The word problem for finitely presented special inverse monoids

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Abstract

In this thesis we investigate the decidability of the word problem for finitely presented special inverse monoids via the prefix membership problem of their maximal group images. The work can be divided into two tranches which use distinct methods.

In the first tranche we approach the problem via a factorisation of the inverse monoid's relators. We show that there are two combinatorial conditions on the factors which when present in conjunction with certain other conditions are sufficient to decide the word problem of the inverse monoid. Further we demonstrate that when these conditions apply to minimal invertible factorisations that there is an equivalence between the word problems of the inverse monoid, its group of units and its maximal group image.

In the second tranche we approach the problem via the Magnus Moldavanskiĭ hierarchy. We show that a family of HNN extensions of the free group have qualities which are algorithmically useful. In particular we are able to demonstrate that this means such HNN extensions have a class of submonoids with decidable word problem. We can then apply these results to solve the prefix membership problem in several examples.

Additionally we are able to provide a number of sufficient conditions for the amalgamated product of two E-unitary inverse monoids to be itself E-unitary.

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1

Introduction

Synopsis

In this chapter we give an overview of the context to the research discussed in later chapters. The primary aim here is not to introduce results, new or old, in their full technical detail but to take a more discursive look both at what has come before and at what we seek to accomplish in later chapters.

Our chief aim within this thesis is to look at the algorithmic properties of inverse monoids. In particular we wish to establish circumstances under which finitely presented special inverse monoids have decidable word problem.

The word problem, the question of whether we can decide if two words represent the same element, is arguably the most fundamental questions we can pose for any object in combinatorial alegbra. A natural place to begin is with the free version of the object and it has been shown that free monoids, free groups [17] and free inverse monoids [26] have decidable word problems. From here, a natural next step would be to assess whether the one-relator version of each structure has decidable word problem in general. This was shown to be the case for groups by Magnus [18] (interestingly this result predated the first examples of finitely presented groups with undecidable word problem by two decades [27] [4]). In contrast, it is still an open question whether all one-relator monoids have decidable word problem, although it is known that special one-relator monoids (those of the form $\text{Mon}\langle X \mid u = 1 \rangle$) have decidable word problem [1]. Further it has been shown that the word problem of all non-special one-relator monoids reduces to being able to decide the cases of the form $\text{Mon}\langle a, b \mid bUa = a \rangle$ and of the form $\text{Mon}\langle X \mid bUa = aVa \rangle$, where $U, V \in \{a, b\}^*$ [2]. Moreover, both of these cases may be embedded in a special inverse monoid either of the form $\text{Inv}\left\langle a, b \mid a^{-1}bUa = 1 \right\rangle$ or $\text{Inv}\left\langle X \mid a^{-1}V^{-1}a^{-1}bUa = 1 \right\rangle$, respectively [11]. Thus a solution to the word problem for special inverse monoids with a single reduced relator would be sufficient to decide the word problem for one-relator monoids.

Until recently the general decidability of one-relator inverse monoids was an open question as well. However, in 2019 a special one-relator inverse monoid which has undecidable word problem was constructed by Gray [7]. By the nature of the construction method employed the sole relator is not even close to being a reduced word. Thus it is still an open question whether there are inverse monoids with a single reduced relator and undecidable word problem. This leaves open the question of whether it is possible to decide the word problem of one-relator monoids via inverse monoids.

If being one-relator is not a sufficient condition for an inverse monoid as it is in groups then this raises a question. What are the circumstances under which the word problems of an inverse monoid and the group with the same presentation to have the same decidability? The results of this thesis can be divided into two main tranches differentiated by alternative approaches to this question. In chapters $3-5$ we will show certain combinatorial conditions lead to such an equivalence and in chapters 6 and 7 we establish that certain inverse monoids with a single cyclically reduced relator have decidable word problem (their maximal group image having decidable word problem via Magnus's theorem).

The cornerstone of our attempts to discern the decidability of the word problem of inverse monoids here is a result of Ivanov, Margolis and Meakin [11]. This result tells us that we may divide the question of whether an inverse monoid has decidable word problem into three "smaller" problems and that a positive answer to all three is sufficient for there to be a positive answer to the whole. These questions are:

- Does the maximal group image have decidable word problem?
- Does the maximal group image have decidable prefix membership problem?
- Is the inverse monoid E-unitary?

It is the second of these questions that will occupy the majority, though not the entirety, of our efforts.

E-unitarity: Before discussing the prefix membership problem more deeply, we will consider the question of when an inverse monoid presentation is E-unitary. Within the same paper that they produced the prefix membership result Ivanov, Margolis and Meakin also contributed a partial answer to this question, showing that all inverse monoids with a single cyclically reduced relator are E-unitary [11, Theorem 4.1].

While a more complete classification of when special one-relator inverse monoids are E-unitary would also be of interest, in some ways the most valuable aspect of the Ivanov, Margolis and Meakin result is that it is a transparent criteria which allows us to easily construct examples of one-relator inverse monoids which are E-unitary. It would be similarly useful to have a result that allows us to construct special finitely presented inverse monoids which are E-unitary.

A natural avenue to producing finitely presented E-unitary inverse monoids is to take the free product of two one-relator inverse monoids are themselves E-unitary. That such a product would indeed be E-unitary was shown by McAlister [24, Theorem 1.3]. The problem with using this method is that the examples it produces will not be more algorithmically interesting than the sum of their parts. For instance, if we can decide the word problem for both the inverse monoids we are taking the free product of then we can decide the word problem for the whole (this follows from Jones's work on normal forms [12]). A question we may then ask is when the free product with amalgamation of two E-unitary inverse monoids is also E-unitary.

A paper of Stephen [30] gives us a sufficient condition for an amalgamated product of two E-unitary inverse monoids to be E-unitary itself. While this condition is general whether it is fulfilled for any particular amalgam is somewhat opaque. In Section 3.1 we show that several more transparent conditions on the amalgamating part are sufficient to fulfill Stephen's condition and produce E-unitary amalgamations. This will then aid us in producing examples for results in Chapters 4 and 5.

The three word problems: Every inverse monoid has a natural association with two groups, its maximal group image and its group of units. Therefore a reasonable question is what conditions on these groups are necessary and which sufficient for the inverse monoid to have decidable word problem.

Such an approach has proved very successful for finitely presented special monoids. It was shown not only that the group of units could be expressed in a presentation with as few relations as the original monoid but that the monoid had decidable word problem if and only if its group of units has decidable word problem; this was shown in the one-relator case by Adjan [1], generalised to multiple relators by Makanin [21] and then later simplified by Zhang [34] [33].

However this simple correspondence does not hold in general for finitely generated special inverse monoids. In fact the structures, and thus the word problems, of the monoid, its maximal group image and its group of units may have a great degree of independence from each other. In fact only one of the possible six implications holds in general. As U_M is a subobject of M the decidability of the latter's word problem implies the decidability of the former's.

If we assume that M is E-unitary then two more of the six implications hold.

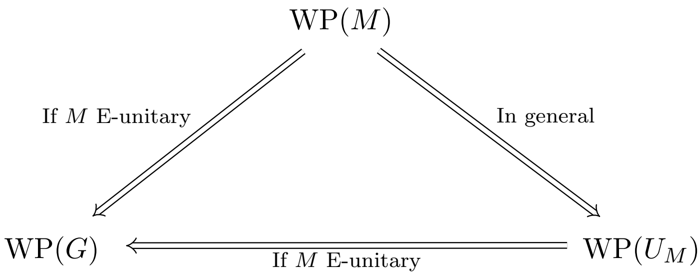

Figure 1.0.1: In this diagram WP() signifies that the object has decidable word problem.

This is because when M is E-unitary its maximal group image G is a subobject of U_M. Thus the word problem of U_M being decidable implies that the word problem of G is decidable. Further G being a subobject of U_M means that it is a subobject of M also. Consequently, an E-unitary M having decidable word problem implies that maximal group image G also has decidable word problem.

In Figure 1.0.1 we illustrate for a finitely presented special inverse monoid M, its maximal group image G and its group of units U_M the limited implications that hold in general between their respective word problems. In Appendix A we lay out a set of examples which demonstrate how none of the other implications can hold. In doing so these examples show that in order to relate the word problem of a special inverse monoid to the word problem of its group of units and maximal group image it will be necessary to impose some conditions on the inverse monoid. It is exactly these kind of results that we produce in Chapters 4 and 5.

Different kinds of factorisations: The most obvious example of when these three word problems are equivalent is when every element is a unit in the inverse monoid. The inverse monoid is thus a group and equal to both its maximal group image and its group of units. This can be viewed through the lens of minimal invertible pieces, the finest division of the relators such that each subword created by the division is a unit (see Definition 3.1.16). In that context the case where every element is a unit can be thought of as the case where the minimal invertible pieces are letters.

We might then ask what more general conditions on the minimal invertible pieces are sufficient to make the word problems of the inverse monoid, its group of units and its maximal group image equivalent. The two that we will investigate here, are when the pieces are uniquely marked, that is each piece has an associated letter which appears in it once and in no other pieces, or the pieces are alphabetically disjoint, that is they are either equal or share no letters. Note that both conditions are generalisations of the case where each piece is a letter.

There are two recent papers dealing with special inverse monoids where the relators factorise in one of these ways. In the first [6] Dolinka and Gray showed that when a word r had a factorisation which was uniquely marked then $\mathrm{Gp}\langle X \mid r = 1 \rangle$ has decidable prefix monoid membership problem. They showed that the prefix membership problem is decidable if r is cyclically reduced and has an alphabetically disjoint factorisation. Both of these results work by using the result of Ivanov, Margolis and Meakin to show that the word problem of the maximal group image being decidable (via Magnus' Theorem) implies the inverse monoid has decidable word problem (if it is E-unitary). In this thesis we shall generalise these results from the one-relator case to the finitely presented case (see Theorem 4.2.2 and Theorem 5.2.2).

The second paper [8], due to Gray and Ruskuc deals with the structure of the group of units of a finitely generated special inverse monoid whose relators have either a uniquely marked or an alphabetically disjoint factorisation. There is also a result of Ivanov, Margolis and Meakin [11, Proposition 4.2] which says that minimal invertible pieces generate the group of units. Combining these results with those discussed in the previous paragraph allows us to formulate circumstances under which the three word problems (those of the inverse monoid, its maximal group image and its group of units) are equivalent (see Corollary 4.3.2 and Corollary 5.3.2).

HNN extensions: The second approach we will use centres around HNN extensions of free groups. HNN extensions of groups are a way of producing more complex structures from simpler ones by “folding” one part of the group onto another. They are well known to have wild algorithmic behaviours, even when the starting group is free. For example, Weiß proved in his PhD thesis that the conjugacy problem is undecidable in general for HNN-extensions of finitely generated free groups [32], and Miller constructed examples of HNN-extensions of free groups with undecidable subgroup membership problem [25]. Even when the defining isomorphism of the HNN extension is an automorphism of a free group there are examples with undecidable submonoid membership problem (see the Burns free-by-cyclic example (Equation 1.0.1) below). So in general, one should not expect to be able to decide membership in finitely generated submonoids of HNN extension of free groups. On the other hand, in order to solve the word problem in various one-relator inverse monoids, by the aforementioned result of Ivanov, Margolis and Meakin [11], it is sufficient to decide membership in certain submonoids of one-relator groups.

The Magnus-Moldavanskiĭ heirarchy: A standard approach to proving results concerning groups with a single cyclically reduced relator is to use the Magnus-Moldavanskiĭ hierarchy. This hierarchy can, in rough terms, be said to classify such groups by how many steps (by HNN extension) they are from being a free group. The approach is then to show that results which hold lower in the hierarchy, in particular in free groups, extend to those higher up.

Of special interest here is that within free groups all rational subsets (see Definition 2.4.3), and thus all submonoids, have decidable membership. This dovetails into our interest in deciding membership in the prefix monoid. Thus a reasonable first question might be whether all groups of the form $\mathrm{Gp}\langle X \mid r = 1 \rangle$ which are a direct HNN extension of a free group have decidable submonoid membership problem. It turns out that this is not the case. Burns, Karrass and Solitar [5] give the following example, which has two possible presentations,

$$\mathrm{Gp}\left\langle\, a, b \mid abba(baab)^{-1} = 1 \,\right\rangle \cong \mathrm{Gp}\left\langle\, x, y, t \mid t^{-1}xt = xy, t^{-1}yt = y \,\right\rangle, \qquad (1.0.1)$$

this has a submonoid with undecidable membership [7], although in this case the prefix membership problem for the presentation on the left is decidable.

Basis preserving isomorphisms: We observe that while the example given in Equation 1.0.1 is an HNN extension whose defining isomorphism maps one basis of the free group $\mathrm{FG}(x, y)$ onto another, these are different bases, i.e. $\{x, y\}$ and $\{xy, y\}$. Therefore we may ask whether HNN extensions whose defining isomorphism maps a free basis onto itself are equally disruly or whether instead they have decidable submonoid membership problem. We are able to give a partial answer to this problem in Theorem 7.1.2 by using novel methods to show that a certain class of submonoids do indeed have decidable membership in such a group.

A result of Dolinka and Gray [6, Theorem 7.2] tells us the following (see Section 2.7 for an explanation of Ξ_w and $\rho_t(w)$):

Theorem 1.0.1. *Let $G = \mathrm{Gp}\langle\, X \mid w = 1 \,\rangle$ be a group such that w is t-sum zero, i.e. the amount of occurences of t and t^{-1} in w are equal. Suppose that*

$$H = \mathrm{Gp}\langle\, \Xi_w \mid \rho_t(w) = 1 \,\rangle$$

is a free group. If w is t-prefix positive, i.e. every prefix of w contains as least as many occurences of t as occurences of t^{-1}, then G has decidable prefix membership problem.

This is, in fact, one way we can see the group in on the left of Equation 1.0.1 has decidable prefix membership problem (note that the presentation on the left in Equation 1.0.1 is b-sum zero and b-prefix positive). The theorem leaves open the question of whether, under the same assumption that the group H is free, groups defined by a relator where the word is t-sum zero but not t-prefix positive have decidable prefix membership problem. Dolinka and Gray [6, Example 7.6] give the following simple example of a group whose prefix membership problem is left open and cannot be solved using their methods (primarily because it is not t-prefix positive).

$$\mathrm{Gp}\left\langle\, a, b, t \mid bt^{-1}at^2bt^{-1}a = 1 \,\right\rangle. \tag{1.0.2}$$

We are able to show that certain one-relator groups which are not t-prefix positive have prefix monoids which take a form where we can apply Theorem 7.1.2, giving us Theorem 7.2.1. In particular, this result allows us to answer the question implicitly left open by Dolinka and Gray and show that $\mathrm{Gp}\left\langle\, a, b, t \mid at^{-1}bt^2at^{-1}b = 1 \,\right\rangle$ has decidable prefix membership problem and thus that $\mathrm{Inv}\left\langle\, a, b, t \mid at^{-1}bt^2at^{-1}b = 1 \,\right\rangle$ has decidable word problem.

A brief summary of results We will now compile a short outline of the remaining chapters and their principle results.

- Chapter 2: This chapter establishes background material and as such has no significant new results.
- Chapter 3: This chapter has two aims. The first is to establish new methods for constructing multiple relator E-unitary inverse monoids, in particular Theorem 3.1.14. The second is to extend a known theory about conservative factorisations to presentations with multiple relators, this is accomplished in Theorem 3.2.8.
- Chapter 4: This chapter focuses on the algorthmic effects of a presentation having a factorisation with uniquely marked pieces. The primary results are Theorem 4.2.2 and Corollary 4.3.2, with the second producing stronger consequences from stricter premises.
- Chapter 5: This chapter focuses on the algorthmic effects of a presentation having a factorisation with alphabetically disjoint pieces. The primary

results are Theorem 5.2.2 and Corollary 5.3.2, with the second producing stronger consequences from stricter premises.

- Chapter 6: This chapter derives numerous lemmas about the properties of a particular class of HNN-extensions of free groups, the value of these lemmas is collective rather than residing in any individual result.

- Chapter 7: This chapter applies the results of the previous chapter to produce two notable results. The first, Theorem 7.1.2, shows that certain kind of submonoid has decidable membership within the class of HNN-extension investigated in the previous chapter. The second, Theorem 7.2.1, applies this to derive a class of one-relator groups which have decidable prefix membership problem (and thus a class of one-relator inverse monoids with decidable word problem).

2

Preliminaries

Synopsis

In this chapter we will provide a great deal of the mathematical background necessary for the rest of this thesis (however, where appropriate, further previously known results will cited in later chapters). Nothing in this chapter should be considered original.

2.1 Words

Let X be an alphabet (a non-empty set of letters) and let X^* denote the *free monoid*, which is the set of words written over this alphabet including the empty word, denoted 1. We use $u \equiv v$ to indicate that two words are equal to each other in the free monoid of the alphabet they are written over.

As we are concerned here with groups and inverse monoids we introduce $\overline{X} = X \cup X^{-1}$, where $X^{-1} = \{x^{-1} \mid x \in X\}$, with the obvious bijective correspondence between X and its 'copy'. This may be extended from letters to words when we say that for $w \equiv x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_k^{\varepsilon_k}$, where $\varepsilon_i \in \{-1, 1\}$ and $x_i \in X$, the corresponding inverse is $w^{-1} \equiv x_k^{-\varepsilon_k} \ldots x_2^{-\varepsilon_2} x_1^{-\varepsilon_1}$. Likewise we may make the obvious extension saying that $\overline{W} = \{w, w^{-1} \mid w \in W\}$ for a set $W \subset \overline{X}^*$.

Example 2.1.1. Let $X = \{a, b, c\}$ be the alphabet. Further let $w_1 \equiv acbaa$, $w_2 \equiv aba^{-1}$ and $w_3 \equiv cc^{-1}$; then $w_1, w_2, w_3 \in \overline{X}^*$, $w_1^{-1} \equiv a^{-1}a^{-1}b^{-1}c^{-1}a^{-1}$,

$w_2^{-1} = ab^{-1}a^{-1}$ and $w_3^{-1} \equiv cc^{-1} \equiv w_3$.

We say that $w \equiv x_1x_2 \dots x_n$, where $x_i \in X$ for $1 \leq i \leq n$, is *reduced* if there is no $1 \leq j < n$ such that $x_{j+1} \equiv x_j^{-1}$. We will use $\mathrm{red}(w)$ at various points throughout to indicate the result of fully reducing a word w, that is iteratively cancelling any xx^{-1} pairs until none remain. We say that w is *cyclically reduced* if it is reduced and also is such that $x_1 \not\equiv x_n^{-1}$.

Example 2.1.2. The word $w_1 \equiv ab^{-1}bc$ is not reduced (and thus not cyclically reduced). The word $w_2 \equiv abbaca^{-1}$ is reduced but not cyclically reduced. The word $w_3 \equiv a^{-1}bcab^{-1}ac$ is cyclically reduced (and thus reduced).

We introduce the following term as a useful way to describe a particular way of dividing up a word.

Definition 2.1.3. Let w be a word. Further let $w_1, w_2, \dots, w_n$ be a set of words such that $w \equiv w_1w_2 \dots w_n$. We call this a *decomposition* of w.

Example 2.1.4. Let $w \equiv ababcb^{-1}b^{-1}a^{-1}$. Further let $w_1 \equiv ab$, $w_2 \equiv ab$, $w_3 \equiv cb^{-1}$ and $w_4 \equiv b^{-1}a^{-1}$. We see that $w \equiv w_1w_2w_3w_4$. We might also indicate such a decomposition by writing $w \equiv (ab)(ab)(cb^{-1})(b^{-1}a^{-1})$.

We use $w(x_1, x_2, \dots, x_k)$ to indicate a specific word written over a certain set of letters and their inverses, i.e. one such that $w \in \overline{\{x_1, x_2, \dots, x_k\}}^*$. Substituting different letters for x_i works as might be expected, see example below.

Example 2.1.5. Let $w(x, y) \equiv xxyx^{-1}$. This means that $w(b, a) \equiv bbab^{-1}$.

In a similar manner, for a set of words $u_1, u_2, \dots, u_k$, we use $w(u_1, u_2, \dots, u_k)$ to indicate that the word w may be written over the pieces u_i and their inverses in the free monoid. Substitutions of pieces works in the same manner as the substituion of letters, again we provide an example.

Example 2.1.6. Let $w(x, y) \equiv xxyx^{-1}$, $u \equiv ab$ and $v \equiv cb^{-1}$. Then

$$w(u, v) \equiv (ab)(ab)(cb^{-1})(ab)^{-1} \equiv ababcb^{-2}a$$

and

$$w(v,u) \equiv (cb^{-1})(cb^{-1})(ab)(cb^{-1})^{-1} \equiv cb^{-1}cb^{-1}ab^2c^{-1}.$$

We also assume that for an individual word $w(x_1,\ldots,x_k)$ that there is no subset of the x_j such that the word may be written over that subset, i.e. for each $1 \le j \le k$ there is at least one appearance of x_j or x_j^{-1}. If we say that there are a set of words $w_i(x_1,\ldots,x_k)$ for $i \in I$ then we only assume that there is no subset of the x_j such that all the w_i may be written over that subset, i.e. for each $1 \le j \le k$ there is at least one word w_i, where $i \in I$, in which x_j or x_j^{-1} appears.

Remark. This notation allows us to succinctly label the decomposition of a word, especially one where subwords are repeated. In doing so it allows us to highlight the structure of the decomposition and when that structure resembles other decompositions, something that might not be obvious simply from writing the words out.

Definition 2.1.7. We say that a set of words $w_1,\ldots,w_n$ *factorises* into the set of *factor words* $u_1,\ldots,u_k$ if $w_1,\ldots,w_n \in \overline{\{u_1,\ldots,u_k\}}^*$. Further, if we have a set of $w_i(x_1,\ldots,x_m)$ such that $w_i \equiv w_i(u_1,\ldots,u_m)$, for $1 \le i \le n$, then we call this specific way of writing the set of the w_i over the u_i a *factorisation*.

Example 2.1.8. Let $w_1 \equiv abcdab$, $w_2 \equiv cdb^{-1}a^{-1}$, $u_1 \equiv ab$ and $u_2 \equiv cd$. If we further let $w_1(x_1,x_2) \equiv x_1x_2x_1$ and $w_2(x_1,x_2) \equiv x_2x_1^{-1}$ then we see that $w_1(u_1,u_2) \equiv (ab)(cd)(ab) \equiv w_1$ and $w_2(u_1,u_2) \equiv (cd)(ab)^{-1} \equiv w_2$.

From here on out we will not explicitly write out the $w(x_1,\ldots,x_n)$ stage in our examples and will instead move immediately on to marking the factorisation out with brackets. (So for Example 2.1.8, we would simply observe that $w_1 \equiv (ab)(cd)(ab)$ and $w_2 \equiv (cd)(ab)^{-1}$, with factor words ab and cd as this is sufficient to define the factorisation).

Remark. It may be noted that we used the singular form of factorisation, both in the definition and the example, when discussing a set of decompositions of words. This is both for linguistic convenience and because there will be far greater value in viewing a set of decompositions collectively as a singular factorisation. For two good examples of this see Example 4.1.3 and Example 5.1.2.

2.2 Presentations

Let $\rho \subseteq \overline{X}^* \times \overline{X}^*$ be a set of pairs of words. We say that this set is reflexive if (w, w) belongs to ρ for all $w \in \overline{X}^*$. We say that the set is symmetric if (u, v) belonging to ρ implies that (v, u) belongs to ρ also for every $u, v \in \overline{X}^*$. We call the set of pairs transitive if (u, v) and (v, w) belonging to ρ implies that (u, w) belongs to ρ. If ρ is reflexive, symmetric and transitive then we say that ρ forms an equivalence relation. If this equivalence relation is also closed under multiplication, i.e. if $(u_1, v_1), (u_2, v_2) \in \rho \Rightarrow (u_1u_2, v_1v_2) \in \rho$, then we call ρ a *congruence*. In this case we may form an object $\overline{X}^*/\rho$ by equating any words $u, v \in \overline{X}^*$ such that $(u, v) \in \rho$.

We may speak of a congruence ρ generated by a set of pairs of words $(u_i, v_i) \subset \overline{X}^* \times \overline{X}^*$ for $i \in I$. When we do this we mean that ρ is the smallest congruence containing all the generating pairs. For an example, let ρ be the congruence generated by $(xx^{-1}, 1)$ and $(x^{-1}x, 1)$ for all $x \in X$. The object produced by the quotient of $\overline{X}^*$ by this congruence is the free group on X, denoted $\mathrm{FG}(X)$.

We say that a set of words $U \subset \overline{X}^*$ is a *free basis* for the free group $FG(X)$ if $\mathrm{Gp}\langle U \rangle = \mathrm{FG}(X)$ and there is no non-trivial $U' \subset U$ such that $\mathrm{Gp}\langle U' \rangle = \mathrm{FG}(X)$. We call $|X|$ the *rank* of the free group $\mathrm{FG}(X)$.

The following is a standard result (found, for instance, in [17, Proposition I.2.7]).

Theorem 2.2.1. *Let F be a free group of finite rank n. The free group F cannot be generated by fewer than n elements and if a set U of n elements generates F then it is a free basis for F.*

Example 2.2.2. Let $G = \mathrm{FG}(a, b, c)$ and $U = \{abc, ab, bc\}$. We can see that $a = (abc)(bc)^{-1} \in \mathrm{Gp}\langle U \rangle$ and $c = (ab)^{-1}(abc) \in \mathrm{Gp}\langle U \rangle$. Hence we can further deduce that $b = (a)^{-1}(abc)(c)^{-1} \in \mathrm{Gp}\langle U\rangle$. Therefore $G \subseteq \mathrm{Gp}\langle U\rangle$ and so by Theorem 2.2.1 U is a free basis for G as $|U| = |\{a, b, c\}|$.

If we have some congruence ρ' generated by $(xx^{-1}, 1)$ and $(x^{-1}x, 1)$ for all $x \in X$, (i.e. the congruence of a free group) and also the set of pairs (u_i, v_i), for $i \in I$, then we may express the object $\overline{X}^*/\rho'$ as

$$G = \mathrm{Gp}\langle\, X \mid u_i = v_i\, (i \in I)\,\rangle$$

which we call a *group presentation*.

In a similar manner we may define presentations for *inverse monoids*, which model partial bijections in the same sense that groups model full bijections. Let ρ be the congruence generated by $(uu^{-1}u, u)$ and $(uu^{-1}vv^{-1}, vv^{-1}uu^{-1})$ for $u, v \in \overline{X}^*$, we call this the *Wagner congruence* on $\overline{X}^*$. Similarly to groups if we let ρ' be the smallest congruence generated by ρ and some set of pairs of words (u_i, v_i) then we may express the object $\overline{X}^*/\rho'$ as

$$M = \mathrm{Inv}\langle\, X \mid u_i = v_i\, (i \in I)\,\rangle$$

which we call an *inverse monoid presentation.* If we have a inverse monoid of the form $\mathrm{Inv}\langle\, X \mid w_i = 1\, (i \in I)\,\rangle$ then we call this a *special inverse monoid.*

An inverse monoid need not necessarily be defined via a presentation, although that is exclusively how we will define them throughout this thesis. It need only be a set of elements with an operation that satisfies one of two equivalent definitions. The first is that every element has a weak inverse (i.e. for every $x \in M$ there is some $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$) and that idempotents commute (observe that this is the basis of the Wagner congruence). The second is that every element has a unique weak inverse. This equivalence is standard, see for example [15, pg. 2].

We call a group or inverse monoid finitely presented if it may be presented using a finite set of generators and defining relations and we will be dealing exclusively with finite presentations throughout.

For an inverse monoid M, we say that $m \in M$ is a *left unit* if it is such that $m^{-1}m = 1$, a *right unit* if $mm^{-1} = 1$ and a *unit* if it is both a left unit and a right unit. We will use L_M, R_M and U_M to denote to the sets of all left units, all right units and all units of M respectively.

We note that if $r_1, r_2 \in R_M$ then $r_1r_2(r_1r_2)^{-1} = r_1r_2r_2^{-1}r_1^{-1} = r_1r_1^{-1} = 1$ and so $r_1r_2 \in R_M$. Therefore R_M is closed under multiplication and is a submonoid of M. Similar arguments may be employed to show that L_M and U_M are also submonoids (and in the U_M case also a subgroup).

We introduce the following standard result (see for instance [8, Lemma 2.2]), which for the sake of completeness we provide a proof of.

Lemma 2.2.3. *Let $M = \text{Inv}\,\langle X \mid R\rangle$ be an inverse monoid. Further let $a, b, c \in M$. If $abb^{-1}c \in R_M$ then $abb^{-1}c = ac$ in M. Consequently, if $w \in \overline{X}^*$ is a right unit of M then $w = \text{red}(w)$ in M.*

Proof. That $abb^{-1}c$ is a right unit of M implies that ab is as well. Hence, in M, $abb^{-1}a^{-1} = 1$ and so $abb^{-1}a^{-1}a = a$. As bb^{-1} and $a^{-1}a$ are idempotents they commute, giving $abb^{-1} = aa^{-1}abb^{-1} = abb^{-1}a^{-1}a = a$ in M. Right multiplication by c then gives $abb^{-1}c = ac$ as desired. □

2.3 The E-unitary Property

For an inverse monoid M there is the notion of a *minimum group congruence* σ which is the smallest congruence such that $G = M/\sigma$ is a group, we call this G the *maximal group image*. Of course, $G = M$ if and only if M is already also a group. We may also use $\sigma()$ to denote the natural homomorphism $M \to G$. A standard fact (see for instance [22, Lemma 1.3]) about the maximal group image is the following.

Lemma 2.3.1. *Let* $M = \mathrm{Inv}\langle X \mid u_i = v_i\,(i \in I)\rangle$ *be an inverse monoid. Its maximal group image can be presented as* $G = \mathrm{Gp}\langle X \mid u_i = v_i\,(i \in I)\rangle$*.*

For a given inverse monoid M, we say that an element $e \in M$ is *idempotent* if $e^2 = e$ and designate the set of all idempotents in M by E_M. Within an inverse monoid any two idempotents $e, f \in E_M$ commute, as such $(ef)^2 = efef$ is equal to $e^2f^2 = ef$ and so E_M is closed under multiplication. Therefore E_M is a submonoid of M.

Definition 2.3.2. An inverse monoid M is said to be *E-unitary* when an element of M is σ congruent to 1 if and only if it is idempotent.

By a slight abuse of notation we may say this definition is equivalent to requiring $\sigma^{-1}(1) = E_M$, where σ^{-1} represents the preimage of the natural homomorphism.

Let M be an inverse monoid, then we may define the *maximal E-unitary image*, the largest homomorphic image of M which is E-unitary. We denote this maximal E-unitary image by M_{EU}. This is concept is analogous to the maximal group image and similarly to that case we observe that $M_{EU} = M$ if and only if M is E-unitary.

Lemma 2.3.3. *Let* $M = \mathrm{Inv}\langle X \mid R\rangle$ *be a inverse monoid and* $G = \mathrm{Gp}\langle X \mid R\rangle$ *be its maximal group image. Further let* $W = \{w \in \overline{X}^* : w = 1 \text{ in } G\}$*. Then*

$$M' = \mathrm{Inv}\left\langle X \mid R, w^2 = w\ (w \in W)\right\rangle$$

is an infinite presentation such that $M' = M$*.*

Proof. We begin by observing that the presentation of M' is indeed infinite as there are infinitely many words in the set W (note $x^i x^{-i} = 1$ in G, for all $i \in \mathbb{Z}$). The maximal group image of M' is $G' = \mathrm{Gp}\left\langle X \mid R, w^2 = w\,(w \in W)\right\rangle$. All the relations of the form $w^2 = w$ in G' are equivalent to relations of the form $w = 1$. However as $w = 1$ in G, such relations must already be a consequence of the set of relations R. Thus the $w^2 = w$ relations in G' are superfluous and may be removed. Therefore $G = G'$ and thus M' shares a maximal group image with M. This in turn means that M' must be E-unitary as all the elements that map to identity in its maximal group image are idempotent. Further, $M' = M_{EU}$ as all the relations in M' not present in M must be true in M_{EU}, which is by definition E-unitary. □

This does not always mean that M_{EU} lacks a finite presentation, in particular if M is E-unitary then $M = M_{EU}$ and so if M is finitely related, so is M_{EU}.

One of our most useful criteria for E-unitarity comes from Ivanov, Margolis and Meakin [11, Theorem 4.1].

Theorem 2.3.4. *Let $M = \text{Inv}\,\langle\, X \mid w = 1 \,\rangle$ be an inverse monoid. If $w \in \overline{X}^*$ is a cyclically reduced word then M is E-unitary.*

Example 2.3.5. The monoid $M = \text{Inv}\left\langle\, a, b, c \mid abacb^{-1}ac^{-1} = 1 \,\right\rangle$ is E-unitary.

In the same paper it is also shown, by the example given below, that all relations being cyclically reduced is insufficient, even in the two relator case, to guarantee the inverse monoid being E-unitary.

Example 2.3.6. Let $M = \text{Inv}\,\langle\, a, b, c, d \mid abc = 1, adc = 1 \,\rangle$ and $G = \text{Gp}\,\langle\, a, b, c, d \mid abc = 1, adc = 1 \,\rangle$. It may easily be found that $bd^{-1} = a^{-1}c^{-1}ca = 1$ in G. Therefore if M is E-unitary then bd^{-1} is idempotent. It may then be shown graphically, as is done in [11], that this is not the case and hence that M is not E-unitary.

We consider the congruence ρ generated by $(aa^{-1}, 1)$, $(a^{-1}a, 1)$, $(bb^{-1}, 1)$, $(b^{-1}b, 1)$ and (b, d). The first four are sufficient to render the submonoid generated by a and b a group. The last and $(c, b^{-1}a^{-1})$ (which may be derived from M's relator $abc = 1$ and those pairs we already used) may then be used to show that all words written over $\overline{\{a, b, c, d\}}$ can be rewritten as a word over $\overline{\{a, b\}}$. Therefore M/ρ is a group. Moreover all five generating components are true in G so ρ is in fact the minimum group congruence σ.

Definition 2.3.7. Let $a, b \in M$, where M is an inverse monoid. We say that $a \sim b$ if and only if both ab^{-1} and $b^{-1}a$ are idempotents. We call this the *compatibility relation.*

The compatibility relation is only transitive in E-unitary inverse monoids. Moreover an inverse monoid is E-unitary if and only if $\sigma =\sim$, that is if its minimum group congruence is the same as the compability relation. This means that for an E-unitary inverse monoid M with maximal group image G, $u = v$ in G if and only if $u \sim v$ in M.

The above paragraph and the rest of this subsection draws on a work of Lawson [15] (available online) which itself draws its material primarily from his book [14]. The following is a standard result, the proof of which may be found in, for instance [15, Lemma 2.10].

Lemma 2.3.8. *Let M be an inverse monoid and let $x, y \in M$. Then the following are equivalent:*

- *That $x = ye$ for some $e \in E_M$.*
- *That $x = fy$ for some $f \in E_M$.*
- *That $x = xx^{-1}y$.*
- *That $x = yx^{-1}x$.*

Definition 2.3.9. Let $x, y \in M$, where M is an inverse monoid. We say that $x \leq y$ if $x = xx^{-1}y$ or equivalently if any of the other conditions laid out in Lemma 2.3.8 are satisfied. This forms the *natural partial order* on M.

For a partial order we can speak of the concept of the *meet* of two elements, denoted $x \wedge y$, which (when it is definable) is the greatest lower bound of the two elements. That is if $z = x \wedge y$ then $z \leq x, y$ and for all $z' \leq x, y$ we also have that $z' \leq z$. It is important to note that $x \wedge y$ is not necessarily defined for all pairs of elements of an inverse monoid. In fact the existence of a meet is tied to the idea of compatibility.

The following result essentially follows from Lawson [15, Lemma 2.16], for completeness we provide a proof.

Lemma 2.3.10. *Let m, n be elements of an E-unitary inverse monoid M. Then $m \wedge n$ exists if and only if $m \sim n$. Moreover if $m \wedge n$ exists then it is equal to all of the following $mm^{-1}n$, $nn^{-1}m$, $mn^{-1}n$ and $nm^{-1}m$.*

Proof. Suppose that $m \wedge n$ exists. Put $z = m \wedge n$. By definition $z \leq m$, which implies that $z = zz^{-1}m$. From this we get that $zm^{-1} = zz^{-1}mm^{-1}$ and that $z^{-1}m = m^{-1}zz^{-1}m$, inspection shows that both of these are idempotents, and so $z \sim m$. Dually $z \sim n$ and as the compatibility relation is transitive in E-unitary monoids we have that $m \sim n$.

Conversely suppose that $m \sim n$, then $m^{-1}n$ is an idempotent. Set $z = mm^{-1}n$, then $z \leq n$ and $z \leq m$, since mm^{-1} and $m^{-1}n$ are idempotent. Let $x \in M$ be such that $x \leq m$ and $x \leq n$. Then $x = xx^{-1}x \leq mm^{-1}n = z$ and so z is the greatest lower bound of m and n, Therefore $m \wedge n$ exists and is equal to $mm^{-1}n$. The other forms may be found by similar arguments. □

2.4 Decidability

We say that a class of true/false questions are *decidable* if there is an algorithm which can determine the answer to any of them in finite time. For instance, given a group presentation $G = \mathrm{Gp}\langle X \mid w_i = 1\,(i \in I)\rangle$ we might ask whether it was decidable if some $w \in \overline{X}^*$ was equal to the empty word 1. This is called the *word problem* for G.

An important result of Magnus [18] on this topic is the following.

Theorem 2.4.1 (Magnus' Theorem)**.** *Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ be a one-relator* *group. The word problem of G is decidable.*

In contrast to this, it was later shown independently by Novikov [27] and Boone [4] that the word problem is undecidable in general for finitely presented groups.

A second class of problems we are interested in are *membership problems.* If we have a group $G = \mathrm{Gp}\langle X \mid R \rangle$ and a set $Q \subset G$ then we can decide membership in Q within G if there is an algorithm that takes a word $w \in \overline{X}^*$ and in finite time outputs yes if w represents an element of Q within G and no if it does not. Importantly this does not require us to know which specific element of Q it may represent.

An important result of this type is also due to Magnus [19].

Theorem 2.4.2 (Freiheitssatz)**.** *Let $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ be a one-relator group where w is cyclically reduced. Let $Y \subset X$ be a subset of the generators. If $w \notin \overline{Y}^*$ then $\mathrm{Gp}\langle Y \rangle \leq G$ is a free group.*

In order to explain the next result we need the following definition.

Definition 2.4.3. We say that a subset of a group G is *rational* if it is a member of $\mathrm{Rat}(G)$ which is the smallest classes that satisfies the following:

- If $W \subseteq G$ is finite then $W \in \mathrm{Rat}(G)$
- If $A, B \in \mathrm{Rat}(G)$ then $A \cup B, A \cap B, A^* \in \mathrm{Rat}(G)$

One of the most important results which deal with membership problems is Benois' Theorem [3].

Theorem 2.4.4 (Benois' Theorem)**.** *Let G be a free group. If $A \in \mathrm{Rat}(G)$ then membership in A is decidable within G.*

All finitely generated submonoids are rational subsets of their group. Hence all finitely generated submonoids (and subgroups) of a free group have decidable membership within that free group.

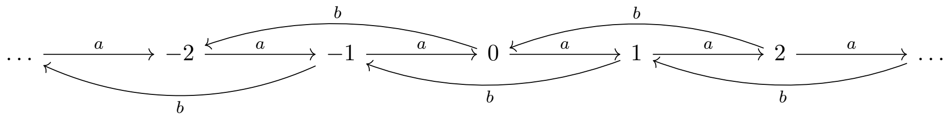

Figure 2.4.1: Cayley graph of the group in Example 2.4.5 with vertices labelled to demonstrate the isomorphism.

For words $w, p \in \overline{X}^*$ we have the natural notion of p being a prefix if $w \equiv pq$ for $q \in \overline{X}^*$. We use $\mathrm{pref}(w)$ to denote the set of prefixes of w. For a group presentation $G = \mathrm{Gp}\langle X \mid w_i = 1\,(i \in I)\rangle$ we call

$$P = \mathrm{Mon}\,\langle \mathrm{pref}(w_i)\,(i \in I)\rangle \leq G$$

the *prefix monoid* of G. It is important to stress here that we do not assume, either in this instance or throughout the rest of this thesis, that the w_i are reduced or cyclically reduced just because they are group relators. Two different presentations of the same group may have different prefix monoids. The following simple example of this is taken from Ivanov, Margolis and Meakin [11, Example after Theorem 3.1].

Example 2.4.5. Let $G_1 = \mathrm{Gp}\langle a, b \mid aab = 1\rangle$ and $G_2 = \mathrm{Gp}\langle a, b \mid aba = 1\rangle$. These groups are isomorphic to each other, in fact both are isomorphic to $\mathbb{Z}$, under the mapping that extends $a \mapsto 1$, $b \mapsto -2$ and $1 \mapsto 0$. However their prefix monoids are different, under our mapping to $\mathbb{Z}$ the prefix monoid of G_1 is sent to $\mathbb{N}$ and the prefix monoid of G_2 is sent to $\mathbb{Z}$.

Given a particular presentation of a group G, we may ask whether membership in the corresponding prefix monoid P within G is decidable. That is for $w \in \overline{X}^*$, is it decidable whether w represents an element of the prefix monoid in G. We call this the *prefix membership problem* and it is dependent on our choice of presentation for G, both in the form of the question, i.e. what monoid are we deciding membership in within G, and what the answer is, i.e. there are groups where one presentation has decidable prefix membership problem and another presentation has undecidable prefix membership problem (see for instance [6, Remark 8.1]).

Another natural class of questions we might ask is: given an inverse monoid $M = \mathrm{Inv}\langle X \mid w_i = 1\,(i \in I)\rangle$, is it decidable whether $u = v$ in M for $u, v \in \overline{X}^*$? This is the word problem for special inverse monoids (note that while we could formulate the word problem for groups in the same manner it would reduce to way we have phrased it above, as $u = v$ is equivalent to $uv^{-1} = 1$ in a group setting). This is a question that has been connected to the two previously mentioned problems by another result of Ivanov, Margolis and Meakin [11, Theorem 3.3].

Theorem 2.4.6. *Let $M = \mathrm{Inv}\langle X \mid w_i = 1\,(i \in I)\rangle$ be an inverse monoid and let $G = \mathrm{Gp}\langle X \mid w_i = 1\,(i \in I)\rangle$ be its maximal group image. If the word problem and prefix membership problem for G are decidable then the word problem for M_{EU} is decidable, in particular M has decidable word problem if it is E-unitary.*

As noted by Ivanov, Margolis and Meakin themselves [11, Theorem 3.1] this result may be simplified in certain one-relator cases.

Theorem 2.4.7. *Let $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ be an inverse monoid and $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ be its maximal group image. If w is cyclically reduced and the prefix membership problem for G is decidable then M has decidable word problem.*

Proof. By Theorem 2.3.4 we know that M is E-unitary and by Magnus' Theorem we know that G has decidable word problem. Consequently the desired result follows from Theorem 2.4.6. □

These results serve to motivate our work here to determine when the prefix membership problem is decidable.

2.5 Free Products

Suppose we have two groups $G_1 = \mathrm{Gp}\langle X_1 \mid r_i = 1\,(i \in I_1)\rangle$ and $G_2 = \mathrm{Gp}\langle X_2 \mid s_i = 1\,(i \in I_2)\rangle$, where $X_1 \cap X_2 = \varnothing$. The *free product* of G_1 and

G_2 is denoted by $G_1 * G_2$ and has the following presentation

$$\mathrm{Gp}\,\langle\, X_1, X_2 \mid r_{i_1} = 1\,(i_1 \in I_1), s_{i_2} = 1\,(i_2 \in I_2)\,\rangle\,.$$

The following is a standard result, see for example [17, Corrollary IV.1.3]

Lemma 2.5.1. *Let G_1 and G_2 be groups with decidable word problem. Their free product $G_1 * G_2$ has decidable word problem.*

Definition 2.5.2. Let G_1 and G_2 be groups with the presentations $\mathrm{Gp}\,\langle\, X_1 \mid r_{i_1} = 1\,(i \in I_1)\,\rangle$ and $\mathrm{Gp}\,\langle\, X_2 \mid s_{i_2} = 1\,(i \in I_2)\,\rangle$. Further let $K_1 \leq G_1$ and $K_2 \leq G_2$ be subgroups which are isomorphic under the map $\phi : K_1 \to K_2$. We can form a new group $G_1 *_{K_1 = K_2} G_2$ with the presentation

$$\mathrm{Gp}\,\langle\, X_1, X_2 \mid r_{i_1} = 1\,(i \in I_1), s_{i_2} = 1\,(i \in I_2),\, k_1 = \phi(k_1)\,(k_1 \in K_1)\,\rangle$$

which we call an *amalgamated free product*.

Throughout the rest of this thesis we will creating amalgamated products where K_1 and K_2 are finitely generated. When we do this we will use $G_1 *_{u_i = v_i} G_2$ to denote the product $G_1 *_{K_1 = K_2} G_2$ where $K_1 = \mathrm{Gp}\,\langle u_1, \ldots u_n \rangle \leq G_1$, $K_2 = \mathrm{Gp}\,\langle v_1, \ldots, v_n \rangle \leq G_2$ and $\phi : K_1 \to K_2$ is the extension of $u_i \mapsto v_i$ for $1 \leq i \leq n$. Thus the group $G_1 *_{u_i = v_i} G_2$ has the presentation

$$\mathrm{Gp}\,\langle\, X_1, X_2 \mid r_{i_1} = 1\,(i_1 \in I_1), s_{i_2} = 1\,(i_2 \in I_2), u_i = v_i\,(1 \leq i \leq n)\,\rangle\,.$$

We cite the following standard result (a proof of which may be found in Lipshutz [16, Lemma 2]).

Lemma 2.5.3. *Let*

$$G_1 = \mathrm{Gp}\,\langle\, X_1 \mid r_{i_1} = 1\,(i_1 \in I_1)\,\rangle \ \textit{and}\ G_2 = \mathrm{Gp}\,\langle\, X_2 \mid s_{i_2} = 1\,(i_2 \in I_2)\,\rangle$$

be finitely presented groups with decidable word problem. Further let

$u_1, \ldots, u_k \in \overline{X_1}^*$ and $v_1, \ldots, v_k \in \overline{X_2}^*$ be sets of words such that the subgroups $\mathrm{Gp}\langle u_1, \ldots, u_k\rangle \leq G_1$ and $\mathrm{Gp}\langle v_1, \ldots, v_k\rangle \leq G_2$ are isomorphic under the mapping $\phi : u_i \mapsto v_i$ and have decidable membership within their respective groups. Then $G_1 *_{u_i=v_i} G_2$ has decidable word problem.

We use $\mathrm{Inv}\langle W\rangle \leq M$ to represent the *inverse submonoid* generated by $W \in \overline{X}^*$ within $M = \mathrm{Inv}\langle\, X \mid R\,\rangle$. This inverse submonoid, like a subgroup but unlike a non-inverse submonoid, satisfies $\mathrm{Inv}\langle W\rangle = \mathrm{Inv}\left\langle W \cup W^{-1}\right\rangle \leq M$ (this is because groups and inverse monoids, unlike monoids, have a native inverse). Thus we can talk of constructing free products of inverse monoids both with or without amalgamation by requiring any amalgamation to be over isomorphic inverse submonoids in the same manner that amalgamated group products require isomorphic subgroups.

2.6 HNN-extensions

If we have a group $H = \mathrm{Gp}\langle\, X \mid R\,\rangle$ with two subgroups A and B which are isomorphic under the map $\phi : A \to B$. Then we can construct a group

$$H^* = H*_{t,\phi:A\to B} = \mathrm{Gp}\left\langle\, X \cup \{t\} \mid R,\, t^{-1}at = \phi(a) \quad (a \in A)\,\right\rangle$$

which we call an *HNN extension* of H, while we call t the *stable letter*, A and B the associated subgroups and ϕ the associated isomorphism.

Remark. Throughout we will use a similar format, where for a given group H, $H*_{t,\phi:A\to B}$ describes what the HNN extension is and H^* is used to denote the resulting HNN extension thereafter.

We observe that if H is finitely presented and A (and equivalently B) finitely generated then H^* will have at least one finite presentation of the form

$$H*_{t,\phi:A\to B} = \mathrm{Gp}\left\langle\, X \cup \{t\} \mid R,\, t^{-1}\alpha_i t = \phi(\alpha_i) \quad (1 \leq i \leq k)\,\right\rangle,$$

where $\alpha_1, \ldots, \alpha_k \in \overline{X}^*$ are a generating set for A.

We say that a word w is in *HNN reduced form* if is of the following form

$$w_0 t^{\varepsilon_1} w_1 \ldots t^{\varepsilon_k} w_k$$

where $\varepsilon_i \in \{-1, 1\}$ for $1 \leq i \leq k$, $w_i \in \overline{X}^*$ for $0 \leq i \leq k$ and no subwords are of the form either $t^{-1} w_i t$ where $w_i \in A$ or $t w_i t^{-1} \in B$.

This allows us to describe one of the fundamental results of the theory of HNN extensions.

Lemma 2.6.1 (Britton's Lemma)**.** *Let H^* be an HNN extension of $H = \mathrm{Gp}\langle X \mid R \rangle$. Further let $w \equiv h_0 t^{n_1} h_1 \ldots t^{n_k} h_k$ is a word in HNN reduced form, where $h_i \in \overline{X}^*$ for $0 \leq i \leq k$. If $k \geq 1$ then $w \neq 1$ in H^*.*

Britton's Lemma can then be applied to produce the following standard result.

Lemma 2.6.2. *Let $H^* = H*_{t,\phi:A \to B}$ be an HNN extension. Then equality between two words in HNN reduced form*

$$g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} g_2 \ldots t^{\varepsilon_n} g_n = h_0 t^{\delta_1} h_1 t^{\delta_2} h_2 \ldots t^{\delta_m} h_m$$

holds in the HNN extension if and only if $n = m$, $\varepsilon_i = \delta_i$ for all $1 \leq i \leq n$ and there exist $1 = \alpha_0, \alpha_1, \ldots, \alpha_n, \alpha_{n+1} = 1 \in A \cup B$ where $\alpha_i \in A$ if $\varepsilon_i = -1$ and $\alpha_i \in B$ if $\varepsilon_i = 1$ such that

$$h_i = \alpha_i^{-1} g_i (t^{\varepsilon_{i+1}} \alpha_{i+1} t^{\varepsilon_{i+1}})$$

for all $0 \leq i \leq n$.

There are also known circumstances under which we can find a reduced form by algorithm, see [17, Pages 184-185] for a description of the process.

Lemma 2.6.3. *Let $H^* = H*_{t,\phi:A \to B}$ where H, A and B are all finitely generated, H has recursively enumerable word problem and membership in A and B within*

H are decidable. Then there is an algorithm which takes a word $w \in \overline{X \cup \{t\}}^*$ as input and returns an HNN reduced form word

$$w' \equiv w_0 t^{\varepsilon_1} w_1 \dots t^{\varepsilon_n} w_n \tag{2.6.1}$$

where $w_i \in \overline{X}^*$ for $0 \leq i \leq n$, $\varepsilon_i \in \{-1, 1\}$ for $1 \leq i \leq n$ and $w = w'$ in H^*.

The following result may be found in Lyndon and Schupp [17, Corollary IV.2.2]

Theorem 2.6.4. *Let $H^* = H*_{t,\phi:A \to B}$ be an HNN extension. If H has decidable word problem, membership in A and B within H is decidable and ϕ and ϕ^{-1} are effectively calculable then H^* has decidable word problem.*

2.7 Hierarchy

Let X be an alphabet. For any given $x \in X$ we can define a function $\sigma_x(w) : \overline{X}^* \to \mathbb{Z}$, which returns the number of appearances of x in $w \in \overline{X}^*$ less the number of appearances of x^{-1} in w. We call this total the *exponent sum* of x in w and say that w has x *exponent sum zero* if $\sigma_x(w) = 0$.

Example 2.7.1. Let $X = \{x, y, z\}$ and $w \equiv xy^2x^{-1}y^{-1}zy^{-1}$. Then

$$\sigma_y(w) = \sigma_y(xy^2x^{-1}y^{-1}zy^{-1}) = 2 - 1 - 1 = 0$$

and so w has y exponent sum zero.

Let X be an alphabet, then we can define a second alphabet,

$$\Xi_X = \{x_i \mid x \in \overline{X}, i \in \mathbb{Z}\}$$

and a function

$$\rho_t : \{w \in (\overline{X} \cup \{t, t^{-1}\})^* \mid w \text{ is cyclically reduced and } \sigma_t(w) = 0\} \to \Xi_X^*$$

for a given t. This is defined as follows; let $S = X \cup \{t\}$, $w \equiv s_1 s_2 \ldots s_n$ where $s_i \in S$, for $1 \leq i \leq n$, and let $p_i \equiv s_1 s_2 \ldots s_i$. If $s_i \in \{t, t^{-1}\}$ then let $s'_i \equiv 1$ and if $s_i \equiv x \in X$ then let $s'_i \equiv x_{-\sigma_t(p_i)}$. Then ρ_t is the map which sends w to $s'_1 s'_2 \ldots s'_n$.

Example 2.7.2. Let $X = \{x, y\}$ and let $w \equiv yt^{-1}xt^2yt^{-1}x \in (\overline{X} \cup \{t, t^{-1}\})^*$. This gives that

$$\rho_t(w) \equiv \rho_t(yt^{-1}xt^2yt^{-1}x) \equiv y_0 x_1 y_{-1} x_0.$$

The function ρ_t^{-1} simply sends each x_i to $t^{-i}xt^i$. It might, arguably, be easier to think of ρ_t as the inverse of this function.

Example 2.7.3. Following on from the previous example, we observe that

$$\rho_t^{-1}(y_0 x_1 y_{-1} x_0) \equiv (t^{-0}yt^0)(t^{-1}xt^1)(t^{-(-1)}yt^{-1})(t^{-0}xt^0). = yt^{-1}xt^2yt^{-1}x.$$

For any particular word w we can define μ_x and m_x to be, respectively, the minimum and maximum values of i such that x_i appears in $\rho_t(w)$. If x does not appear at all in w then we say that $\mu_x = 0 = m_x$. For instance in Example 2.7.2 $\mu_y = -1$ and $m_y = 0$.

Using these we can define a subset of Ξ_X,

$$\Xi_w = \{x_i \in \Xi_X \mid \mu_x \leq i \leq m_x\}$$

so in our running example $\Xi_w = \{x_0, x_1, y_{-1}, y_0\}$. However in general it need not be the case that all the letters of Ξ_X appear in $\rho_t(w)$, for instance if $X = \{x, y, z\}$ and $w = yxt^{-2}xt^2$ we have $\Xi_w = \{x_0, x_1, x_2, y_0, z_0\}$.

With these ideas in place we introduce the following result, originally due to Moldavanskiĭ, a proof can be found within [17, Theorem IV.5.1] which uses the methods of McCool and Schupp.

Theorem 2.7.4. *Let $w \in (\overline{X} \cup \{t, t^{-1}\})^*$ be a word with t exponent sum zero such* *that $\rho_t(w)$ is cyclically reduced and let $G = \mathrm{Gp}\langle\, X \mid w = 1 \,\rangle$ be a group. Then G is an HNN extension of the group*

$$H = \mathrm{Gp}\langle\, \Xi_w \mid \rho_t(w) = 1 \,\rangle$$

where the associated subgroups A and B are free and generated by $\Xi_w \setminus \{x_{m_x} \mid x \in X\}$ and $\Xi_w \setminus \{x_{\mu_x} \mid x \in X\}$ respectively and the associated isomorphism is defined by $\phi(x_i) = x_{i+1}$ for $x \in X$ and $\mu_x \leq i < m_x$.

This is easiest to understand through examples

Example 2.7.5. If $X = \{x, y\}$ and $w \equiv yt^{-1}xt^2yt^{-1}x$ then

$$H = \mathrm{Gp}\langle\, x_0, x_1, y_{-1}, y_0 \mid y_0x_1y_{-1}x_0 = 1 \,\rangle,$$

$A = \mathrm{Gp}\langle x_0, y_{-1}\rangle$, $B = \mathrm{Gp}\langle x_1, y_0\rangle$ and ϕ is the extension of the function sending x_0 to x_1 and y_{-1} to y_0.

Example 2.7.6. If $X = \{x, y, z\}$ and $w \equiv yxt^{-2}xt^2x$ then

$$H = \mathrm{Gp}\langle\, x_0, x_1, x_2, y_0, z_0 \mid y_0x_0x_2x_0 = 1 \,\rangle,$$

$A = \mathrm{Gp}\langle x_0, x_1\rangle$, $B = \mathrm{Gp}\langle x_1, x_2\rangle$ and ϕ is the extension of the function sending x_0 to x_1 and x_1 to x_2.

3

New results on E-unitary amalgamated products and conservative factorisations

Synopsis

This chapter is divided into two sections establishing two sets of results which we will deploy in the succeeding chapters. The results of the first section concern when the amalgamated product of two E-unitary inverse monoids is itself E-unitary; these come from analysing the consequences of work by Stephen. The results of the second section concern when factorisations of a group's relators are conservative; these results extend work by Dolinka and Gray from one-relator groups to finitely presented ones.

3.1 E-unitary amalgamated products

Please recall the notions $x \leq y$ and $x \wedge y$ were defined in Definition 2.3.9 and the subsequent paragraph, respectively. They will be used throughout this section.

In 1998, Stephen [30] introduced the following notion.

Definition 3.1.1. Let S be an E-unitary inverse semigroup and U be an inverse subsemigroup. We say that S is *upwardly directed* into U if for all $s \in S$ and

$u \in U$ then $s \wedge u$ being defined implies that there exists $v \in U \cup \{1\}$ such that $s, u \leq v$.

Stephen then proved the following theorem [30, Theorem 6.5] which uses this definition.

Theorem 3.1.2. *Let S_1 and S_2 be E-unitary inverse semigroups and let U be an inverse subsemigroup of S_1 and S_2. If both S_1 and S_2 are upwardly directed into U then $S_1 *_U S_2$ is E-unitary.*

Which, as he noted, immediately gives

Corollary 3.1.3. *Let S be an E-unitary inverse semigroup and let U be an inverse subsemigroup of S. If S is upwardly directed into U then $S *_U S$ is E-unitary.*

Here we will be restricting our use of this result to inverse monoids rather than the broader class of subsemigroups. We may do so because the result is not specific to the presentation of the inverse semigroup and because any finitely presented inverse monoid may be finitely presented as an inverse semigroup. This latter is the case because an inverse monoid with the presentation $\text{Inv}\langle X \mid R \rangle$ has the following inverse semigroup presentation

$$\text{InvSem}\langle X, e \mid R', e^2 = e, xe = x, x = ex \, (x \in X) \rangle$$

where R' is the result of replacing any relations of the form $r = 1$ with $r = e$.

Recalling that inverse submonoids contain the identity element we may observe that the following definition is a consequence of the broader one above.

Definition 3.1.4. An E-unitary inverse monoid S is *upwardly directed* into an inverse submonoid U of S if for all $s \in S$ and $u \in U$ then $s \wedge u$ being defined implies that there exists $v \in U$ such that $s, u \leq v$.

It is worth showing that an inverse monoid is not always upwardly directed into its inverse submonoids, even when the inverse monoid is E-unitary.

Example 3.1.5. We let

$$M = \mathrm{Inv}\langle a, b \mid aa^{-1} = a^{-1}a,\ bb^{-1} = b^{-1}b,$$
$$ab = bb,\ ab^{-1} = bb^{-1},\ a^{-1}b = b^{-1}b,\ a^{-1}b^{-1} = b^{-1}b^{-1}\rangle.$$

Suppose we have an element of M written exclusively over a and a^{-1}, through repeated application of $aa^{-1}a = a$, $a^{-1}aa^{-1} = a^{-1}$ and $aa^{-1} = a^{-1}a$, we can write this element either as aa^{-1} or a^i, where $i \in \mathbb{Z} \setminus \{0\}$. Suppose we have an element of M not written exclusively over a and a^{-1}, then either this is identity or we can use the second row of relations to rewrite it so as to be exclusively written over b and b^{-1}. If we have an element of M written exclusively over b and b^{-1} then by a similar logic to above we may rewrite this as bb^{-1} or b^i, where $i \in \mathbb{Z} \setminus \{0\}$. Thus any element of M may be written in one of the following forms a^i, b^i, aa^{-1}, bb^{-1} or 1, where $i \in \mathbb{Z} \setminus \{0\}$. Clearly the only ones of these which are mapped to 1 by $\sigma : M \to G$ are aa^{-1}, bb^{-1} and 1 are precisely the idempotents. Therefore M is E-unitary.

Further let $U = \mathrm{Inv}\,\langle b\rangle \leq M$, $m \in M$ and $u \in U$. By Lemma 2.3.10 if $m \wedge u$ is defined then it is equal to $uu^{-1}m$ and $um^{-1}m$. Thus when $m \wedge u$ is defined $\sigma(u) = \sigma(um^{-1}m) = \sigma(uu^{-1}m) = \sigma(m)$. Therefore if M is upwardly directed into U then $\sigma(m) \in \sigma(U)$ and there exists $v \in U$ such that $v \geq m$.

However $\sigma(a) = b \in \sigma(U)$, which implies there exists $v \in U = \mathrm{Inv}\,\langle b\rangle$ such that $v \geq a$. This would mean $a = vaa^{-1}$ by Lemma 2.3.8. If $v = 1$ then this is obviously false. If $v = bb^{-1}$ then

$$vaa^{-1} = b(b^{-1}a)a^{-1} = bb^{-1}(ba^{-1}) = bb^{-1}bb^{-1}$$

which is not equal to a. A similar process will show that any v of the form b^i, where $i \in \mathbb{Z} \setminus \{0\}$, produces $vaa^{-1} = b^iaa^{-1} = b^ibb^{-1} = b^i \neq a$. So any choice of

v produces a contradiction.

Therefore we have established that M cannot be upwardly directed into U.

We will now demonstrate that certain classes of inverse submonoids are such that their inverse monoids are always upwardly directed into them.

Lemma 3.1.6. *Let M be an E-unitary inverse monoid and N be an inverse submonoid of M. If $N \subseteq U_M$ then M is upwardly directed into N.*

Proof. Suppose that $N \subseteq U_M$. Let $m \in M$ and $n \in N$. By Lemma 2.3.10 we know that if $m \wedge n$ is defined then it is equal to both $nn^{-1}m$ and $mm^{-1}n$. As n is a unit the former is equal to m and so we may say that $m = mm^{-1}n$. Thus $m \leq n$ as mm^{-1} is idempotent. Therefore as $n, m \leq n \in N$, we have shown that M is upwardly directed into N. □

Example 3.1.7. Let $M = \text{Inv}\left\langle\, a, b, c, d \mid abcabcbc^{-1}d = 1, dcabcb^{-1}d^{-1}ab = 1 \,\right\rangle$. It may be found that both ab and d are units in M as both are the prefix to one of the relators and the suffix to another. Therefore M is upwardly directed into $N = \text{Inv}\,\langle ab, d\rangle \subseteq U_M$.

Lemma 3.1.8. *Let M be an E-unitary inverse monoid and N be an inverse submonoid of M. If $N \subseteq E_M$ then M is upwardly directed into N.*

Proof. Suppose that $N \subseteq E_M$. Let $m \in M$ and $n \in N$ be such that $m \wedge n$ is defined. As n is an idempotent it may easily be deduced that $1 \sim n$. Moreover as the meet of m and n is defined this means by Lemma 2.3.10 we know that they are compatible. Thus we may write $m \sim n$. As M is E-unitary, compatibility is transitive and so $1 \sim m$, which in turn implies that $m \in E_M$. Therefore $m, n \leq 1 \in N$ and so M is upwardly directed into N. □

Example 3.1.9. Let $M = \text{Inv}\left\langle\, a, b, c \mid a^2 = a, b^2 = b, c^2 = c, abacbc = 1 \,\right\rangle$. It follows immediately from the relations that a and b are idempotents. Therefore M is upwardly directed into $N = \text{Inv}\,\langle a, b\rangle \subseteq E_M$.

Lemma 3.1.10. *Let M be an E-unitary inverse monoid and let $N = \text{Inv}\langle r \rangle \leq M$ where $r \in R_M$. The inverse submonoid M is upwardly directed into N.*

Proof. Suppose that $N = \text{Inv}\langle r \rangle \leq M$ where $r \in R_M$. Let $n \in N$ and $m \in M$ be such that $m \wedge n$ is defined. As $n \in \{r, r^{-1}\}^*$ and $rr^{-1} = 1$, n must be of the form $r^{-k_1}r^{k_2}$ for some $k_i \geq 0$. Moreover as $r^{-i}r^i$ is idempotent for all i, we may write $n = er^k$ for some k and idempotent $e \in N$. It may be then be seen that $n \sim r^k$ (as $nr^{-k} = er^kr^{-k}$ and $r^{-k}n = r^{-k}er^k$ are both idempotents). By Lemma 2.3.10 the existence of $m \wedge n$ means that $m \sim n$. Thus $m \sim r^k$ and m may be written fr^k for some idempotent $f \in N$ (as $r^k \in R_M$ is maximal in the partial order by $R_M \cap E_M = \{1\}$). Therefore $m, n \leq r^k \in N$ and so M is upwardly directed into N. □

Example 3.1.11. Let $M = \text{Inv}\left\langle a, b, c, d \mid abcba^{-1} = 1, dcbac^{-1} = 1 \right\rangle$. Looking at the relations we can see that a and d are right units and hence ad is also a right unit. Therefore M is upwardly directed into $N = \text{Inv}\langle ad \rangle$.

Lemma 3.1.12. *Let M be an E-unitary inverse monoid and let $N = \text{Inv}\langle l \rangle \leq M$ where $l \in L_M$. The inverse submonoid M is upwardly directed into N.*

Proof. The proof of this is dual to Lemma 3.1.10. □

Remark. If $l \in L_M$ and we let $r = l^{-1}$ then $r \in R_M$. Further it follows that

$$N = \text{Inv}\langle l \rangle = \text{Inv}\left\langle l, l^{-1} \right\rangle = \text{Inv}\left\langle r^{-1}, r \right\rangle = \text{Inv}\langle r \rangle.$$

So, Lemma 3.1.10 and Lemma 3.1.12 are not just dual but are in fact the same result.

Example 3.1.13. By token of the above remark, if we let $M = \text{Inv}\left\langle a, b, c, d \mid abcba^{-1} = 1, dcbac^{-1} = 1 \right\rangle$ then $d^{-1}a^{-1} = (ad)^{-1} \in L_M$ and so M is upwardly directed into $N = \text{Inv}\left\langle d^{-1}a^{-1} \right\rangle = \text{Inv}\langle ad \rangle$.

Using the first of these cases we will now produce a theorem which will prove useful later in the construction of examples.

Theorem 3.1.14. *Let $M = M_1 *_N M_2$ where M_1 and M_2 are E-unitary inverse monoids and N is an inverse submonoid of both M_1 and M_2. If $N \subseteq U_{M_1} \cap U_{M_2}$ then M is E-unitary. Further if M_1 and M_2 have finite special presentations and N is finitely generated then there is a finite special presentation of M.*

Proof. By Lemma 3.1.6 $N \subseteq U_{M_1}$ implies that M_1 is upwardly directed into N. Dually M_2 is upwardly directed into N. By Theorem 3.1.2 we have that M_1 and M_2 are E-unitary and that both M_1 and M_2 are upwardly directed into N implies that $M = M_1 *_N M_2$ is E-unitary.

Suppose that M_1 and M_2 are finitely presented and N is finitely generated. Let $M_1 = \mathrm{Inv}\langle X_1 \mid r_i = 1 (i \in I) \rangle$ and $M_2 = \mathrm{Inv}\langle X_2 \mid s_j = 1(j \in J) \rangle$. Further let N be generated by $u_1, \ldots u_k$ in M_1 and correspondingly by $v_1, \ldots, v_k$ in M_2. Then M may be written

$$\mathrm{Inv}\langle X_1, X_2 \mid r_i = 1(i \in I), s_j = 1(j \in J), u_1 = v_1, \ldots, u_k = v_k \rangle$$

It is a general fact about inverse monoids that $st^{-1} = 1$ implies $s = t$ (as the former gives $t^{-1}st^{-1} = t^{-1}$ and $st^{-1}s = s$ and therefore the uniqueness of inverses implies $s^{-1} = t^{-1}$). Thus $u_i v_i^{-1} = 1$ implies $u_i = v_i$, the reverse implication however does require us to use the assumption that $u_i \in U_M$. This allows us to find the following.

$$M = \mathrm{Inv}\left\langle X_1, X_2 \mid r_i = 1(i \in I), s_j = 1(j \in J), u_1 v_1^{-1} = 1, \ldots, u_k v_k^{-1} = 1 \right\rangle$$

which is a finite special presentation.

□

The next deinition uses the notation, $r(u, v)$, in a manner indentical to the notation, $w(u, v)$, which was exhibited in Example 2.1.6 and explained in the paragraph preceding it.

Definition 3.1.15. Let $M = \mathrm{Inv}\langle X \mid r_i(u_1, \ldots, u_k) = 1 \,(i \in I) \rangle$ be an inverse monoid. We say that the factorisation of the r_i into u_j is *unital* when each u_j is a unit in M.

Remark. Our use of "unital" here takes its lead from the language of Dolinka and Gray [6].

Definition 3.1.16. We say that a unital factorisation is into *minimal invertible pieces* when every factor u of the factorisation is such that for any $pq \equiv u$, p is invertible if and only if $p \equiv 1$ or $p \equiv u$.

There are a number of methods for determining a unital factorisations, the simplest of which is iteratively finding words which are both prefixes and suffixes of known units starting with the relators (this method is sufficient to find the minimal invertible pieces of a one-relator non-inverse monoid). However no method is currently known is guaranteed to produce a decomposition of the relators of an inverse monoid into their minimal invertible pieces, even in the one-relator case, for more on all this see [13],[8] and [28]. We can use this notion of unital factorisations to find applications for Theorem 3.1.14.

Example 3.1.17. Consider the inverse monoid

$$M = \mathrm{Inv}\langle x_1, y_1, z_1, x_2, y_2, z_2 \mid (z_1)(x_1^2y_1)^2(z_1) = 1, (z_2)(x_2^2y_2)^2(z_2) = 1, \\ (z_1)(z_2)^{-1} = 1\rangle$$

for which the brackets mark a unital factorisation. This is the product of two inverse monoids $M_i = \mathrm{Inv}\left\langle x_i, y_i, z_i \mid (z_i)(x_i^2y_i)^2(z_i) = 1 \right\rangle$ (for $i = 1, 2$) amalgamated over $z_1 = z_2$. As M_1 and M_2 are copies of each other the inverse submonoids generated by z_1 and z_2 are clearly isomorphic (and finitely generated). Moreover as each M_i is defined by a single cyclically reduced relator we know that they are both E-unitary by Theorem 2.3.4. Therefore by Theorem 3.1.14 M is E-unitary.

Example 3.1.18. For $i = 1, 2$, let

$$M_i = \mathrm{Inv}\left\langle a_i, b_i, c_i \mid (a_i^2b_i^3)(c_i^5)^2(a_i^2b_i^3)(c_i^5)(a_i^2b_i^3) = 1 \right\rangle$$

be two inverse monoids. Then we can find that $u_i \equiv a_i^2 b_i^3$ and $v_i \equiv c_i^5$ are unital pieces. Therefore we can show that

$$\begin{aligned} M = \mathrm{Inv}\langle a_1, b_1, c_1, a_2, b_2, c_2 \mid (a_1^2 b_1^3)(c_1^5)^2(a_1^2 b_1^3)(c_1^5)(a_1^2 b_1^3) = 1,& \\ (a_2^2 b_2^3)(c_2^5)^2(a_2^2 b_2^3)(c_2^5)(a_2^2 b_2^3) = 1,& \\ (a_1^2 b_1^3)(a_2^2 b_2^3)^{-1} = 1\rangle& \end{aligned}$$

is E-unitary in a similar manner to above.

3.2 Conservative Factorisations

Definition 3.2.1. Let $w \equiv x_1 x_2 \ldots x_k$ be a word written over $\overline{X}$ such that $x_i \in X \cup X^{-1}$. The *prefixes* of w are 1 and the words $p_i \equiv x_1 x_2 \ldots x_i$, for $1 \leq i \leq k$, (note $w \equiv p_k$). We denote the set of all prefixes of w by $\mathrm{pref}(w)$.

Definition 3.2.2. Let $G = \mathrm{Gp}\,\langle\, X \mid r_i = 1\,(i \in I)\,\rangle$ be a group presentation. We call the submonoid generated by the prefixes of the relators the *prefix monoid*. Thus we may write the prefix monoid $\mathrm{Mon}\,\langle \mathrm{pref}(r_i)\,(i \in I)\rangle \leq G$.

It is important to understand that even in the one-relator case two different presentations of the same group may have different prefix monoids (see Example 2.4.5).

We introduce the following elementary result.

Lemma 3.2.3. *Let M be an inverse monoid. For any subset of elements $N \subseteq M$, we have $N \cdot E_M = E_M \cdot N$.*

Proof. Let $m \in N \cdot E_M$, this means there are some $n \in N$ and $e \in E_M$ such that $m = ne$. It follows that $m = nn^{-1}ne$, further as $n^{-1}n$ is an idempotent it commutes with e, giving us $m = nen^{-1}n$. The element nen^{-1} is also idempotent and so $m = (nen^{-1})n \in E_M \cdot N$. This means that $N \cdot E_M \subseteq E_M \cdot N$.

That $E_M \cdot N \subseteq N \cdot E_M$ follows by a dual argument.

□

There is a strong link between the right units of a special inverse monoid and the prefix monoid of the maximal group image. This link is even stronger when the monoid in question is E-unitary.

Lemma 3.2.4. *Let* $M = \mathrm{Inv}\,\langle\, X \mid s_i = 1\,(i \in I)\,\rangle$ *and* $G = \mathrm{Gp}\,\langle\, X \mid s_i = 1\,(i \in I)\,\rangle$ *be an inverse monoid and its corresponding maximal group image. Further let* σ *be the map from* M *to* G *induced by identity on* $\overline{X}$.

If P *is the prefix monoid of* G *then* $P = \sigma(R_M)$ *and furthermore if* M *is E-unitary then* $\sigma^{-1}(P) = R_M \cdot E_M = E_M \cdot R_M$.

Proof. Let τ and π be the natural maps from $\overline{X}^*$ to M and G respectively.

Let $p \in P$, by definition of the prefix monoid this means that we may find some $w_p \equiv w_1 w_2 \ldots w_k$ such that $\pi(w_p) = p$ and each w_j is a prefix of one of the s_i. As prefixes of the relators are right invertible in a special monoid we also know that each $\tau(w_j)$ is right invertible in M. As the product of right invertible elements is right invertible we have that $\tau(w_p) \in R_M$. Therefore $p = \sigma(\tau(w_p)) \in \sigma(R_M)$ and so $P \subseteq \sigma(R_M)$.

Let $m \in R_M$. By a geometric argument of Ivanov, Margolis and Meakin [11, Paragraph 2 of Lemma 4.2], every right unit is a product of prefixes. Therefore we may find some $w_m \equiv w_1 w_2 \ldots w_k$ such that $\tau(w_m) = m$ and each w_j is a prefix of one of the s_i. This means that

$$\sigma(m) = \sigma(\tau(w_m)) = \sigma(\tau(w_1 w_2 \ldots w_k)) = \sigma(\tau(w_1))\sigma(\tau(w_2)) \ldots \sigma(\tau(w_k)) \in P$$

as each $\sigma(\tau(w_i))$ will be a generator of P. Therefore $\sigma(R_M) \subseteq P$ and so $P = \sigma(R_M)$.

We now supppose that M is E-unitary.

Let $m \in \sigma^{-1}(P)$. By the prior result this means that $m \in \sigma^{-1}(\sigma(R_M))$. Therefore there is some $r \in R_M$ such that $\sigma(m) = \sigma(r)$. It follows that $\sigma(r^{-1}m) = \sigma(1)$ and, as M is E-unitary, that $r^{-1}m \in E_M$. Therefore, as r is right invertible, $m = r \cdot r^{-1}m \in R_M \cdot E_M$ and so $\sigma^{-1}(P) \subseteq R_M \cdot E_M$.

Let $m \in R_M \cdot E_M$, then there exist $e \in E_M$ and $r \in R_M$ such that $m = er$. By the E-unitarity of M we may see that $\sigma(m) = \sigma(re) = \sigma(r)$ and so $m \in \sigma^{-1}(\sigma(R_M))$ which means by the above that $m \in \sigma^{-1}(P)$. Therefore $R_M \cdot E_M \subseteq \sigma^{-1}(P)$ and so $\sigma^{-1}(P) = R_M \cdot E_M$.

Finally we know that $R_M \cdot E_M = E_M \cdot R_M$ by Lemma 3.2.3.

□

Remark. An easy consequence of the word problem for an inverse monoid M is decidable is being able to decide membership in both E_M and R_M within M. However, it is not clear whether another consequence is being able to decide membership in $E_M \cdot R_M$ within M. We observe that when M is E-unitary this is equivalent, by Lemma 3.2.4, to being able to decide membership in the prefix monoid of M's maximal group image. This is notable as whether an E-unitary special inverse monoid having decidable word problem implies its maximal group image has decidable prefix membership problem is an open problem [10].

Definition 3.2.5. Let $G = \mathrm{Gp}\langle X \mid r_i(u_1, \ldots, u_k) = 1\,(i \in I)\rangle$ be a group. Further let $V \subseteq \overline{\{u_1, \ldots, u_k\}}$ be minimal such that $r_i \in V^*$ for every $i \in I$. The factorisation into $u_1, \ldots, u_k$ is said to be *conservative* in G if

$$\mathrm{Mon}\left\langle \bigcup_{i \in I} \mathrm{pref}(r_i) \right\rangle = \mathrm{Mon}\left\langle \bigcup_{v \in V} \mathrm{pref}(v) \right\rangle \leq G$$

that is, if the submonoid generated by the prefixes of the factors is equal to the prefix monoid, the monoid generated by the prefixes of the relators.

Remark. Similarly to unital, our use of the word "conservative" in relation to a factorisation is in line with how it was used by Dolinka and Gray in [6]

It may be noticed that we were particular about whether a certain factor appeared in its positive or negative form in our definition but that we use the phrase a conservative factorisation into $u_1, u_2, \ldots, u_k$ without reference to whether their inverses are or aren't used. The reason for that may be seen in the next lemma.

Lemma 3.2.6. *Let $G = \mathrm{Gp}\langle X \mid r_i(u_1, \ldots, u_k) = 1\,(i \in I)\rangle$ be a group and let the set $\Gamma \subseteq \{(j, \varepsilon) \,|\, 1 \leq j \leq k, \varepsilon \in \{-1, 1\}\}$ be minimal such that $r_i \in U^*$, for every $i \in I$, where $U = \{u_j^{\varepsilon} \mid (j, \varepsilon) \in \Gamma\}$. Then*

$$\mathrm{Mon}\left\langle \bigcup_{(i,\varepsilon)\in\Gamma} \mathrm{pref}(u_j^{\varepsilon}) \right\rangle = \mathrm{Mon}\left\langle \bigcup_{1\leq j\leq k} \bigcup_{\varepsilon\in\{-1,1\}} \mathrm{pref}(u_j^{\varepsilon}) \right\rangle \leq G \tag{3.2.1}$$

Consequently, if the r_i have a conservative factorisation into $u_1, u_2, \ldots, u_k$ then

$$P = \mathrm{Mon}\left\langle \bigcup_{1\leq j\leq k} \bigcup_{\varepsilon\in\{-1,1\}} \mathrm{pref}(u_j^{\varepsilon}) \right\rangle \leq G$$

is equal to the prefix monoid of G.

Proof. Let Q be equal to the submonoid on the left hand side of Equation 3.2.1 and suppose there is some j such that $u_j^{-1} \notin U$. As each u_j must appear in either its positive or its negative form at least once, there must be some r_i such that $r_i \equiv v_p u_j v_s$ where $v_p, v_s \in U^*$. In G, $r_i = 1$ which implies that $u_j^{-1} = v_s v_p$. Thus $u_j^{-1} \in Q$ in G and so $\mathrm{pref}(u_j^{-1}) = u_j^{-1} \cdot \mathrm{pref}(u_j) \in Q$. Dually in any case where u_j does not appear in the factorisation we may show that $\mathrm{pref}(u_j) \in Q$.

This shows that the right hand side of Equation 3.2.1 is a subset of the left. The reverse is obvious and so we have shown equality. The second part is then immediate from the definition of a conservative factorisation.

□

Example 3.2.7. Let $G = \mathrm{Gp}\langle a, b, c, d \mid (ab)(cd)(ab) = 1\rangle$. The prefix monoid may be written

$$P = \text{Mon}\,\langle 1, a, ab, abc, abcd, abcda, abcdab\rangle$$

which may be immediately simplified to

$$P = \text{Mon}\,\langle a, ab, abc, abcd\rangle$$

by removing those generators which are equal to 1 or concatenations of other generators. We next observe that $(abcd)(abc) \equiv (abcdab)(c) = c$ and likewise $(abcd)(abcd) = cd$. Hence

$$P = \text{Mon}\,\langle a, ab, abc, abcd, c, cd\rangle$$

from which we may remove the now extraneous generators to produce

$$P = \text{Mon}\,\langle a, ab, c, cd\rangle\,,$$

thus showing that the factorisation is conservative.

Further, by Lemma 3.2.6 we know that the prefix monoid may be written

$$P = \text{Mon}\,\left\langle a, ab,\, b^{-1}, b^{-1}a^{-1},\, c, cd,\, d^{-1}, d^{-1}c^{-1}\right\rangle$$

The next result establishes a connection between unital and conservative factorisations of special presentations, extrapolating the result of [6] from the one-relator case to finite presentations.

Theorem 3.2.8. *Let* $M = \text{Inv}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ *be an inverse monoid and* $G = \text{Gp}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ *be its maximal group image. If the factorisation of the* r_i *is unital in* M *then it is conservative in* G*. Furthermore if* M *is E-unitary then it is also true that the factorisation of the* r_i *being conservative in* G *implies it is a unital factorisation in* M*.*

Proof. Suppose that the factorisation given by the u_j is unital in M. Let $q \in \text{pref}(u_j^\varepsilon)$ for some u_j^ε that appears exactly in at least one of the r_i. Such an r_i will have a prefix p of the form $p \equiv w \cdot q$ for some $w \in \overline{\{u_1, u_2, \ldots, u_k\}}^*$. As w is a concatenation of words which are unital in M it will also be a unit when considered as an element of M. Moreover as $r_i = 1$ we can see p is a right unit, thus q is a right unit in M and so by Lemma 3.2.4 belongs to the prefix monoid. Therefore for all $1 \leq j \leq k$ and $\varepsilon \in \{-1, 1\}$ such that u_j^ε appears exactly in the factorisation the prefixes of u_j^ε belong to the prefix monoid. By Lemma 3.2.6 this means that the factorisation into u_j is conservative.

Suppose that M is E-unitary and that the factorisation given by the u_j is conservative in G. Let r_i be an arbitary relator and $r_i \equiv v_1 v_2 \ldots v_\ell$ where $v_j \in \overline{\{u_1, u_2, \ldots, u_k\}}$ and the factorisation of r_i into v_j is in line with the overall factorisation. First we observe that $r_i = 1$ which means $v_1^{-1} = v_2 \ldots v_\ell$ in G. Further as the factorisation is conservative each factor, including the v_j, belongs to the prefix monoid. Thus we have that $v_1^{-1} = p_1 \ldots p_n$ in G where each p_j is a prefix to one of the r_i. As M is E-unitary this means that $v_1^{-1} \sim p_1 \ldots p_n$, which in turn means that $p_1 \ldots p_n v_1 \in E_M$. This is a product of right invertible elements and so is itself right invertible, and there is only one right invertible idempotent, the identity element. So $p_1 \ldots p_n v_1 = 1$ in M, which means v_1 is left invertible and so as we already know it to be right invertible we have shown v_1 is invertible. Hence we deduce that $v_2 v_3 \ldots v_\ell v_1 = 1$ and from here we may repeat the process to show that $v_2, \ldots, v_\ell$ are also invertible. As our choice of r_i was arbitary this means that we can show any factor word u_j is invertible and therefore the factorisation is unital.

□

Example 3.2.9. Let

$$M = \text{Inv}\,\langle\, a, b, x, y, z \mid (xy)(zy)(zzyz) = 1, (ab)(xy)(ab)(xy) = 1 \,\rangle\,.$$

We can find that ab, xy and zyz are units and that thus that the brackets denote a unital factorisation in M. By Theorem 3.2.8 this means that in the group

$$G = \text{Gp}\langle\, a, b, x, y, z \mid (xy)(zy)(zzyz) = 1, (ab)(xy)(ab)(xy) = 1 \,\rangle$$

the brackets denote a conservative factorisation.

Example 3.2.10. Let

$$G_1 = \text{Gp}\langle\, a, b, c, d \mid (ab)(cd)(ab) = 1 \,\rangle \quad G_2 = \text{Gp}\langle\, t, x, y, z \mid (xy)(zt)(xy) = 1 \,\rangle .$$

These are copies of Example 3.2.7 and therefore the brackets denote a conservative factorisation. Further let

$$G = \text{Gp}\langle\, a, b, c, d, t, x, y, z \mid (ab)(cd)(ab) = 1, (xy)(zt)(xy) = 1, (xy)(ab) = 1 \,\rangle ,$$

this is the amalgamated product $G_1 *_{y^{-1}x^{-1}=ab} G_2$. Finally we let

$$M_1 = \text{Inv}\langle\, a, b, c, d \mid (ab)(cd)(ab) = 1 \,\rangle \quad M_2 = \text{Inv}\langle\, t, x, y, z \mid (xy)(zt)(xy) = 1 \,\rangle .$$

These are such that G_1 and G_2 are, respectively, their maximal group images.

By Theorem 2.3.4 M_1 and M_2 are E-unitary. Further as $y^{-1}x^{-1}$ and ab are both right units, M_1 and M_2 are upwardly directed into the inverse submonoids $\text{Inv}\langle y^{-1}x^{-1}\rangle$ and $\text{Inv}\langle ab\rangle$ respectively by Lemma 3.1.10. Moreover these inverse submonoids are isomorphic to each other due to M_1 and M_2 being copies of each other. Therefore $M_1 *_{y^{-1}x^{-1}=ab} M_2$ is E-unitary, it may be presented

$$M = \text{Inv}\langle\, a, b, c, d, t, x, y, z \mid (ab)(cd)(ab) = 1, (xy)(zt)(xy) = 1, (xy)(ab) = 1 \,\rangle .$$

As the brackets mark conservative factorisations for G_1 and G_2 their prefix monoids may be written

$$P_1 = \text{Mon}\langle \text{pref}(ab), \text{pref}(cd)\rangle \leq G_1 \quad P_2 = \text{Mon}\langle \text{pref}(xy), \text{pref}(zt)\rangle \leq G_2.$$

We observe that $\mathrm{Mon}\langle P_1 \cup P_2 \rangle \leq G$ is sufficient to generate all the prefixes of G's relators. Further as G_1 is a subgroup of G and $(ab)(cd)(ab)$ is a relator of G as well as G_1 we may conclude that P_1 is a submonoid of the prefix monoid of G. Similarly P_2 is a submonoid of the prefix monoid of G. Thus the bracketing of the relators of G denote a conservative factorisation.

The factorisation of G is conservative and M is E-unitary so by Theorem 3.2.8 this means that the bracketed factorisation of M is unital.

4

Factorisations which are uniquely marked

Synopsis

This chapter focuses on the consequences of a presentation having a factorisation with uniquely marked pieces. The first section will deal with structural implications for a group fitting this criterion. In the second section we will show that certain additional conditions are sufficient for such a group to have decidable prefix membership problem. In the third section we will show that the minimal invertible pieces of a special inverse monoid being uniquely marked is sufficient to render the word problems of the inverse monoid, its maximal group image and its group of units equivalent. Finally in the fourth section we will show that there are certain special inverse monoids not possessing a uniquely marked unital factorisation which have an alternative presentation that does possess such a factorisation.

4.1 Basic Properties of Groups with Uniquely Marked Factorisations

Definition 4.1.1. We call the factorisation of a set of words *uniquely marked* if the factor words, $u_1, u_2, \ldots, u_k \in \overline{X}^*$, are such that each u_i has a corresponding $x_i \in \overline{X}$ which appears precisely once in u_i and not at all in any of the u_j or u_j^{-1} where $j \neq i$. We call each x_i the *marker letter* of the corresponding u_i.

For any set of words the factorisation into letters is uniquely marked.

Example 4.1.2. Consider the words $aabc^{-1}$ and $cbaa$. We may factorise these into $(a)(a)(b)(c^{-1})$ and $(c)(b)(a)(a)$, i.e. with factor words $u_1 \equiv a$, $u_2 \equiv b$ and $u_3 \equiv c$. This is a uniquely marked factorisation as each factor word consists of one letter which acts as the marker letter. Note that the factor word c appears in both its positive and negative form.

It is important to observe that it is the entire set of factor words that must be uniquely marked. It is not sufficient for each word taken indiviually to have a set of factor words that can be uniquely marked. This is perhaps made clearest through an example.

Example 4.1.3. Consider the words $axaby^{-1}$ and $byaxa$. The factorisation $(axa)(by^{-1})$ of $axaby^{-1}$ is uniquely marked, if we take $u_1 \equiv axa$, $u_2 \equiv by^{-1}$, $x_1 \equiv x$ and $x_2 \equiv y$. Likewise, the factorisation $(by)(axa)$ of $byaxa$ is uniquely marked, if we take $u_1 \equiv axa$, $u_3 \equiv by$, $x_1 \equiv x$ and $x_3 \equiv y$. However the factorisation $(axa)(by^{-1})$ and $(by)(axa)$ of $axaby^{-1}$ and $byaxa$ is not uniquely marked, because there is no way to pick unique marker letters for factor words $u_1 \equiv axa$, $u_2 \equiv by^{-1}$ and $u_3 \equiv by$ (in particular y and y^{-1} do not count as different marker letters).

We introduce the following lemma describing how a uniquely marked factorisation effects the structure of a group. It is implicit in the work of Dolinka and Gray [6] and Gray and Ruskuc [8], but for the sake of completeness we include an explicit proof here.

Lemma 4.1.4. *Let* $G = \mathrm{Gp}\langle X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\rangle$ *be a group where* $u_1, u_2, \ldots, u_k \in \overline{X}^*$ *are a set of uniquely marked words. Then* G *is isomorphic to the free product of* $H = \mathrm{Gp}\langle z_1, z_2, \ldots, z_k \mid r_i(z_1, z_2, \ldots, z_k) = 1\,(i \in I)\rangle$ *and a free group. Furthermore* H *is isomorphic to* $\mathrm{Gp}\langle u_1, u_2, \ldots, u_k\rangle \leq G$ *under the mapping that sends* z_i *to* u_i*.*

Proof. We will begin with a series of Tietze transformations (see [20, Section 1.5]) which will give us a new form for the group. We equate each factor word, u_j, with a new generator, z_j, giving

$$\mathrm{Gp}\langle X, Z \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I), u_j = z_j\,(1 \leq j \leq k)\rangle$$

where $Z = \{z_1, z_2, \ldots, z_k\}$.

As $u_1, u_2, \ldots, u_k$ are uniquely marked we may divide the alphabet X into two parts, Y, the unique marker letters and $X' = X \setminus Y$. For each factor we may write $u_j \equiv p_j y_j q_j$ where $y_j \in \overline{Y}$ and $p_j, q_j \in \overline{X'}^*$. Applying this we may easily derive the following

$$\mathrm{Gp}\left\langle X, Z \mid r_i(z_1, z_2, \ldots, z_k) = 1\,(i \in I), y_j = p_j^{-1} z_j q_j^{-1}\,(1 \leq j \leq k)\right\rangle$$

from the above.

From this it is clear that the y_j generators are now redundant, because as unique marker letters they will not have appeared in p_j and q_j which are written over $X' = X \setminus Y$. This gives the presentation

$$\mathrm{Gp}\left\langle X', Z \mid r_i(z_1, z_2, \ldots, z_k) = 1\,(i \in I)\right\rangle$$

which has no relations on x'. Therefore we may write

$$\mathrm{Gp}\,\langle\, Z \mid r_i(z_1, z_2, \ldots, z_k) = 1\,(i \in I)\,\rangle * \mathrm{FG}(X').$$

This is isomorphic to G under the mapping induced by sending z_j to u_j and x_j to x_j for $z_j \in Z$ and $x_j \in X'$. Hence $\mathrm{Gp}\,\langle\, Z \mid r_i(z_1, z_2, \ldots, z_k) = 1\,(i \in I)\,\rangle$ is embedded within G.

□

Example 4.1.5. Let $G = \mathrm{Gp}\,\langle\, a, b, x, y \mid axabyaxaby = 1\,\rangle$. Its sole relation may be factorised into axa and by, which are uniquely marked by x and y respectively, giving $(axa)(by)(axa)(by)$. This means if we let $H_1 = \mathrm{Gp}\,\langle\, z_1, z_2 \mid z_1z_2z_1z_2 = 1\,\rangle$ then by Lemma 4.1.4 we have that $G \cong H_1 * \mathrm{FG}(a, b)$ and that $H_1 \cong \mathrm{Gp}\,\langle axa, by\rangle \leq G$.

Alternatively we might choose $axaby$ as a single factor word which is uniquely marked by b. In this case we would have $H_2 = \mathrm{Gp}\left\langle\, z \mid z^2 = 1\,\right\rangle$ and by Lemma 4.1.4, that $G \cong H_2 * \mathrm{FG}(a, x, y)$ and that $H_2 \cong \mathrm{Gp}\,\langle axaby\rangle \leq G$.

Remark. Let $G = \mathrm{Gp}\,\langle\, X \mid r_i = 1\,(i \in I)\,\rangle$ be a group. It was previously noted that every word has a trivial uniquely marked factorisation into its component letters. If we take this factorisation and apply Lemma 4.1.4 then the result is an H which is just a relabelling of G. This renders the observations made about H by the lemma trivial (they would say that $G \cong H * \mathrm{FG}(\varnothing)$ and that $G \cong H \cong \mathrm{Gp}\,\langle X\rangle \leq G$).

Example 4.1.6. Let

$$G = \mathrm{Gp}\left\langle\, a, b, c, d, x, y \mid ada^{-1}bxcy = 1, bxbx = 1, ada^{-1}y^{-1}c^{-1} = 1\,\right\rangle,$$

this group's relators have a factorisation into ada^{-1}, bx and cy uniquely marked by the letters d, x and c respectively. Applying Lemma 4.1.4 gives $H = \mathrm{Gp}\left\langle\, z_1, z_2, z_3 \mid z_1z_2z_3 = 1, z_2^2 = 1, z_1z_3^{-1} = 1\,\right\rangle$ where $G \cong H * \mathrm{FG}(a, b, y)$

and $H \cong \mathrm{Gp}\left\langle ada^{-1}, bx, cy \right\rangle \leq G$.

The following may be found by application of Lemma 4.1.4.

Lemma 4.1.7. *Let $G = \mathrm{Gp}\langle X \mid r = 1 \rangle$ be a group. If there is a letter $x \in X$ such that either x or x^{-1} appears precisely once in r then G is a free group of rank $|X| - 1$.*

4.2 Uniquely Marked Unital Factorisations

We introduce a theorem of Dolinka and Gray [6, Theorem A] regarding the decidability of membership in certain submonoids of amalgamated group products.

Theorem 4.2.1. *Let $G = B *_A C$ be a group, where A, B and C are finitely generated groups such that both B and C have decidable word problem and that membership in A is decidable within both B and C. Let M be a submonoid of G such that the following hold:*

1. *both $M \cap B$ and $M \cap C$ are finitely generated and*

$$M = \mathrm{Mon}\langle (M \cap B) \cup (M \cap C) \rangle \leq G;$$

2. *membership in $M \cap B$ is decidable within B;*
3. *membership in $M \cap C$ is decidable within C;*
4. *$A \subseteq M$.*

Then membership in M within G is decidable.

We can now apply this and our previous structural result, Lemma 4.1.4, to find conditions under which the prefix membership problem is decidable.

Theorem 4.2.2. *Let* $M = \mathrm{Inv}\langle X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\rangle$ *be an inverse monoid and let* $G = \mathrm{Gp}\langle X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\rangle$ *be its maximal group image. If the words* $u_1, u_2, \ldots, u_k \in \overline{X}^*$ *are uniquely marked and form a conservative factorisation in* G *(in particular if the factorisation is unital in* M*) and* G *has decidable word problem then* G *has decidable prefix membership problem and* M_{EU} *has decidable word problem (in particular* M *has decidable word problem if it is E-unitary).*

Proof. We carry over the notation from Lemma 4.1.4.

By Lemma 3.2.6 we know that the prefix monoid takes the form

$$P = \mathrm{Mon}\left\langle \bigcup_{1\leq j\leq k} \bigcup_{\varepsilon\in\{-1,1\}} \mathrm{pref}(u_j^{\varepsilon}) \right\rangle \leq G$$

We saw in Lemma 4.1.4 that $G \cong H * \mathrm{FG}(X')$. We now consider the monoid $P' \leq H * \mathrm{FG}(X')$ which isomorphic to $P \leq G$ under the mapping sending u_i to z_i. We do this by looking at where the generators of P are sent. Recalling that $u_j \equiv p_j y_j q_j$, we may see that

$$\mathrm{pref}(u_j) = \mathrm{pref}(p_j) \cup p_j y_j \cdot \mathrm{pref}(q_j).$$

As our mapping from G to $H * \mathrm{FG}(X')$ sends u_j to z_j it also sends y_j to $p_j^{-1} z_j p_j^{-1}$. Using this we see that $\mathrm{pref}(u_j)$ is sent to

$$\mathrm{pref}(u_j) = \mathrm{pref}(p_j) \cup z_j q_j^{-1} \cdot \mathrm{pref}(q_j) = \mathrm{pref}(p_j) \cup z_j \cdot \mathrm{pref}(q_j^{-1})$$

Dually, we may find that $\mathrm{pref}(u_j^{-1})$ is sent to $\mathrm{pref}(q_j^{-1}) \cup z_j^{-1} \cdot \mathrm{pref}(p_j)$. Thus if we define the monoid

$$Q = \mathrm{Mon}\left\langle \bigcup_{1\leq j\leq k} \left(\mathrm{pref}(p_j) \cup \mathrm{pref}(q_j^{-1})\right) \right\rangle \leq \mathrm{FG}(X')$$

we may write that

$$P' = \text{Mon}\left\langle \overline{Z} \cup Q \right\rangle \leq H * \text{FG}(X').$$

As P' is the submonoid of a free product we can apply Theorem 4.2.1. Further, as this free product has no amalgamation, the conditions in the Theorem are trivially satisfied. It is clear that $P' \cap H = \text{Mon}\left\langle \overline{Z} \right\rangle = H$ and $P \cap \text{FG}(X') = \text{Mon}\langle Q \rangle$, and that both of these are finitely generated. It follows therefore that $P' = \text{Mon}\langle (P' \cap H) \cup (P \cap \text{FG}(X')) \rangle$, satisfying the first condition. As membership in H as a submonoid of itself is trivially decidable and as all submonoids have decidable membership in the free group $\text{FG}(X')$ by Benois's Theorem [3] the second and third conditions are also satisfied. Thus P' has decidable membership in $H * \text{FG}(X')$ which by isomorphism means that P has decidable membership in G. Therefore G has decidable prefix membership problem (as well as decidable word problem) and so by Theorem 2.4.6 the word problem of M_{EU} is decidable.

□

The next lemma, which is due to Magnus, may be found in Lyndon and Schupp [17, Theorem IV.5.3], though it's given form here has been restricted to the case of finitely generated groups.

Lemma 4.2.3. *Let $G = \text{Gp}\langle X \mid w = 1 \rangle$ be a group where w is a cyclically reduced word and let $X' \subseteq X$. Then membership in $\text{Gp}\langle X' \rangle$ within G is decidable.*

We now give an example of how Theorem 4.2.2 may be applied to solve the word problem for a finitely presented special inverse monoid.

Example 4.2.4. Consider the inverse monoid

$$M = \text{Inv}\left\langle X \mid (z_1)(x_1^2 y_1)^2 (z_1) = 1, (z_2)(x_2^2 y_2)^2 (z_2) = 1, (z_1)(z_2)^{-1} = 1 \right\rangle$$

and its maximal group image

$$G = \text{Gp}\left\langle X \mid (z_1)(x_1^2 y_1)^2 (z_1) = 1, (z_2)(x_2^2 y_2)^2 (z_2) = 1, (z_1)(z_2)^{-1} = 1 \right\rangle.$$

The group G is the product of $G_i = \mathrm{Gp}\left\langle x_i, y_i, z_i \mid (z_i)(x_i^2y_i)^2(z_i) = 1 \right\rangle$ (for $i = 1, 2$) amalgamated over $z_1 = z_2$. Thus to show G has decidable word problem it suffices to show that the G_i have decidable word problem and that membership in $\mathrm{Gp}\langle z_i \rangle$ within G_i is decidable. The G_i are one-relator groups and so have decidable word problem by Magnus's Theorem. The membership questions are decidable by Lemma 4.2.3. We have already seen by Example 3.1.17 that M is E-unitary and the brackets demarcate a uniquely marked unital factorisation. Therefore we may apply Theorem 4.2.2 and find that G has decidable prefix membership problem and M has decidable word problem.

Remark. It is important to note here that given inverse monoids M_1 and M_2 with maximal group images G_1 and G_2 that two words u_1 and u_2 generating isomorphic subgroups in G_1 and G_2 respectively does not in general imply that they will generate isomorphic inverse submonoids in M_1 and M_2 respectively. Likewise u_1 and u_2 generating isomorphic inverse submonoids in M_1 and M_2 respectively does not in general imply that they will generate isomorphic subgroups in G_1 and G_2. We illustrate both of these cases below with simple examples. However in the cases we will deal with here this is not a concern as the u_i being units in their respective special inverse monoids this is sufficient to say $\mathrm{Inv}\langle u_i \rangle \leq M_i$ is equivalentto $\mathrm{Gp}\langle u_i \rangle \leq G_i$.

Example 4.2.5. Let $M_1 = \mathrm{Inv}\left\langle a \mid aa^{-1} = 1 \right\rangle$, $M_2 = \mathrm{Inv}\left\langle x \mid x^{-1}x = 1 \right\rangle$, $u_1 \equiv a^{-1}a$ and $u_2 = x^{-1}x$. As u_1 is a non-trivial idempotent in M_1 and u_2 is an identity element in M_2 they will not generate isomorphic inverse submonoids. However $G_1 = \mathrm{Gp}\left\langle a \mid aa^{-1} = 1 \right\rangle$ and $G_2 = \mathrm{Gp}\left\langle x \mid x^{-1}x = 1 \right\rangle$ are both free, further u_1 and u_2 are identity in their respective groups and therefore generate isomorphic subgroups.

Example 4.2.6. Let $M_1 = \mathrm{Inv}\left\langle a, b \mid aba^{-1} = 1 \right\rangle$, $M_2 = \mathrm{Inv}\langle x, y \mid \rangle$, $u_1 = b$ and $u_2 = x$. That u_2 generates a free inverse submonoid of M_2 is obvious and it may be found that u_1 also generates a free inverse submonoid. This may be seen by, for example, thinking about the Schützenberger graphs (for the background on Schützenberger graphs see [29] [31]). Free inverse monoids with a single generator are isomorphic. However, u_1 is the identity element in $G_1 = \mathrm{Gp}\left\langle a, b \mid aba^{-1} = 1 \right\rangle$ and u_2 is not the identity element in $G_2 = \mathrm{Gp}\langle x, y \mid \rangle$ thus they do not generate isomorphic subgroups in their respective groups.

4.3 Uniquely Marked Minimal Invertible Pieces

When the factorisation is minimal and the inverse monoid is E-unitary we can strengthen the results of Theorem 4.2.2 but to do this we need to introduce the following lemma.

Lemma 4.3.1. *Let $M = \mathrm{Inv}\langle X \mid r_i = 1\, (i \in I) \rangle$ be an inverse monoid. Then the minimal invertible pieces $u_1, \ldots, u_k$ generate the group of units U_M.*

The above is found in Dolinka and Gray [6, Proposition 3.1], and is a slight generalisation of arguments found in Ivanov, Margolis and Meakin [11, Proposition 4.2].

Corollary 4.3.2. *Let $M = \mathrm{Inv}\langle X \mid r_i(u_1, u_2, \ldots, u_k)\, (i \in I) \rangle$ be an E-unitary inverse monoid, $G = \mathrm{Gp}\langle X \mid r_i(u_1, u_2, \ldots, u_k)\, (i \in I) \rangle$ be its maximal group image and U_M be its group of units. If $u_1, u_2, \ldots, u_k$ are the minimal invertible pieces of M and are uniquely marked then the following are equivalent:*

1. *The word problem of G is decidable.*
2. *The word problem of M is deciable.*
3. *The word problem of U_M is decidable.*

Proof. The minimal invertible pieces inherently form a unital factorisation and so (1) implies (2) by Theorem 4.2.2. As U_M is a submonoid of M, (2) implies (3). We know by Lemma 4.1.4 that G is isomorphic to the free product of a free group and the subgroup generated by the u_i. By Lemma 4.3.1 the u_i being minimal invertible pieces implies that they generate the subgroup U_M. Thus G is the free product of U_M and a free group. By Lemma 2.5.1 the free product of two groups with decidable word problem has decidable word problem. Therefore (3) implies (1) and so we have equivalence. □

4.4 A method of refactorisation and an application

We now introduce a theorem from a paper of Gray and Ruskuc [8, Theorem 3.2].

Theorem 4.4.1. *Let* $M = \mathrm{Inv}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ *be an inverse monoid, where the* u_i *are all reduced words and form a unital factorisation. Further, let*

$$\phi : \mathrm{FG}(Y) \to \mathrm{Gp}\,\langle u_1, u_2, \ldots, u_k\rangle \leq \mathrm{FG}(X)$$

be an epimorphism and let $u'_1, u'_2 \ldots, u'_k \in \mathrm{FG}(Y)$ *be such that* $\phi(u'_i) = u_i$ *in* $\mathrm{FG}(X)$*, for* $1 \leq i \leq k$*. Finally, let* $\psi : \overline{Y}^* \to \overline{X}^*$ *be the homomorphism produced by extending* $y \mapsto \phi(y)\,(y \in \overline{Y})$*. Then*

$$M = \mathrm{Inv}\left\langle\, X \mid r_i(\psi(u'_1), \psi(u'_2), \ldots, \psi(u'_k)) = 1\,(i \in I)\,\right\rangle$$

and $\{\psi(y) \mid y \in \overline{Y}\}$ *is a set of invertible pieces of* M *which generates the same subgroup of* M *as* $u_1, \ldots, u_k$*.*

The statement of this theorem is technical. The key point is that there is sometimes a way to find an alternative presentation for an inverse monoid which decomposes into a more convenient set of unital factors. For convenience we will render the particular way we intend to use it as a corollary.

Corollary 4.4.2. *Let* $M = \mathrm{Inv}\,\langle\, X \mid r_i(u_1, \ldots, u_k) = 1\,(i \in I)\,\rangle$ *be an inverse monoid and let the* r_i *have a unital factorisation into a set of reduced words* $U = \{u_1, \ldots, u_k\} \subset \overline{X}^*$*. Let* $V = \{v_1, \ldots, v_l\} \subset \overline{X}^*$ *be a set of words such that*

$\mathrm{Gp}\langle V\rangle = \mathrm{Gp}\langle U\rangle \leq \mathrm{FG}(X)$. *Then there are* $r_i' \in \overline{X}^*$ *such that*

$$M = \mathrm{Inv}\left\langle\, X \mid r_i' = 1\,(i \in I)\,\right\rangle,$$

that each r_i' *reduces to* r_i *and that the* r_i' *have a unital factorisation into* V*.*

Proof. Let the set of words $w_j(v_1,\ldots,v_l) \in \overline{X}^*$ be such that $w_j(v_1,\ldots,v_l) = u_j$ in $\mathrm{FG}(X)$, for $1 \leq j \leq k$. Let $Y = \{y_1,\ldots,y_l\}$ and let $u_j' \equiv w_j(y_1,\ldots,y_l)$. Finally, let

$$\phi : \mathrm{FG}(Y) \to \mathrm{Gp}\langle U\rangle \leq \mathrm{FG}(X) \text{ and } \psi : \overline{Y}^* \to \overline{X}^*$$

be the maps sending y_i to v_i as elements and words respectively.

We can see that $\phi(\mathrm{FG}(Y)) = \mathrm{Gp}\langle V\rangle \leq \mathrm{FG}(X)$ by our definition of ϕ. By assumption $\mathrm{Gp}\langle V\rangle = \mathrm{Gp}\langle U\rangle \leq \mathrm{FG}(X)$ and so ϕ is an epimorphism. Moreover

$$\phi(u_j') = \phi(w_j(y_1,\ldots,y_l)) = w_j(v_1,\ldots,v_l) = u_j$$

in $\mathrm{FG}(X)$ and $\overline{V} = \{\psi(y) \mid y \in \overline{Y}\}$. Therefore we satisfy all the conditions of Theorem 4.4.1.

Hence if we let

$$r_i' = r_i(\psi(u_1'),\ldots,\psi(u_k')) = r_i(w_1(v_1,\ldots,v_k),\ldots,w_k(v_1,\ldots,v_k))$$

we have that $M = \mathrm{Inv}\langle\, X \mid r_i' = 1\,(i \in I)\,\rangle$. Moreover we have that the set $\overline{V}$ is composed of invertible pieces in M and therefore the r_i' have a unital factorisation into the V in M. We also observe that as each u_j is reduced it must be the case that if $w_j(v_1,\ldots,v_l) = u_j$ then $\mathrm{red}(w_j(v_1,\ldots,v_l)) \equiv u_j$, thus r_i' reduces to r_i. □

Remark. Though each r_i' must reduce to its respective r_i, we do not strictly require that the r_i are reduced words.

Example 4.4.3. Let

$$M = \mathrm{Inv}\left\langle\, a,b,c,x \mid (abcxa^{-1})(acxa^{-1})(axa^{-1})(abbcxa^{-1})(acxa^{-1}) = 1 \,\right\rangle.$$

The brackets mark a unital factorisation into the factors $U = \{abcxa^{-1}, acxa^{-1}, axa^{-1}, abbcxa^{-1}\}$ (this may be shown by applying the arguments used in [23, Example 4] which deals with a very similar monoid). Let $V = \{aba^{-1}, aca^{-1}, axa^{-1}\}$. That $\mathrm{Gp}\,\langle U\rangle \subseteq \mathrm{Gp}\,\langle V\rangle \leq \mathrm{FG}(a,b,c,x)$ is obvious. That $\mathrm{Gp}\,\langle V\rangle \subseteq \mathrm{Gp}\,\langle U\rangle \leq \mathrm{FG}(a,b,c,x)$ follows from $aba^{-1} = (abbcxa^{-1})(abcxa^{-1})^{-1}$ and $aca^{-1} = (acxa^{-1})(axa^{-1})^{-1}$. Thus we can apply Corollary 4.4.2 to find that

$$\begin{aligned} M = \mathrm{Inv}\langle a,b,c,x \mid {} & (aba^{-1})(aca^{-1})(axa^{-1})(aca^{-1})(axa^{-1})(axa^{-1}) \\ & (aba^{-1})(aba^{-1})(aca^{-1})(axa^{-1})(aca^{-1})(axa^{-1}) = 1\rangle. \end{aligned}$$

The relator in the second presentation reduces to the relator in the first presentation but that relator is not fully reduced.

In particular, this allows us to identify a family of special inverse monoids whose initial presentations do not possess a uniquely marked unital factorisation but which may be differently presented so that there is a unital and uniquely marked factorisation. This generalises a result of Dolinka and Gray [6, Theorem 5.4].

Theorem 4.4.4. *Let $M = \mathrm{Inv}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ be an inverse monoid and the factorisation into u_j be unital. Further let the maximal group image of M be $G = \mathrm{Gp}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ and suppose it has decidable word problem. Finally, for $1 \leq j \leq \ell$, let there be $x_j, z_j \in X_j \subseteq X$, $Y_j = X_j \setminus \{x_j, z_j\}$ and $W_j \subset \overline{Y_j}^*$ such that the following hold:*

1. *The sets X_{j_1} and X_{j_2} are equal if and only if $j_1 = j_2$ and are disjoint otherwise.*
2. *The sets $\{u_1, u_2, \ldots, u_k\}$ and $\bigcup_{j=1}^{\ell}\{x_j \cdot w \cdot z_j \mid w \in W_j\}$ are equal.*

3. *The empty word belongs to each W_j, for $1 \leq j \leq \ell$.*

4. *For each $y \in Y_j$ there is some $w_y \in \overline{W_j}^*$ such that $y \equiv \mathrm{red}(w_y)$, for $1 \leq j \leq \ell$.*

Then G has decidable prefix membership problem and M_{EU} has decidable word problem (in particular M has decidable word problem if it is E-unitary).

Proof. Let $V = \bigcup_{j=1}^{\ell}\{x_j y x_j^{-1} \mid y \in Y_j\} \cup \{x_j z_j\}$ and $U = \{u_1, u_2, \ldots, u_k\}$. We know that each u_i is of the form $x_j \cdot w \cdot z_j$ for some $w \in W_j \subset \overline{Y}^*$. Therefore if $w \equiv y_1 y_2 \ldots y_n$ we can say that u_i is equal in the free group $\mathrm{FG}(X_j)$ to $x_j y_1 x_j^{-1} \cdot x_j y_2 x_j^{-1} \cdot \ldots \cdot x_j y_n x_j^{-1} \cdot x_j z_j$. Thus $\mathrm{Gp}\langle U \rangle \subseteq \mathrm{Gp}\langle V \rangle \leq \mathrm{FG}(X)$.

Each $y \in Y_j$ has a corresponding $w_y \in \overline{W_j}^*$ which reduces to y. This means any $v \equiv x_j y x_j^{-1}$ is equal to $x_j w_y x_j^{-1}$ in the free group $\mathrm{FG}(X_j)$. In the free group this $x_j w_y x_j^{-1}$ will be a product of $x_j w x_j^{-1}$ for $w \in W$. Thus as $x_j w x_j^{-1} = x_j w z_j (x_j z_j)^{-1} \in \mathrm{Gp}\langle U \rangle$, it follows that $x_j y x_j^{-1} \in \mathrm{Gp}\langle U \rangle$. The only $v \in V$ not of the form $x_j y x_j^{-1}$ is $x_j z_j$ which already amongst the u_i. Thus $\mathrm{Gp}\langle V \rangle \subseteq \mathrm{Gp}\langle U \rangle \leq \mathrm{FG}(X)$ and therefore $\mathrm{Gp}\langle U \rangle = \mathrm{Gp}\langle V \rangle \leq \mathrm{FG}(X)$.

Therefore we can apply Corollary 4.4.2. This means that there is a set of words $r_i' \in \overline{X}^*$ which satisfy $M' = \mathrm{Inv}\langle X \mid r_i' = 1\,(i \in I)\rangle$ is equal to M and which factorise into V. Further it means that this factorisation is unital in M (and therefore M'). As the elements of V are uniquely marked we can now decide membership in the prefix monoid of the group $G' = \mathrm{Gp}\langle X \mid r_i' = 1\,(i \in I)\rangle$ by Theorem 4.2.2.

Suppose that some word $w \in \overline{X}^*$ belongs to this monoid when considered as an element of G'. By Lemma 3.2.4 we know that this means w is right invertible as an element of M'. As $M' = M$ this means that w is right invertible as an element of M too, therefore by a second application of Lemma 3.2.4 w belongs to the prefix monoid of G. Thus we can decide the prefix membership problem for G and therefore by Theorem 2.4.6 the word problem for M_{EU}.

□

We give will now give an example of applying the above result.

Example 4.4.5. For $i = 1, 2$, let

$$M_i = \mathrm{Inv}\,\langle\, a_i, b_i, c_i, d_i \mid (a_ib_ic_id_i)(a_ic_id_i)(a_id_i)(a_ib_ib_ic_id_i)(a_ic_id_i) = 1 \,\rangle\,.$$

These are copies of the *O'Hare monoid* (see [23] for more, including an implicit proof of why the above factorisation is unital). These monoids are both E-unitary by Theorem 2.3.4 as they have one cyclically reduced relator. Let G_1 and G_2 be the maximal group images of M_1 and M_2 respectively, they will be one-relator and thus have decidable word problem by Magnus's Theorem. Further let $M = M_1 *_{a_1b_1b_1c_1d_1=a_2d_2} M_2$, by Theorem 3.1.14 this is E-unitary and has a special finite presentation

$$\mathrm{Inv}\langle a_i, b_i, c_i, d_i \mid (a_ib_ic_id_i)(a_ic_id_i)(a_id_i)(a_ib_ib_ic_id_i)(a_ic_id_i) = 1 \text{ for } i = 1, 2, \\ (a_1b_1b_1c_1d_1)(a_2d_2)^{-1} = 1\rangle$$

The maximal group image of M has decidable word problem by Lemma 2.5.3 as it is the amalgamation of two groups with decidable word problem, G_1 and G_2, over finitely generated subgroups with decidable membership within their respective groups, $\mathrm{Gp}\,\langle a_1b_1b_1c_1d_1\rangle \leq G_1$ and $\mathrm{Gp}\,\langle a_2d_2\rangle \leq G_2$ (these are decidable by Benois' Theorem as the G_i are free, see [6, Remark 5.5]).

If we let $X_i = \{a_i, b_i, c_i, d_i\}$, $Y_i = \{b_i, c_i\}$ and $W_i = \{b_ic_i, c_i, 1, b_ib_ic_i\}$ then it is easily seen that M satisfies the conditions for Theorem 4.4.4 and therefore has decidable word problem.

5

Factorisations which are alphabetically disjoint

Synopsis

This chapter examines the consequences of a presentation having a factorisation with alphabetically disjoint pieces. The first section will deal with structural implications for a group fitting this criteria. In the second section we will show that certain additional conditions are sufficient for such a group to have decidable prefix membership problem. In the third section we will show that the minimal invertible pieces of a special inverse monoid being alphabetically disjoint is, when supplemented by a couple of other conditions, sufficient to render the word problems of the inverse monoid, its maximal group image and its group of units equivalent. Finally in the fourth section we will focus on the specific structural effects of a special inverse monoid possesing factorisation into a single factor (such a factorisation being, in a trivial sense, alphabetically disjoint).

5.1 Basic Properties of Groups with Alphabetically Disjoint Factorisations

Definition 5.1.1. We say that a factorisation of a set of words w_i is *alphabetically disjoint* if the factor words, $u_1, u_2, \ldots, u_k \in \overline{X}^*$, are such that for all $1 \leq j_1, j_2 \leq k$ if $j_1 \neq j_2$ then the factors u_{j_1} and u_{j_2} have no letters in common (where one including x and the other including x^{-1} counts as a common letter).

It is important to note that the set of factors must be alphabetically disjoint across the entire set of words being factorised, not just for each word individually. We illustrate this with the following example.

Example 5.1.2. Consider the words abc and adc. The factorisation $(a)(bc)$ of abc as an individual word is alphabetically disjoint as a and bc share no letters. Similarly, the factorisation $(a)(dc)$ of adc as an individual word is also alphabetically disjoint. However the factorisation $(a)(bc)$ and $(a)(dc)$ of the set of words abc and adc is not an alphabetically disjoint factorisation as bc and dc share a common letter c.

Every set of words has a trivial alphabetically disjoint factorisation, i.e. the factorisation into letters.

Example 5.1.3. Consider the words aab and acc^{-1}. Let $(a)(a)(b)$ and $(a)(c)(c)^{-1}$ be the factorisation. This has factor words a, b and c, as each of these factor words is only a letter and so they share no letters in common, thus it forms an alphabetically disjoint factorisation.

There might be multiple alphabetically disjoint factorisations of the same set of words.

Example 5.1.4. Consider the word $abcdab$. This has a number of alphabetically disjoint factorisations, namely $(ab)(cd)(ab)$ and the two trivial options $(a)(b)(c)(d)(a)(b)$ and $(abcdab)$.

Remark. Our definition of alphabetically disjoint includes $k = 1$, the case where every word in the set may be written over a single factor word. This is in contrast to Dolinka and Gray [6, Theorem 5.7] who restricted to $k \geq 2$ but in line with Gray and Ruskuc [8]. In the last section of this chapter we will investigate inverse monoids whose relators have only a single factor word (the $k = 1$ case) and see how this condition imposes further structure and thus improved algorithmic results.

We now consider the implications of a group possessing such a factorisation. Similarly to the first lemma of the previous section, this lemma compiles some results implicit in [6] and [8] and provides a proof for the sake of completeness.

Lemma 5.1.5. *Let $G = \mathrm{Gp}\langle X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\rangle$ be a group where the factorisation into $u_1, u_2, \ldots, u_k \in \overline{X}^*$ is alphabetically disjoint. Further let $X_1, \ldots, X_k \subset X$ be sets where each X_j is the minimal alphabet such that $u_j \in \overline{X_j}^*$ and let $X_0 = X \setminus \bigcup_{j=1}^{k} X_j$.*

Let

$$G_j = \mathrm{Gp}\langle X_0, z_1, \ldots, z_j, X_{j+1}, \ldots, X_k \mid r_i(z_1, \ldots z_j, u_{j+1}, \ldots, u_k) = 1\,(i \in I)\rangle$$

for $0 \leq j \leq k$. Then the order of u_j is the same in all G_i where $0 \leq i < j$ and also equal to the order of z_j in all G_i where $j \leq i \leq k$.

For $0 \leq j \leq k$, let m_j be the order of u_j in G_{j-1} if this is finite and 0 otherwise. Further let $B_j = \mathrm{Gp}\left\langle X_j \mid u_j^{m_j} = 1 \right\rangle$, and thus $B_j = \mathrm{FG}(X_j)$ if u_j is of infinite order, be a second series of groups.

These are such that $G = G_0$, $G_k = H * \mathrm{FG}(X_0)$ where

$$H = \mathrm{Gp}\langle z_1, \ldots, z_k \mid r_i(z_1, \ldots, z_k) = 1\,(i \in I)\rangle,$$

and that

$$G_{j-1} = G_j *_{A_j} B_j$$

where A_j represents amalgamation by $u_j = z_j$ over the isomorphic subgroups

$\mathrm{Gp}\langle u_j \rangle \le B_j$ *and* $\mathrm{Gp}\langle z_j \rangle \le G_j$*. Consequently* H *embeds naturally into* G *under the mapping* z_i *to* u_i*.*

Proof. The claims that $G_0 = G$ and $G_k = H * \mathrm{FG}(X_0)$ are immediate consequences of the definitions.

We know that G_{j-1} takes the form

$$\mathrm{Gp}\left\langle X_0, z_1, \ldots z_{j-1}, X_j, \ldots, X_k \mid r_i(z_1, \ldots, z_{j-1}, u_j, \ldots u_k) = 1\, (i \in I) \right\rangle.$$

We now introduce a new generator z_j and set it equal to the word u_j. We can also introduce the relation $u_j^{m_j} = 1$ without changing the group as the order of u_j is already m_j by assumption. Together these changes give

$$\begin{aligned}&\mathrm{Gp}\langle X_0, z_1, \ldots z_{j-1}, X_j, \ldots, X_k \mid \\ &\qquad r_i(z_1, \ldots, z_{j-1}, u_j, \ldots u_k)\,(i \in I), z_j = u_j, u_j^{m_j} = 1\rangle\end{aligned}$$

which may be separated out into the amalgamated product

$$\begin{aligned}&\mathrm{Gp}\left\langle X_j \mid u_j^{m_j} = 1 \right\rangle *_{u_j = z_j} \mathrm{Gp}\langle X_0, z_1, \ldots z_j, X_{j+1}, \ldots, X_k \mid \\ &\qquad\qquad r_i(z_1, \ldots, z_j, u_{j+1}, \ldots u_k)\,(i \in I)\rangle\end{aligned}$$

The first part of this is equal G_j, the second part to B_j and the amalgamation is over the subgroups generated by z_j and u_j respectively, matching A_j. By [17, Theorem IV.5.2] if a group of the form $\mathrm{Gp}\left\langle X_j \mid u_j^{m_j} = 1 \right\rangle$ then u_j has order precisely m_j if m_j finite. The order of z_j in G_j will be m_j as a consequence of the relations $r_i(z_1, \ldots z_j, u_{j+1}, \ldots, u_k) = 1$ just as u_j was of order m_j in G_{j-1} as a consequence of the relations $r_i(z_1, \ldots z_{j-1}, u_j, \ldots, u_k) = 1$. Therefore z_j and u_j generate isomorphic subgroups in G_j and B_j repectively and so the amlgamation is valid.

Finally, we note that as G_j embeds in G_{j-1}, for $1 \le j \le k$, under the embedding $x \mapsto x$, for $x \in X \setminus \left(\Sigma_{i=1}^{j} X_i\right)$, $z_i \mapsto z_i$, for $1 \le i \le j-1$ and $z_j \mapsto u_j$ it follows that the u_j (and z_j) have consistent order across the G_i in the sense described. In particular this order is either m_j if $m_j > 0$ or is not of finite order otherwise.

$\square$

Example 5.1.6. Let

$$G = \mathrm{Gp}\langle a, b, c, d, x, y | (a)^4 = 1, (a)^3(b^{-1}c^2b)^7 = 1,$$
$$(b^{-1}c^2b)(xyxyx^{-1})(b^{-1}c^2b)^{-1}(xyxyx^{-1})^{-1} = 1\rangle.$$

The relators of G have an alphabetically disjoint factorisation into a, $b^{-1}c^2b$ and $xyxyx^{-1}$. In the language of Lemma 5.1.5, our H presents as

$$\mathrm{Gp}\left\langle\, \alpha, \beta, \gamma \mid \alpha^4 = 1, \alpha^3\beta^7 = 1, \beta\gamma\beta^{-1}\gamma^{-1} = 1 \,\right\rangle.$$

By combining the first two relations we may see that $\beta^7 = \alpha$. Thus every element of H may be written over β and γ. Further the third relation tells us that β and γ commute. Therefore every element of H may be written in the form $\beta^i\gamma^j$ and is equal to identity if and only if $\beta^i = 1$ and $\gamma^j = 1$.

No substitution based on the third relation of H can change the difference between the number of instances of β and the number of instances of β^{-1} in a word. Thus a word of the form β^i is only equal to identity if it can be shown to be so using the first two relations of H. Either of the first two relations of H may be rewritten exclusively over β and in both cases this gives $\beta^{28} = 1$. Therefore β is of order 28 in H and consequently α has order 4 in H.

Similarly to β, we can argue that the third relation cannot be used to show that $\gamma^j = 1$ when $j \neq 0$. As γ appears in no other relations, we may deduce that γ is not of finite order in H.

Lemma 5.1.5 tells us that the orders of α, β and γ in H are, respectively, also the orders of a, $b^{-1}c^2b$ and $xyxyx^{-1}$ in G. Further the Lemma tells us that we may make the following sequence of decompositions

$$G_3 = H * \mathrm{FG}(d),$$
$$G_2 = G_3 *_{\gamma = xyxyx^{-1}} \mathrm{FG}(x, y),$$
$$G_1 = G_2 *_{\beta = b^{-1}c^2b} \mathrm{Gp}\left\langle\, b, c \mid (b^{-1}c^2b)^{28} = 1 \,\right\rangle,$$
$$G_0 = G_1 *_{\alpha = a} \mathrm{Gp}\left\langle\, a \mid a^4 = 1 \,\right\rangle$$

and thus, by $G = G_0$, this tells us that our original group has the following structure

$$\begin{aligned} G = (((H * \mathrm{FG}(d)) \\ &*_{\gamma = xyxyx^{-1}} \mathrm{FG}(x, y)) \\ &*_{\beta = b^{-1}c^2b} \mathrm{Gp}\left\langle\, b, c \mid (b^{-1}c^2b)^{28} = 1 \,\right\rangle) \\ &*_{\alpha = a} \mathrm{Gp}\left\langle\, a \mid a^4 = 1 \,\right\rangle. \end{aligned}$$

5.2 Alphabetically Disjoint Unital Factorisations

Before we can approach the prefix membership problem we first need to introduce another result [6, Lemma 5.6].

Lemma 5.2.1. *Let $G = B *_A C$ and let U be a finite subset of $B \cup C$ such that $M = \mathrm{Mon}\,\langle U \rangle$ contains A. Then $M \cap B$ and $M \cap C$ are generated by $(U \cap B) \cup A$ and $(U \cap C) \cup A$ respectively.*

Similarly to the previous section, we now apply our understanding of the structure to provide a set of conditions on the prefix monoid of a group with such a factorisation.

Theorem 5.2.2. *Let $M = \mathrm{Inv}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$, where $k > 1$, be an inverse monoid and let $G = \mathrm{Gp}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ be its maximal group image.*

If all the following hold

- *The factorisation into* $u_1, u_2, \dots, u_k \in \overline{X}^*$ *is alphabetically disjoint and conservative in* G *(in particular if they form a unital factorisation in* M*);*
- *The word problem for* G *is decidable;*
- *For every factor* u_j *of finite order,* m_j*, in* G*, the group* $\mathrm{Gp}\left\langle X \mid u_j^{m_j} = 1 \right\rangle$ *has decidable prefix membership problem;*
- *For every factor* u_j *not of finite order in* G*, the subgroup generated by* z_j *has decidable membership within* G_j *(where* z_j *and* G_j *are as defined in Lemma 5.1.5)*

then G *has decidable prefix membership problem. Furthermore* M_{EU} *has decidable word problem (in particular* M *has decidable word problem if it is E-unitary).*

Proof. We carry over notation from Lemma 5.1.5.

Let $U_j = \{z_{j_1}^{\varepsilon}, \mathrm{pref}(u_{j_2}^{\varepsilon}) \mid 1 \le j_1 \le j < j_2 \le k, \varepsilon \in \{-1, 1\}\}$ for $0 \le j \le k$, also let $M_j = \mathrm{Mon}\,\langle U_j \rangle \le G_j$. We claim the following:

1. That $\mathrm{Mon}\,\langle U_{j-1} \rangle = \mathrm{Mon}\left\langle U_j \cup \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1}) \right\rangle \le G_j *_{A_j} B_j$
2. That $M_{j-1} \cap G_j = M_j \le G_j *_{A_j} B_j = G_{j-1}$
3. That $M_{j-1} \cap B_j = \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1}) \le G_j *_{A_j} B_j$

for $1 \le j \le k$.

That $U_{j-1} \subseteq U_j \cup \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1})$ follows immediately from the definition of U_j. As $z_j^{\varepsilon} = u_j^{\varepsilon} \in \mathrm{pref}(u_j^{\varepsilon})$ in $G_j *_{A_j} B_j$, we have that

$$U_j = U_j \cup \{z_j, z_j^{-1}\} \supseteq U_j \cup \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1}).$$

Therefore $U_{j-1} = U_j \cup \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1})$ in G_{j-1} and (1) is an immediate consequence.

By (1), $M_{j-1} = \text{Mon}\left\langle U_j \cup \text{pref}(u_j) \cup \text{pref}(u_j^{-1})\right\rangle$ in $G_j *_{A_j} B_j$. Further by Lemma 5.2.1 we have that $M_{j-1} \cap G_j$ is generated by

$$\left(\left(U_j \cup \text{pref}(u_j) \cup \text{pref}(u_j^{-1})\right) \cap G_j\right) \cup A_j.$$

The generators $p \in \text{pref}(u_j^{\varepsilon})$ where $p \neq u_j^{\varepsilon} = z_j^{\varepsilon}$ do not belong to G_j and A_j is generated by $z_j \in U_j$. Hence $M_{j-1} \cap G_j$ is the submonoid of G_j generated by U_j and is therefore equal to M_j, which proves (2).

Similarly we may apply Lemma 5.2.1 to say that $M_{j-1} \cap B_j$ is generated by

$$\left(\left(U_j \cup \text{pref}(u_j) \cup \text{pref}(u_j^{-1})\right) \cap B_j\right) \cup A_j.$$

The only members of $U_j \cap B_j$ are $z_j = u_j$ and $z_j^{-1} = u_j^{-1}$ and these are also the generators of A_j. Therefore $M_{j-1} \cap B_j$ is the submonoid generated by $\text{pref}(u_j)$ and $\text{pref}(u_j^{-1})$, which proves (3).

We shall now apply these claims to show inductively that each M_j has decidable membership within its respective G_j. Firstly, the monoid M_k is equal to H, the group generated by all the z_j, and as $G_k = H * \text{FG}(X_0)$ membership is decidable.

Suppose that $M_j \leq G_j$ is decidable. We will now show this implies that $M_{j-1} \leq G_{j-1}$ has decidable membership via use of Theorem 4.2.1.

Firstly, we see that

$$M_{j-1} = \text{Mon}\left\langle (M_{j-1} \cap G_j) \cup (M_{j-1} \cap B_j)\right\rangle \leq G_j *_{A_j} B_j = G_{j-1}$$

as a consequence of the claims we proved above.

The group G_j has decidable word problem as it embeds naturally in G, which has decidable word problem by assumption. The group B_j is either a free group or a one-relator group both classes which always have decidable word problem. Both are finitely generated.

The membership of A_j in B_j is decidable as either B_j is a free group, in which case we may apply Benois's Theorem, or u_j has a finite order and as such A_j, which is generated u_j alone, is a finite subgroup of B_j and so B_j having a decidable word problem suffices to decide membership. The membership of A_j in G_j is decidable if the order of the u_j is finite by the same argument as for B_j. If u_j is not of finite order then as A_j is the subgroup of G_j generated by z_j which has decidable membership by assumption.

The monoid $M_{j-1} \cap G_j$ is equal to the monoid M_j which has decidable membership in G_j by inductive hypothesis. The monoid $M_{j-1} \cap B_j$ is equal to the monoid $\mathrm{Mon}\left\langle \mathrm{pref}(u_j) \cup \mathrm{pref}(u_j^{-1}) \right\rangle$. If u_j is of finite order in G then $B_j = \mathrm{Gp}\left\langle X_j \mid u_j^{m_j} = 1 \right\rangle$ and $(u_j)(u_j)\dots(u_j)$ is a conservative factorisation of the relator. Therefore by Lemma 3.2.6 the monoid $M_{j-1} \cap B_j$ is the prefix monoid of B_j and which has decidable membership by assumption. If u_j is not of finite order then B_j is a free group and all submonoids have decidable membership as a consequence of Benois' theorem. Thus we have staisfied the conditions of Theorem 4.2.1 and M_{j-1} has decidable membership within G_{j-1}.

Therefore by induction M_0 has decidable membership in $G_0 = G$ and as M_0 is generated by the prefixes of both the factors and the inverses of the factors of a conservative factorisation of G it is equal to the prefix monoid by Lemma 3.2.6. Thus G has decidable prefix membership problem. □

Example 5.2.3. If we let $Y_i = \{a_i, b_i, c_i\}$, $u_i \equiv a_i^2 b_i^3$ and $v_i \equiv c_i^5$ for $i = 1, 2$. Then we may define the inverse monoid

$$M = \mathrm{Inv}\left\langle Y_1, Y_2 \mid u_i v_i^2 u_i v_i u_i = 1\,(i = 1, 2), u_1 u_2^{-1} = 1 \right\rangle$$

(this is the same presentation as Example 3.1.18) with maximal group image

$$G = \mathrm{Gp}\left\langle Y_1, Y_2 \mid u_i v_i^2 u_i v_i u_i = 1\,(i = 1, 2), u_1 u_2^{-1} = 1 \right\rangle.$$

It may easily be seen that the factorisation into u_i and v_i is alphabetically disjoint;

that it is also unital in M may be shown by applying the Adjan algorithm (see [6, Section 3] for a fuller description but in this case it suffices to observe that the relators are units and then make repeated use of the rule that $w_1w_2, w_2w_3 \in U_M$ implies $w_1, w_2, w_3 \in U_M$).

If we let $K_i = \mathrm{Gp}\left\langle Y_i \mid u_iv_i^2u_iv_iu_i = 1 \right\rangle$ (for $i = 1, 2$) then as the factorisation into u_i and v_i is alphabetically disjoint the subgroups $\mathrm{Gp}\langle u_i\rangle$ and $\mathrm{Gp}\langle v_i\rangle$ have decidable membership within K_i by Lemma 4.2.3 and the groups K_i themselves have decidable word problem by Magnus's Theorem. Therefore $G = K_1 *_{u_1=u_2} K_2$ has decidable word problem by Lemma 2.5.3.

The words u_i and v_i are not of finite order within K_i by the Freiheitssatz. It follows from K_i embedding in G that the words are not of finite order in G either. Therefore there are no factors of finite order.

The amalgamated parts of K_1 and K_2 are the groups generated by u_1 and u_2 respectively. We have seen above that membership in these groups is decidable within their respective K_i. By Theorem 4.2.1 (or by use of the normal form theorem for amalgamated free products, see for instance [17, Theorem IV.2.6]) this is sufficient to decide membership in K_i within G. From there we can then decide membership in $\mathrm{Gp}\langle u_i\rangle$ or $\mathrm{Gp}\langle v_i\rangle$ within K_i and thus within G. By Lemma 5.1.5 the above is sufficient to decide membership in the relevant $\mathrm{Gp}\langle z_j\rangle$ within G_j, as this problem will embed in the larger one.

Thus we may apply Theorem 5.2.2 and find that G has decidable prefix membership problem and, as we have already seen in Example 3.1.18 that M is E-unitary, that M has decidable word problem.

Remark. We note that the original theorem [6, Theorem 5.7] may be recovered from this result. Suppose we have some monoid $M = \mathrm{Inv}\left\langle X \mid w = 1 \right\rangle$ for a cyclically reduced w which has an alphabetically disjoint factorisation into two or more factors. Each factor of w is written over a strict subset of the alphabet that w is written over, so by the Freiheitsatz we know that no factor can have finite order. As w is cyclically reduced, the condition on factors of non-finite order is satisfied by Lemma 4.2.3. Then we know that its maximal group image has decidable word problem by Magnus's Theorem and that it is E-unitary by Theorem 2.3.4 as w is cyclically reduced. Therefore by Theorem 5.2.2 M has decidable word problem.

5.3 Alphabetically Disjoint Minimal Invertible Pieces

We now consider the relationship between the word problems of the G_j from Lemma 5.1.5 (with the notation of that lemma being carried over).

Lemma 5.3.1. *Let* $G = \mathrm{Gp}\,\langle\, X \mid r_i(u_1, u_2, \ldots, u_k) = 1\,(i \in I)\,\rangle$ *be a group with alphabetically disjoint factorisation. For* $1 \leq j \leq k$*, let*

$$G_j = \mathrm{Gp}\,\langle\, X_0, z_1, \ldots, z_j, X_{j+1}, \ldots, X_k \mid r_i(z_1, \ldots z_j, u_{j+1}, \ldots, u_k) = 1\,(i \in I)\,\rangle$$

and let $1 \leq n, m \leq k$ *and suppose that* G_n *has decidable word problem.*

Then the word problem of G_m *is decidable if:*

- *Either* $n \leq m$
- *Or* $m < n$ *and all* u_j*, where* $m \leq j < n$*, of not of finite order are such that* $\mathrm{Gp}\,\langle z_j \rangle \leq G_j$ *has decidable membership.*

Consequently, if $\mathrm{Gp}\,\langle u_j \rangle \leq G$ *has decidable membership for all factors,* u_j*, of non-finite order then* $H = \mathrm{Gp}\,\langle\, z_1, \ldots, z_k \mid r_i(z_1, \ldots, z_k) = 1\,(i \in I)\,\rangle$ *having decidable word problem implies that* G *has decidable word problem.*

Proof. Suppose that $n \leq m$. Then G_m embeds naturally into G_n and so deciding the latter's word problem is sufficient to decide the former.

Suppose that $m < n$. It suffices to demonstrate the case for $m = n - 1$, as the full result follows by induction. Recall that $G_{n-1} = G_n *_{A_n} B_n$. By Lemma 2.5.3

to show that this group amalgam has decidable word problem it will suffice to demonstrate that the two groups being amalgamated have decidable problem and that membership in the amalgamting subgroup is decidable within each of these two groups.

As B_n is either free or a one-relator group it has decidable word problem. If u_n has finite order in B_n, this means that A_n which is generated by u_n has finitely many members. Otherwise B_n is free and membership in A_n is decidable by Benois' Theorem.

By assumption G_n has decidable word problem. If the order of z_n, which (by Lemma 5.1.5) is equal to the order of u_n, is finite then as above A_n is finite and thus has decidable membership within G_n. If however the order of z_n is not finite the decidability of membership in $\mathrm{Gp}\langle z_n \rangle$ within G_n is decidable by assumption.

Therefore G_{n-1} has decidable word problem and inductively so does G_m.

Hence if H has decidable word problem, which is equivalent to the word problem of $G_k = H * \mathrm{FG}(X_0)$, and each of the non-finite order factors generate a subgroup with decidable membership within G then G has decidable word problem.

□

We can now once again consider the word problem for the group of units of a suitably factorised inverse monoid.

Corollary 5.3.2. *Let* $M = \mathrm{Inv}\langle X \mid r_i(u_1, u_2, \ldots, u_k)\ (i \in I) \rangle$ *be an E-unitary inverse monoid, let* $G = \mathrm{Gp}\langle X \mid r_i(u_1, u_2, \ldots, u_k)\ (i \in I) \rangle$ *be its maximal group image and let* U_M *be its group of units. If the factor words* $u_1, u_2, \ldots, u_k$ *are the minimal invertible pieces of* M *and alphabetically disjoint and:*

- *For every factor* u_j *of finite order,* m_j*, in* G *the group* $\mathrm{Gp}\left\langle X \mid u_j^{m_i} = 1 \right\rangle$ *has decidable prefix membership problem;*
- *For every factor* u_j *which does not have finite order in* G *the membership*

of $\mathrm{Gp}\langle u_j \rangle \leq G$ *is decidable;*

then the following are equivalent:

1. *The word problem of* G *is decidable.*
2. *The word problem of* M *is deciable.*
3. *The word problem of* U_M *is decidable.*

Proof. All the conditions of Theorem 5.2.2 are satisfied, so (1) implies (2). As U_M is a submonoid of M, (2) implies (3). By Lemma 4.3.1 the u_i being minimal invertible pieces means that they generate U_M. Moreover as they are invertible the structure of U_M will be identical to the subgroup generated by the u_i within G. By Lemma 5.1.5 this is isomorphic to

$$H = \mathrm{Gp}\langle\, z_1, \ldots, z_k \mid r_i(z_1, \ldots, z_k) = 1\, (i \in I)\,\rangle$$

and so by Lemma 5.3.1 we may say that (3) implies (1).

□

Before we discuss the $k = 1$ case in more detail in the next section, we shall lay out some questions that these results raise.

We showed in Corollary 4.3.2 that the minimal invertible pieces of an E-unitary special inverse monoid being uniquely marked is sufficient to render the word problems of the monoid, its group of units and its maximal group image equivalent. However in Corollary 5.3.2 we were not able to show that the minimal invertible pieces being alphabetically disjoint was on its own sufficient to yield a similar equivalency. We had to place two further conditions on the pieces to get our desired result.

First, Corollary 5.3.2 requires that for every factor u_j of finite order m_j the group $\mathrm{Gp}\left\langle X \mid u_j^{m_j} = 1 \right\rangle$ has decidable prefix membership problem. As $u_j^{m_j} = 1$ is a consequence of the defining relations of M, u_j being a minimal invertible piece in M implies that it is a minimal invertible piece in $\mathrm{Inv}\left\langle\, X \mid u_j^{m_j} = 1 \,\right\rangle$ also. Thus it would remove a condition from Corollary 5.3.2 if there was a positive answer to the following question:

Question 5.3.3. *Does the group* $\mathrm{Gp}\langle\, X \mid u^m = 1 \,\rangle$, *where* $m \geq 1$, *have decidable prefix membership problem if the only minimal invertible piece of the corresponding inverse monoid* $\mathrm{Inv}\langle\, X \mid u^m = 1 \,\rangle$ *is* u*?*

The second additional restriction which Corollary 5.3.2 imposes is that for every factor u_j not of finite order in G the subgroup $\mathrm{Gp}\langle u_j \rangle$ has decidable membership within G. In the case of a G with a single cyclically reduced relator we already know by Lemma 4.2.3 that a subgroup generated by some subset of the relators has decidable membership. Thus we might ask:

Question 5.3.4. *For a group* $G = \mathrm{Gp}\langle\, X \mid r_i = 1\, (i \in I) \,\rangle$ *with decidable word problem and* $x \in X$*, under what conditions is membership in* $\mathrm{Gp}\langle x \rangle$ *within* G *decidable?*

More generally, by answering the above questions or otherwise, can we find an answer to the following:

Question 5.3.5. *Is the word problem of an E-unitary special inverse monoid equivalent to the word problems of its maximal group image and group of units if the minimal invertible pieces of its relators are alphabetically disjoint?*

Finally, sets of words being uniquely marked or alphabetically disjoint are just specific examples of properties which are what Gray and Ruskuc [8, Definition 3.6] called being *free for substitution*, so we might question whether this more general property is sufficient:

Question 5.3.6. *Let* $M = \mathrm{Inv}\langle\, X \mid r_i(u_1, \ldots, u_k) = 1\, (i \in I) \,\rangle$ *be an E-unitary inverse monoid and let* $G = \mathrm{Gp}\langle\, X \mid r_i(u_1, \ldots, u_k) = 1\, (i \in I) \,\rangle$ *be its maximal group image. Further let* $u_1, \ldots, u_k$ *be the minimal invertible pieces and suppose* *that*

$$\mathrm{Gp}\,\langle\, z_1, \ldots, z_k \mid r_i(z_1, \ldots, z_k) = 1\,(i \in I)\,\rangle \cong \mathrm{Gp}\,\langle u_1, \ldots, u_k\rangle \leq G.$$

Are the word problems of G, M and U_M equivalent?

5.4 Trivially Disjoint Alphabets

We shall now return to the case where the factorisation contains only one factor and is thus alphabetically disjoint by default. We will first prove a structural result which shows that such monoids have a very limited range of forms. Then we will discuss the implications of this.

Lemma 5.4.1. *Let $M = \mathrm{Inv}\,\langle\, X \mid r_i = 1\,(i \in I)\,\rangle$ be an inverse monoid, where $r_i \in \{u, u^{-1}\}^*$ for $i \in I$ and some $u \in \overline{X}^*$. There is an alternative presentation for M which takes one of the following forms, $\mathrm{Inv}\left\langle\, X \mid uu^{-1} = 1\,\right\rangle$, $\mathrm{Inv}\left\langle\, X \mid u^{-1}u = 1\,\right\rangle$, $\mathrm{Inv}\left\langle\, X \mid uu^{-1} = 1, u^{-1}u = 1\,\right\rangle$ or $\mathrm{Inv}\left\langle\, X \mid u^{K} = 1\,\right\rangle$, where $K \in \mathbb{N}$. Moreover the factorisation into u is unital if and only if M has a presentation of the third or fourth form. Also if u has finite order in M then it takes the fourth form and K is equal to the order of u.*

Proof. Let r_j be a relator of M. As $r_j \in \{u, u^{-1}\}^*$, if $r_j \not\equiv 1$ then it must be reducible to u^{k_j}, where $k_j \in \mathbb{Z}$.

Suppose that $k_i = 0$ for all $i \in I$. Let $a_M \equiv uu^{-1}$ if $u \in R_M$ and $a_M \equiv 1$ otherwise. Similarly, let $b_M \equiv u^{-1}u$ if $u \in L_M$ and $b_M \equiv 1$ otherise. Adding the relations $a_M = 1$ and $b_M = 1$ to M does not change the inverse monoid, as these statements are already consequences of the defining relations of M. So

$$\mathrm{Inv}\,\langle\, X \mid r_i = 1\,(i \in I), a_M = 1, b_M = 1\,\rangle$$

is an alternative presentation of M. We define

$$M' = \text{Inv}\,\langle\, X \mid a_M = 1, b_M = 1\,\rangle\,.$$

Let r_j be a relator of M. If r_j begins with a u then u must have been a right unit of M, so $a_M \equiv uu^{-1}$ and u is a right unit of M' also. If this is not the case then r_j must begin with a u^{-1} and so u^{-1} must have been a right unit of M, $b_M \equiv u^{-1}u$ and u^{-1} be a right unit of M' also. Further, as $k_j = 0$ we know that r_j freely reduces to 1. Therefore by Lemma 2.2.3 we know that $r_j = \text{red}(r_j) = 1$ in M'. Thus

$$\text{Inv}\,\langle\, X \mid r_i = 1\,(i \in I), a_M = 1, b_M = 1\,\rangle$$

is an alternative presentation of M' also and thus $M' = M$.

Suppose that there is some $k_j \neq 0$ and let $r'_j \equiv u^{k_j}$. By Lemma 2.2.3 we know that r'_j being a reduction from the right unit r_j implies $r'_j = r_j = 1$ in M. Therefore

$$\text{Inv}\left\langle\, X \mid r_i = 1\,(i \in I), r'_j = 1\,\right\rangle$$

is an alternative presentation of M. Which we may then extrapolate to

$$\text{Inv}\left\langle\, X \mid r_i = 1, r'_i = 1\,(i \in I)\,\right\rangle$$

being an alternative presentation of M, where $r'_i \equiv \text{red}(r_i)$ for $i \in I$. We define

$$M' = \text{Inv}\left\langle\, X \mid r'_i = 1\,(i \in I)\,\right\rangle.$$

We have already supposed that there is some $j \in I$ such that $k_j \neq 0$. This means that $r'_j \equiv u^{k_j}$ so we know that $u \in U_{M'}$. As $r_i \in \{u, u^{-1}\}^*$ for all $i \in I$ we deduce that $r_i \in U_{M'}$. Thus as r_i reduces to r'_i it follows that $r_j = r'_j = 1$ in M'. Therefore

$$\text{Inv}\left\langle\, X \mid r_i = 1, r'_i = 1\,(i \in I)\,\right\rangle$$

is an alternative presentation of M' also and so $M' = M$.

Let $K > 0$ be the greatest common divisor of the set $\{k_i \,|\, i \in I\}$. By the methods of Euclid's algorithm there are integers $\lambda_i \in \mathbb{Z}$ such that

$$K = \Sigma_{i \in I} \lambda_i k_i$$

and therefore in M'

$$u^K = u^{\Sigma_{i\in I}\lambda_i k_i} = \Pi_{i\in I}(u^{k_i})^{\lambda_i} = \Pi_{i\in I}(r_i')^{\lambda_i} = 1,$$

with the second equality following from u being invertible in M'. Further as K divides all k_i where $i \in I$, $u^K = 1$ implies $r_i' = u^{k_i} = 1$. Thus

$$M = M' = \text{Inv}\left\langle X \mid r_i' = 1\,(i \in I) \right\rangle = \text{Inv}\left\langle X \mid u^K = 1 \right\rangle$$

as required. □

Corollary 5.4.2. *Let $M = \text{Inv}\langle X \mid r_i = 1\,(i \in I)\rangle$ be an inverse monoid, where $r_i \in \{u, u^{-1}\}^*$ for $i \in I$ and some cyclically reduced $u \in \overline{X}^*$. If u is of finite order, then M is E-unitary.*

Proof. By Lemma 5.4.1 that u has finite order implies $M = \text{Inv}\left\langle X \mid u^K = 1 \right\rangle$ for some K. That u is cyclically reduced implies u^K is cyclically reduced. Further, Theorem 2.3.4 tells us that M is E-unitary as it may be presented with a single cyclically reduced relator. □

We introduce a result of Margolis and Meakin [22, Theorem 3.2].

Theorem 5.4.3. *Let $M = \text{Inv}\langle X \mid e_i = f_i\,(i \in I)\rangle$ be an inverse monoid, where all the e_i and f_i are idempotents in $\text{FIM}(X)$. The word problem of M is decidable.*

If we apply this in conjunction with the previous result we get the following.

Corollary 5.4.4. *The class of inverse monoids of the form $\text{Inv}\langle X \mid r_i = 1\,(i \in I)\rangle$, where the $r_i \in \{u, u^{-1}\}^*$ for $i \in I$ and some $u \in \overline{X}^*$,* *have decidable word problem if and only if the class of inverse monoids of the form* $\mathrm{Inv}\langle X \mid u^K = 1 \rangle$*, where* $K \in \mathbb{N}$*, have decidable word problem.*

Of particular note is that if we let $K = 1$ then the class $\mathrm{Inv}\langle X \mid u^K = 1 \rangle$ includes all one-relator inverse monoids. This is of course to be expected as all one-relator inverse monoids have a trivial (unital) factorisation with one factor. Thus it is natural to ask whether the restriction to requiring a single factor is more useful in the case where we assume the factorisation is minimal.

Corollary 5.4.5. *Let* $M = \mathrm{Inv}\langle X \mid r_i(u) = 1\,(i \in I) \rangle$ *be an E-unitary inverse monoid where the factorisation into minimal invertible pieces has only one factor word* u *with order* $K > 0$*. The inverse monoid* M*, its maximal group image and its group of units all have decidable word problem if the group* $G = \mathrm{Gp}\langle X \mid u^K = 1 \rangle$ *has decidable prefix membership problem.*

Proof. We know by Lemma 5.4.1 that $M = \mathrm{Inv}\langle X \mid u^K = 1 \rangle$. That the three word problems are equivalent then follows as a consequence of Corollary 5.3.2. Further $M = \mathrm{Inv}\langle X \mid u^K = 1 \rangle$ implies that G is the maximal group image. By Theorem 2.4.1 that G is a one-relator group implies that it has decidable word problem. Therefore equivalently M and its group of units also have decidable word problem. □

Corollary 5.4.6. *Let* $M = \mathrm{Inv}\langle X \mid r_i(u) = 1\,(i \in I) \rangle$ *be an inverse monoid where the factorisation into minimal invertible pieces has only one factor word* u *which is cyclically reduced and has order* $K > 0$ *in* M*. The inverse monoid* M *is E-unitary. Furthermore the inverse monoid* M*, its maximal group image and its group of units all have decidable word problem if the group* $G = \mathrm{Gp}\langle X \mid u^K = 1 \rangle$ *has decidable prefix membership problem.*

Proof. The inverse monoid M fulfils the requirements of Corollary 5.4.2 and is thus E-unitary. The rest then follows by Corollary 5.4.5. □

6

Elementary Operations within HNN extensions of free groups

Synopsis

In this chapter we will examine the structure of HNN extensions of finite presentations of free groups. In particular we will look at those HNN extensions where the defining isomorphism maps one subset of a free basis of the original maps bijectively onto another subset of that same free basis. In the first section we will lay out the definitions and preliminary results that inform the rest of the chapter. In the second section we will look at how the different words representing the same element within the HNN extension may be transformed into each other. In the third section we use the results of the second to show that every element has a finite set of "most reduced" words which possess a number of useful properties.

6.1 Defining Elementary Operations

Throughout this section we will assume that H is a free group over an alphabet X which has a free basis $U_H \subset \overline{X}^*$. We will also assume that A and B are isomorphic subgroups of H which are generated by $U_A, U_B \subseteq U_H$ respectively and that the isomorphism between them, which we call ϕ, produces a bijection between $\overline{U_A}$ and $\overline{U_B}$. Finally we will also be assuming that $H^* = H*_{t,\phi:A\to B}$.

We begin by noting an algorithmic property of such groups.

Theorem 6.1.1. *The word problem for H^* is decidable.*

Proof. As H is a free group it has decidable word problem. By Benois' theorem that H is free also means that A and B have decidable subgroup membership within H. Any word w in $\overline{X}^*$ is equal to a reduced word w' written over $\overline{U_H}$ in H and this may be found by algorithm. Further if w belongs to A then this w' will be written over $\overline{U_A}$. Thus $\phi(w') = \phi(w)$ is calculable via algorithm by replacing each $u_a \in U_A$ by the corresponding $\phi(u_a) \in U_B$. Therefore ϕ is effectively calculable and ϕ^{-1} is effectively calculable by a dual argument. Hence by Theorem 2.6.4 $H^* = H*_{t,\phi:A\to B}$ has decidable word problem. □

Next we make the following observations.

Lemma 6.1.2. *Let $V_G \subseteq U_H$ and $V_M \subseteq \overline{U_H}$ be non-empty sets. Then we have the following:*

1. *The sets $V_M \cap \overline{V_G}$ and $V_M \cap \mathrm{Gp}\langle V_G\rangle$ respresent the same elements in H*
2. *The sets $(V_M \cap \overline{V_G})^*$ and $\mathrm{Mon}\langle V_M\rangle \cap \mathrm{Gp}\langle V_G\rangle$ represent the same elements in H*

Proof. (1): Suppose there exists some $v \in (V_M \cap \mathrm{Gp}\langle V_G\rangle) \setminus (V_M \cap \overline{V_G})$. This may easily be found to be equivalent to $v \in (V_M \cap \mathrm{Gp}\langle V_G\rangle) \setminus \overline{V_G}$. That $v \in \mathrm{Gp}\langle V_G\rangle$ implies that there is some word $w \in \overline{V_G}^*$ such that $w = v$ in H. However $v \in V_M \setminus \overline{V_G}$, so this implies that an element of the free basis U_H can be written in terms of different members of the free basis in H, which is a contradiction. Therefore $V_M \cap \mathrm{Gp}\langle V_G\rangle$ represent a subset of the elements that $V_M \cap \overline{V_G}$ do in H. The converse is obvious from the definitions.

(2): Suppose that $x \in \mathrm{Mon}\langle V_M\rangle \cap \mathrm{Gp}\langle V_G\rangle$. That $x \in \mathrm{Mon}\langle V_M\rangle$ implies there is a word $w_M \in V_M^*$ such that $w_M = x$ in H. Similarly, that $x \in \mathrm{Gp}\langle V_G\rangle$

implies there is a word $w_G \in \overline{V_G}^*$ such that $w_G = x$ in H. As w_H and w_G are words written over $\overline{U_H}$, U_H is a free basis for H and $w_M = x = w_G$ in H we may conclude that both w_M and w_G reduce over U_H to the same word written over $\overline{U_H}$. The resulting word, w, will satisfy $w \in (V_M \cap \overline{V_G})^*$. Thus we may conclude that $\mathrm{Mon}\langle V_M \rangle \cap \mathrm{Gp}\langle V_G \rangle \subseteq (V_M \cap \overline{V_G})^*$. The converse is obvious from the defintions.

$\square$

Example 6.1.3. Let $H = \mathrm{Gp}\langle\, a, b, c, d \mid abcd = 1\,\rangle$, $U_H = \{abc, ab, bc\}$, $V_M = \{abc, (ab)^{-1}, bc, (bc)^{-1}\}$ and $V_G = \{ab, bc\}$. It may be easily seen that H is equal to the free group $\mathrm{FG}(a, b, c)$ and that U_H is a free basis. Thus Lemma 6.1.2 is applicable and tells us two things. First, that

$$\begin{aligned}
&\{abc, (ab)^{-1}, bc, (bc)^{-1}\} \cap \mathrm{Gp}\langle ab, bc\rangle \\
=&\{abc, (ab)^{-1}, bc, (bc)^{-1}\} \cap \{ab, (ab)^{-1}, bc, (bc)^{-1}\} \\
=&\{(ab)^{-1}, bc, (bc)^{-1}\}
\end{aligned}$$

and second, that

$$\mathrm{Mon}\left\langle abc, (ab)^{-1}, bc, (bc)^{-1}\right\rangle \cap \mathrm{Gp}\langle ab, bc\rangle = \{(ab)^{-1}, bc, (bc)^{-1}\}^*.$$

Remark. In the notation used here $\varnothing^*$ consists solely of the empty word. Hence for part (2) in the case where the intersection of V_M and $\overline{V_G}$ is empty the set $(V_M \cap \overline{V_G})^*$ represents only the identity element in H.

If we let $V_G = U_A$ and $V_M = U_H$ then part (1) of Lemma 6.1.2 tells us that $U_H \cap A = U_H \cap \overline{U_A} = U_A$. Similarly we may conlude that $U_H \cap B = U_B$.

Suppose we have a word w written over $\overline{X} \cup \{t, t^{-1}\}$ in the following form

$$w \equiv w_0 t^{n_1} w_1 \dots t^{n_k} w_k,$$

where $w_i \in \overline{X}^*$ and $n_i \in \mathbb{Z}$. As U_H is a free basis for $\mathrm{FG}(X)$ every w_i may be uniquely rewritten as a word $w_i' \in \overline{U_H}^*$ such that $w_i' = w_i$ in H and that w_i' is reduced in terms of U_H. Applying this to every w_i of w yields a unique rewriting

$$w' \equiv w_0' t^{n_1} w_1' \ldots t^{n_k} w_k'.$$

Further, this is such that if our initial w is in HNN reduced form in H^* then so is w'. Thus together with Lemma 2.6.3 we can algorithmically rewrite any word over $X \cup \{t, t^{-1}\}$ as a word over $\overline{U_H} \cup \{t, t^{-1}\}$ which is in reduced form in H^*.

Definition 6.1.4. We list three *elementary operations*, which can be applied to a given word written over $\overline{U_H} \cup \{t, t^{-1}\}$ to produce another word written over $\overline{U_H} \cup \{t, t^{-1}\}$.

- *Elementary Addition*, the insertion of some pair uu^{-1} into the word, where $u \in \overline{U_H}$, in such a way that the result is still written over $\overline{U_H} \cup \{t, t^{-1}\}$;
- *Elementary Cancellation*, the removal of some pair uu^{-1} from the word, where $u \in \overline{U_H}$;
- *Elementary Semicommutation*, either substitution of $u_a t$ for $t u_b$ or vice versa or the substitution of $t^{-1} u_a$ for $u_b t^{-1}$ or vice versa, where $u_a \in U_A$, $u_b \in U_B$ and $\phi(u_a) = u_b$.

6.2 Applying Elementary Operations

We will now demonstrate that these are sufficient under our assumptions to serve as the discrete units to describe a transformation between any two words both equal and in HNN reduced form in H^*.

Lemma 6.2.1. *Let $w, w' \in \overline{U_H \cup \{t\}}^*$ be two words which are in HNN reduced form in H^*. The words w and w' are equal in H^* if and only if there is a sequence of elementary operations which turns w into w'.*

Proof. Let Y be a set of letters of size $|U_H|$. Further let $\theta : U_H \to Y$ be a bijection. As U_H is a basis for H this extends to a bijection $\theta : H \to \mathrm{FG}(Y)$.

If

$$H = \mathrm{Gp}\langle X \mid r_1 = 1, \ldots, r_n = 1 \rangle$$

then we may write that

$$H^* = \mathrm{Gp}\left\langle X, t \mid r_1 = 1, \ldots, r_n = 1, t^{-1}ut = \phi(u) \quad (u \in U_A) \right\rangle.$$

If we let $Y_A = \theta(U_A)$ and $Y_B = \theta(U_B)$ then we may likewise define the following group

$$K = \mathrm{Gp}\left\langle Y, s \mid s^{-1}\theta(u)s = \theta(\phi(u)) \quad (u \in U_A) \right\rangle$$

as an extension of $\mathrm{FG}(Y)$. This K is then isomorphic to H^* under an extension of θ which also sends t to s.

That θ is an isomorphism implies $\theta(w) = \theta(w')$ in K if and only if $w = w'$ in H^*. In K it is obvious that two words in HNN reduced form are equal if and only if some sequence of composed of: insertions of pairs of the form yy^{-1} where $y \in \overline{Y}$; cancellations of pairs of the form yy^{-1} where $y \in \overline{Y}$ and substitutions of the form $y_a t$ for ty_b or $t^{-1}y_a$ for $y_b t^{-1}$ or vice versa where $\phi(\theta^{-1}(y_a)) = \theta^{-1}(y_b)$, $y_a \in Y_A$ and $y_b \in Y_B$. Such a sequence easily translates to a sequence of elementary operations in H^*. Therefore $w = w'$ in H^* if and only if there is a sequence of elementary operations between w and w'. □

Example 6.2.2. Let $H = \mathrm{Gp}\langle a, b, c, d \mid abcd = 1 \rangle$ and $U_H = \{abc, ab, bc\}$, these form a free group and a free basis. Further let $U_A = \{abc, ab\}$, let $U_B = \{ab, bc\}$ and let ϕ be the isomorphism between them which extends $abc \mapsto ab$ and $ab \mapsto bc$. This gives us the HNN extension $H^* = H*_{t,\phi:A\to B}$, where $A = \mathrm{Gp}\langle U_A \rangle$ and $B = \mathrm{Gp}\langle U_B \rangle$, with the following presentation

$$\mathrm{Gp}\left\langle a, b, c, d, t \mid abcd = 1, t^{-1}abct = ab, t^{-1}abt = bc \right\rangle.$$

We let $Y = \{x, y, z\}$ and further let θ be the bijection which extends $abc \mapsto x$, $ab \mapsto y$ and $bc \mapsto z$ to a map between H and $\mathrm{FG}(Y)$.

We can then let

$$K = \mathrm{Gp}\left\langle\, x, y, z, s \mid s^{-1}xs = y, s^{-1}ys = z \,\right\rangle$$

and this will be isomorphic to H^* under an extension of θ that also sends t to s. So in this example θ is the map acts on the letters of H^* as follows $\theta(a) = xz^{-1}$, $\theta(b) = zx^{-1}y$, $\theta(c) = y^{-1}x$, $\theta(d) = x^{-1}$ and $\theta(t) = s$.

Remark. At this stage it is worth observing that all the remaining results in this section and Theorem 7.1.2 could be proved in some K and then transferred by the isomorphism θ^{-1} to a corresponding H^*. However, Theorem 7.2.1, which is one of the main results of the thesis, needs to work even when the factorisation is into subwords which are not letters. With this in mind, we feel it is clearer to assume throughout that our elementary operations are acting on the words of a free basis rather than letters.

It is usual practice, as we have done so far, to represent a word w in reduced HNN form in the following form $w_0 t^{\varepsilon_1} w_1 \dots t^{\varepsilon_n} w_n$, where each $w_i \in \overline{X}^*$ may be empty but the ε_i are either 1 or -1. However for the remainder of this section we will instead use

$$w \equiv t^{n_0} u_1 t^{n_1} \dots u_j t^{n_j} \dots u_k t^{n_k},$$

where $u_i \in \overline{U_H}$ for $1 \leq k$ and $n_j \in \mathbb{Z}$ for $0 \leq j \leq k$, thus prioritising consistent labelling of the individual elements of the free basis over consistent labelling of the individual stable letters.

Remark. Though all semicommutations can be described as a replacement of some $t^{n_{j-1}} u_j t^{n_j}$ with either $t^{n_{j-1}+1} \phi(u_j) t^{n_j-1}$ or $t^{n_{j-1}-1} \phi^{-1}(u_j) t^{n_j+1}$ this **does not** mean that all such replacements represent semicommutations. For instance, the replacement of $t^0 u_a t^0$ with $t\phi(u_a)t^{-1}$, for some $u_a \in \overline{U_A}$ does not represent an elementary semicommutation. Note that the former is in HNN reduced form while the latter is not.

We will use $v \sim v'$ to indicate that there is a finite sequence of elementary semicommutations between v and v'. It is important to note that this is **not** the same as the compatability relation in Definition 2.3.7, and that we will **exclusively** use $\sim$ to indicate a finite sequence of elementary semicommutations in this chapter.

Lemma 6.2.3. *Suppose we have two words, $w, w' \in \overline{U_H \cup \{t\}}^*$, in HNN reduced form that are written*

$$w \equiv t^{n_0} u_1 t^{n_1} \dots u_j t^{n_j} \dots u_k t^{n_k}$$

and

$$w' \equiv t^{n'_0} u'_1 t^{n'_1} \dots u'_j t^{n'_j} \dots u'_{k'} t^{n'_{k'}}.$$

where $u_j, u'_j \in \overline{U_H}$ and $n_j, n'_j \in \mathbb{Z}$,

If $w \sim w'$ then $k = k'$ and $u'_j \equiv \phi^{z_j}(u_j)$ where $z_j = \Sigma_{i=0}^{j-1}(n'_i - n_i)$, for $1 \le j \le k$. Furthermore this means that the set $W = \{w' \mid w' \sim w\}$ is finite and can be found by algorithm.

Proof. As semicommutation does not change the number of basis words present we immediately have that $k = k'$.

Suppose that difference between w and w' is a single elementary semicommutation centred around the jth position of w. Then looking at the definition we can see that it must have one of two effects, either $t^{n_{j-1}} u_j t^{n_j}$ is replaced by $t^{n_{j-1}+1} \phi(u_j) t^{n_j - 1}$ or by $t^{n_{j-1}-1} \phi^{-1}(u_j) t^{n_j+1}$.

We can see that $z_i = z_{i-1} + n'_i - n_i$ for $1 < i \le k$. Therefore $z_i = z_{i-1} = 0$ for all $i < j - 1$ as we have not changed anything left of $t^{n_{j-1}}$. Looking at our two cases we see that $n'_{j-1} - n_{j-1}$ will be equal to -1 and 1 respectively as required.

Further as $n_{j-1} + n_j = n'_{j-1} + n'_j$ in both cases we can see that

$$z_j = z_{j-2} + (n'_{j-1} - n_{j-1}) + (n'_j - n_j) = z_{j-2} = 0.$$

As we have also not changed anything to the right of t^{n_j} either we can also conclude that $z_i = 0$ for $j < i$ as well. Thus we see that the formula correctly finds that $u'_i = \phi^0(u_i) \equiv u_i$ when $i \neq j - 1$.

Suppose that we have a sequence of K elementary semicommutations between w and w' for which the lemma's statement holds. Further suppose we have w'' which is a single elementary semicommutation away from w'. We say that

$$w'' \equiv t^{n''_0} u''_1 t^{n''_1} \ldots u''_j t^{n''_j} \ldots u''_k t^{n''_k}.$$

We know by the previous section of the proof that $u''_j \equiv \phi^{z'_j}(u'_j)$ where $z'_j = \Sigma_{i=0}^{j-1}(n''_i - n'_i)$. Further we know by inductive assumption that $u'_j \equiv \phi^{z_j}(u_j)$ where $z_j = \Sigma_{i=0}^{j-1}(n'_i - n_i)$. Combining this we can see that $u''_j \equiv \phi^{z_j + z'_j}(u_j)$ and that

$$z_j + z'_j = \Sigma_{i=0}^{j-1}(n'_i - n_i) + \Sigma_{i=0}^{j-1}(n''_i - n'_i) = \Sigma_{i=0}^{j-1}(n''_i - n_i),$$

which means the sequence between w and w'', and therefore all sequences of length $K + 1$, satisfy the lemma's statement.

Thus we can conclude by induction that the first part of the lemma's statement holds for sequences of any finite length.

Let N be the total number of instances of t and t^{-1} in w. This means that $N = \Sigma_{i=0}^{k}|n_i|$. We have established above that the entirety of any word $w' \in W$ can be determined by knowing the n'_i values. As by assumption w and w' are both in HNN reduced form the total number of instances of t^{-1} and t must be the same in both. Therefore $N = \Sigma_{i=0}^{k}|n'_i|$ also, for every $w' \in W$. There are only a finite number of ways to partition N into $n'_0, n'_1, \ldots, n'_k$ and then by multiplying this figure by a finite amount, to account for choice of sign, it is possible to produce a finite limit on the number of possible sets of choices for n'_i. Thus there a finite number of w' such that $w' \sim w$. As only a finite number of elementary semicommutations are possible from any particular w' it is possible for an algorithm to find every member of W by an exhaustive search.

□

Example 6.2.4. Let $H = \text{Gp}\langle a, b, c, t \mid abc = 1 \rangle$, $U_H = \{a, b\}$, $U_A = \{a\}$, $U_B = \{b\}$ and ϕ which extends $a \mapsto b$. This gives us the HNN extension $H^* = \text{Gp}\left\langle a, b, c, t \mid abc = 1, t^{-1}at = b \right\rangle$. Consider the word $w \equiv bt^{-2}at^{-1}a$. We may write this as $w \equiv t^0bt^{-2}at^{-1}at^0$. One semicommutation transforms this to $t^0bt^{-2+1}\phi(a)t^{-1-1}at^0 = t^0bt^{-1}bt^{-2}at^0$. While a second consectutive semicommutation gives $t^{0-1}\phi^{-1}(b)t^{-1+1}bt^{-2}at^0 = t^{-1}at^0bt^{-2}at^0$.

Now that we have seen there is a limit to the extent a word, and in particular its basis words, can change under a sequence of elementary semicommutations we will now examine to what extent elementary cancellations are invariant under sequences of elementary semicommutations.

Before this we define the following pieces of notation:

Let $w, v \in \overline{U_H \cup \{t\}}^*$ be two words in HNN reduced form. We will use $w \to_m v$ to specify that there is a single elementary cancellation between the mth and $m+1$th basis words of w which transforms it into v. That is if

$$w \equiv t^{n_0}u_1t^{n_1} \ldots t^{n_{m-1}}u_mt^{n_m}u_{m+1}t^{n_{m+1}} \ldots u_kt^{n_k}$$

then $u_{m+1} \equiv u_m^{-1}$, $n_m = 0$ and

$$v \equiv t^{n_0}u_1t^{n_1} \ldots t^{n_{m-1}+n_{m+1}} \ldots u_kt^{n_k}.$$

Lemma 6.2.5. *Let $w, w', v, v' \in (U_H \cup \{t, t^{-1}\})^*$ be words in HNN reduced form. If $v \leftarrow_m w \sim w' \to_m v'$ then $v \sim v'$.*

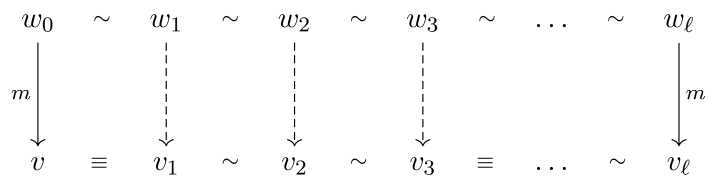

Figure 6.2.1: The solid arrows represent cancellation at m, same as in the text. The dashed arrows denote the "cancellation" between the w_i and their respective v_i. The particular sequence of $\equiv$ and $\sim$ between the v_i is purely illustrative.

Proof. As $w \sim w'$ there must be a finite sequence $w_0, w_1, \dots, w_k$ such that $w \equiv w_0$, $w' \equiv w_k$ and the difference between w_i and w_{i+1} is a single elementary semicommutation, for $0 \leq i < k$. Suppose that

$$w_i \equiv t^{n_{i,0}} u_{i,1} t^{n_{i,1}} \dots t^{n_{i,m-1}} u_{i,m} t^{n_{i,m}} u_{i,m+1} t^{n_{i,m+1}} \dots u_{i,k} t^{n_{i,k}}$$

and define

$$v_i \equiv t^{n_{i,0}} u_{i,1} t^{n_{i,1}} \dots t^{n_{i,m-1}+n_{i,m}+n_{i,m+1}} \dots u_{i,k} t^{n_{i,k}}.$$

Under this definition $v \equiv v_0$ and $v' \equiv v_k$.

Further we claim that, for $0 \leq i < k$, either v_i and v_{i+1} are equal or there is a single elementary semicommutation between them. The semicommutation between w_i and w_{i+1} can be represemted by the replacement of $t^{n_{i,j-1}} u_{i,j} t^{n_{i,j}}$ with $t^{n_{i,j-1}+\varepsilon} \phi^{\varepsilon}(u_{i,j}) t^{n_{i,j}-\varepsilon}$ for some $1 \leq j \leq k$ and $\varepsilon \in \{-1, 1\}$. If $0 \leq j < m$ or $m+1 < j \leq k$ then the same replacement can be performed to get from v_i to v_{i+1} by a single elementary semicommutation. If $j = m$ then $v_i \equiv v_{i+1}$ as $(n_{i,m-1} + \varepsilon) + (n_{i,m} - \varepsilon) + n_{i,m+1} = n_{i,m-1} + n_{i,m} + n_{i,m+1}$. Similarly $v_i \equiv v_{i+1}$ if $j = m+1$.

As each entry in the sequence $v_0, v_1, \dots, v_k$ is either equal to or a single semicommutation away from the next one it follows that $v \equiv v_0 \sim v_k \equiv v'$.

We include a representation of this proof in Figure 6.2.1.

□

Let $w, v \in \overline{U_H \cup \{t\}}^*$ be words in HNN reduced form. We will use $w \rightsquigarrow_m v$ to indicate that there are HNN reduced form words $w', v' \in \overline{U_H \cup \{t\}}^*$ such that $w \sim w' \rightarrow_m v' \sim v$.

Lemma 6.2.6. *Let $v, w \in \overline{(U_H \cup \{t\})}^*$ be words in HNN reduced form such that $w \rightsquigarrow_m v$. If*

$$w \equiv t^{n_0} u_1 t^{n_1} \dots t^{n_{m-1}} u_m t^{n_m} u_{m+1} t^{n_{m+1}} \dots u_k t^{n_k}$$

where $u_j \in \overline{U_H}$ and $n_j \in \mathbb{Z}$, then $u_{m+1} \equiv \phi^{n_m}(u_m^{-1})$.

Proof. That $w \rightsquigarrow_m v$ implies that there are $w', v' \in \overline{U_H \cup \{t\}}^*$ in HNN reduced form such that $w \sim w' \rightarrow_m v' \sim v$. We may write

$$w' \equiv t^{n'_0} u'_1 t^{n'_1} \dots u'_j t^{n'_j} \dots u'_k t^{n'_k}.$$

and as cancellation is possible at m we know that $(u'_m)^{-1} \equiv u'_{m+1}$ and $n'_m = 0$. Further we know by Lemma 6.2.3 that $u'_j \equiv \phi^{z_j}(u_j)$ where $z_j = \Sigma_{i=0}^{j-1}(n'_i - n_i)$.

Combining these facts, we have that

$$\phi^{z_m}(u_m^{-1}) \equiv (u'_m)^{-1} \equiv u'_{m+1} \equiv \phi^{z_{m+1}}(u_{m+1})$$

and thus $u_{m+1} \equiv \phi^{z_m - z_{m+1}}(u_m^{-1})$. Moreover, we have that

$$z_m - z_{m+1} = n_m - n'_m = n_m - 0 = n_m$$

and therefore that $u_{m+1} \equiv \phi^{n_m}(u_m^{-1})$.

□

Remark. We observe that the choice of v is irrelevant to Lemma 6.2.6. What is necessary is that some v satisfies $w \rightsquigarrow_m v$ and thus that there is some $w' \sim w$ where cancellation is possible at position m.

Two simple consequences of this are:

Lemma 6.2.7. *Let $w \in (U_Q \cup \{t, t^{-1}\})^*$, where $U_Q \subseteq \overline{U_H}$, and $v \in \overline{(U_H \cup \{t\})}^*$ be words in HNN reduced form such that $w \rightsquigarrow_m v$. Then there is a word $v' \in (U_Q \cup \{t, t^{-1}\})^*$ in HNN reduced form such that $w \rightsquigarrow_m v'$.*

Proof. By Lemma 6.2.6 we know that we may write w in the form

$$t^{n_0} u_1 t^{n_1} \dots t^{n_{m-1}} u_m t^{n_m} \phi^{n_m}(u_m^{-1}) t^{n_{m+1}} \dots u_k t^{n_k}.$$

By a finite sequence of elementary semicommutations we can transform this into

$$t^{n_0} u_1 t^{n_1} \dots t^{n_{m-1}} u_m u_m^{-1} t^{n_m + n_{m+1}} \dots u_k t^{n_k}$$

which after an elementary cancellation gives

$$t^{n_0} u_1 t^{n_1} \dots t^{n_{m-1} + n_m + n_{m+1}} \dots u_k t^{n_k}.$$

This word is written over a subset of the basis words used in w as the u_j where $j \neq m, m+1$ are untouched. Thus if we designate it v' it satisfies our conditions.

□

Lemma 6.2.8. *Let $v, v', w, w' \in \overline{U_H \cup \{t\}}^*$ be words in HNN reduced form such that $v \leftarrow_m w \sim w' \rightarrow_{m'} v'$. There is some $w'' \in \overline{U_H \cup \{t\}}^*$ such that $w \sim w''$ and elementary cancellations are possible at both m and m' in w''.*

Proof. We may assume without loss of generality that $m \leq m'$. If we write

$$w \equiv t^{n_0} u_0 t^{n_1} \dots u_m t^{n_m} u_{m+1} \dots u_{m'} t^{n_{m'}} u_{m'+1} \dots u_k t^{n_k}$$

then we know by Lemma 6.2.6 that $w \rightsquigarrow_{m'} v'$ implies that $u_{m'+1} \equiv \phi^{n_{m'}}(u_{m'}^{-1})$. Therefore as we already know that $n_m = 0$ and $u_{m+1} \equiv u_m^{-1}$ by $v \leftarrow_m w$ we can write

$$w \equiv t^{n_0} u_0 t^{n_1} \dots u_m u_m^{-1} \dots u_{m'} t^{n_{m'}} \phi^{n_{m'}}(u_{m'}^{-1}) t^{n_{m'+1}} \dots u_k t^{n_k}$$

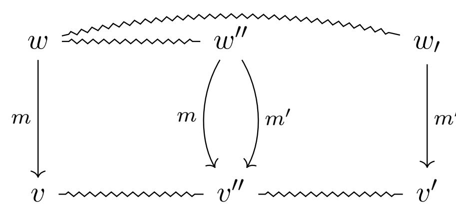

Figure 6.2.2: The squiggly lines represent a sequence of elementary semicommutations in the same way $\sim$ does in the body of the text.

which is a finite sequence of elementary semicommutations away from

$$w'' \equiv t^{n_0} u_0 t^{n_1} \dots u_m u_m^{-1} \dots u_{m'} u_{m'}^{-1} t^{n_{m'}+n_{m'+1}} \dots u_k t^{n_k}$$

which has possible cancellations at both m and m'.

$\square$

Remark. A similar remark to the one that followed Lemma 6.2.6 may be made here about the choice of v and v'.

Using this we can strengthen Lemma 6.2.5.

Lemma 6.2.9. *Suppose we have words $v, v', w, w' \in \overline{U_H \cup \{t\}}^*$ in reduced HNN form such that $v \leftarrow_m w \sim w' \rightarrow_{m'} v'$. Either $\mid m - m' \mid \leq 1$ in which case $v \sim v'$ or $\mid m - m' \mid > 1$ in which case there are some $v^\flat, v'', v^\sharp \in \overline{U_H \cup \{t\}}^*$ in HNN reduced form such that $v \sim v^\flat \rightarrow_m v'' \leftarrow_{m'} v^\sharp \sim v'$.*

Proof. Suppose that $\mid m - m' \mid = 0$. Then $m = m'$ and this case has already been taken care of by Lemma 6.2.5.

Suppose that $\mid m - m' \mid = 1$. By Lemma 6.2.8 we know that there exists some $w'' \in \overline{U_H \cup \{t\}}^*$ such that cancellations at both m and m' are possible and $w \sim w''$. We assume without loss of generality that $m' = m + 1$. Thus we know that if we write $w'' \equiv t^{n_0} u_1 t^{n_1} \dots u_m t^{n_m} u_{m+1} t^{n_{m+1}} u_{m+2} \dots u_k t^{n_k}$ then $u_m \equiv u_{m+1}^{-1} \equiv u_{m+2}$ and $n_m = 0 = n_{m+1}$. Therefore the cancellations at m and m' both produce the same result which we call v''. This means that $v \leftarrow_m w \sim w'' \rightarrow_m v''$ and so by

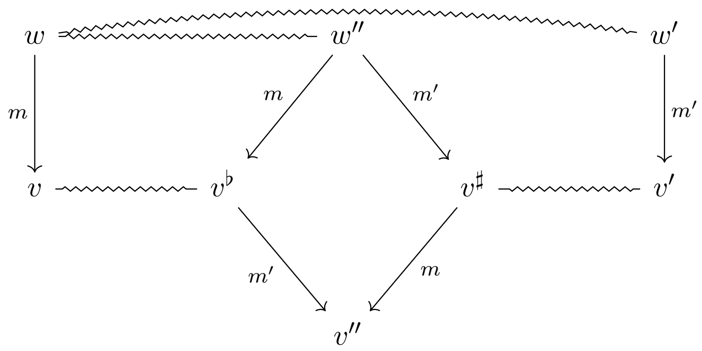

Figure 6.2.3: The squiggly lines represent a sequence of elementary semicommutations in the same way $\sim$ does in the body of the text.

Lemma 6.2.5 $v \sim v''$. Similarly we can argue that $v'' \sim v'$. As $\sim$ is transitive we have that $v \sim v'$. See Figure 6.2.2 for an illustration of this.

Suppose that $\mid m - m' \mid > 1$. By Lemma 6.2.8 we know that there exists some $w'' \in \overline{U_H \cup \{t\}}^*$ such that $w \sim w''$ and cancellations at both m and m' are possible. As $\mid m - m' \mid > 1$ we know that the two possible cancellations do not overlap and thus that unlike the previous cases it is possible to perform them sequentially (and that either order will produce the same product). Let $v^\flat, v'', v^\sharp \in \overline{U_H \cup \{t\}}^*$ be such that $w'' \rightarrow_m v^\flat \rightarrow_{m'} v''$ and $w'' \rightarrow_{m'} v^\sharp \rightarrow_m v''$. This means that $v \leftarrow_m w \sim w'' \rightarrow_m v^\flat$ and so by Lemma 6.2.5 we know that $v \sim v^\flat$. Similarly we can find that $v^\sharp \sim v'$. Therefore $v \sim v^\flat \rightarrow_m v'' \leftarrow_{m'} v^\sharp \sim v'$ as required. See Figure 6.2.3 for an illustration of this.

□

We use $w \rightsquigarrow v$ to indicate that $w \rightsquigarrow_m v$ for some m.

Lemma 6.2.10. *Suppose we have words $w, v, v' \in \overline{U_H \cup \{t\}}^*$ in HNN reduced form such that $v \leftsquigarrow w \rightsquigarrow v'$. Either $v \sim v'$ or there exists $v'' \in \overline{U_H \cup \{t\}}^*$ in HNN reduced form such that $v \rightsquigarrow v'' \leftsquigarrow v'$.*

Proof. The hypotheses in the statement of the lemma imply that there are HNN reduced form words $v_1, v_2, w_1, w_2 \in \overline{U_H \cup \{t\}}^*$ and numbers $m, m' \in \mathbb{N}$ such that $v \sim v_1$, $v_2 \sim v'$, $w_1 \sim w \sim w_2$ and $v_1 \leftarrow_m w_1 \sim w_2 \rightarrow_{m'} v_2$. The rest follows immediately from Lemma 6.2.9 and the definition of $\rightsquigarrow$.

□

We can further show that any two words equal in H^* possess a, not necessarily unique, "sink" word that can be reached by a finite sequence of elmentary cancellations and semicommutations from either starting word.

Lemma 6.2.11. *Let $v_0, v_1, \dots, v_k \in \overline{(U_H \cup \{t\})}^*$ be a set of words in HNN reduced form such that $v_i \leftrightsquigarrow v_{i+1}$ or $v_i \rightsquigarrow v_{i+1}$, for $0 \leq i < k$.*

There exists a sequence of words $v'_0, v'_1, \dots, v'_{k'} \in \overline{(U_H \cup \{t\})}^*$ in HNN reduced form such that $v'_0 \equiv v_0$, $v'_{k'} \equiv v_k$, $k' \leq k$, $v'_i \rightsquigarrow v'_{i+1}$ for $0 \leq i < m$ and $v'_i \leftrightsquigarrow v'_{i+1}$ for $m \leq i \leq k'$, for some $0 \leq m \leq k'$.

Proof. If $k = 0$ then the statement is trivial.

If $k = 1$ then either $v \equiv v_0 \leftrightsquigarrow v_1 \equiv v'$ or $v \equiv v_0 \rightsquigarrow v_1 \equiv v'$ and in either case this fulfils the statement.

If $k = 2$, then either the statement is fulfilled immediately, e.g $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow v_2$, or our initial sequence of words takes the form $v_0 \leftrightsquigarrow v_1 \rightsquigarrow v_2$. In this case by Lemma 6.2.10 tells us that there exists a word $v'_1 \in \overline{(U_H \cup \{t\})}^*$ sequence $v_0 \rightsquigarrow v'_1 \leftrightsquigarrow v_2$ which fulfils the statement.

Assume that $k = K + 1 \geq 3$ and that for all sequences with $k \leq K$ arrows the Lemma's statement is satisfied. This means that we can find a sequence of words $v'_0, \dots, v'_{K'} \in \overline{U_H \cup \{t\}}^*$, for $K' \leq K$, such that $v_0 \equiv v'_0$, $v_K \equiv v'_{K'}$ and that there exists some $0 \leq m \leq K'$ such that $v'_j \rightsquigarrow v'_{j+1}$ if $0 \leq j < m$ and $v'_j \leftrightsquigarrow v'_{j+1}$ if $m \leq j \leq K'$. If $K' < K$ then there is a sequence of $K' + 1 \leq K$ arrows between v_0 and v_{K+1} and the Lemma's statement is satisfied by inductive assumption. So going forward we assume $K' = K$.

Suppose $v_K \leftsquigarrow v_{K+1}$ then we can say that

$$v_0 \rightsquigarrow v'_1 \rightsquigarrow v'_2 \rightsquigarrow \ldots \rightsquigarrow v'_m \leftsquigarrow \ldots \leftsquigarrow v'_{K-1} \leftsquigarrow v_K \leftsquigarrow v_{K+1}$$

and have a sequence of arrows satisfying the necessary conditions. Otherwise $v_K \rightsquigarrow v_{K+1}$ and we have

$$v_0 \rightsquigarrow v'_1 \rightsquigarrow v'_2 \rightsquigarrow \ldots \rightsquigarrow v'_m \leftsquigarrow \ldots \leftsquigarrow v'_{K-1} \leftsquigarrow v_K \rightsquigarrow v_{K+1}$$

In this case we can apply Lemma 6.2.10 to $v'_{K'-1} \leftsquigarrow v'_{K'} \rightsquigarrow v_{K+1}$. This means either that $v'_{K'-1} \sim v_{K+1}$ or that there is some word $v''_K \in \overline{(U_H \cup \{t\})}^*$ in HNN reduced form such that $v'_{K-1} \rightsquigarrow v''_K \leftsquigarrow v_{K+1}$. If the former holds then we have a sequence of $K' - 1 + 1 = K' \leq K$ arrows between v_0 and v_{K+1} (and as above we can satisfy the necessary conditions by inductive assumption). If the latter holds then we have that

$$v_0 \rightsquigarrow v'_1 \rightsquigarrow v'_2 \rightsquigarrow \ldots \rightsquigarrow v'_j \leftsquigarrow \ldots \leftsquigarrow v'_{K-1} \rightsquigarrow v''_K \leftsquigarrow v_{K+1}.$$

We can apply the inductive assumption to the subsequence of arrows going from v_0 to v''_K in a similar manner to above and get a sequence of the form

$$v_0 \rightsquigarrow v''_1 \rightsquigarrow v''_2 \rightsquigarrow \ldots \rightsquigarrow v''_{m'} \leftsquigarrow \ldots \leftsquigarrow v''_{K''-1} \leftsquigarrow v''_{K''} \leftsquigarrow v_{K+1},$$

for some m', which satisfies the necessary conditions. For a compiled diagram illustrating the proof see Figure 6.2.4.

□

We use $w \gtrsim v$ to indicate that either $w \sim v$ or there is a sequence of words $w_0, w_1, \ldots, w_k \in \overline{(U_H \cup \{t\})}^*$ in HNN reduced form such that $w \equiv w_0$, $v \equiv w_k$ and $w_i \rightsquigarrow w_{i+1}$ for $0 \leq i < k$.

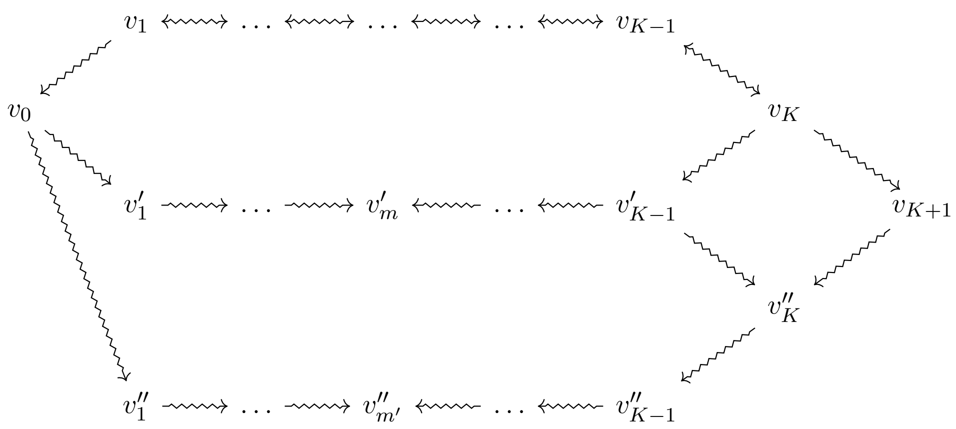

Figure 6.2.4: Double ended squiggly arrows indicate an indifference to direction. This demonstrates a case where the most extensive method is require.

Lemma 6.2.12. *Let $v, v' \in \overline{U_H \cup \{t\}}^*$ be words which are in HNN reduced form in H^*. If $v = v'$ in H^* then there exists some word $v'' \in \overline{U_H \cup \{t\}}^*$ such that $v \gtrsim v'' \lesssim v'$.*

Proof. By Lemma 6.2.1 $v = v'$ in H^* implies that there is a finite sequence of elementary operations between them. This sequence can only contain a finite number of elementary additions and elementary cancellations, thus there is a sequence of words $v_0, v_1, \ldots, v_k \in \overline{(U_H \cup \{t\})}^*$ such that $v_0 = v$, $v_k = v'$ and $v_i \leftsquigarrow v_{i+1}$ or $v_i \rightsquigarrow v_{i+1}$ for $0 \leq i < k$. The rest follows from Lemma 6.2.11.

□

We may also strengthen Lemma 6.2.7.

Lemma 6.2.13. *Let $v_1, v_k \in \overline{U_H \cup \{t\}}^*$ be words which are in HNN reduced form in H^*. Further let $V_H \subseteq \overline{U_H}$ be such that $v_1 \in (V_H \cup \{t, t^{-1}\})^*$. If $v_1 \gtrsim v_k$ then there exists $v'_k \in (V_H \cup \{t, t^{-1}\})^*$ such that $v_k \sim v'_k$.*

Proof. As $v_1 \gtrsim v_k$ there is a sequence of HNN reduced form words $v_i \in \overline{U_H \cup \{t\}}^*$, for $1 \leq i \leq k$, such that $v_i \rightsquigarrow_{m_i} v_{i+1}$, for $1 \leq i < k$.

Let $i = 1$. As $v'_1 \equiv v_1$, then trivially we have $v_1 \sim v'_1$ and $v'_1 \in (V_H \cup \{t, t^{-1}\})^*$.

Let $i = K < k$ and suppose we have $v'_K \in (V_H \cup \{t, t^{-1}\})^*$ such that $v'_K \sim v_K$. This means that $v'_K \rightsquigarrow_{m_K} v_{K+1}$. So by Lemma 6.2.7, we know that there exists $v'_{i+1} \in (V_H \cup \{t, t^{-1}\})^*$ such that $v'_K \rightsquigarrow_{m_K} v'_{K+1}$. Thus we have $v_{K+1} \leftsquigarrow_{m_K} v'_K \rightsquigarrow_{m_K} v'_{K+1}$ which, by Lemma 6.2.9, means that $v'_{K+1} \sim v_{K+1}$.

Therefore by induction we can find $v'_k \in (V_H \cup \{t, t^{-1}\})^*$ such that $v'_k \sim v_k$.

□

6.3 Most Reduced Forms

We shall say that a word $v \in \overline{U_H \cup \{t\}}^*$ is in its *most reduced form* if it is in HNN reduced form and there is no $v' \in \overline{U_H \cup \{t\}}^*$ such that $v \rightsquigarrow v'$. We can then define

$$\mathrm{MRF}(w) = \{v \in \overline{(U_H \cup \{t\})}^* \mid v \precsim w, \ v \text{ is in its most reduced form}\}$$

to be the set of words which are most reduced forms of the defining word. We will now prove a series of results about sets of this form, which will be useful in the next section. In particular these results relate to the content and algorithmic properties of such sets.

Lemma 6.3.1. *Let $w \in \overline{U_H \cup \{t\}}^*$ be a word in HNN reduced form. The set* $\mathrm{MRF}(w)$ *is non-empty.*

Proof. The word w is made up of a finite concatenation of words from the sets $\{t, t^{-1}\}$ and $\overline{U_H}$. Each elementary cancellation reduces the amount of basis words from the set $\overline{U_H}$ used. Thus every sequence of elementary cancellations and semicommutations can only contain finitely many elementary cancellations. Thus there must be a most reduced form for w. □

Lemma 6.3.2. *Let $w \in \overline{U_H \cup \{t\}}^*$ be a word in HNN reduced form. Let $v_1 \in$* $\mathrm{MRF}(w)$ *then $v_2 \in$* $\mathrm{MRF}(w)$ *if and only if $v_1 \sim v_2$.*

Proof. Suppose that $v_1 \sim v_2$. As $v_2 \sim v_1 \precsim w$ we have that $v_2 \precsim w$. Thus if $v_2 \notin \mathrm{MRF}(w)$ there must be some $v_3 \in \overline{U_H \cup \{t\}}^*$ such that $v_2 \rightsquigarrow v_3$. However this implies that $v_1 \sim v_2 \rightsquigarrow v_3$ and hence that $v_1 \rightsquigarrow v_3$, which contradicts $v_1 \in \mathrm{MRF}(w)$. Therefore $v_2 \in \mathrm{MRF}(w)$.

Suppose that $v_2 \in \mathrm{MRF}(w)$. That $v_2 \precsim w \succsim v_1$ means that there is a sequence of elementary operations between them. By Lemma 6.2.1 this means that $v_1 = v_2$ and thus by Lemma 6.2.12 there is some v_3 such that $v_2 \succsim v_3 \precsim v_1$. By assumption v_2 is in most reduced form, therefore $v_2 \succsim v_3$ implies that $v_2 \sim v_3$ as no cancellations are possible. Similarly $v_3 \sim v_1$ and thus $v_1 \sim v_2$. □

Lemma 6.3.3. *Let $w, w' \in \overline{U_H \cup \{t\}}^*$ be words in HNN reduced form. Either $\mathrm{MRF}(w) = \mathrm{MRF}(w')$ or $\mathrm{MRF}(w) \cap \mathrm{MRF}(w') = \varnothing$. In particular $v \in \mathrm{MRF}(w)$ implies that $\mathrm{MRF}(v) = \mathrm{MRF}(w)$.*

Proof. Suppose $v \in \mathrm{MRF}(w) \cap \mathrm{MRF}(w')$. If $v' \in \mathrm{MRF}(w')$ then, as $v \in \mathrm{MRF}(w')$, by Lemma 6.3.2 we know that $v \sim v'$. Thus by a second application of Lemma 6.3.2, as $v \in \mathrm{MRF}(w)$, we know that $v' \in \mathrm{MRF}(w)$. Therefore $\mathrm{MRF}(w') \subseteq \mathrm{MRF}(w)$. Dually $\mathrm{MRF}(w) \subseteq \mathrm{MRF}(w')$ and so $\mathrm{MRF}(v) = \mathrm{MRF}(v')$.

Thus we have shown that any overlap forces equality between the sets. □

Lemma 6.3.4. *Let $w, w' \in \overline{U_H \cup \{t\}}^*$ be words in HNN reduced form. Then $\mathrm{MRF}(w) = \mathrm{MRF}(w')$ if and only if $w = w'$ in H^*. In particular, $v \in \mathrm{MRF}(w)$ implies that $v = w$.*

Proof. Suppose that $\mathrm{MRF}(w) \equiv \mathrm{MRF}(w')$. Let $v \in \mathrm{MRF}(w) \equiv \mathrm{MRF}(w')$. This means that $w \succsim v \precsim w'$. Thus there is a sequence of elementary operations between w and w' and so by Lemma 6.2.1 $w = w'$ in H^*.

Suppose that $w = w'$ in H^*. By Lemma 6.2.12 there exists some $w'' \in \overline{U_H \cup \{t\}}^*$ such that $w \succsim w'' \precsim w'$. We know by Lemma 6.3.1 there is some word $v \in \mathrm{MRF}(w'')$. As $v \precsim w''$ it follows that $v \precsim w$ also. Thus as we already know that v is in most reduced form it follows that $v \in \mathrm{MRF}(w)$. Similarly we may also deduce that $v \in \mathrm{MRF}(w')$. By Lemma 6.3.3 we know that these sets are either equal or disjoint, therefore as their intersection is non-empty we know that $\mathrm{MRF}(w) \equiv \mathrm{MRF}(w')$. □

Lemma 6.3.5. *Let $w \in \overline{U_H \cup \{t\}}^*$ be a word in HNN reduced form. The set $\mathrm{MRF}(w)$ is finite and can be found by algorithm.*

Proof. Let $w_1 \equiv w$. By Lemma 6.2.3 we know that the set $W_1 = \{w_1' \mid w_1' \sim w_1\}$ is finite and can be found by algorithm. If we find some $w_1' \in W_1$ where an elementary cancellation is possible then we perform that cancellation and call the result w_2.

If we repeat this process until we find some w_k such that no member of $W_k = \{w_k' \mid w_k' \sim w_k\}$ has any possible elementary cancellations (this must happen as w is of finite length and each w_{i+1} is strictly shorter than the preceding w_i). As $w_i \rightsquigarrow w_{i+1}$ we have that $w_k \precsim w$ and so $w_k \in \mathrm{MRF}(w)$. Moreover, by Lemma 6.3.2 we have that $\mathrm{MRF}(w) = W_k$ and thus by Lemma 6.2.3 the set is finite and can be found by algorithm. □

Lemma 6.3.6. *Let $V \subseteq \overline{U_H}$ be such that $w \in (V \cup \{t, t^{-1}\})^*$, then there is some word $v' \in (V \cup \{t, t^{-1}\})^*$ such that $v' \in \mathrm{MRF}(w)$.*

Proof. By Lemma 6.3.1 there is some $v \in \mathrm{MRF}(w)$. As $v \precsim w$, by Lemma 6.2.13 there exists some $v' \in (V \cup \{t, t^{-1}\})^*$ such that $v' \sim v$. By Lemma 6.3.2 this means that $v' \in \mathrm{MRF}(w)$. □

Lemma 6.3.7. *Let $w \in \overline{U_H \cup \{t\}}^*$ be a word in HNN reduced form and let $V \subseteq \overline{U_H}$. It may be determined by algorithm whether or not there is some word $w' \in (V \cup \{t, t^{-1}\})^*$ in HNN reduced form such that $w = w'$.*

Proof. Suppose there is some $w' \in \overline{V \cup \{t\}}^*$ such that $w = w'$ then by Lemma 6.3.6 there is some $v' \in (V \cup \{t, t^{-1}\})^*$ such that $v' \in \mathrm{MRF}(w')$. Then by Lemma 6.3.4 we know $v' = w'$ and thus $v' = w$. Therefore MRF(w) contains no words written over $V \cup \{t, t^{-1}\}$ only if there is no $w' = w$ such that $w' \in (V \cup \{t, t^{-1}\})^*$.

If there is a word $w' \in V \cup \{t, t^{-1}\}$ such that $w' \in$ MRF(w) then by Lemma 6.3.4 we have $w' = w$. Thus MRF(w) contains a word written over $V \cup \{t, t^{-1}\}$ if and only there is some $w' \in (V \cup \{t, t^{-1}\})^*$ such that $w' = w$.

By Lemma 6.3.5 there are a finite number of words in MRF(w) and these can be found by algorithm. When given a finite set of words written over $U_H \cup \{t, t^{-1}\}$ we can decide if any are written over $V \cup \{t, t^{-1}\}$ by algorithm. Consequently we can determine by algorithm if there is some $w' \in (V \cup \{t, t^{-1}\})^*$ such that $w' = w$. $\square$

Taken together these results mean that for any word that is already in HNN reduced form we can find a set of words which collectively act as something like a normal form. In the next section we will see how this can be applied to solve membership problems.

7

Membership in submonoids of HNN extensions of free groups

Synopsis

In this chapter we will apply the results of the previous chapter. In the first section we will use them to show that there is a particular form of submonoid in which we can always decide membership within the kind of HNN extension being discussed. In the second section we will demonstrate that there is a form of one-relator group presentation where the first section's result may be used to decide the prefix membership problem, and consequently the word problem of the corresponding inverse monoid.

7.1 A family of decidable submonoids

Suppose we have a group H and an HNN extension of that group $H^* = H*_{t,\phi:A\to B}$. If we have a submonoid Q of H then we may also consider the submonoid $P = \mathrm{Mon}\left\langle Q \cup \{t, t^{-1}\}\right\rangle$ of H^*. We observe that P contains $t^{-k}qt^k$, for $k \in \mathbb{Z}$ and $q \in Q$, and that therefore wherever $\phi^k(q)$ is a valid construction, we have that $\phi^k(q) \in P \cap H$. This leads us to produce the following lemma which associates a restriction on the associated isomorphism of the HNN extension, ϕ, to a restriction on the relationship between P and Q.

Lemma 7.1.1. *Let $H^* = H*_{t,\phi:A\to B}$ be an HNN extension of a group $H = \mathrm{Gp}\langle X \mid R\rangle$. Further let $Q \leq H$ be a monoid and let $P = \mathrm{Mon}\left\langle Q \cup \{t, t^{-1}\}\right\rangle \leq H^*$. Then $Q = P \cap H$ if and only if $\phi(Q \cap A) = Q \cap B$.*

Proof. "⇒": Suppose that $Q = P \cap H$. Let $q \in Q \cap A$. It is clear from $q \in Q$ and the definition of P that $t^{-1}qt \in P$. Additionally $q \in A$ implies that $t^{-1}qt = \phi(q) \in B$. So $\phi(q) \in P \cap B = P \cap (H \cap B) = (P \cap H) \cap B = Q \cap B$, which means that $\phi(Q \cap A) \subseteq Q \cap B$. A dual argument gives that $\phi^{-1}(Q \cap B) \subseteq Q \cap A$, which is equivalent to $\phi(Q \cap A) \supseteq Q \cap B$ and therefore $\phi(Q \cap A) = Q \cap B$.

"⇐": Suppose that $\phi(Q \cap A) = Q \cap B$. Let $p \in P \cap H$. We know by the definition of P that we can write this in the form $q_0 t^{\varepsilon_1} q_1 \ldots t^{\varepsilon_n} q_n$, where $q_i \in Q$ for $0 \leq i \leq n$ and $\varepsilon_i \in \{-1, 1\}$ for $1 \leq i \leq n$. Further as $p \in H$ we know that the sum of ε_i must be 0 and that the form is only HNN reduced if $n = 0$. We know that if a word is not HNN reduced then there is either a sequence of the form $t^{-1}at$ where $a \in A$ or one of the form tbt^{-1} where $b \in B$. This means that there is either an i such that $\varepsilon_i = -1$, $q_i \in A$ and $\varepsilon_{i+1} = 1$ or an i such that $\varepsilon_i = 1$, $q_i \in B$ and $\varepsilon_{i+1} = -1$.

If the first is true then $t^{\varepsilon_i} q_i t^{\varepsilon_{i+1}} = \phi(q_i)$, this being a well defined use of ϕ due to $q_i \in A$. Moreover as $q_i \in Q \cap A$ we know that $\phi(q_i) \in Q \cap B$. This means that we can write p in the form

$$q_0 t^{\varepsilon_1} \ldots t^{\varepsilon_{i-1}} q_{i-1} \phi(q_i) q_{i+1} t^{\varepsilon_{i+2}} \ldots t^{\varepsilon_n} q_n$$

which is composed of elements belonging to Q and $\{t, t^{-1}\}$ only and has two fewer occurences of t and t^{-1} collectively. A dual argument can be made for a similar reduction in the second case.

This means that if $n \neq 0$ then we can inductively reduce p to a form where $n = 0$ while preserving that the non-t parts belong to Q, so $p \in Q$ and therefore $P \cap H \subseteq Q$. That $Q \subseteq P \cap H$ is obvious from the definitions and so we can conclude that $Q = P \cap H$.

□

We can now further deduce the following, which is one of the main results of this thesis.

Theorem 7.1.2. *Let* $H^* = H*_{t,\phi:A\to B}$ *be an HNN extension of a free group* $H = \mathrm{Gp}\langle X \mid R \rangle$ *with free basis* $U_H \subset \overline{X}^*$*. Further let this free basis be such that there are* $U_A, U_B \subset \overline{U_H}$ *where* $A = \mathrm{Gp}\langle U_A \rangle$*,* $B = \mathrm{Gp}\langle U_B \rangle$ *and* $\phi(\overline{U_A}) = \overline{U_B}$*. Finally let* Q *be a submonoid of* H *generated by some* $U_Q \subseteq \overline{U_H}$*.*

Membership in the submonoid $P = \mathrm{Mon}\left\langle Q \cup \{t, t^{-1}\} \right\rangle$ *within* H^* *is decidable if* $\phi(Q \cap A) = Q \cap B$*.*

Proof. Suppose we have a word $w \in \overline{X \cup \{t\}}^*$. By Lemma 2.6.3 we can algorithmically find a word equal to w in H^* which is in HNN reduced form. Therefore we can assume without loss of generality that w in HNN reduced form. We can also assert without loss of generality that the parts of w written over $\overline{X}^*$ are written over the free basis $\overline{U_H}$.

If the element in H^* represented by w belongs to P then there is some word $w' \in (U_Q \cup \{t, t^{-1}\})^*$ which is equal to w in H^*. Moreover, we saw in the proof of Lemma 7.1.1 that we can find an HNN reduced form of any word written over $U_Q \cup \{t, t^{-1}\}$ which is still written over $U_Q \cup \{t, t^{-1}\}$. Therefore if there exists some word $w' \in (U_Q \cup \{t, t^{-1}\})^*$ which is equal to w in H^*, we can assume without loss of generality that it is in HNN reduced form.

By Lemma 6.3.7 we can determine algorithmically if there is some w' written over $U_Q \cup \{t, t^{-1}\}$ in HNN reduced form such that $w' = w$. Hence we can decide whether $w \in P$ within H^*. □

Remark. We may observe that the conditions of Theorem 7.1.2 almost fulfil the requirements of a result of Dolinka and Gray [6, Theorem C] as $Q = P \cap H$, by Lemma 7.1.1, and Q has decidable membership within G, by Benois' Theorem.

However the key difference is that to be able to apply their result we would require that $A \cup B \subseteq P$, which is not necessarily the case.

Example 7.1.3. We claim the monoid

$$P = \mathrm{Mon}\left\langle t, t^{-1}, b_0, b_0 a_1, b_{-1}, a_1^{-1} b_0^{-1} \right\rangle$$

has decidable membership within

$$H^* = \mathrm{Gp}\left\langle\, a_0, a_1, b_{-1}, b_0 \mid b_0 a_1 b_{-1} a_0 = 1,\ t^{-1} b_0 a_1 t = (b_0 a_1)^{-1},\ t^{-1} b_{-1} t = b_0 \,\right\rangle.$$

Let $H = \mathrm{Gp}\langle\, a_0, a_1, b_{-1}, b_0 \mid b_0 a_1 b_{-1} a_0 = 1 \,\rangle$, this is a free group for which $U_H = \{b_0, b_0 a_1, b_{-1}\} \subseteq \overline{X}^*$ forms a free basis. Further let $U_A = \{b_0 a_1, b_{-1}\}$, $U_B = \{b_0, b_0 a_1\}$, $A = \mathrm{Gp}\langle U_A \rangle$, $B = \mathrm{Gp}\langle U_B \rangle$ and let ϕ be the map extending $b_{-1} \mapsto b_0$ and $b_0 a_1 \mapsto (b_0 a_1)^{-1}$. (Observe that $\phi(a_0) = a_1$ may be easily deduced from these.) Then H^* is the HNN extension $H{*}_{t,\phi:A\to B}$. If we let $U_Q = \{b_0, b_0 a_1, b_{-1}, a_1^{-1} b_0^{-1}\}$ and $Q = \mathrm{Mon}\langle U_Q \rangle$ then it may be found that

$$\phi(Q \cap A) = \mathrm{Mon}\left\langle b_0 a_1, (b_0 a_1)^{-1} \right\rangle = Q \cap B$$

and $P = \mathrm{Mon}\left\langle t, t^{-1}, U_Q \right\rangle$. Therefore the conditions of Theorem 7.1.2 are fulfilled justifying our claim.

Remark. The main result of the next section, Theorem 7.2.1, can be viewed as a generalisation of how Theorem 7.1.2 is applied above in Example 7.1.3 (see also Example 7.2.2 and remark which follows).

Remark. Though the next section is focused on applying these results to one-relator HNN extensions of one-relator groups, it is important to note that nothing in this section nor the preceding chapter is restricted to one-relator HNN extensions nor to HNN extensions of one-relator groups. In all cases we have only assumed that the HNN extension and its basis are finitely presented (in addition, of course, to whatever the various lemmas and theorems explicitly require).

7.2 Application to the Prefix Membership Problem

In this section we show how the results of the previous section can be applied to the prefix membership problem of a family of groups. We can then use this to decide the word problem of the inverse monoids with presentations corresponding to the family of groups by Theorem 2.4.7. We note that the family we identify in Theorem 7.2.1 below has the property of being defined by a single t-sum zero word whose prefixes are a mixture of t-sum positive and t-sum negative, and that they have decidable prefix membership problem cannot be shown by the methods in Dolinka and Gray [6].

Theorem 7.2.1. *Let* $M = \mathrm{Inv}\left\langle X \cup \{t\} \mid wt^{-2\sigma_t(w)}w = 1 \right\rangle$ *be an inverse monoid and let* $G = \mathrm{Gp}\left\langle X \cup \{t\} \mid wt^{-2\sigma_t(w)}w = 1 \right\rangle$ *be its maximal group image, where the word* $w \in \overline{X \cup \{t\}}^*$ *meets the following requirements:*

- *That* $w \equiv x_0^{\varepsilon_0} t^{n_1} x_1^{\varepsilon_1} \dots t^{n_k} x_k^{n_k}$*, such that* $n_i \in \mathbb{Z}$ *for* $1 \leq i \leq k$*,* $\varepsilon_i \in \{-1, 1\}$ *and* $x_i \in X$ *for* $0 \leq i \leq k$ *and* $x_i \not\equiv x_j$ *if* $i \neq j$*;*
- *That* $\sigma_t(w) = n_1 + \dots + n_k \neq 0$*.*

Both the prefix membership problem for G *and the word problem for* M *are decidable.*

Proof. Recall, that in Section 2.7, we defined $\sigma_t(w)$ to be the t-exponent sum of a word w (the number of instances of t in w less the number of instances of t^{-1}). Throughout this proof we will use $\sigma(w)$ as shorthand for $\sigma_t(w)$, to aid legibility.

We will assume throughout that $\sigma(w) \geq 0$ and that all $\varepsilon_i = 1$, we may do this without loss of generality as all words are only a relabelling away from such a case.

It may be seen that the sole relator word of M has a unital factorisation into w and t. This means by Theorem 3.2.8 that in G the factorisation is conservative and so the prefix monoid of G can take the form

$$P = \text{Mon}\left\langle \text{pref}(w), t, t^{-1} \right\rangle.$$

All prefixes of w have the form $p \equiv x_0 t^{n_1} x_1 \ldots x_i t^j$ where $i \leq k$ and either $0 \leq j \leq n_{i+1}$ if $n_{i+1} \geq 0$ or $n_{i+1} \leq j \leq 0$ if $n_{i+1} \leq 0$. However as we have both t and t^{-1} in our generating set for P any prefixes of w ending in t or t^{-1} are extraneous. So we may write

$$P = \text{Mon}\left\langle x_0, x_0 t^{n_1} x_1, \ldots, x_0 t^{n_1} x_1 \ldots t^{n_k} x_k, t, t^{-1} \right\rangle.$$

The overall t-exponent sum of G's sole relator is 0, therefore we can apply Theorem 2.7.4 to G and acquire a new group H which G can be viewed as a HNN extension of.

The group H takes the form

$$H = \text{Gp}\left\langle\, \Xi_r \mid x_{0,s_0} x_{1,s_1} \ldots x_{k,s_k} x_{0,s_0+\sigma(w)} x_{1,s_1+\sigma(w)} \ldots x_{k,s_k+\sigma(w)} = 1 \,\right\rangle$$

where $s_i = n_1 + \ldots + n_i$ and Ξ_r is the alphabet consisting of all $x_{i,j}$ such that $0 \leq i \leq k$ and $s_i \leq j \leq s_i + \sigma(w)$. There is a unique letter in the sole relator of H, so Lemma 4.1.7 tells us that H is a free group of rank $|\Xi_r| - 1$.

We claim that every $x_{i,j}$ in the defining relation of H is distinct. As the x_i in w are defined as distinct letters, we know that for all $0 \leq i \leq k$, and $x_{i,j} \equiv x_{i',j'}$ if and only if $i = i'$ and $j = j'$, the letters x_{i,s_i} are distinct from each other. By the same logic the letters $x_{i,s_i+\sigma(w)}$ are distinct from each other. Moreover, as $\sigma(w) \neq 0$ we know that $x_{i,s_i} \not\equiv x_{i,s_i+\sigma(w)}$ and so the two sets of letters are disjoint.

This group is such that if we take ϕ to be the mapping $\phi(x_{i,j}) = x_{i,j+1}$, A to be the subgroup generated by $\Xi_r \setminus \{x_{i,s_i} \text{ for } 0 \leq i \leq k\}$ and B to be the subgroup generated by $\Xi_r \setminus \{x_{i,s_i+\sigma(w)}$ for $0 \leq i \leq k\}$ then the HNN extension $H^* = H*_{t,\phi:A\to B}$ is isomorphic to G under the mapping $x_{i,j} \mapsto t^{-j}x_i t^j$.

Applying the preimage of such a mapping to our most recent generating set for P we get

$$P^* = \text{Mon}\left\langle x_{0,s_0},\, x_{0,s_0}x_{1,s_1},\, \ldots,\, x_{0,s_0}x_{1,s_1}\ldots x_{k,s_k}, t, t^{-1}\right\rangle \tag{7.2.1}$$

such that P^* within H^* is isomorphic to P within G.

We define $\gamma_{i,j} = x_{0,s_0+j}x_{1,s_1+j}\ldots x_{i,s_i+j}$ for $0 \leq i \leq k$ and $0 \leq j \leq \sigma(w)$. We shall refer to the set of all such $\gamma_{i,j}$ as Γ.

We claim that $U_H = \Gamma \setminus \{\gamma_{k,\sigma(w)}\}$ is a free basis of H. First we note that $\gamma_{k,0}^{-1} = \gamma_{k,\sigma(w)}$ in H and so we can immediately recover the discarded element of Γ. Secondly, $x_{0,s_0+j} = \gamma_{0,j}$ and $x_{i,s_i+j} = \gamma_{i-1,j}^{-1}\gamma_{i,j}$ for $0 < i \leq k$. Therefore $\overline{\Gamma \setminus \{\gamma_{k,\sigma(w)}\}}$ generates all of Ξ_r and consequently all of H. Finally, as there is an obvious one-to-one correspondence between the elements of Γ and Ξ_r the two sets are of the same size. This means that $U_H = \Gamma \setminus \{\gamma_{k,\sigma(w)}\}$ is of size $\mid \Xi_r \mid -1$ which is the rank of H as a free group and so by Theorem 2.2.1 U_H is a free basis of H.

We can now rewrite our previous expression over the new basis to produce

$$P^* = \text{Mon}\left\langle \gamma_{0,0}, \gamma_{1,0}, \ldots, \gamma_{k,\sigma(w)}, t, t^{-1}\right\rangle.$$

As $t^{-1}\gamma_{i,j}t = \gamma_{i,j+1}$ for $0 \leq j < \sigma(w)$, the generators above are sufficient to produce the rest of Γ. Thus we can write

$$P^* = \text{Mon}\left\langle \Gamma \cup \{t, t^{-1}\}\right\rangle$$

as we may add superfluous generators without consequence. We may then find the form

$$P^* = \text{Mon}\left\langle \left(\Gamma \setminus \{\gamma_{k,\sigma(w)}\}\right) \cup \{\gamma_{k,0}^{-1}\} \cup \{t, t^{-1}\}\right\rangle$$

by making the substitution of $\gamma_{k,0}^{-1}$ for $\gamma_{k,\sigma(w)}$ as these are equal in H.

We now let $U_Q = U_H \cup \{\gamma_{k,0}^{-1}\} = \big(\Gamma \setminus \{\gamma_{k,\sigma(w)}\}\big) \cup \{\gamma_{k,0}^{-1}\}$ and $Q = \text{Mon}\,\langle U_Q \rangle$. We seek to show that $\phi(Q \cap A) = Q \cap B$. In H we know that $\gamma_{k,\sigma(w)} = \gamma_{k,0}$, thus $U_Q = \Gamma$ in H.

All the defined mappings of ϕ and ϕ^{-1} send individual elements of Γ to other individual elements of Γ, and thus we may deduce that $\phi(\Gamma \cap A) = \Gamma \cap B$. Furthermore as U_Q is a relabelling Γ this means that $\phi(U_Q \cap A) = U_Q \cap B$. By the first part of Lemma 6.1.2 we can see that by setting $V_M = U_Q$ and $V_G = U_A$ that $U_Q \cap A = U_Q \cap \overline{U_A}$, dually we can see that $U_Q \cap B = U_Q \cap \overline{U_B}$. Further by the second part of the same Lemma we see that any element of $q_a \in Q \cap A = \text{Mon}\,\langle U_Q \rangle \cap \text{Gp}\,\langle U_A \rangle$ can be written as a word $w_a \in (U_Q \cap \overline{U_A})$. This means that $\phi(q_a)$ is sent to an element which can be written over

$$\phi(U_Q \cap \overline{U_A}) = \phi(U_Q \cap A) = U_Q \cap B = U_Q \cap \overline{U_B}.$$

We can then use the second part of Lemma 6.1.2 to say that a word written over $U_Q \cap \overline{U_B}$ represents an element of $Q \cap B$ in H. Thus we have shown that $\phi(Q \cap A) = Q \cap B$.

We take the sets of words $U_A = U_H \setminus \{\gamma_{0,\sigma(w)}, \gamma_{1,\sigma(w)}, \ldots, \gamma_{k-1,\sigma(w)}\}$ and $U_B = U_H \setminus \{\gamma_{0,0}, \gamma_{1,0}, \ldots, \gamma_{k-1,0}\}$ (recall that $\gamma_{k,0} = \gamma_{k,\sigma(w)}^{-1}$ and that therefore we can apply ϕ^{-1}) to be the bases for the groups A and B respectively. Thus we have a setup which fulfills all the conditions of Theorem 7.1.2 and therefore membership in P^* within H^* is decidable. This is equivalent to being able to decide membership in P within G and so G has decidable prefix membership problem.

Further as $wt^{-2\sigma(w)}w$ is cyclically reduced we can apply Theorem 2.4.7 and thus decide the word problem for M.

□

Example 7.2.2. An example of the class described in Theorem 7.2.1 is the group $G = \mathrm{Gp}\left\langle\, a, b, t \mid bt^{-1}at^2bt^{-1}a = 1 \,\right\rangle$, thus G has decidable prefix membership problem.

The reason for particular interest in this group and its prefix monoid is that Dolinka and Gray [6] noted it as example to which they could not apply the methods they had developed in that paper for assessing the prefix membership problem and thus left open the question of whether the problem was indeed decidable.

Remark. The prefix membership problem of G in Example 7.2.2 is equivalent to the membership problem of P within H^* in Example 7.1.3.

This all raises two questions. The first is whether we can extend the statement of Theorem 7.2.1.

Question 7.2.3. *Do all one-relator HNN extensions of free groups have decidable prefix membership problem?*

The second question is whether the machinery developed in the past two chapters can be taken further. The class of group described at the start of Chapter 6 and examined throughout that chapter seem to have a certain "freeness", therefore we may ask:

Question 7.2.4. *Let $H^* = H*_{t,\phi:A\to B}$ be an HNN extension of a finitely generated free group H with free basis U_H. Further let there be $U_A, U_B \subseteq U_H$ such that $\mathrm{Gp}\,\langle U_A\rangle = A$, $\mathrm{Gp}\,\langle U_B\rangle = B$ and $\phi(\overline{U_A}) = \overline{U_B}$. Does H^* have decidable submonoid membership problem?*

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A

Examples of independence

In this appendix we will lay out a series of sketch proofs for why the word problems of a special inverse monoid M, its maximal group image G and its group of units U_M are, excepting the implication $\mathrm{WP}(M) \Rightarrow \mathrm{WP}(U_M)$, independent in general.

The first two results in this appendix, A.1 and A.2, are immediate consequences of the referenced paper of Gray [7]. The remaining three, A.3, A.4 and A.5, are derived via examples original to this thesis.

A.1 $\mathrm{WP}(G) \not\Rightarrow \mathrm{WP}(M)$

It has been shown in recent years that there is a one-relator inverse monoid with undecidable word problem [7]. It is also known that the maximal group image of any one-relator inverse monoid has a presentation as a one-relator group and all one-relator groups have decidable word problem. Therefore, the maximal group image having decidable word problem does not imply the inverse monoid having decidable word problem.

A.2 $\mathrm{WP}(U_M) \not\Rightarrow \mathrm{WP}(M)$

The one-relator inverse monoid with undecidable word problem devised by Gray is also E-unitary [7]. When an inverse monoid is E-unitary the natural map between the inverse monoid and its maximal group image is injective with respect to the group of units [22, Lemma 1.5] [31, Theorem 3.8]. Further, by Lemma 4.3.1 it may be seen that as a finitely presented special inverse monoid, M has a finitely generated group of units. Therefore the inverse monoid constructed by Gray has a group of units which is a finitely generated subgroup of the maximal group image. As we already know that the maximal group image has decidable word problem (see the previous section) this is sufficient to show that this the group of units has decidable word problem as well. Thus, unlike in the non-inverse case a special inverse monoid's group of units having decidable word problem does not imply that the monoid itself has decidable word problem.

A.3 $\mathrm{WP}(G) \not\Rightarrow \mathrm{WP}(U_M)$

Consider an inverse monoid of the following form

$$M = \mathrm{Inv}\langle a_1, \dots, a_k, t_1, \dots, t_k \,|\, r_1 = 1, \dots, r_n = 1, \\ a_i a_i^{-1} = 1, a_i^{-1} a_i = 1, t_i^{-1} a_i t_i = 1 \, (1 \leq i \leq k) \rangle$$

where $r_1, \dots, r_n \in \{a_1, \dots, a_k\}^*$. It can be seen that the t_i are not units as there is a homomorphic mapping from M to a bicyclic monoid of the form $\mathrm{Inv}\left\langle t_i \mid t_i^{-1} t_i = 1 \right\rangle$ for every $1 \leq i \leq k$. Since neither t_i nor t_i^{-1} are units it follows that the only invertible prefixes of $t_i^{-1} a_i t_i$ are the trivial ones (1 and $t_i^{-1} a_i t_i$, both of which are equal to identity in M). Clearly every a_i is a unit and each r_j, for $1 \leq j \leq n$, is written over the a_i. Thus all the minimal invertible pieces of M are either one of the a_i or equal to identity. By Lemma 4.3.1 we know that M's group of units is generated by the minimal invertible pieces. Therefore, we may deduce that M's group of units is $\mathrm{Inv}\,\langle a_1, \dots, a_k \rangle$.

A result of Stephen [31] tells us that two words are equal in an inverse monoid if their Schützenberger graphs are equal. Schützenberger graphs are birooted graphs, consisting of a labelled digraph and two roots, the initial root (the vertex where defining word starts) and the terminal root (the vertex where the defining word terminates). As all words written over the a_i are units their Schützenberger graphs share a labelled digraph and an initial root (identity). Moreover, their Schützenberger graphs will have a copy of the Cayley graph of the group

$$H = \mathrm{Gp}\,\langle\, a_1, \ldots, a_k \mid r_1 = 1, \ldots, r_k = 1 \,\rangle$$

attached at the initial root. This means that the terminal root, which will be at the vertex found by reading a path labelled the generating word of the Schützenberger graph, will also be in this subgraph (as we are only considering words written over the a_i). Thus two words written over the a_i are only equal in M if they are equal in the presentation H. So we may deduce that the group of units has the presentation

$$U_M = \mathrm{Gp}\,\langle\, a_1, \ldots, a_k \mid r_1 = 1, \ldots, r_k = 1 \,\rangle\,.$$

Thus, as not all finitely presented groups have decidable word problem, we may choose r_j such that U_M has undecidable word problem. However if we then consider the maximal group image

$$G = \mathrm{Gp}\langle a_1, \ldots, a_k, t_1, \ldots, t_k \,|\, r_1 = 1, \ldots, r_n = 1, \\ a_i a_i^{-1} = 1, a_i^{-1} a_i = 1, t_i^{-1} a_i t_i = 1\,(1 \leq i \leq k)\rangle.$$

then the relations of the form $t_i^{-1} a_i t_i = 1$ imply that $a_i = 1$ in G. Therefore G reduces to $\mathrm{FG}(t_1, \ldots, t_k)$, which has decidable word problem. Hence, the maximal group image of an inverse monoid having decidable word problem does not imply that corresponding the group of units has decidable word problem.

A.4 $\mathrm{WP}(U_M) \not\Rightarrow \mathrm{WP}(G)$

For this example, we use a similar construction to the previous one. Consider

$$M = \mathrm{Inv}\langle X, t_1, \ldots, t_k \,|\, t_1^{-1} w_1 t_1 = 1, \ldots, t_n^{-1} w_n t_n = 1, \\ xx^{-1} = 1, x^{-1}x = 1\,(x \in X)\rangle.$$

Using the same logic as the previous example (i.e. application of [11, Lemma 4.2]) it may be deduced that M's group of units is generated by X. Further it may be shown that $\mathrm{Inv}\,\langle X\rangle \leq M$ is isomorphic to $\mathrm{FG}(X)$ by considering the Schützenberger graphs of words written over X (using similar arguments as in the previous example). As free groups have decidable word problem this means that M's group of units has decidable word problem. The maximal group image of M on the other hand may be written

$$G = \mathrm{Gp}\,\langle\, X, t_1, \ldots, t_k \;|\; w_1 = 1, \ldots, w_n = 1 \,\rangle\,,$$

and we may choose w_i such that G has undecidable word problem. Thus the group of units having decidable word problem does not imply that the maximal group image has decidable word problem.

A.5 $\mathrm{WP}(M) \not\Rightarrow \mathrm{WP}(G)$

Consider an inverse monoid of the form

$$M = \mathrm{Inv}\langle X, t_1, \ldots, t_k \,|\, t_1^{-1} w_1 t_1 = 1, \ldots, t_n^{-1} w_n t_n = 1\rangle.$$

It may be deduced, independent of the choice of w_i, that all Schützenberger graphs of an inverse monoid of this form has are quasi-isometric to trees. This means that if we look at the strongly connected components of the Cayley graph of M then on a large scale their geometry resembles that of tree graphs. This has algorithmic consequences, in particular, a result of Gray, Silva and Szákacs [9, Theorem 4.15], M has decidable word problem.

The maximal group image of M is

$$G = \mathrm{Gp}\langle X, t_1, \ldots, t_k \,|\, t_1^{-1} w_1 t_1 = 1, \ldots, t_n^{-1} w_n t_n = 1 \rangle.$$

which is isomorphic to

$$\mathrm{Gp}\left\langle\, X, t_1, \ldots, t_k \mid w_1 = 1, \ldots, w_k = 1 \,\right\rangle.$$

Thus we may choose w_i such that G has undecidable word problem. Therefore we may conclude that, an inverse monoid having decidable word problem does not imply that its maximal group image also has decidable word problem.