completeness of MM^{++} for the theory of $L(\mathcal{P}(\omega_1))$ under forcing into a mathematical fact, thus rendering MM^{++} even more natural.

And both $(*)$ and MM^{++} were known to imply $2^{\aleph_0} = \aleph_2$.

5 High forcing axioms. Given the success of classical forcing axioms (for meeting $\aleph_1$-many dense sets) culminating in MM^{++}, it is natural to inquire whether a comparable theory can be developed for higher forcing axioms (i.e., for meeting more than $\aleph_1$-many dense sets).

The answer is No. For example:

- Π_2 maximality for $H(\omega_3)$ fails: CH and $\neg$CH are both expressible over $H(\omega_3)$ by Π_2 sentences, and both can be forced.
- There are serious problems with building models of strong high forcing axioms. In fact, many forcing axiom candidates for meeting families of $\aleph_2$-many dense, even for fairly modest classes Γ of forcing notions, are just false.

The conclusion is that the success of MM^{++} cannot be replicated, at the level of high forcing axioms, to yield a natural axiom implying for example $2^{\aleph_0} = \aleph_3$.

This completes the argument for $2^{\aleph_0} = \aleph_2$.

Still, there are interesting open questions in the higher forcing axioms camp. We will now present some of them.

QUESTION 5.1. *Do reasonable LC axioms imply that there is no partition $\mathcal{P} \in L(\mathbb{R})$ of $\mathbb{R}$ into $\aleph_3$-many pieces?*

A positive answer to this question would show that sufficiently strong large cardinals block the existence of counterexamples to $2^{\aleph_0} \leq \aleph_2$ which are *effectively given,* in a natural precise sense sense, and would thus contribute to the naturalness of the statement $2^{\aleph_0} \leq \aleph_2$. It should be noted that, by a result of Woodin, it is consistent to have, together with any consistent large cardinal axiom, that there is a partition $\mathcal{P} \in L(\mathbb{R})$ of $\mathbb{R}$ into $\aleph_2$-many pieces. In fact, the existence of such a partition follows from both MM and $(*)$.

QUESTION 5.2. *Do reasonable LC axioms imply that one can force some Σ_2 statement A which is complete for the theory of $H(\omega_3)$ modulo forcing preserving A?*

The point of Question 5.2 is that, by Theorem 4.6, $(*)$ is such an axiom for $H(\omega_2)$.

QUESTION 5.3. *Is there any Π_2 sentence σ such that the following hold?*

1. ZFC *proves that if $H(\omega_3)$ satisfies σ, then $2^{\aleph_0} = \aleph_3$.*
2. *For some reasonable LC axiom A,* ZFC+ *A proves that it is forcible that $H(\omega_3)$ satisfies σ.*

This question is motivated by two facts. The first fact is that Π_2 sentences about a suitable $H(\kappa)$ are precisely the type of sentences that forcing axioms are designed to imply.[11] The second fact is that, for the classical case, the proof that a forcing axiom $\mathrm{FA}_{\aleph_1}(\Gamma)$ implies $2^{\aleph_0} = \aleph_2$ proceeds usually by showing that $\mathrm{FA}_{\aleph_1}(\Gamma)$ implies that $H(\omega_2)$ satisfies some Π_2 sentence σ, where this statement in turn implies $2^{\aleph_0} = \aleph_2$.

6 A competing view. Woodin has championed an alternative view [11]. This is the content of the *Ultimate-L program.* This is an ambitious version of the *inner model program,* one of whose main goals is to construct an L-like subuniverse accommodating all possible LC axioms.

The axiom $V =$ Ultimate-L would provide a complete picture modulo forcing for the entire universe. And it implies $2^{\aleph_0} = \aleph_1$.

On the down side, $V =$ Ultimate-L is a difficult axiom to work with. And it is not clear at present to what extent the program can be implemented.

At any rate, the implications and ramifications of the program are extremely deep.

7 Back to MM^{++}: A challenge. By our theorem, MM^{++} implies that $L(\mathcal{P}(\omega_1))$ is a homogeneous extension of the AD-model $L(\mathbb{R})$ and hence a canonical model. And by forcing with homogeneous forcing notions over stronger AD-models it is possible to obtain models of $\mathrm{MM}^{++}(2^{\aleph_0})$ (i.e., MM^{++} for forcings of size $2^{\aleph_0}$) and even stronger fragments of MM^{++}. A major challenge nowadays is therefore the following.

Challenge:

1. Obtain a model of full MM^{++}, or of stronger fragments of MM^{++}, by (homogeneous) forcing over strong models of the Axiom of Determinacy.
2. Obtain a model of MM^{++} in which $(*)^{++}$ holds. $(*)^{++}$ says that there is $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ and a $\mathbb{P}_{\max}$-generic filter G over $L(\mathbb{R}, \Gamma)$ such that $\mathcal{P}(\mathbb{R}) \subseteq L(\mathbb{R}, \Gamma)[G]$.

Remark 7.1. In the presence of large cardinals (for example a proper class of Woodin cardinals), the ability to obtain a model of $(*)^{++}$ by set-forcing would give a positive answer to Question 5.2.

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