Frobenius reciprocity for multifiat 2-categories

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Abstract

This thesis has a single goal, namely the establishment of a form of Frobenius reciprocity for finitary birepresentations of multifiat 2-categories.

Multifiat 2-categories are the 2-categorical analogues of finite dimensional associative algebras with involution. Our first novel result is an explicit form of correspondence between birepresentations, morphisms of birepresentations and modifications on the one hand, and coalgebras, bicomodules and morphisms of bicomodules on the other hand. In the latter context, there is a natural definition of induction of birepresentations along a pseudofunctor, while in the former, there is a natural notion of restriction, completely analogous to other representation theories. We use our correspondence to define both in the same setting, that of coalgebras and bicomodules.

We show that restriction and induction are adjoint as pseudofunctors, given some technical assumptions.

Contents

Acknowledgements

A bi cassowary, an example of bi representation. Artist: Sonja Klisch

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Fy nghariad i Cymru a'r Cymraeg; i bawb sy'n breuddwydio am fyd heb ffiniau; ac i'r rhai a gynnau tân fel y tân yn Llŷn.

Introduction

In this thesis, we recover a form of Frobenius reciprocity for finitary birepresentations of multifiat 2-categories; that is, given a pseudofunctor $\mathscr{F}$ between multifiat 2-categories, we find an biadjunction, in the bicategorical sense, between the induction and restriction pseudofunctors along $\mathscr{F}$, under some technical conditions on $\mathscr{F}$.

Representation theory, the study of how algebraic objects act on other - usually geometric - objects, has been a staple of abstract algebra since at least 1896, with the work of Ferdinand Frobenius in [Fro96]. Within a decade of Frobenius' paper, acclaimed results such as Schur's lemma ([Sch05]) and Burnside's theorem ([Bur04]) appeared, placing representation theory in a position of central importance in the study of finite groups. Later, representation theory was the major tool of the so-called Enormous Theorem, also known as the classification of finite simple groups ([Gor79]). The tools of representation theory have been used in most fields of algebra including, for example, Lie groups and algebras ([FH91]), associative algebras ([AF92]), and p-adic groups ([Fin22]).

Categorification - the process of replacing sets with categories, functions with functors and so forth, and studying the new richer structure obtained - has become an essential tool in the study of algebras, used from as early as 2008 to understand properties of symmetric groups; see, for example, [CR04]. 2-representations, first defined in [Rou08], serve the role of representations from the classical setting and are equally ubiquitous in understanding the rich structure of categorified algebras. Naturally, one might expect fundamental representation-theoretic results to have analogies in the 2-categorical setting, and already several have been recovered: in [MM16a], the authors find a 2-analogue of Schur's Lemma for a nice class of simple transitive 2-representations, which are the analogue of simple representations; in [MM16b], the authors find a Jordan-Hölder theory for finitary 2-categories, the analogue of finite-dimensional algebras. Frobenius reciprocity is another fundamental result that naturally suggests a 2-analogue, but the particulars are somewhat subtle. For finite groups, Frobenius reciprocity is the result that restriction and induction are adjoint on both sides (often confusingly called biadjoint), while for modules over associative algebras induction is only left adjoint to restriction: this latter, weaker result is what we recover for multifiat 2-categories.

To find such a result, we need a number of ingredients. Firstly, a pair of pseudofunctors $\mathcal{R}$ and $\mathcal{P}$ called induction and restriction, constructed in Chapters 3 and 4 respectively. Secondly, a pair of 2-natural transformations $\eta : 1_{\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}} \to \mathcal{R}\mathcal{P}$ and $\epsilon : \mathcal{P}\mathcal{R} \to 1_{\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{D}}}}$, called the unit and counit, constructed in Chapters 5 and 6 respectively. Finally, a pair of modifications τ and σ called the triangulators, satisfying the swallowtail diagrams, which are covered in Chapter 7. For full details of the construction, see Definition 1.1.6.

In Chapter 1, we cover the basic ideas used throughout the thesis. Bicategories and their associated notions are presented with some fundamental results, though some familiarity with their properties is assumed. (Multi)fiat 2-categories, originally introduced in [MM11], are described here, being the natural 2-analogue of cellular algebras (alternatively, of finite dimensional associative algebras with involution). These are the 2-categories for which we recover Frobenius reciprocity. We define injective abelianisations following the approach of [MMM+21], a construction which lets us locally move between structure-rich abelian categories and their well-behaved injective objects, even when our original 2-category is far from locally abelian. We define coalgebra 1-morphisms and internal cohoms, originally introduced in [MMMT19] and used as the basis for our definition of induction. We also lay the foundation for the main result of Chapter 2.

In Chapter 2, we re-examine Theorem 4.26 from [MMM+21], where $\overline{\mathscr{C}}$ is the projective abelianisation as defined (in dual form) in Proposition 1.2.11:

Theorem 0.0.1 (Biequivalence between cyclic birepresentations and algebra 1-morphisms)**.** Given a (multi)fiat 2-category $\mathscr{C}$, there is a biequivalence between the bicategory of biprojective bimodule 1-morphisms over $\overline{\mathscr{C}}$, and the bicategory of cyclic representations of $\mathscr{C}$.

This theorem has an obvious dual, which is the version more useful to us:

Theorem 0.0.2 (Biequivalence between cyclic birepresentations and coalgebra 1-morphisms)**.** Given a (multi)fiat 2-category $\mathscr{C}$, there is a biequivalence between the bicategory of biinjective bicomodule 1-morphisms over $\underline{\mathscr{C}}$, and the bicategory of cyclic representations of $\mathscr{C}$.

Moreover, as a novel result, we construct an explicit form for this equivalence, which allows us to smoothly move between these settings.

Chapter 3 defines induction along a pseudofunctor $\mathscr{F}$, and shows that induction is itself pseudofunctorial. Induction was first considered in [MMM+21] Lemma 3.11, though not by that name.

Induction and restriction are most naturally constructed in different settings. Induction has no direct definition for representations; indeed, without the equivalence between cyclic representations and representations over a coalgebra 1-morphism, it is unclear how one could define it. However, it can be straightforwardly defined for coalgebra and bicomodule 1-morphisms. Meanwhile, restriction has a simple presentation in the context of birepresentations but a more opaque definition for coalgebra 1-morphisms. We choose to work in the bicomodule setting, which requires us to use Theorem 0.0.2 to move between settings.

Chapter 4 defines restriction along $\mathscr{F}$ and shows that it is also a pseudofunctor. When working in the bicomodule setting, some subtle technical conditions arise. In particular, we move from $\mathscr{C}$ to a construction called $\mathscr{C}^{\oplus}$, defined in Proposition 4.1.3, and take care to show this does not reduce the generality of our result. We also introduce the simplifying assumption of essential 1-surjectivity for $\mathscr{F}$ to prove a technical lemma. This assumption makes the constructions of Chapter 5 more tractable.

Chapters 5-7 are concerned with constructing the adjunction between restriction and induction. The bulk of these chapters are centred on the many coherence requirements for a 2-adjunction.

Together, these prove our main result:

Theorem 0.0.3 (Frobenius reciprocity for fiat 2-categories)**.** If $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an essentially 1-surjective $\mathbb{k}$-linear pseudofunctor of (multi)fiat 2-categories, then induction and restriction along $\mathscr{F}$ are biadjoint as pseudofunctors.

1

Preliminaries

In this section, we discuss the fundamentals used through the rest of the thesis, and fix some of our notation. We start with a rundown of bicategories, following [Lei98], and provide a proof for a common folk theorem. We next look at (multi)fiat categories, first introduced in [MM11]. Then we discuss coalgebra and (bi)comodule 1-morphisms, which were first used to study birepresentations in [MMMT19]; we loosely follow the expositional structure of [MMM+21, Section 3]. After this, we discuss birepresentations, followed by the internal cohom construction, which was first introduced in [MMMT19] as a dual to a construction from [EGNO15]; in [MMMT19], this construction was called the internal hom, but we follow the convention of later papers in calling it the internal cohom.

Throughout, we let $\Bbbk$ be a field.

1.1 Basic Bicategories

Definition 1.1.1 (Bicategory)**.** A bicategory $\mathscr{C}$ consists of the following data:

- A collection of objects $ob(\mathscr{C})$, written $i, j, k, \ldots$;
- For each pair of objects i, j, a category $\mathscr{C}(i, j)$, whose objects are *1-morphisms* $F, G, H, ..$ and morphisms are *2-morphisms* $\alpha, \beta, \gamma, \ldots$;
- Functors

$$\begin{aligned} c_{ijk} : \mathscr{C}(j,k) \times \mathscr{C}(i,j) &\to \mathscr{C}(i,k) \\ (G, F) &\mapsto G \circ F = GF \\ (\alpha, \beta) &\mapsto \alpha * \beta \end{aligned}$$

and

$$I_i : 1 \to \mathscr{C}(i,i),$$

where 1 is here the one-object, one-morphism category;

- Natural isomorphisms

$$\begin{tikzcd}
\mathscr{C}(k,m)\times\mathscr{C}(j,k)\times\mathscr{C}(i,j) \arrow[r, "1\times c_{ijk}"] \arrow[d, "c_{jkm}\times 1"'] \arrow[dr, phantom, "\Rightarrow a_{ijkm}"] & \mathscr{C}(k,m)\times\mathscr{C}(i,k) \arrow[d, "c_{ikm}"] \\
\mathscr{C}(j,m)\times\mathscr{C}(i,j) \arrow[r, "c_{ijm}"'] & \mathscr{C}(i,m)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{C}(i,j)\times 1 \arrow[d, "1\times I_i"] \arrow[dr, equal] \arrow[dr, phantom, "\Rightarrow r_{ij}"'] & \\
\mathscr{C}(i,j)\times\mathscr{C}(i,i) \arrow[r, "c_{iij}"] & \mathscr{C}(i,j)
\end{tikzcd}$$

$$\begin{tikzcd}
1\times\mathscr{C}(i,j) \arrow[d, "I_i\times 1"] \arrow[dr, equal] \arrow[dr, phantom, "\Rightarrow l_{ij}"'] & \\
\mathscr{C}(j,j)\times\mathscr{C}(i,j) \arrow[r, "c_{ijj}"] & \mathscr{C}(i,j)
\end{tikzcd}$$

and thus 2-morphisms

$$a_{HGF} : (HG)F \to H(GF)$$
$$r_F : F \circ I_i \to F$$
$$l_F : I_j \circ F \to F.$$

These are such that the following diagrams commute:

$$\begin{tikzcd}
& ((KH)G)F \arrow[rr, "a*1"] \arrow[dl, "a"'] & & (K(HG))F \arrow[dr, "a"] & \\
(KH)(GF) \arrow[drr, "a"'] & & & & K((HG)F) \\
& & K(H(GF)) \arrow[urr, "1*a"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
(GI)F \arrow[rr, "a"] \arrow[dr, "r*1"'] & & G(IF) \arrow[dl, "1*l"] \\
& GF &
\end{tikzcd}$$

We say that $\mathscr{C}$ is a *2-category* if the *associators* a and *unitors* l, r are identities.

In every example we look at, the associator a is trivial, so the pentagram diagram automatically commutes. In a 2-category, both diagrams automatically commute.

Any bicategory satisfies the *interchange law*: $(F_1 * F_2) \circ (G_1 * G_2) = (F_1 \circ G_1) * (F_2 \circ G_2)$.

$\triangleleft$

We work with 2-categories $\mathscr{C}, \mathscr{D}$ etc. unless otherwise specified; it is possible that results still hold for the bicategory case, if appropriate associators and unitors are added in, but we do not check this.

On the other hand, we work with colax functors (at least initially). We give the definition below (in the case where associators are trivial) for reference:

Definition 1.1.2 (Colax functor)**.** Given bicategories $\mathscr{C}$ and $\mathscr{D}$, a colax functor from $\mathscr{C}$ to $\mathscr{D}$ consists of the following data:

- A map $\mathcal{F} : ob(\mathscr{C}) \to ob(\mathscr{D})$
- Functors $\mathcal{F} : \mathscr{C}(i,j) \to \mathscr{D}(\mathcal{F}(i), \mathcal{F}(j))$
- Natural transforms

$$\begin{tikzcd}
\mathscr{C}(j,k) \times \mathscr{C}(i,j) \arrow[r, "c_{ijk}"] \arrow[d, "\mathcal{F}\times\mathcal{F}"'] \arrow[dr, phantom, "\swarrow \mathcal{F}^2_{ijk}"] & \mathscr{C}(i,k) \arrow[d, "\mathcal{F}"] \\
\mathscr{D}(\mathcal{F}(j), \mathcal{F}(k)) \times \mathscr{D}(\mathcal{F}(i), \mathcal{F}(j)) \arrow[r, "c_{\mathcal{F}(i)\mathcal{F}(j)\mathcal{F}(k)}"'] & \mathscr{D}(\mathcal{F}(i), \mathcal{F}(k))
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[r, "I_i"'] \arrow[d, equal] \arrow[dr, phantom, "\mathcal{F}^0_i \Downarrow"] & \mathscr{C}(i,i) \arrow[d, "\mathcal{F}_{i,i}"'] \\
1 \arrow[r, "I_{\mathcal{F}(i)}"] & \mathscr{D}(\mathcal{F}(i), \mathcal{F}(i))
\end{tikzcd}$$

(and hence 2-morphisms $\mathcal{F}^2_{G,F} : \mathcal{F}(G \circ F) \to \mathcal{F}(G) \circ \mathcal{F}(F)$ and $\mathcal{F}^0_i : \mathcal{F}(1_i) \to 1_{\mathcal{F}(i)}$)

such that the following commute:

$$\begin{array}{ccc}
\mathcal{F}(HGF) & \xrightarrow{\mathcal{F}^2} & \mathcal{F}(H)\mathcal{F}(GF) \\
\downarrow \mathcal{F}^2 & & \downarrow 1*\mathcal{F}^2 \\
\mathcal{F}(HG)\mathcal{F}(F) & \xrightarrow{\mathcal{F}^2*1} & \mathcal{F}(H)\mathcal{F}(G)\mathcal{F}(F)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{F}(F \circ I_i) & \xrightarrow{\mathcal{F}^2} & \mathcal{F}(F)\mathcal{F}(I_i) & \xrightarrow{1*\mathcal{F}^0} & \mathcal{F}(F)I_{\mathcal{F}(i)} \\
\downarrow \mathcal{F}(r) & & & & \downarrow r \\
\mathcal{F}(F) & & = & & \mathcal{F}(F)
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{F}(I_j \circ F) & \xrightarrow{\mathcal{F}^2} & \mathcal{F}(I_j)\mathcal{F}(F) & \xrightarrow{\mathcal{F}^0*1} & I_{\mathcal{F}(j)}\mathcal{F}(F) \\
\downarrow \mathcal{F}(l) & & & & \downarrow l \\
\mathcal{F}(F) & & = & & \mathcal{F}(F)
\end{array}$$

We call the first of these three diagrams the coassociativity diagram, and the last two the counitality diagrams. Together, we call these the *higher coherences for colax 2-functors*.

If the $\mathcal{F}^2_{ijk}$ and $\mathcal{F}^0_i$ are natural isomorphisms, then we say $\mathcal{F}$ is a *pseudofunctor*; if they are identity natural transformations, we say $\mathcal{F}$ is a *strict 2-functor*, or just a *2-functor*. We write $\mathcal{F}^{-2}_{ijk}$ for $(\mathcal{F}^2_{ijk})^{-1}$, and $\mathcal{F}^{-0}_i$ for $(\mathcal{F}^0_i)^{-1}$.

$\triangleleft$

Definition 1.1.3 (Colax 2-natural transform)**.** A colax 2-natural transform

$$\Gamma : \mathcal{F} \to \mathcal{G} : \mathscr{C} \to \mathscr{D},$$

where $\mathcal{F}, \mathcal{G}$ are colax functors, consists of the following data:

- For each object $i \in \mathscr{C}$, a 1-morphism $\Gamma_i \in \mathscr{D}(\mathcal{F}(i), \mathcal{G}(i))$;
- For each pair of objects $i, j \in \mathscr{C}$, a natural transform

$$\begin{tikzcd}
\mathscr{C}(i,j) \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{G}"'] \arrow[dr, phantom, "\Swarrow \Gamma_{ij}"] & \mathscr{D}(\mathcal{F}(i), \mathcal{F}(j)) \arrow[d, "(\Gamma_j)_*"] \\
\mathscr{D}(\mathcal{G}(i), \mathcal{G}(j)) \arrow[r, "(\Gamma_i)^*"'] & \mathscr{D}(\mathcal{F}(i), \mathcal{G}(j))
\end{tikzcd}$$

and therefore 2-cells $\Gamma_F : \Gamma_j \circ \mathcal{F}(F) \to \mathcal{G}(F) \circ \Gamma_i$.

These are such that the following diagrams commute:

$$\begin{tikzcd}
\Gamma_k \circ \mathcal{F}(GF) \arrow[rr, "\Gamma_{GF}"'] \arrow[d, "1*\mathcal{F}^2"] & & \mathcal{G}(GF) \circ \Gamma_i \arrow[d, "\mathcal{G}^2*1"'] \\
\Gamma_k \circ \mathcal{F}(G) \circ \mathcal{F}(F) \arrow[r, "\Gamma_G*1"] & \mathcal{G}(G) \circ \Gamma_j \circ \mathcal{F}(F) \arrow[r, "1*\Gamma_F"] & \mathcal{G}(G) \circ \mathcal{G}(F) \circ \Gamma_i
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_i \circ \mathcal{F}(1_i) \arrow[rr, "\Gamma_{1_i}" description] \arrow[d, "1*\mathcal{F}^0" description] & & \mathcal{G}(1_i) \circ \Gamma_i \arrow[d, "\mathcal{G}^0*1" description] \\
\Gamma_i \circ 1_{\mathcal{F}(i)} \arrow[dr, "r" description] & & 1_{\mathcal{G}(i)} \circ \Gamma_i \arrow[dl, "l" description] \\
& \Gamma_i &
\end{tikzcd}$$

We call these the *higher coherences for colax 2-natural transforms*

If the Γ_{ij} are natural isomorphisms, then this is a *strong 2-natural transform*; when they are identities it is a *strict 2-natural transform*.

$\lhd$

Definition 1.1.4 (Adjoint equivalence in a bicategory)**.** Given objects i, j of a bicategory $\mathscr{C}$, we say i and j are equivalent if there are 1-morphisms $F : i \to j$ and $G : j \to i$ such that $GF \cong 1_i$ and $FG \cong 1_j$. We call F an equivalence (in $\mathscr{C}$), and similarly for G. In particular, we say small categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if they are equivalent as objects in the bicategory of small categories.

We say i and j are adjoint equivalent if there are 2-morphisms $\gamma : 1_i \to GF$ and $\zeta : FG \to 1_j$ such that $(\zeta * 1_F) \circ (1_F * \gamma) : F \to FGF \to F$ and $(1_G * \zeta) \circ (\gamma * 1_G) : G \to GFG \to G$ are the identities on F and G respectively. $\lhd$

Definition 1.1.5 (Modification)**.** A modification

$$\gamma : \Gamma \to \Lambda : \mathcal{F} \to \mathcal{G} : \mathscr{C} \to \mathscr{D}$$

between colax 2-natural transforms Γ, Λ consists of 2-morphisms $\gamma_i : \Gamma_i \to \Lambda_i$ such that, for any 1-morphism $F : i \to j$, the following diagram commutes:

$$\begin{tikzcd}
\Gamma_j \circ \mathcal{F}(F) \arrow[r, "\gamma_j * 1"] \arrow[d, "\Gamma_F"] & \Lambda_j \circ \mathcal{F}(F) \arrow[d, "\Lambda_F"] \\
\mathcal{G}(F) \circ \Gamma_i \arrow[r, "1 * \gamma_i"] & \mathcal{G}(F) \circ \Lambda_i
\end{tikzcd}$$

$\triangleleft$

Finally, we define the central object of the paper:

Definition 1.1.6 (Biadjunction)**.** Given 2-categories $\mathscr{C}$ and $\mathscr{D}$, $\mathcal{F} : \mathscr{C} \to \mathscr{D}$ is said to be left biadjoint to $\mathcal{G} : \mathscr{D} \to \mathscr{C}$ ($\mathcal{G}$ right biadjoint to $\mathcal{F}$; $\mathcal{F}$ and $\mathcal{G}$ biadjoint), written $\mathcal{F} \dashv \mathcal{G}$, if there are strong 2-natural transforms $\eta : 1_{\mathscr{C}} \to \mathcal{G}\mathcal{F}$ and $\epsilon : \mathcal{F}\mathcal{G} \to 1_{\mathscr{D}}$, respectively called the *unit* and *counit*, along with modifications

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathcal{F}(\eta)"] \arrow[dr, equal] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, "\epsilon_{\mathcal{F}}"] \\
& \mathcal{F} \arrow[u, phantom, shift right=6, "\stackrel{\sigma}{\Rightarrow}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, "\eta_{\mathcal{G}}"] \arrow[dr, equal] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, "\mathcal{G}(\epsilon)"] \\
& \mathcal{G} \arrow[u, phantom, shift right=6, "\stackrel{\tau}{\Leftarrow}"]
\end{tikzcd}$$

called *triangulators*, satisfying the *swallowtail diagrams*, given below:

$$\begin{tikzcd}
1_{\mathscr{C}} \arrow[r, "\eta"] \arrow[d, "\eta"] & \mathcal{G}\mathcal{F} \arrow[d, "\mathcal{G}\mathcal{F}(\eta)"] \arrow[dl, Rightarrow, "\eta_\eta"'] \arrow[ddr, equal, bend left] & \\
\mathcal{G}\mathcal{F} \arrow[r, "\eta_{\mathcal{G}\mathcal{F}}"] \arrow[drr, equal, bend right] & \mathcal{G}\mathcal{F}\mathcal{G}\mathcal{F} \arrow[dr, "\mathcal{G}(\epsilon_{\mathcal{F}})"] \arrow[d, phantom, "\Downarrow \tau_{\mathcal{F}}"] \arrow[r, phantom, "\Leftarrow \mathcal{G}(\sigma)"] & {} \\
& {} & \mathcal{G}\mathcal{F}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
1_{\mathscr{C}} \arrow[dr, "\eta"] & \\
& \mathcal{G}\mathcal{F}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{FG} \arrow[dr, "\mathcal{F}(\eta_{\mathcal{G}})"] \arrow[drr, bend left, equal] \arrow[ddr, bend right=40, equal] & \arrow[d, phantom, "\Uparrow \mathcal{F}(\tau)"] & \\
\arrow[r, phantom, "\sigma_{\mathcal{G}} \Rightarrow"] & \mathcal{FGFG} \arrow[r, "\mathcal{FG}(\epsilon)"] \arrow[d, "\epsilon_{\mathcal{FG}}"] & \mathcal{FG} \arrow[d, "\epsilon"] \\
& \mathcal{FG} \arrow[r, "\epsilon"] \arrow[ur, Rightarrow, "\epsilon_{\epsilon}"] & 1_{\mathscr{D}}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{FG} \arrow[dr, "\epsilon"] & \\
& 1_{\mathscr{D}}
\end{tikzcd}$$

Note that, since $\eta_i : i \to \mathcal{GF}(i)$ is a 1-morphism for an object i, $\eta_{\eta_i} : \mathcal{GF}(\eta_i) \circ \eta_i \to \eta_{\mathcal{GF}(i)} \circ \eta_i$ is a 2-morphism, and similarly for $\varepsilon_{\varepsilon_i}$. So these diagrams make sense.

$\lhd$

We record the following useful propositions:

Proposition 1.1.7. Suppose $\Gamma : \mathcal{F} \to \mathcal{G} : \mathscr{C} \to \mathscr{D}$ is a strong 2-natural transform of pseudofunctors, and suppose further that for each i an object of $\mathscr{C}$, Γ_i is an equivalence in $\mathscr{D}$. Then Γ is an adjoint equivalence of functors, i.e. an adjoint equivalence in the 2-category $[\mathscr{C}, \mathscr{D}]$ (pseudofunctors $\mathscr{F} : \mathscr{C} \to \mathscr{D}$, strong transforms and modifications).

Proof : Since each Γ_i is an equivalence, recall that equivalent 1-categories are adjoint equivalent. So WLOG, there is a 1-cell Γ_i^{-1} in $\mathscr{D}$, along with a pair of invertible 2-cells $\gamma_i : 1_{\mathcal{F}(i)} \to \Gamma_i^{-1}\Gamma_i$ and $\zeta_i : \Gamma_i \Gamma_i^{-1} \to 1_{\mathcal{G}(i)}$ satisfying

$$\begin{tikzcd}
\mathcal{F}(i) \arrow[r, "\Gamma_i"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\gamma_i \Rightarrow"] & \mathcal{G}(i) \arrow[d, "\Gamma_i^{-1}"] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\zeta_i \Rightarrow"] & \\
& \mathcal{F}(i) \arrow[r, "\Gamma_i"] & \mathcal{G}(i)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{F}(i) \arrow[dr, "\Gamma_i"] & \\
& \mathcal{G}(i)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{G}(i) \arrow[r, "\Gamma_i^{-1}"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\zeta_i \Rightarrow"] & \mathcal{F}(i) \arrow[d, "\Gamma_i"] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\gamma_i \Rightarrow"] & \\
& \mathcal{G}(i) \arrow[r, "\Gamma_i^{-1}"] & \mathcal{F}(i)
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{G}(i) \arrow[dr, "\Gamma_i^{-1}"] & \\
& \mathcal{F}(i)
\end{tikzcd}$$

Moreover, since we have 2-cells $\Gamma_F : \Gamma_j \circ \mathcal{F}(F) \to \mathcal{G}(F) \circ \Gamma_i$ for each $F : i \to j$, we can define $(\Gamma^{-1})_F : \Gamma_j^{-1} \circ \mathcal{G}(F) \to \mathcal{F}(F) \circ \Gamma_i^{-1}$ as the following composition:

$$\begin{tikzcd}
\mathcal{G}(i) \arrow[rr, equal] \arrow[dd, "\Gamma_i^{-1}"'] \arrow[dr, phantom, "\Downarrow (\zeta_i)^{-1}"] & & \mathcal{G}(i) \arrow[rr, "\mathcal{G}(F)"] & & \mathcal{G}(j) \arrow[dd, "\Gamma_j^{-1}"] \\
& {} \arrow[rr, phantom, "\Rightarrow (\Gamma_F)^{-1}"] & & {} \arrow[dr, phantom, "\Downarrow (\gamma_i)^{-1}"] & \\
\mathcal{F}(i) \arrow[uurr, "\Gamma_i"] \arrow[rr, "\mathcal{F}(F)"'] & & \mathcal{F}(j) \arrow[uurr, "\Gamma_j"] \arrow[rr, equal] & & \mathcal{F}(j)
\end{tikzcd}$$

We need to show that the Γ_i^{-1} together with the $(\Gamma^{-1})_F$ form a 2-natural transform $\Gamma : \mathcal{G} \to \mathcal{F}$; and that there are invertible modifications $\gamma : 1_{\mathcal{F}} \to \Gamma^{-1}\Gamma$ and $\zeta : \Gamma\Gamma^{-1} \to 1_{\mathcal{G}}$ such that

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\Gamma"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Uparrow\gamma"] & \mathcal{G} \arrow[d, "\Gamma^{-1}"] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\Uparrow\zeta"] & \\
& \mathcal{F} \arrow[r, "\Gamma"'] & \mathcal{G}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{F} \arrow[dr, "\Gamma"] & \\
& \mathcal{G}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\Gamma^{-1}"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Uparrow\zeta"] & \mathcal{F} \arrow[d, "\Gamma"] \arrow[dr, equal] \arrow[dr, phantom, shift right=3, "\Uparrow\gamma"] & \\
& \mathcal{G} \arrow[r, "\Gamma^{-1}"'] & \mathcal{F}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{G} \arrow[dr, "\Gamma^{-1}"] & \\
& \mathcal{F}
\end{tikzcd}$$

We start with the first claim.

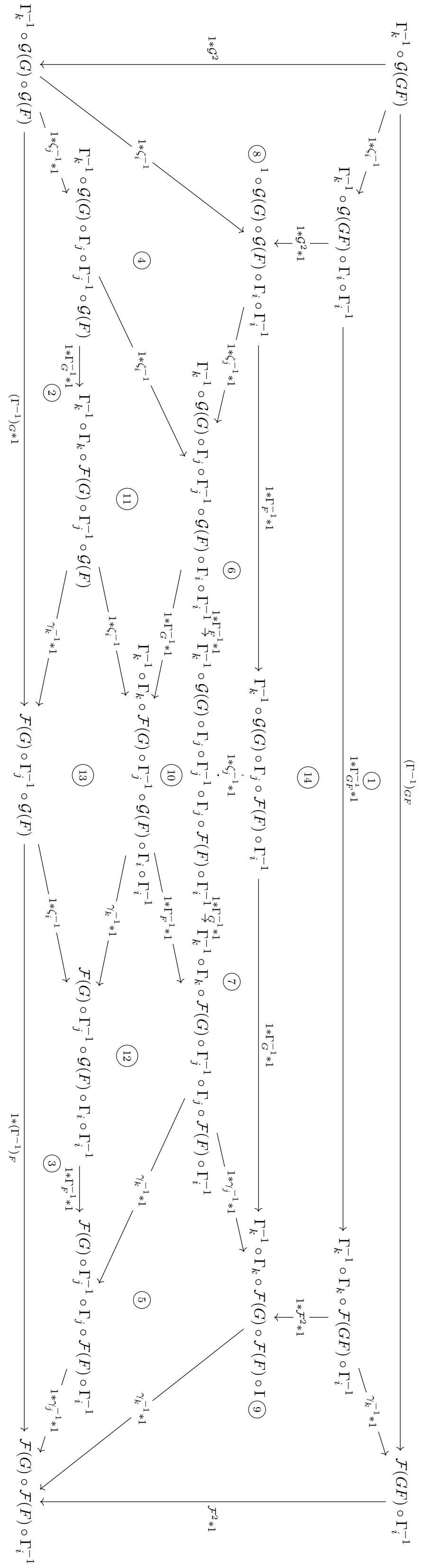
To show that Γ^{-1} is a 2-natural transformation, we first show that $(\Gamma^{-1})_F$ is natural in F. But $(\Gamma_F)^{-1}$ is natural in F since Γ_F is, and neither $(\zeta_i)^{-1}$ nor $(\gamma_i)^{-1}$ depend on F. So this is immediate.

Next, we show that the following diagrams commute:

$$\begin{tikzcd}
\Gamma_k^{-1} \circ \mathcal{G}(GF) \arrow[rr, "(\Gamma^{-1})_{GF}"] \arrow[d, "1 * \mathcal{G}^2"'] \arrow[drr, phantom, "\boxed{1}"] & & \mathcal{F}(GF) \circ \Gamma_i^{-1} \arrow[d, "\mathcal{F}^2 * 1"] \\
\Gamma_k^{-1} \circ \mathcal{G}(G) \circ \mathcal{G}(F) \arrow[r, "(\Gamma^{-1})_G * 1"'] & \mathcal{F}(G) \circ \Gamma_j^{-1} \circ \mathcal{G}(F) \arrow[r, "1 * (\Gamma^{-1})_F"'] & \mathcal{F}(G) \circ \mathcal{F}(F) \circ \Gamma_i^{-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_i^{-1} \circ \mathcal{G}(1_i) \arrow[rr, "(\Gamma^{-1})_{1_i}"] \arrow[d, "1 * \mathcal{G}^0"'] \arrow[drr, phantom, "\boxed{2}"] & & \mathcal{F}(1_i) \circ \Gamma_i^{-1} \arrow[d, "\mathcal{F}^0 * 1"] \\
\Gamma_i^{-1} \circ 1_{\mathcal{G}(i)} \arrow[dr, "r"] & & 1_{\mathcal{F}(i)} \circ \Gamma_i^{-1} \arrow[dl, "l"'] \\
& \Gamma_i^{-1} &
\end{tikzcd}$$

To see that $\boxed{1}$ commutes, consider the following diagram:



1-3 commute by definition of $(\Gamma^{-1})_H$;

4-13 commute by the interchange law;

and 14 commutes by coherence for Γ.

Therefore the outer rectangle commutes, that is, $\boxed{1}$ commutes.

To see that $\boxed{2}$ commutes, consider the following diagram:

$$\begin{tikzcd}
\Gamma_i^{-1}\circ\mathcal{G}(1_i) \arrow[rrrrrr, "(\Gamma^{-1})_{1_i}"] \arrow[drr, "1*\varsigma_i^{-1}"] \arrow[dd, "1*\mathcal{G}^0"'] & & & & & & \mathcal{F}(1_i)\circ\Gamma_i^{-1} \arrow[dd, "\mathcal{F}^0*1"] \\
& & \Gamma_i^{-1}\circ\mathcal{G}(1_i)\circ\Gamma_i\circ\Gamma_i^{-1} \arrow[rr, "1*\Gamma^{-1}_{1_i}*1"] \arrow[d, "1*\mathcal{G}^0*1"] \arrow[dll, phantom, "2"] & \arrow[u, phantom, "1"] & \Gamma_i^{-1}\circ\Gamma_i\circ\mathcal{F}(1_i)\circ\Gamma_i^{-1} \arrow[urr, "\gamma_i^{-1}*1"] \arrow[d, "1*\mathcal{F}^0*1"] \arrow[drr, phantom, "3"] & & \\
\Gamma_i^{-1}\circ 1_{\mathcal{G}(i)} \arrow[rr, "1*\varsigma_i^{-1}"] \arrow[ddr, "r"'] & & \Gamma_i^{-1}\circ 1_{\mathcal{G}(i)}\circ\Gamma_i\circ\Gamma_i^{-1} \arrow[dr, "1*l*1=r*1"] \arrow[ddl, phantom, "4"] \arrow[rr, phantom, "6"] & & \Gamma_i^{-1}\circ\Gamma_i\circ 1_{\mathcal{F}(i)}\circ\Gamma_i^{-1} \arrow[rr, "\gamma_i^{-1}*1"] \arrow[dl, "1*r*1=1*l"'] \arrow[ddr, phantom, "5"] & & 1_{\mathcal{F}(i)}\circ\Gamma_i^{-1} \arrow[ddl, "l"] \\
& & & \Gamma_i^{-1}\circ\Gamma_i\circ\Gamma_i^{-1} \arrow[drr, "\gamma_i^{-1}*1"] \arrow[d, phantom, "7"] & & & \\
& \Gamma_i^{-1} \arrow[urr, "1*\varsigma_i^{-1}"] \arrow[rrrr, equal] & & \; & & \Gamma_i^{-1} &
\end{tikzcd}$$

1 commutes by definition of $(\Gamma^{-1})_H$;

2-5 commute by the interchange law;

6 commutes by higher coherence for Γ;

7 commutes by our choice of γ and ζ.

Therefore, the outer diagram commutes, that is, $\boxed{2}$ commutes. So Γ^{-1} is a 2-natural transform, and is clearly strong by construction of $(\Gamma^{-1})_F$.

Next, we show that the γ_i and ζ_i assemble to modifications $\gamma : 1_{\mathcal{F}} \to \Gamma^{-1}\Gamma$ and $\zeta : \Gamma\Gamma^{-1} \to 1_{\mathcal{G}}$, that is, that the following diagrams commute:

$$\begin{tikzcd}
1_{\mathcal{F}(j)} \circ \mathcal{F}(F) \arrow[rr, "\gamma_j * 1"] \arrow[d, equal] \arrow[drr, phantom, "\boxed{3}"] & & (\Gamma^{-1}\Gamma)_j \circ \mathcal{F}(F) \arrow[d, "(\Gamma^{-1}\Gamma)_F"] \\
\mathcal{F}(F) \circ 1_{\mathcal{F}(i)} \arrow[rr, "1 * \gamma_i"] & & \mathcal{F}(F) \circ (\Gamma^{-1}\Gamma)_i
\end{tikzcd}$$

$$\begin{tikzcd}
(\Gamma\Gamma^{-1})_j \circ \mathcal{G}(F) \arrow[rr, "\zeta_j * 1"] \arrow[d, "(\Gamma\Gamma^{-1})_F"] \arrow[drr, phantom, "\boxed{4}"] & & 1_{\mathcal{G}(j)} \circ \mathcal{G}(F) \arrow[d, equal] \\
\mathcal{G}(F) \circ (\Gamma\Gamma^{-1})_i \arrow[rr, "1 * \zeta_i"] & & \mathcal{G}(F) \circ 1_{\mathcal{G}(i)}
\end{tikzcd}$$

To see that $\boxed{3}$ commutes, consider the following diagram:

$$
\begin{tikzcd}[column sep=small]
1_{\mathcal{F}(j)} \circ \mathcal{F}(F) \arrow[rrrr, "\gamma_j * 1"] \arrow[dddd, equal] \arrow[drr, phantom, "1"] & & & & (\Gamma^{-1}\Gamma)_j \circ \mathcal{F}(F) \arrow[dl, dashed, "1 * \Gamma_F"'] \arrow[dddd, "(\Gamma^{-1}\Gamma)_F"] \arrow[dddlll, equal, bend right] \arrow[dd, phantom, "6"'] \\
& & & \Gamma_j^{-1} \circ \mathcal{G}(F) \circ \Gamma_i \arrow[dl, equal] \arrow[d, dashed, "1 * \zeta_i^{-1} * 1"] \arrow[l, phantom, "2"] & \\
& & \Gamma_j^{-1} \circ \mathcal{G}(F) \circ \Gamma_i \arrow[r, "1 * \gamma_i"'] \arrow[dl, dashed, "1 * \Gamma_F^{-1}"'] \arrow[ur, phantom, "3"] \arrow[d, phantom, "4"] & \Gamma_j^{-1} \circ \mathcal{G}(F) \circ \Gamma_i \circ \Gamma_i^{-1} \circ \Gamma_i \arrow[d, dashed, "1 * \Gamma_F^{-1} * 1"] & \\
& \Gamma_j^{-1} \circ \Gamma_j \circ \mathcal{F}(F) \arrow[rr, "1 * \gamma_i"] \arrow[dl, dashed, "\gamma_j^{-1} * 1"'] \arrow[drr, phantom, "5"] & & \Gamma_j^{-1} \circ \Gamma_j \circ \mathcal{F}(F) \circ \Gamma_i^{-1} \circ \Gamma_i \arrow[dr, dashed, "\gamma_j^{-1} * 1"] & \\
\mathcal{F}(F) \circ 1_{\mathcal{F}(i)} \arrow[rrrr, "1 * \gamma_i"] & & & & \mathcal{F}(F) \circ (\Gamma^{-1}\Gamma)_i
\end{tikzcd}
$$

1-3 commute trivially;

4-5 commute by the interchange law;

and 6 commutes by definition of $(\Gamma^{-1}\Gamma)_F$.

So the outer diagram commutes, that is, $\boxed{3}$ commutes. So γ is a modification. A similar proof shows that $\boxed{4}$ commutes, so ζ is a modification. The remaining properties of γ and ζ - that they are invertible and satisfy the unit-counit diagrams for an adjoint equivalence - are immediate from our choice of γ_i and ζ_i.

So $(\Gamma, \Gamma^{-1}, \gamma, \zeta)$ are the data of an adjoint equivalence in $[\mathscr{C}, \mathscr{D}]$.

□

The 1-categorical analogue of this result is the classical result that, if a natural transformation is locally an isomorphism, then it is a natural isomorphism.

The second of our propositions is a technical lemma used only once in this thesis, to show that restriction - as we define it - is a pseudofunctor. Morally, it says that something "2-naturally isomorphic" to a pseudofunctor is also a pseudofunctor.

Proposition 1.1.8. Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a pseudofunctor. Suppose we have some data $\mathscr{G}$ with $\mathscr{G}(i)$ an object of $\mathscr{D}$ for each object i of $\mathscr{C}$, $\mathscr{G}(F : i \to j) = \mathscr{G}(F) : \mathscr{G}(i) \to \mathscr{G}(j)$ a 1-morphism of $\mathscr{D}$ for each 1-morphism F of $\mathscr{C}$, $\mathscr{G}(\alpha : F \to G) = \mathscr{G}(\alpha) : \mathscr{G}(F) \to \mathscr{G}(G)$ a 2-morphism of $\mathscr{D}$ for each 2-morphism α of $\mathscr{C}$, $\mathscr{G}^2_{F,G} : \mathscr{G}(GF) \to \mathscr{G}(G)\mathscr{G}(F)$ a 2-morphism in $\mathscr{D}$ for each composable pair F, G of 1-morphisms in $\mathscr{C}$, and $\mathscr{G}^0_i : \mathscr{G}(1_i) \to 1_{\mathscr{G}(i)}$ a 2-morphism in $\mathscr{D}$ for each object i of $\mathscr{C}$. Suppose further that for each object i of $\mathscr{C}$, there is an epimorphism $\Gamma_i : \mathscr{F}(i) \twoheadrightarrow \mathscr{G}(i)$ of $\mathscr{D}$; that for each 1-morphism F of $\mathscr{C}$, there is an epimorphism $\Gamma_F : \Gamma_j\mathscr{F}(F) \twoheadrightarrow \mathscr{G}(F)\Gamma_i$; and that the following diagrams all commute for any $F, F' : i \to j$, $G : j \to k$, $\alpha : F \to F'$ in $\mathscr{C}$:

$$\begin{tikzcd}
\Gamma_j\mathscr{F}(F) \arrow[r, "\Gamma_F"] \arrow[d, "1*\mathscr{F}(\alpha)"'] \arrow[dr, phantom, "\boxed{1}"] & \mathscr{G}(F)\Gamma_i \arrow[d, "\mathscr{G}(\alpha)"] \\
\Gamma_j\mathscr{F}(F') \arrow[r, "\Gamma_{F'}"'] & \mathscr{G}(F')\Gamma_i
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_k\mathscr{F}(GF) \arrow[rr, "\Gamma_{GF}"'] \arrow[d, "1*\mathscr{F}^2"] \arrow[drr, phantom, "\boxed{2}"] & & \mathscr{G}(GF)\Gamma_i \arrow[d, "\mathscr{G}^2*1"] \\
\Gamma_k\mathscr{F}(G)\mathscr{F}(F) \arrow[r, "\Gamma_G*1"] & \mathscr{G}(G)\Gamma_j\mathscr{F}(F) \arrow[r, "1*\Gamma_F"] & \mathscr{G}(G)\mathscr{G}(F)\Gamma_i
\end{tikzcd}$$

$$\begin{tikzcd}
\Gamma_i\mathscr{F}(1_i) \arrow[rr, "\Gamma_{1_i}"] \arrow[d, "1*\mathscr{F}^0"] \arrow[drr, phantom, "\boxed{3}"] & & \mathscr{G}(1_i)\Gamma_i \arrow[d, "\mathscr{G}^0*1"] \\
\Gamma_i 1_{\mathscr{F}(i)} \arrow[dr, "r"] & & 1_{\mathscr{G}(i)}\Gamma_i \arrow[dl, "l"'] \\
& \Gamma_i &
\end{tikzcd}$$

Then $\mathscr{G}$ is a pseudofunctor, and Γ is a 2-natural transformation of functors.

Proof :

We need to show three things: that $\mathscr{G}$ is locally functorial, that $\mathscr{G}^2$ is natural in F and G, and that $\mathscr{G}^2, \mathscr{G}^0$ are coherent.

To show that $\mathscr{G}$ is locally functorial, we need to show that $\mathscr{G}(\beta\alpha) = \mathscr{G}(\beta)\mathscr{G}(\alpha)$ for any compatible 2-morphisms α, β in $\mathscr{C}$, and that $\mathscr{G}(1_F) = 1_{\mathscr{G}(F)}$ for any 1-morphism F in $\mathscr{C}$.

First, consider the following diagram:

$$\begin{tikzcd}
\Gamma_j\mathscr{F}(F) \arrow[r, "\Gamma_F"] \arrow[d, "1*\mathscr{F}(\alpha)"] \arrow[dd, bend right=60, "1*\mathscr{F}(\beta\alpha)"'] \arrow[dr, phantom, "2"] & \mathscr{G}(F)\Gamma_i \arrow[d, "\mathscr{G}(\alpha)*1"] \arrow[dd, bend left=60, "\mathscr{G}(\beta\alpha)*1"] \\
\Gamma_j\mathscr{F}(G) \arrow[r, "\Gamma_G"] \arrow[d, "1*\mathscr{F}(\beta)"] \arrow[l, phantom, "1"] \arrow[dr, phantom, "3"] & \mathscr{G}(G)\Gamma_i \arrow[d, "\mathscr{G}(\beta)*1"] \arrow[r, phantom, "4"] & {} \\
\Gamma_j\mathscr{F}(H) \arrow[r, "\Gamma_H"] & \mathscr{G}(H)\Gamma_i
\end{tikzcd}$$

1 commutes because $\mathcal{F}$ is locally functorial;

and 2, 3 and the outer diagram commute by our assumption $\boxed{1}$ on Γ.

So we find that $((\mathscr{G}(\beta)\mathscr{G}(\alpha)) * 1_{\Gamma_i})\Gamma_F = (\mathscr{G}(\beta\alpha) * 1_{\Gamma_i})\Gamma_F$. Since Γ_i and Γ_F are epimorphisms, we get that $\mathscr{G}(\beta)\mathscr{G}(\alpha) = \mathscr{G}(\beta\alpha)$, as we wanted.

Next, consider the following diagram:

$$\begin{tikzcd}
\Gamma_j\mathscr{F}(F) \arrow[rr, "\Gamma_F"] \arrow[d, "\mathscr{F}(1_F)"] \arrow[d, bend right=60, "1_{\Gamma_j\mathscr{F}(F)}"'] \arrow[d, phantom, shift right=4ex, "1"] & \arrow[d, phantom, "2"] & \mathscr{G}(F)\Gamma_i \arrow[d, "\mathscr{G}(1_F)"'] \arrow[d, bend left=60, "1_{\mathscr{G}(F)\Gamma_i}"] \arrow[d, phantom, shift left=4ex, "3"] \\
\Gamma_j\mathscr{F}(F) \arrow[rr, "\Gamma_F"] & & \mathscr{G}(F)\Gamma_i
\end{tikzcd}$$

1 commutes because $\mathscr{F}$ is locally functorial;

2 commutes by our assumption $\boxed{1}$ on Γ;

and the outer diagram commutes trivially.

So $(\mathscr{G}(1_F)*1_{\Gamma_i})\Gamma_F = (1_{\mathscr{G}(i)}*1_{\Gamma_i})\Gamma_F$. Since Γ_F and Γ_i are epimorphisms, we get that $\mathscr{G}(1_F) = 1_{\mathscr{G}(i)}$, as we wanted.

So $\mathscr{G}$ is locally functorial.

To see that $\mathscr{G}^2$ is natural in F, G, let $\alpha : F \to F'$ and $\beta : G \to G'$ be 2-morphisms in $\mathscr{C}$, and consider the following diagram:

$$\begin{tikzcd}[column sep=small]
\Gamma_k\mathscr{F}(GF) \arrow[rrrr, "\Gamma_{GF}"] \arrow[dr, "1*\mathscr{F}(\beta*\alpha)"] \arrow[ddd, "1*\mathscr{F}^2"'] & & \arrow[d, phantom, "3"] & & \mathscr{G}(GF)\Gamma_i \arrow[dl, "\mathscr{G}(\beta*\alpha)*1"'] \arrow[ddd, "\mathscr{G}^2*1"] \\
\arrow[d, phantom, "1"] & \Gamma_k\mathscr{F}(G'F') \arrow[rr, "\Gamma_{G'F'}"] \arrow[d, "1*\mathscr{F}^2"] & \arrow[d, phantom, "2"] & \mathscr{G}(G'F')\Gamma_i \arrow[d, "\mathscr{G}^2*1"] & \arrow[d, phantom, "6"] \\
& \Gamma_k\mathscr{F}(G')\mathscr{F}(F') \arrow[r, "\Gamma_{G'}*1"] & \mathscr{G}(G')\Gamma_j\mathscr{F}(F') \arrow[r, "1*\Gamma_{F'}"] & \mathscr{G}(G')\mathscr{G}(F')\Gamma_i & \\
\Gamma_k\mathscr{F}(G)\mathscr{F}(F) \arrow[rr, "\Gamma_G*1"] \arrow[ur, "1*\mathscr{F}(\beta)*\mathscr{F}(\alpha)"] & \arrow[u, phantom, "4"] & \mathscr{G}(G)\Gamma_j\mathscr{F}(F) \arrow[rr, "1*\Gamma_F"] \arrow[u, "\mathscr{G}(\beta)*1*\mathscr{F}(\alpha)"'] & \arrow[u, phantom, "5"] & \mathscr{G}(G)\mathscr{G}(F)\Gamma_i \arrow[ul, "\mathscr{G}(\beta)*\mathscr{G}(\alpha)*1"']
\end{tikzcd}$$

1 commutes because $\mathscr{F}^2$ is a natural transformation;

2 and the outer diagram commute by our assumption $\boxed{2}$ on Γ;

3-5 commute by our assumption $\boxed{1}$ on Γ;

and Γ_{GF} is an epimorphism.

So 6 commutes. But since Γ_i is an epimorphism, this tells us that that $\mathscr{G}^2$ is natural in F and G.

To see that $\mathscr{G}$ satisfies the coassociativity diagram, consider the following diagram:

$$
\begin{tikzcd}
& \Gamma_l\mathscr{F}(H)\mathscr{F}(G)\mathscr{F}(F) \arrow[rr, "\Gamma_H*1"] \arrow[drr, phantom, "8"] \arrow[dl, phantom, "6"] & & \mathscr{G}(H)\Gamma_k\mathscr{F}(G)\mathscr{F}(F) \arrow[rrr, "1*\Gamma_G*1"] \arrow[drrr, phantom, "5"] & & & \mathscr{G}(H)\mathscr{G}(G)\Gamma_j\mathscr{F}(F) \arrow[dd, "1*\Gamma_F"] \\
& \Gamma_l\mathscr{F}(H)\mathscr{F}(GF) \arrow[u, "1*\mathscr{F}^2"'] \arrow[rr, "\Gamma_H*1"] \arrow[dl, "1*\mathscr{F}^2"'] & & \mathscr{G}(H)\Gamma_k\mathscr{F}(GF) \arrow[u, "1*\mathscr{F}^2"'] \arrow[rr, "1*\Gamma_{GF}"] \arrow[d, phantom, "4"] & & \mathscr{G}(H)\mathscr{G}(GF)\Gamma_i \arrow[dr, "1*\mathscr{G}^2*1"] & \\
\Gamma_l\mathscr{F}(H)\mathscr{F}(G)\mathscr{F}(F) \arrow[uur, equal, bend left] \arrow[dd, "\Gamma_H*1"] \arrow[rr, phantom, "1"] & & \Gamma_l\mathscr{F}(HGF) \arrow[ul, "1*\mathscr{F}^2"'] \arrow[dl, "1*\mathscr{F}^2"] \arrow[rr, "\Gamma_{HGF}"] & & \mathscr{G}(HGF)\Gamma_i \arrow[ur, "\mathscr{G}^2*1"] \arrow[dr, "\mathscr{G}^2*1"'] \arrow[rr, phantom, "10"] & & \mathscr{G}(H)\mathscr{G}(G)\mathscr{G}(F)\Gamma_i \arrow[ddl, equal, bend left] \\
& \Gamma_l\mathscr{F}(HG)\mathscr{F}(F) \arrow[ul, "1*\mathscr{F}^2*1"] \arrow[rr, "\Gamma_{HG}*1"] \arrow[d, phantom, "2"] & & \mathscr{G}(HG)\Gamma_j\mathscr{F}(F) \arrow[u, phantom, "3"] \arrow[d, "\mathscr{G}^2*1"'] \arrow[rr, "1*\Gamma_F"] \arrow[drr, phantom, "9"] & & \mathscr{G}(HG)\mathscr{G}(F)\Gamma_i \arrow[ur, "\mathscr{G}^2*1"'] \arrow[d, "\mathscr{G}^2*1"'] \arrow[dr, phantom, "7"] & \\
\mathscr{G}(H)\Gamma_k\mathscr{F}(G)\mathscr{F}(F) \arrow[rrr, "1*\Gamma_G*1"] & & & \mathscr{G}(H)\mathscr{G}(G)\Gamma_j\mathscr{F}(F) \arrow[rr, "1*\Gamma_F"] & & \mathscr{G}(H)\mathscr{G}(G)\mathscr{G}(F)\Gamma_i &
\end{tikzcd}
$$

1 commutes because $\mathscr{F}$ satisfies the coassociativity diagram;

2-5 commute by our assumption $\boxed{2}$ on Γ;

6-7 and the outer diagram commute trivially;

and 8-9 commute by our assumption $\boxed{1}$ on Γ.

So 10 commutes when Γ_{HGF} is precomposed. But Γ_{HGF} and Γ_i are epimorphisms. So $\mathscr{G}$ satisfies the coassociativity diagram.

Finally, we check that $\mathscr{G}$ satisfies the counitality diagrams. We check only one, the other being completely analogous.

To see that it does, consider the following diagram:

$$\begin{tikzcd}
\Gamma_j\mathscr{F}(F1_i) \arrow[rrrr, "1*\mathscr{F}(r)"] \arrow[drr, "\Gamma_{F1_i}"] \arrow[dd, "1*\mathscr{F}^2"] \arrow[ddrr, phantom, "1"] & & & & \Gamma_j\mathscr{F}(F) \arrow[dl, "\Gamma_F"'] \arrow[ddddd, equal] \\
& & \mathscr{G}(F1_i)\Gamma_i \arrow[r, "\mathscr{G}(r)*1"] \arrow[d, "\mathscr{G}^2*1"] \arrow[urr, phantom, "2"] & \mathscr{G}(F)\Gamma_i \arrow[dd, equal] \arrow[ddr, phantom, "3"] & \\
\Gamma_j\mathscr{F}(F)\mathscr{F}(1_i) \arrow[r, "\Gamma_F*1"'] \arrow[ddd, "1*\mathscr{F}^0"] \arrow[ddr, phantom, "6"] & \mathscr{G}(F)\Gamma_i\mathscr{F}(1_i) \arrow[r, "1*\Gamma_{1_i}"'] \arrow[dd, "1*\mathscr{F}^0"] \arrow[dr, phantom, "5"] & \mathscr{G}(F)\mathscr{G}(1_i)\Gamma_i \arrow[d, "1*\mathscr{G}^0*1"] \arrow[dr, phantom, "8"] & & \\
& & \mathscr{G}(F)1_{\mathscr{G}(i)}\Gamma_i \arrow[r, bend left, "r*1"] \arrow[r, bend right, "1*l"'] \arrow[r, phantom, "4"] & \mathscr{G}(F)\Gamma_i & \\
& \mathscr{G}(F)\Gamma_i 1_{\mathscr{F}(i)} \arrow[urr, "1*r"] \arrow[drr, phantom, "7"] & & & \\
\Gamma_j\mathscr{F}(F)1_{\mathscr{F}(i)} \arrow[ur, "\Gamma_F*1"] \arrow[rrrr, "1*r"'] & & & & \Gamma_j\mathscr{F}(F) \arrow[uul, "\Gamma_F"']
\end{tikzcd}$$

1 commutes by our assumption $\boxed{2}$ on Γ;

2 commutes by our assumption $\boxed{1}$ on Γ;

3 commutes trivially;

4 commutes because left and right unitors commute;

5 commutes by our assumption $\boxed{3}$ on Γ;

6 commutes by the interchange law;

7 commutes by naturality of the unitors;

and the outer diagram commutes because $\mathscr{F}$ satisfies the counitality diagrams.

So 8 commutes when Γ_{F1_i} is precomposed. But Γ_{F1_i} and Γ_i are epimorphisms. So $\mathscr{G}$ satisfies the counitality diagrams, as required.

So $\mathscr{G}$ is a psuedofunctor, and Γ is immediately a 2-natural transformation. $\square$

1.2 (Multi)fiat 2-categories and injective abelianisations

The particular categories we often work with are called (multi)fiat 2-categories. To explain what this means, we need some set-up.

Definition 1.2.1 (Split idempotent)**.** We say a morphism e in a category $\mathcal{C}$ is an *idempotent* if $e \circ e = e$.

We say an idempotent $e : i \to i$ is *split* if there are morphisms $s : i \to j$ and $t : j \to i$ such that $t \circ s = e$ and $s \circ t = 1_j$.

We say a $\mathcal{C}$ *has split idempotents* if every idempotent is split. $\lhd$

Definition 1.2.2 (Additive category)**.** We call a category *preadditive* if it is enriched over the category of abelian groups.

We call a category *additive* if it is preadditive and has all finite products (in particular, including the empty product). $\lhd$

The following proposition can be found in a standard text on abelian categories, for example as Theorem 2.35 in [Fre64]

Proposition 1.2.3. In a preadditive category, every product of objects is also a biproduct on the same objects. The empty product, where it exists, is a zero object.

Definition 1.2.4 ($\mathbb{k}$-linear category)**.** We say a category is $\mathbb{k}$-*linear* if it is enriched over the category of $\mathbb{k}$-vector spaces (so in particular, it is preadditive). We say a functor (or natural transform) is $\mathbb{k}$-linear if it is an enriched functor (respectively, natural transform) over the category of $\mathbb{k}$-vector spaces. $\lhd$

Definition 1.2.5 (Indecomposable objects)**.** We say an object X in a category $\mathcal{C}$ is indecomposable if it cannot be expressed as a non-trivial coproduct, that is, whenever $X \cong \coprod_{i=1}^{n} X_i$, there is a unique $1 \leq i \leq n$ such that $X_i \cong X$, and $X_j \cong 0$ for $j \neq i$. $\lhd$

Definition 1.2.6 (Finitary category)**.** We say an additive $\mathbb{k}$-linear category is finitary if it has split idempotents, finitely many isomorphism classes of indecomposable objects, and the morphism sets are finite dimensional as $\mathbb{k}$-vector spaces. $\lhd$

Definition 1.2.7 (2-Category of finitary categories)**.** We denote by $\mathfrak{A}_{\Bbbk}^{f}$ the 2-category of finitary categories, additive $\Bbbk$-linear functors and natural transformations. ◁

Definition 1.2.8 ((Multi)finitary 2-category)**.** We call a 2-category $\mathscr{C}$ *multifinitary* if:

- $\mathscr{C}$ has finitely many objects;
- each hom-category $\mathscr{C}(i, j)$ is a finitary category;
- and composition is additive and $\Bbbk$-linear in each argument.

If, moreover, all identity 1-morphisms are indecomposable, then we say $\mathscr{C}$ is finitary. ◁

Definition 1.2.9 (Fiat 2-category)**.** We say a 2-category $\mathscr{C}$ is (multi)fiat if it is a (multi)finitary 2-category with a weak involutive anti-equivalence $*$ (reversing both 1- and 2-morphisms) such that, for every 1-morphism F, the pair (F, F^*) is an adjoint pair of 1-morphisms. ◁

We make a short note on the injective abelianisation of a (multi)finitary 2-category. For full details, one can look at e.g. [MMMT19] Section 3. In particular, the two given propositions are immediate from Section 3.1.

Definition 1.2.10 (Injective objects)**.** An object Q in a category $\mathcal{C}$ is called injective if, for every morphism $f : X \to Q$, and every monomorphism $g : X \to Y$, there's an $h : Y \to Q$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, hook, "g"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "h"] \\ Q & \end{tikzcd}$$

In an abelian category, this is equivalent to the hom-functor $\mathrm{Hom}_{\mathcal{C}}(-, Q)$ being exact. ◁

Proposition 1.2.11. Given a finitary category $\mathcal{C}$, there is an abelian category $\underline{\mathcal{C}}$ for which $\mathcal{C}$ embeds into the injective objects of $\underline{\mathcal{C}}$, and is equivalent via this embedding to the subcategory of injective objects in $\underline{\mathcal{C}}$. Moreover, this construction is natural in $\mathcal{C}$, and in particular we can define $\underline{\mathcal{F}} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ for $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ a $\Bbbk$-linear functor.

We call this an *injective abelianisation* of $\mathcal{C}$.

Moreover, given a (multi)finitary 2-category $\mathscr{C}$, we can define the *injective abelianisation* $\underline{\mathscr{C}}$ of $\mathscr{C}$, which has the same objects as $\mathscr{C}$, and for which $\underline{\mathscr{C}}(i, j) = \underline{\mathscr{C}(i, j)}$. This construction is also natural in $\mathscr{C}$.

There are actually several such constructions natural in $\mathscr{C}$; we present the diagrammatic injective abelianisation below, but note that essentially any injective abelianisation would work.

Proposition 1.2.12. Let $\mathcal{C}$ be a finitary category. Define the diagrammatic injective abelianisation $\underline{\mathcal{C}}$ as follows:

- Objects of $\underline{\mathcal{C}}$ are morphisms $F : i \to j$ in $\mathcal{C}$;
- Morphisms in $\underline{\mathcal{C}}$ are equivalence classes of solid commutative diagrams (that is, without Q)

$$\begin{tikzcd} i \arrow[r, "F"] \arrow[d, "G"] & j \arrow[d, "H"] \arrow[dl, dashed, "Q"] \\ i' \arrow[r, "F'"'] & j' \end{tikzcd}$$

  modulo the ideal generated by diagrams with a 'homotopy', that is, a Q as above for which $G = QF$.
- Identity morphisms are given by diagrams in which G and H are both identities, and composition is given by vertical composition of diagrams.

Then the diagrammatic injective abelianisation is indeed an abelianisation, as characterised in the previous proposition.

In particular, a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ extends to a functor $\underline{\mathcal{F}} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in the obvious way. □

Henceforth, injective abelianisation will refer to diagrammatic injective abelianisation.

1.3 Coalgebra 1-morphisms

Of central importance is the notion of a coalgebra 1-morphism and related concepts. Throughout this section, let $\mathscr{C}$ be a finitary 2-category.

Definition 1.3.1 (Coalgebra 1-morphism)**.** A coalgebra 1-morphism (at an object i of $\mathscr{C}$) consists of the following data:

- A 1-morphism $C : i \to i$ in $\mathscr{C}$;
- A 2-morphism $\Delta : C \to CC$ in $\mathscr{C}$;
- A 2-morphism $\varepsilon : C \to 1_i$ in $\mathscr{C}$;

which satisfy the coalgebra axioms:

$$\begin{tikzcd} C \arrow[r, "\Delta"'] \arrow[d, "\Delta"'] & CC \arrow[d, "\Delta*1"] \\ CC \arrow[r, "1*\Delta"'] & CCC \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, equal] & CC \arrow[d, "\varepsilon*1"] \\ CC \arrow[r, "1*\varepsilon"'] & C \end{tikzcd}$$

◁

Definition 1.3.2 (Coalgebra homomorphism)**.** A homomorphism of coalgebra 1-morphisms in $\mathscr{C}$, θ : $(C, \Delta, \varepsilon) \to (D, \Delta', \varepsilon')$, is a 2-morphism $\theta : C \to D$ in $\mathscr{C}$ making the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[d, "\Delta"] & D \arrow[d, "\Delta'"] \\ CC \arrow[r, "\theta*\theta"] & DD \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\theta"] \arrow[dr, "\varepsilon"'] & & D \arrow[dl, "\varepsilon'"] \\ & 1 & \end{tikzcd}$$

◁

Definition 1.3.3 (Comodule 1-morphism)**.** A (right) comodule 1-morphism (at j) for a coalgebra 1-morphism $C = (C, \Delta, \varepsilon)$ (at i) is a pair of a 1-morphism $M : i \to j$, the comodule, and a 2-morphism $\delta_{M,C} : M \to MC$, the coaction, in $\mathscr{C}$, such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\delta"] \arrow[d, "\delta"] & MC \arrow[d, "1*\Delta"] \\ MC \arrow[r, "\delta*1"] & MCC \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\delta"] \arrow[dr, equal] & MC \arrow[d, "1*\varepsilon"] \\ & M \end{tikzcd}$$

Similarly, we can define a left C-comodule 1-morphism $(M, \delta_{C,M})$, and then a C-D-bicomodule 1-morphism is a left C-comodule and a right D-comodule, such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\delta_{C,M}"] \arrow[d, "\delta_{M,D}"] & CM \arrow[d, "1*\delta_{M,D}"] \\ MD \arrow[r, "\delta_{C,M}*1"] & CMD \end{tikzcd}$$

◁

Definition 1.3.4 (Biinjective bicomodule 1-morphism)**.** A bicomodule 1-morphism ${}_CM_D$ is biinjective if it's injective as both a left C- and right D-comodule 1-morphism.

◁

Definition 1.3.5 (Bicomodule homomorphism)**.** A homomorphism of C-D-bicomodule 1-morphisms in $\mathscr{C}$, $\phi : (N, \delta_{C,N}, \delta_{N,D}) \to (M, \delta_{C,M}, \delta_{M,D})$, is a 2-morphism $\phi : M \to N$ in $\mathscr{C}$ such that the following diagrams commute:

$$\begin{tikzcd} N \arrow[r, "\phi"] \arrow[d, "\delta_{C,N}"] & M \arrow[d, "\delta_{C,M}"] \\ CN \arrow[r, "1*\phi"] & CM \end{tikzcd} \qquad \begin{tikzcd} N \arrow[r, "\phi"] \arrow[d, "\delta_{N,D}"] & M \arrow[d, "\delta_{M,D}"] \\ ND \arrow[r, "\phi*1"] & MD \end{tikzcd}$$

If ϕ satisfies the left diagram, we say it is a left C-comodule homomorphism; if it satisfies the right diagram, we say it is a right D-comodule homomorphism.

We define comod$_{\underline{\mathscr{C}}}(C)_j$ to be the category whose objects are right C-comodule 1-morphisms at j, and whose morphisms are right C-comodule homomorphisms. Similarly, ${}_j$comod$_{\underline{\mathscr{C}}}(C)$ is the category of left C-comodule 1-morphisms at j, and left C-comodule homomorphisms. ◁

The left- and right-injective comodule 1-morphisms have a nice characterisation in fiat 2-categories:

Lemma 1.3.6. If $\mathscr{C}$ is a fiat 2-category, $C \in \underline{\mathscr{C}}(i,i)$ a coalgebra 1-morphism, then C is biinjective as a C-C-bicomodule 1-morphism in $\underline{\mathscr{C}}$.

Proof : As a corollary of Example 1.5.2, we will show that there is an adjunction $\mathrm{Hom}_{comod_{\underline{\mathscr{C}}}}(-, C) \cong \mathrm{Hom}_{\underline{\mathscr{C}}}(-, \mathbb{1}_i)$. Since $\mathbb{1}_i$ is injective in $\underline{\mathscr{C}}$ by construction, the right functor is exact, so the left functor is exact, so C is injective as a right comodule 1-morphism. Similarly, C is injective as a left comodule 1-morphism, so a biinjective bicomodule 1-morphism. □

Lemma 1.3.7. If $\mathscr{C}$ is a fiat 2-category, $C \in \underline{\mathscr{C}}(i,i)$ a coalgebra 1-morphism, then the right-injective C-comodule 1-morphisms at j of $\underline{\mathscr{C}}$ are precisely the additive closure of $\{FC | F \in \mathscr{C}(i,j)\}$. Similarly, the left-injective C-comodule 1-morphisms at j of $\mathscr{C}$ are precisely the additive closure of $\{CF | F \in \mathscr{C}(j,i)\}$.

Proof : Because $\mathscr{C}$ is fiat, every $F \in \mathscr{C}(i,j)$ has adjoints, so the mapping of 1-morphisms $M \mapsto FM$ is exact. So since C is an injective right C-comodule 1-morphism, FC is an injective right C-comodule 1-morphism. Summands of injective objects are injective, so every object in the additive closure of $\{FC | F \in \mathscr{C}(i,j)\}$ is injective.

Now, if X is a right-injective comodule 1-morphism in $\mathrm{comod}_{\underline{\mathscr{C}}}(C)_j$, we note that $\delta_{X,C} : X \to XC$ is a monic comodule homomorphism (by counitality), and that X has an injective presentation $X_0 \to X_1$ for X_i 1-morphisms in $\mathscr{C}$; so $X \hookrightarrow XC \hookrightarrow X_0C$ is an embedding of X into an object in $\{FC | F \in \mathscr{C}(i,i)\}$, from which the first claim follows.

The characterisation of the left-injective comodule 1-morphisms is similar. □

The following definition can be found in Section 0 of [Tak77]

Definition 1.3.8 (Cotensor product)**.** The cotensor product of a C-D-bicomodule 1-morphism M and a D-E-bicomodule 1-morphism over D in $\mathscr{C}$, where it exists, is the equalizer $(M \underset{D}{\circledast} N, t^D_{M,N})$ of the pair of morphisms

$$\begin{tikzcd} M \underset{D}{\circledast} N \arrow[r, dashed, "t^D_{M,N}"'] & MN \arrow[r, shift left, "\delta_{M,D} * 1"] \arrow[r, shift right, "1 * \delta_{D,N}"'] & MDN \end{tikzcd}$$

The cotensor product of a pair of bicomodule homomorphisms $\phi_M : M \to M'$, $\phi_N : N \to N'$, is the unique map $\phi_M \underset{D}{\circledast} \phi_N$ making the following diagram commute:

$$\begin{tikzcd} M \underset{D}{\circledast} N \arrow[r, "t^D_{M,N}"'] \arrow[d, dashed, "\phi_M \underset{D}{\circledast} \phi_N"'] & MN \arrow[d, "\phi_M * \phi_N"] \\ M' \underset{D}{\circledast} N' \arrow[r, "t^D_{M',N'}"'] & M'N' \end{tikzcd}$$

$M \underset{D}{\circledast} N$ becomes an C-E-bicomodule 1-morphism with the maps

$$\delta_{C, M \underset{D}{\circledast} N} := \delta_{C,M} \underset{D}{\circledast} 1_N, \delta_{M \underset{D}{\circledast} N, E} := 1_M \underset{D}{\circledast} \delta_{N,E};$$

$t^D_{M,N}$ is then a bicomodule homomorphism.

When $\mathscr{C}$ is abelian, for instance if we work in the injective abelianisation $\underline{\mathscr{C}}$, the cotensor product always exists.

◁

Lemma 1.3.9. The following diagram commutes:

$$
\begin{tikzcd}
M \underset{C}{\boxast} N \underset{D}{\boxast} L \arrow[r, "t^D_{M \boxast N, L}"] \arrow[d, "t^C_{M, N \boxast L}"'] & M \boxast NL \arrow[d, "t^C_{M, NL}"] \\
MN \boxast L \arrow[r, "t^D_{MN, L}"] & MNL
\end{tikzcd}
$$

Proof : First, by inspection of the following diagram, it's immediate that $t^C_{M,N} * 1 = t^C_{M,NL}$.

$$
\begin{tikzcd}
M \boxast NL \arrow[r, "t^C_{M,N} * 1"] & MNL \arrow[r, shift left, "\delta_{M,C} * 1"] \arrow[r, shift right, "1 * \delta_{C,NL} = 1 * \delta_{C,N} * 1"'] & MCNL
\end{tikzcd}
$$

Next, by inspection of the following diagram, it's clear that $t^C_{M,N} \boxast 1 = t^C_{M, N \boxast L}$:

$$
\begin{tikzcd}
M \boxast N \boxast L \arrow[r, "t^C_{M,N} \boxast 1"] & MN \boxast L \arrow[r, shift left, "\delta_{M,C} * 1"] \arrow[r, shift right, "1 * \delta_{C, N \boxast L} = 1 * \delta_{C,N} \boxast 1"'] & MCN \boxast L
\end{tikzcd}
$$

Finally, our result follows from the definition of $t^C_{M,N} \boxast 1$ $\square$

Proposition 1.3.10. If we define, for coalgebra 1-morphisms C and D, $\mathscr{B}\mathsf{icom}_{\mathscr{C}}(C, D)$ to be the category whose objects are C-D-bicomodule 1-morphisms, and whose morphisms are bicomodule homomorphisms, then for coalgebra 1-morphisms C, D and E,

$$
\begin{aligned}
- \underset{D}{\boxast} - : \mathscr{B}\mathsf{icom}_{\mathscr{C}}(D, E) \times \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C, D) &\to \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C, E) \\
(N, M) &\mapsto M \boxast N \\
(\psi, \phi) &\mapsto \phi \underset{D}{\boxast} \psi
\end{aligned}
$$

is a functor when it exists.

Proof : First, we note that it is clear that for coalgebra 1-morphisms C and D, $\mathscr{B}\mathsf{icom}_{\mathscr{C}}(C, D)$ is indeed a category.

We assume that $- \underset{D}{\boxast} -$ exists, that is, that all relevant cotensor products exist. For $- \underset{D}{\boxast} -$ to be a functor, it must respect composition of bicomodule homomorphisms, and identity bicomodule homomorphisms. To see that it respects composition, let $\phi : M \to M'$ and $\phi' : M' \to M''$ be C-D-bicomodule homomorphisms, $\psi : N \to N'$ and $\psi' : N' \to N''$ be D-E-bicomodule homomorphisms.

We need

$$\phi'\phi \underset{D}{\circledast} \psi'\psi = (\phi' \underset{D}{\circledast} \psi') \circ (\phi \underset{D}{\circledast} \psi).$$

Consider the following diagram:

$$\begin{tikzcd}
M \underset{D}{\circledast} N \arrow[r, "t^D_{M,N}"] \arrow[d, "\phi \underset{D}{\circledast} \psi"'] \arrow[dr, phantom, "1"] & MN \arrow[d, "\phi * \psi"] \\
M' \underset{D}{\circledast} N' \arrow[r, "t^D_{M',N'}"] \arrow[d, "\phi' \underset{D}{\circledast} \psi'"'] \arrow[dr, phantom, "2"] & M'N' \arrow[d, "\phi' * \psi'"] \\
M'' \underset{D}{\circledast} N'' \arrow[r, "t^D_{M'',N''}"] & M''N''
\end{tikzcd}$$

Here, 1 and 2 commute by definition of $\phi \underset{D}{\circledast} \psi$ and $\phi' \underset{D}{\circledast} \psi'$ respectively, so the outer diagram commutes. But the outer diagram is

$$\begin{tikzcd}
M \underset{D}{\circledast} N \arrow[r, "t^D_{M,N}"] \arrow[d, "(\phi' \underset{D}{\circledast} \psi') \circ (\phi \underset{D}{\circledast} \psi)"'] & MN \arrow[d, "\phi'\phi * \psi'\psi"] \\
M'' \underset{D}{\circledast} N'' \arrow[r, "t^D_{M'',N''}"] & M''N''
\end{tikzcd}$$

and by definition, the unique map which can be on the left-hand side when this diagram commutes is $\phi'\phi \underset{D}{\circledast} \psi'\psi$. So we must have $\phi'\phi \underset{D}{\circledast} \psi'\psi = (\phi' \underset{D}{\circledast} \psi') \circ (\phi \underset{D}{\circledast} \psi)$, as required.

To see that $- \underset{D}{\circledast} -$ respects identity bicomodule homomorphisms, we must have that $1_M \underset{D}{\circledast} 1_N = 1_{M \circledast N}$. Consider the following diagram:

$$\begin{tikzcd}
M \circledast N \arrow[r, "t"] \arrow[d, "1_{M \circledast N}"'] & MN \arrow[d, "1_M * 1_N = 1_{MN}"] \\
M \circledast N \arrow[r, "t"] & MN
\end{tikzcd}$$

It's clear that this commutes. But by construction $1_M \underset{D}{\circledast} 1_N$ is the unique map making this diagram commute, so we must have that $1_M \underset{D}{\boxast} 1_N = 1_{M \boxast N}$ as required.

So $- \underset{D}{\boxast} -$ is a functor. $\square$

Proposition 1.3.11. We can define natural isomorphisms $\varepsilon^l_{C,D} : C \underset{C}{\boxast} - \to 1_{\mathscr{B}\mathsf{icom}_{\mathscr{C}}(C,D)}$ and $\varepsilon^r_{C,D}$: $- \underset{D}{\boxast} D \to 1_{\mathscr{B}\mathsf{icom}_{\mathscr{C}}(C,D)}$ at the C-D-bicomodule 1-morphism M by

$$(\varepsilon^l_{C,D})_M = \varepsilon^l_M := (\varepsilon * 1) \circ t^C_{C,M} : C \underset{C}{\boxast} M \to M,$$

$$(\varepsilon^r_{C,D})_M = \varepsilon^r_M := (1 * \varepsilon) \circ t^D_{M,D} : M \underset{D}{\boxast} D \to M.$$

Proof : Fix C and D, and write $\varepsilon^l := \varepsilon^l_{C,D}$, and similarly ε^r. Showing that ε^l and ε^r are natural isomorphisms is the same thing as showing that they are pointwise invertible and the following diagrams commute for all C-D-bicomodule 1-morphisms M, M' and bicomodule homomorphisms $\phi : M \to M'$:

$$\begin{tikzcd}
C \underset{C}{\boxast} M \arrow[r, "\varepsilon^l_M"] \arrow[d, "1 \boxast \phi"'] & M \arrow[d, "\phi"] \\
C \underset{C}{\boxast} M' \arrow[r, "\varepsilon^l_{M'}"] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \underset{D}{\boxast} D \arrow[r, "\varepsilon^r_M"] \arrow[d, "\phi \boxast 1"'] & M \arrow[d, "\phi"] \\
M' \underset{D}{\boxast} D \arrow[r, "\varepsilon^r_{M'}"] & M'
\end{tikzcd}$$

We first consider the left of these two diagrams (noting that the right diagram can be shown to commute in essentially the same way). Look at the following diagram:

$$\begin{tikzcd}
C \underset{C}{\boxast} M \arrow[r, "t^C_{C,M}"] \arrow[d, "1 \boxast \phi"'] \arrow[dr, phantom, "1"] & CM \arrow[r, "\varepsilon * 1"] \arrow[d, "1 * \phi"] \arrow[dr, phantom, "2"] & M \arrow[d, "\phi"] \\
C \underset{C}{\boxast} M' \arrow[r, "t^C_{C,M'}"] & CM' \arrow[r, "\varepsilon * 1"] & M'
\end{tikzcd}$$

1 commutes by definition of $1 \boxast \phi$, and 2 commutes by the interchange law. Therefore the outer diagram commutes, which is precisely our left-hand diagram above.

We show that ε^l is invertible as follows (noting that ε^r is similar). First, consider the following diagram:

$$M \xrightarrow{\delta_{C,M}} CM \underset{1*\delta_{C,M}}{\overset{\Delta_C*1}{\rightrightarrows}} CCM$$

By the axioms of a left C-comodule 1-morphism, both possible compositions must be equal, that is, $\delta_{C,M}$ equalises the two right-hand maps. By the universal property of $C \underset{C}{\circledast} M$ as an equaliser, there must therefore be a unique map $\delta^l_M : M \to C \underset{C}{\circledast} M$ such that $t^C_{C,M} \circ \delta^l_M = \delta_{C,M}$. We can then compute

$$\begin{aligned}\varepsilon^l_M \circ \delta^l_M &= (\varepsilon_C * 1) \circ t^C_{C,M} \circ \delta^l_M \\ &= (\varepsilon_C * 1) \circ \delta_{C,M} \\ &= 1_M\end{aligned}$$

using the definitions of ε^l_M and δ^l_M, and the axioms for a left C-comodule 1-morphism. So ε^l_M is split epic. Next, we consider the following diagram:

$$\begin{tikzcd} C \underset{C}{\circledast} M \arrow[r, "t^C_{C,M}"] & CM \arrow[r, shift left, "\Delta_C*1"] \arrow[r, shift right, "1*\delta_{C,M}"'] \arrow[d, "\varepsilon*1"] & CCM \arrow[d, "\varepsilon*1*1"] \\ & M \arrow[r, "\delta_{C,M}"] & CM \end{tikzcd}$$

With respect to the lower of the two top right morphisms, the square commutes by the interchange law. The composition of the upper of these two morphisms with the right-hand vertical morphism is the identity, by counitality. But then $t^C_{C,M}$ equalises the top two morphisms by definition. So any path through this diagram from $C \underset{C}{\circledast} M$ to the bottom right CM is equal, and equal to $t^C_{C,M}$, which is monic. In particular, ε^l_M is a right factor of a monomorphism, and thus monic.

Therefore, ε^l_M is split epic and monic, and thus an isomorphism (with inverse δ^l_M). So $\varepsilon^l_{C,D}$ is a natural isomorphism, and similarly so is $\varepsilon^r_{C,D}$. □

As an immediate corollary of this result, we get the following nice property of the cotensor product:

Corollary 1.3.12. Suppose $\mathscr{C}$ is a fiat 2-category. If C, D, E are coalgebra 1-morphisms in $\underline{\mathscr{C}}$, ${}_CN_D$ and ${}_DM_E$ are right injective bicomodule 1-morphisms, then $N \circledast M$ is right injective. Similarly, if M and N are left injective, then $N \circledast M$ is left injective.

If M and N are biinjective, then $N \circledast M$ is biinjective.

Proof : Since the cotensor is additive, we need only check this for $N = FD$, $M = GE$ for some $F \in \mathscr{C}(i,j)$, $G \in \mathscr{C}(j,k)$, by Lemma 1.3.7. But by the previous proposition, $N \underset{D}{\boxast} M = FD \underset{D}{\boxast} GE \cong FGE$, which is in $\{HE | H \in \mathscr{C}(i,k)\}$, and thus injective as a right E-comodule 1-morphism, as required.

The left-injective case is similar, and the biinjective case follows from these. $\square$

These results let us build a pair of nice bicategories:

Definition 1.3.13 (Bicategory of (right injective, biinjective) bicomodule 1-morphisms)**.** For any category $\mathscr{C}$ for which all relevant cotensor products exist, the *bicategory of (right injective, biinjective) bicomodule 1-morphisms* of $\mathscr{C}$, written $\mathscr{B}\mathrm{icom}_{\mathscr{C}}$ ($\mathscr{R}\mathscr{B}\mathrm{icom}_{\mathscr{C}}$, $\mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{C}}$), consists of the following data:

- Objects are coalgebra 1-morphisms in $\mathscr{C}$;
- 1-morphisms from C to D are (right injective, biinjective) C-D-bicomodule 1-morphisms in $\mathscr{C}$;
- 2-morphisms are bicomodule homomorphisms in $\mathscr{C}$;
- Horizontal composition is given by the cotensor product;
- Vertical composition is given by vertical composition in $\mathscr{C}$;
- The identity 1-morphism on an object C is given by C viewed as a C-C-bicomodule 1-morphism;
- The identity 2-morphism on a 1-morphism M is 1_M, the identity 2-morphism of M as a 1-morphism in $\mathscr{C}$;
- The associator is induced by the identity, and is omitted;
- The left and right unitors are ε^l and ε^r respectively, cf. Proposition 1.3.11.

$\triangleleft$

Proposition 1.3.14. When $\mathscr{C}$ has all relevant cotensor products, $\mathscr{B}\mathrm{icom}_{\mathscr{C}}$ is a bicategory. When $\mathscr{C}$ is a fiat 2-category, $\mathscr{R}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ and $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ are bicategories.

Proof : By Proposition 1.3.10, we know that horizontal composition is functorial for $\mathscr{B}\mathrm{icom}_{\mathscr{C}}$.

To see that this functor restricts to the $\mathscr{R}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$, $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ setting when $\mathscr{C}$ is a fiat 2-category, we need only show that pairs of (right injective, biinjective) bicomodule 1-morphisms get sent to a (right injective, biinjective) bicomodule 1-morphism, since $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}(M,N)$ is a full subcategory of $\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}(M,N)$. But this is precisely Corollary 1.3.12.

By Lemma 1.3.6, C is a biinjective C-C bicomodule 1-morphism when $\mathscr{C}$ is fiat, so the identity 1-morphisms are well-defined for $\mathscr{R}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$, $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$.

Finally, since the left and right unitors are natural isomorphisms by Proposition 1.3.11 (and noting that the restriction of a natural isomorphism to a full subcategory is again a natural isomorphism), we need only check that they commute. To see this is true, let M be a C-D-bicomodule 1-morphism, N be a D-E-bicomodule 1-morphism, and consider the following diagram:

$$\begin{tikzcd}
M \underset{D}{\boxast} D \underset{D}{\boxast} N \arrow[r, "1\boxast t"] \arrow[d, "t\boxast 1"'] \arrow[dr, phantom, "1"] & M \underset{D}{\boxast} DN \arrow[d, "t"] \arrow[ddr, "1\boxast\epsilon_D*1"] \arrow[dr, phantom, "2"] & \\
MD \underset{D}{\boxast} N \arrow[r, "t"] \arrow[drr, "1*\epsilon_D\boxast 1"'] \arrow[dr, phantom, "3"] & MDN \arrow[d, "1*\epsilon*1"] & \\
 & MN & M \underset{D}{\boxast} N \arrow[l, "t"']
\end{tikzcd}$$

By Lemma 1.3.9, 1 commutes.

2 and 3 commute by definition of $1 \boxast \epsilon_D$ and $\epsilon_D \boxast 1$ respectively.

Therefore, the top and bottom arrows, when post-composed with t, are equal. But t is monic. So in fact, the top and bottom arrows are equal. But this is precisely the statement that the unitors commute.

Therefore, both statements of the proposition are proved.

□

We also have some simple results about coalgebra homomorphisms, which we record here:

Lemma 1.3.15. Let C, C', D, D' be coalgebra 1-morphisms in a 2-category $\mathscr{C}$. Let $(N, \delta_{C,N}, \delta_{N,D})$ be a C-D-bicomodule 1-morphism, $\theta_C : C \to C'$ and $\theta_D : D \to D'$ be coalgebra homomorphisms. Define $\delta_{C',N} := (\theta_C * 1_N) \circ \delta_{C,N}$ and $\delta_{N,D'} := (1_N * \theta_D) \circ \delta_{N,D}$. Then $(N, \delta_{C',N}, \delta_{N,D'})$ is a C'-D'-bicomodule 1-morphism, which we write as ${}^{\theta_C}N^{\theta_D}$. Moreover, if $\rho_C : C' \to C''$ and $\rho_D : D' \to D''$ are coalgebra homomorphisms, then ${}^{\rho_C}({}^{\theta_C}N^{\theta_D})^{\rho_D} = {}^{\rho_C\circ\theta_C}N^{\rho_D\circ\theta_D}$.

Proof : We first show that $(N, \delta_{C',N})$ is a left C'-comodule 1-morphism. Consider the following diagram:

$$
\begin{tikzcd}
N \arrow[r, "\delta_{C,N}"] \arrow[d, "\delta_{C,N}"'] \arrow[dr, phantom, "1"] & CN \arrow[r, "\theta_C * 1"] \arrow[d, "1*\delta_{C,N}"] \arrow[dr, phantom, "2"] & C'N \arrow[d, "1*\delta_{C,N}"] \\
CN \arrow[r, "\Delta_C * 1"] \arrow[d, "\theta_C * 1"'] \arrow[dr, phantom, "3"] & CCN \arrow[r, "\theta_C * 1 * 1"] \arrow[d, "\theta_C * \theta_C * 1"] \arrow[dr, phantom, "4"] & C'CN \arrow[d, "1*\theta_C*1"] \\
C'N \arrow[r, "\Delta_{C'} * 1"] & C'C'N \arrow[r, equal] & C'C'N
\end{tikzcd}
$$

(1) commutes since $(N, \delta_{C,N})$ is a left C-comodule 1-morphism;

(2) and (4) commute by the interchange law;

(3) commutes since θ_C is a coalgebra homomorphism.

Therefore, the outer square commutes, that is, the left C' action on N commutes with the comultiplication on C'.

Next, consider the following diagram:

$$
\begin{tikzcd}
N \arrow[r, "\delta_{C,N}"] \arrow[dr, equal] & CN \arrow[r, "\theta_C * 1"] \arrow[d, "\varepsilon_C * 1"] \arrow[dl, phantom, "1"] \arrow[dr, phantom, "2"] & C'N \arrow[dl, "\varepsilon_{C'} * 1"] \\
& N &
\end{tikzcd}
$$

(1) commutes since $(N, \delta_{C,N})$ is a left C-comodule 1-morphism;

(2) commutes since θ_C is a coalgebra homomorphism.

Therefore, the outer triangle commutes, that is, N satisfies the second axiom for being a left C'-comodule 1-morphism, hence N is a left C'-comodule 1-morphism.

Similarly, N is a right D'-comodule 1-morphism.

Finally, consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, "\delta_{C,N}"] \arrow[d, "\delta_{N,D}"'] \arrow[dr, phantom, "1"] & CN \arrow[r, "\theta_C * 1"] \arrow[d, "1*\delta_{N,D}"] \arrow[dr, phantom, "2"] & C'N \arrow[d, "1*\delta_{N,D}"] \\
ND \arrow[r, "\delta_{C,N}*1"] \arrow[d, "1*\theta_D"'] \arrow[dr, phantom, "3"] & CND \arrow[r, "\theta_C*1*1"] \arrow[d, "1*1*\theta_D"] \arrow[dr, phantom, "4"] & C'ND \arrow[d, "1*1*\theta_D"] \\
ND' \arrow[r, "\delta_{C,N}*1"] & CND' \arrow[r, "\theta_C*1*1"] & C'ND'
\end{tikzcd}$$

1 commutes since $(N, \delta_{C,N}, \delta_{N,D})$ is a C-D-bicomodule 1-morphism;

2, 3 and 4 commute by the interchange law.

Therefore, the outer square commutes, that is, the left C' and right D' action commute.

So N is an C'-D'-bicomodule 1-morphism, as required.

The final claim of the lemma is immediate from construction.

□

Remark. The lemma discusses "twisting" both actions of a bicomodule 1-morphism by coalgebra homomorphisms simultaneously, but by taking one or other to be the identity, we can "twist" on one side only. This is regularly done, and where it is, we omit writing the "twist" by the identity, that is, we write ${}^{1_C}N^{\theta_D} = N^{\theta_D}$ and ${}^{\theta_C}N^{1_D} = {}^{\theta_C}N$. It is straightforward that $({}^{\theta_C}N)^{\theta_D} = {}^{\theta_C}(N^{\theta_D}) = {}^{\theta_C}N^{\theta_D}$.

Lemma 1.3.16. Let C, C', D, D', θ_C and θ_D be as above. Let N, M be C-D-bicomodule 1-morphisms, $\phi : N \to M$ a C-D-bicomodule homomorphism. Then ϕ is also a C'-D'-bicomodule homomorphism.

Proof : Consider the following diagram:

$$\begin{tikzcd}
N \arrow[r, "\phi"] \arrow[d, "\delta_{C,N}"'] \arrow[dr, phantom, "(1)"] & M \arrow[d, "\delta_{C,M}"] \\
CN \arrow[r, "1*\phi"] \arrow[d, "\theta_C*1"'] \arrow[dr, phantom, "(2)"] & CM \arrow[d, "\theta_C*1"] \\
C'N \arrow[r, "1*\phi"] & C'M
\end{tikzcd}$$

(1) commutes since ϕ is a left C-comodule homomorphism;

(2) commutes by the interchange law.

Therefore the outer diagram commutes, that is, ϕ is a left C'-comodule homomorphism. Similarly, ϕ is a right D'-comodule homomorphism, so a C'-D'-bicomodule homomorphism.

$\square$

Lemma 1.3.17. Suppose we have an C-D-bicomodule 1-morphism N, and a D-E-bicomodule 1-morphism M. Suppose further that we have a coalgebra homomorphism $\theta : D \to D'$. Then there is a bicomodule homomorphism $\theta\restriction^{(N,M)}: N \underset{D}{\boxast} M \to N^\theta \underset{D'}{\boxast} {}^\theta M$. For $\rho : D' \to D''$, $\rho \restriction^{(M,N)} \circ\theta \restriction^{(M,N)}= (\rho \circ \theta) \restriction^{(M,N)}$. Moreover, when θ is monic, $\theta \restriction^{(N,M)}$ is an isomorphism.

Proof : Consider the following diagram:

$$\begin{tikzcd}
N \underset{D}{\boxast} M \arrow[r, "t^D_{N,M}"] \arrow[d, dashed, "\theta\restriction^{(N,M)}"'] & NM \arrow[r, shift left, "\delta_{N,D}*1"] \arrow[r, shift right, "1*\delta_{D,M}"'] \arrow[d, equal] & NDM \arrow[d, equal] & \\
N^\theta \underset{D'}{\boxast} {}^\theta M \arrow[r, "t^{D'}_{N^\theta,{}^\theta M}"] & NM \arrow[r, shift left, "\delta_{N,D}*1"] \arrow[r, shift right, "1*\delta_{D,M}"'] & NDM \arrow[r, "1*\theta*1"] & ND'M
\end{tikzcd}$$

The top line is the equalizer diagram for $N \underset{D}{\boxast} M$; the bottom line is the equalizer diagram for $N^\theta \underset{D'}{\boxast} {}^\theta M$. It is obvious that $t^D_{N,M}$ equalizes $\delta_{N,D'} * 1$ and $1 * \delta_{D',M}$, so we obtain a morphism $\theta \restriction^{(N,M)}: N \underset{D}{\boxast} M \to N^\theta \underset{D'}{\boxast} {}^\theta M$ with $t^{D'}_{N^\theta,{}^\theta M} \circ \theta \restriction^{(N,M)}= t^D_{N,M}$, as required.

To see that $\theta \restriction^{(M,N)}$ is a bicomodule homomorphism, consider the following diagram:

$$\begin{tikzcd}
 & & MN \arrow[d, equal] \arrow[drr, "\delta_{C,M} * 1"] & & \\
M \underset{D}{\boxast} N \arrow[urr, "t"] \arrow[d, "\theta\restriction^{(M,N)}"] \arrow[drr, "\delta_{C,M} \underset{D}{\boxast} 1"] & & M^{\theta\theta}N \arrow[drr, "\delta_{C,M^\theta} * 1"] & & CMN \arrow[d, equal] \\
M_\theta \underset{D'}{\boxast} {}_\theta N \arrow[urr, "t"] \arrow[drr, "\delta_{C,M^\theta} \underset{D'}{\boxast} 1"] & & CM \underset{D}{\boxast} N \arrow[urr, "1*t"] \arrow[d, "1*\theta\restriction^{(M,N)}"] & & CM^{\theta\theta}N \\
 & & CM^\theta \underset{D'}{\boxast} {}^\theta N \arrow[urr, "1*t"] & &
\end{tikzcd}$$

The top and bottom faces commute by definition of the cotensor;

the back left and front right faces commute by definition of $\theta \restriction^{(M,N)}$;

the back right face commutes trivially;

$1 * t$ is monic.

By a diagram chase, we see that

$$(1 * t) \circ (1 * \theta \restriction^{(M,N)}) \circ (\delta_{C,M} \underset{D}{\boxast} 1) = (1 * t) \circ (\delta_{C,M} \underset{D'}{\boxast} 1) \circ \theta \restriction^{(M,N)},$$

so that since $1 * t$ is monic,

$$(1 * \theta \restriction^{(M,N)}) \circ (\delta_{C,M} \underset{D}{\boxast} 1) = (\delta_{C,M} \underset{D'}{\boxast} 1) \circ \theta \restriction^{(M,N)},$$

that is, the front left face commutes, so $\theta \restriction^{(M,N)}$ is a left C-comodule homomorphism. Similarly it is a right E-comodule homomorphism, so a C-E-bicomodule homomorphism as required.

Given $\rho : D' \to D''$, one can see that by construction, $\rho \restriction^{(M,N)} \circ \theta \restriction^{(M,N)} = (\rho \circ \theta) \restriction^{(M,N)}$.

When θ is monic, we can apply the first part of our argument in reverse to find an inverse for $\theta \restriction^{(N,M)}$, proving the final claim.

□

Lemma 1.3.18. $\theta \restriction^{(N,M)}$ is natural in N and M, that is, the following diagram is a natural transformation of functors:

$$\begin{tikzcd}
\mathscr{B}\mathsf{icom}_{\mathscr{C}}(D,E) \times \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C,D) \arrow[r, "- \underset{D}{\boxast} -"] \arrow[d, "({}^{\theta_D}(-)^{\theta_E}, {}^{\theta_C}(-)^{\theta_D})"'] \arrow[dr, phantom, "\Downarrow \theta_D \restriction^{(-,-)}"] & \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C,E) \arrow[d, "{}^{\theta_C}(-)^{\theta_E}"] \\
\mathscr{B}\mathsf{icom}_{\mathscr{C}}(D',E') \times \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C',D') \arrow[r, "- \underset{D'}{\boxast} -"'] & \mathscr{B}\mathsf{icom}_{\mathscr{C}}(C',E')
\end{tikzcd}$$

Proof : Suppose $\phi_N : N \to N'$ and $\phi_M : M \to M'$ are bicomodule homomorphisms. Consider the following diagram:

$$\begin{tikzcd}
 & & NM \arrow[d, "\phi_N * \phi_M"] \arrow[drr, equal] & & \\
N \underset{D}{\boxast} M \arrow[urr, "t"] \arrow[d, "\phi_N \underset{D}{\boxast} \phi_M"'] \arrow[drr, "\theta\restriction^{(N,M)}"] & & N'M' \arrow[drr, equal] & & NM \arrow[d, "\phi_N * \phi_M"] \\
N' \underset{D}{\boxast} M' \arrow[urr, "t"] \arrow[drr, "\theta\restriction^{(N',M')}"'] & & N^{\theta} \underset{D'}{\boxast} {}^{\theta}M \arrow[urr, "t"] \arrow[d, "\phi_N \underset{D'}{\boxast} \phi_M"] & & N'M' \\
 & & N'^{\theta} \underset{D'}{\boxast} {}^{\theta}M' \arrow[urr, "t"'] & &
\end{tikzcd}$$

Then the top and bottom faces commute by definition of $\theta \restriction^{(-,-)}$;

the back left and front right faces commute by definition of $\phi_N \boxast \phi_M$;

the back right face commutes trivially;

and $t : N'^{\theta} \underset{D'}{\boxast} {}^{\theta}M' \to N'M'$ is monic.

Therefore, by a previous argument, the front left face commutes, i.e. $\theta \restriction^{(N,M)}$ is natural in N and M.

□

Lemma 1.3.19. Let M be a right C-comodule 1-morphism, N be a C-D bicomodule 1-morphism, L be a left D-comodule 1-morphism. Let $\theta_C : C \to C'$ and $\theta_D : D \to D'$ be coalgebra homomorphisms. Then $1_M \boxast \theta_D \restriction^{(N,L)} = \theta_D \restriction^{(M \boxast N, L)}$ and $\theta_C \restriction^{(M,N)} \boxast 1_L = \theta_C \restriction^{(M, N \boxast L)}$

Proof : Consider the following diagram:

$$\begin{tikzcd}
M \underset{C}{\circledast} N \underset{D}{\circledast} L \arrow[r, "\theta_C\restriction^{(M,N\circledast L)}"] \arrow[d, "t"] & M^{\theta_C} \underset{C'}{\circledast} {}^{\theta_C}N \underset{D}{\circledast} L \arrow[d, "t"] \\
M \underset{C}{\circledast} NL \arrow[r, "\theta_C\restriction^{(M,NL)}=\theta_C\restriction^{(M,N)}*1"'] & M^{\theta_C} \underset{C'}{\circledast} {}^{\theta_C}NL
\end{tikzcd}$$

This commutes by naturality of $\theta_C \restriction^{(-,-)}$. But by definition, $\theta_C \restriction^{(M,N)} \circledast 1$ is the unique arrow making this square commute. □

Corollary 1.3.20. With M, N, L, θ_C, θ_D as above, the following square commutes:

$$\begin{tikzcd}
M \underset{C}{\circledast} N \underset{D}{\circledast} L \arrow[r, "\theta_D\restriction^{(M\circledast N,L)}"] \arrow[d, "\theta_C\restriction^{(M,N\circledast L)}"] & M \underset{C}{\circledast} N^{\theta_D} \underset{D'}{\circledast} {}^{\theta_D}L \arrow[d, "\theta_C\restriction^{(M,N\circledast L)}"] \\
M^{\theta_C} \underset{C'}{\circledast} {}^{\theta_C}N \underset{D}{\circledast} L \arrow[r, "\theta_D\restriction^{(M\circledast N,L)}"] & M^{\theta_C} \underset{C'}{\circledast} {}^{\theta_C}N^{\theta_D} \underset{D'}{\circledast} {}^{\theta_D}L
\end{tikzcd} \quad \square$$

Proof : Immediate from naturality of $\theta_C \restriction^{(-,-)}$. □

1.4 Categories of representations

Throughout this section, let $\mathscr{C}$ be a finitary 2-category.

We can now define some particular bicategories we work with. First, what do we mean by a birepresentation?

Definition 1.4.1 (2-Category of finitary birepresentations)**.** The 2-category of finitary birepresentations of $\mathscr{C}$, written $\mathscr{C} - afmod$, is the 2-category $[\mathscr{C}, \mathfrak{A}_{\mathbb{k}}^{f}]$ of pseudofunctors from $\mathscr{C}$ to $\mathfrak{A}_{\mathbb{k}}^{f}$. Explicitly, it consists of the following data:

- Objects are $\mathbb{k}$-linear pseudofunctors $\mathbb{M} : \mathscr{C} \to \mathfrak{A}_{\mathbb{k}}^{f}$ (called a *finitary birepresentation* of $\mathscr{C}$);
- 1-morphisms are $\mathbb{k}$-linear strong 2-natural transforms of pseudofunctors (called a *morphism of birepresentations*, and written $\Phi : \mathbb{M} \to \mathbb{M}'$);
- 2-morphisms are modifications, written $\sigma : \Phi \to \Psi : \mathbb{M} \to \mathbb{M}'$.

If a finitary birepresentation is a strict 2-functor, we say it is a 2-representation. If a morphism of birepresentations is a strict 2-natural transform, we say it is a strict morphism.

Given a finitary birepresentation $\mathbb{M}$, we can define $\underline{\mathbb{M}}$ by letting $\underline{\mathbb{M}}(i) = \underline{\mathbb{M}(i)}$, the injective abelianisation of the category $\mathbb{M}(i)$, and inheriting the action of $\mathscr{C}$.

We say a morphism of birepresentations $\Phi : \mathbb{M} \to \mathbb{M}'$ is an *exact morphism of representations* if the component functors $\Phi_i : \mathbb{M}(i) \to \mathbb{M}'(i)$ extend to exact functors $\underline{\Phi_i} : \underline{\mathbb{M}(i)} \to \underline{\mathbb{M}'(i)}$.

If $\mathbb{M}$ is a finitary birepresentation of $\mathscr{C}$, i an object of $\mathscr{C}$, we write the objects of $\mathbb{M}(i)$ as X, Y, Z,...; and the morphisms of $\mathbb{M}(i)$ as $f, g, h, \ldots$.

$\lhd$

Example 1.4.2. If i is an object in $\mathscr{C}$, then $\mathbb{P}_i^{\mathscr{C}} = \mathscr{C}(i, -)$ defines a strict 2-representation, sometimes written $\mathbb{P}_i$, explicitly given by

$$\mathbb{P}_i : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^f$$

$$\mathbb{P}_i(j) = \mathscr{C}(i, j)$$

$$\mathbb{P}_i(F : j \to k) = F \circ - : \mathbb{P}_i(j) \to \mathbb{P}_i(k)$$

$$\mathbb{P}_i(\alpha : F \to G) = \alpha * 1 : \mathbb{P}_i(F) \to \mathbb{P}_i(G)$$

which we call the i^{th} principal 2-representation of $\mathscr{C}$.

Similarly, if $F : i \to j$ is a 1-morphism in $\mathscr{C}$, then $\mathbb{P}_F = \mathscr{C}(F, -)$ defines a strict morphism of 2-representations $\mathbb{P}_F : \mathbb{P}_j \to \mathbb{P}_i$.

If $\alpha : F \to G$ is a 2-morphism in $\mathscr{C}$, then $\mathbb{P}_\alpha = \mathscr{C}(\alpha, -) : \mathbb{P}_F \to \mathbb{P}_G$ is a modification. $\lll$

We're particularly interested in those finitary birepresentations that have generators:

Definition 1.4.3 (Representation generated by an object)**.** Let $\mathbb{M}$ be a finitary birepresentation of $\mathscr{C}$, and let $X \in \mathbb{M}(i)$ be an object. Define $\mathbb{M} \cdot X$ as follows:

- $(\mathbb{M} \cdot X)(j) = \text{add}\{\mathbb{M}(F)X | F \in \mathscr{C}(i, j)\}$, that is, the additive subcategory of $\mathbb{M}(j)$ generated by the objects $\mathbb{M}(F)X$;
- $\mathbb{M} \cdot X$ acts on $(\mathbb{M} \cdot X)(j)$ as $\mathbb{M}$.

This gives a new finitary birepresentation, the *birepresentation generated by* X, which has an obvious embedding $\mathbb{M} \cdot X \hookrightarrow \mathbb{M}$.

We say X is a *generator* for $\mathbb{M}$ if this embedding is an equivalence of birepresentations.

If $\mathbb{M}$ has a generator, we say that it is a *cyclic birepresentation*.

$\lhd$

The following equivalent characterisation is immediate from Proposition 1.1.7.

Proposition 1.4.4. Let $\mathbb{M}$ be a finitary birepresentation of $\mathscr{C}$. Then for X an object in $\mathbb{M}(i)$, X is a generator for $\mathbb{M}$ if and only if the embedding

$$\mathrm{add}\{\mathbb{M}(F)X|F \in \mathscr{C}(i,j)\} \hookrightarrow \mathbb{M}(j)$$

is an equivalence for every j an object of $\mathscr{C}$.

Definition 1.4.5 (Categories of cyclic birepresentations)**.** We denote the full subcategory of $\mathscr{C} - afmod$ consisting of cyclic birepresentations by $\mathscr{C} - cfmod$.

Similarly, denote by $\mathscr{C} - cfmod^*$ the category of *birepresentations with generator*, that is, pairs $(\mathbb{M}, X)$, where X is a generator for $\mathbb{M}$.

1-morphisms in this category are morphisms of the underlying birepresentations.

2-morphisms are modifications.

Write $\mathscr{C} - cfmod^*_{ex}$ for the sub-2-category of $\mathscr{C} - cfmod^*$ with only exact morphisms of birepresentations. $\lhd$

Of particular interest are those finitary birepresentations associated to coalgebra 1-morphisms.

Definition 1.4.6 (Internal birepresentations)**.** Let $C : i \to i$ be a coalgebra 1-morphism in $\underline{\mathscr{C}}$. The internal birepresentation of C in $\mathscr{C}$ is the finitary birepresentation $\mathbb{M}_C$ given as follows:

- For j an object in $\mathscr{C}$, $\mathbb{M}_C(j) = \mathrm{inj}_{\underline{\mathscr{C}}}(C)_j$ is the category of injective right C-comodule 1-morphisms $M : i \to j$ in $\underline{\mathscr{C}}$, along with right C-comodule homomorphisms;
- For $F : j \to k$ a 1-morphism in $\mathscr{C}$, the functor $\mathbb{M}_C(F) = F \circ - : \mathrm{inj}_{\underline{\mathscr{C}}}(C)_j \to \mathrm{inj}_{\underline{\mathscr{C}}}(C)_k$ is left composition with F;
- For $\alpha : F \to G$ a 2-morphism in $\mathscr{C}$, the natural transformation $\mathbb{M}_C(\alpha) = \alpha * 1$.

Moreover, if N is a right injective C-D-bicomodule 1-morphism in $\underline{\mathscr{C}}$, then the internal morphism associated with N is $- \boxast N : \mathbb{M}_C \to \mathbb{M}_D$, and if $\phi : N \to L$ is a bicomodule homomorphism, then the internal modification associated to ϕ is $- \boxast \phi : - \boxast N \to - \boxast L$.

$\lhd$

Example 1.4.7. By Lemma 1.3.7, C generates $\mathbb{M}_C$.

Proposition 1.4.8. Let $\mathscr{C}$ be a fiat 2-category. Given C, D, N, L, ϕ as above, $\mathbb{M}_C$ is a 2-representation; $- \circledast N : \mathbb{M}_C \to \mathbb{M}_D$ is a morphism of birepresentations and is exact when N is biinjective; $- \circledast \phi$ is a modification. C is a generator for $\mathbb{M}_C$.

Proof : It's immediate that the data of $\mathbb{M}_C$ are well-defined. Moreover, $\mathbb{M}_C(GF) = \mathbb{M}_C(G)\mathbb{M}_C(F)$, and $\mathbb{M}_C(1) = 1$, so $\mathbb{M}_C$ is a 2-representation. $(- \circledast N)_i$, viewed as a functor, is well-defined on objects by Propositions 1.3.10 and 1.3.12, and clearly well-defined on morphisms. Since $(XM) \circledast N \cong X(M \circledast N)$, and these isomorphisms are trivially natural in X, $- \circledast N$ is a strong morphism of birepresentations. By Corollary 1.3.12, when N is biinjective cotensoring with N is (isomorphic to) a summand of the regular action of F for some $F \in \mathscr{C}$, which is exact, so $- \circledast N$ is exact.

Finally, we show that $- \circledast \phi$ is a modification.

First, to see that $(- \circledast \phi)_i : (- \circledast N)_i \to (- \circledast L)_i$ is a natural transformation of functors, we note that if $\xi : M \to M'$ is a morphism of right C-comodule 1-morphisms, the following square commutes by the interchange law:

$$\begin{tikzcd}
M \circledast N \arrow[r, "1 \circledast \phi"] \arrow[d, "\xi \circledast 1"'] & M \circledast L \arrow[d, "\xi \circledast 1"] \\
M' \circledast N \arrow[r, "1 \circledast \phi"'] & M' \circledast L
\end{tikzcd}$$

Next, it's clear that the following diagram commutes:

$$\begin{tikzcd}
(XM) \circledast N \arrow[r, "\sim"] \arrow[d, "(1*1) \circledast \phi"] & X(M \circledast N) \arrow[d, "1*(1 \circledast \phi)"'] \\
(XM) \circledast L \arrow[r, "\sim"] & X(M \circledast L)
\end{tikzcd}$$

That is, the following diagram commutes:

$$\begin{tikzcd}
(- \circledast N)_j \circ \mathbb{M}_C(X) \arrow[r, "\sim"] \arrow[d, "(- \circledast \phi)_j * 1"] & \mathbb{M}_C(X) \circ (- \circledast N)_i \arrow[d, "1 * (- \circledast \phi)_i"'] \\
(- \circledast L)_j \circ \mathbb{M}_C(X) \arrow[r, "\sim"] & \mathbb{M}_C(X) \circ (- \circledast L)_i
\end{tikzcd}$$

So $- \circledast \phi$ is a modification.

That C generates $\mathbb{M}_C$ is Lemma 1.3.7. $\square$

This defines an "inclusion" of $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ into $\mathscr{C}-cfmod^*_{ex}$. We want to show that this "inclusion" is an equivalence of bicategories.

We spell this out in three propositions below:

Proposition 1.4.9. Suppose $\mathbb{M}$ is a cyclic birepresentation of a fiat 2-category $\mathscr{C}$. Then for some coalgebra 1-morphism C of $\underline{\mathscr{C}}$, there is an adjoint equivalence between $\mathbb{M}$ and $\mathbb{M}_C$.

We defer the proof of this proposition for now; it is proved as Proposition 2.1.1.

Proposition 1.4.10. Suppose $\Phi : \mathbb{M}_C \to \mathbb{M}_D$ is a morphism of birepresentations of some fiat 2-category $\mathscr{C}$, where C, D are coalgebra 1-morphisms in $\underline{\mathscr{C}}$. Then $\Phi_i(C) \in \mathbb{M}_D(i)$ has the structure of a right-injective C-D-bicomodule 1-morphism. There is an invertible modification $\sigma : \Phi \to - \boxast \Phi_i(C)$. When Φ is exact, $\Phi_i(C)$ is biinjective.

Proof : First, note that when viewed as a 1-morphism in $\mathscr{C}$, $\mathbb{M}_C(C) = \mathbb{M}_D(C) = C \circ -$. Write $\Phi(C) = \Phi_i(C)$. By definition, $\Phi(C)$ is an injective right D-comodule 1-morphism. The following composition of maps gives a left coaction, which we write $\delta_{C,\Phi(C)}$:

$$\Phi(C) \xrightarrow[\Phi(\Delta)]{} \Phi(CC) \xrightarrow[\Phi_C]{} C\Phi(C)$$

To see that this makes $\Phi(C)$ a left C-comodule 1-morphism, we need the following diagrams to commute:

$$\begin{tikzcd}
\Phi(C) \arrow[r, "\delta_{C,\Phi(C)}"] \arrow[d, "\delta_{C,\Phi(C)}"'] \arrow[dr, phantom, "\boxed{1}"] & C\Phi(C) \arrow[d, "\Delta_C * 1"] \\
C\Phi(C) \arrow[r, "1*\delta_{C,\Phi(C)}"] & CC\Phi(C)
\end{tikzcd}$$

$$\begin{tikzcd}
\Phi(C) \arrow[r, "\delta_{C,\Phi(C)}"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\boxed{2}"] & C\Phi(C) \arrow[d, "\varepsilon_C * 1"] \\
 & \Phi(C)
\end{tikzcd}$$

We first examine $\boxed{1}$. Consider the following diagram:

$$\begin{tikzcd}
\Phi(C) \arrow[r, "\Phi(\Delta)"] \arrow[d, "\Phi(\Delta)"'] \arrow[dr, phantom, "1"] & \Phi(CC) \arrow[r, "\Phi_C"] \arrow[d, "\Phi(\Delta*1)"] \arrow[dr, phantom, "2"] & C\Phi(C) \arrow[dd, "\Delta*1"] \\
\Phi(CC) \arrow[r, "\Phi(1*\Delta)"] \arrow[d, "\Phi_C"'] \arrow[dr, phantom, "3"] & \Phi(CCC) \arrow[d, "\Phi_C"] \arrow[dr, "\Phi_{CC}"] \arrow[dr, phantom, "4"'] & \\
C\Phi(C) \arrow[r, "1*\Phi(\Delta)"'] & C\Phi(CC) \arrow[r, "1*\Phi_C"'] & CC\Phi(C)
\end{tikzcd}$$

Here, 1 commutes by coassociativity for C;

2 and 3 commute by naturality of Φ;

and 4 commutes by higher coherence for Φ. Therefore the outer diagram commutes, that is, $\boxed{1}$ commutes. Next, consider the following diagram:

$$\begin{tikzcd}
\Phi(C) \arrow[r, "\Phi(\Delta)"] \arrow[dr, equal] & \Phi(CC) \arrow[r, "\Phi_C"] \arrow[d, "\Phi(\varepsilon*1)"] \arrow[l, phantom, "1"'] & C\Phi(C) \arrow[dl, "\varepsilon*1"] \arrow[l, phantom, "2"] \\
& \Phi(C) &
\end{tikzcd}$$

1 commutes by counitality for C;

and 2 commutes by naturality of Φ.

Therefore, the outer diagram commutes, that is, $\boxed{2}$ commutes. So $\Phi(C)$ is a left C-comodule 1-morphism.

For $\Phi(C)$ to be a bicomodule 1-morphism, we want the left C- and right D-coactions to commute. Rephrasing, we want the left C-coaction to be a right D-comodule homomorphism. But $\Phi(\Delta)$ is the image of a right C-comodule homomorphism, so by definition is a right D-comodule homomorphism; and Φ_C, as a morphism in $\mathbb{M}_D(i)$, is by definition a right D-comodule homomorphism. So the composition, $\delta_{C,\Phi(C)}$, is a right D-comodule homomorphism. So $\Phi(C)$ is a C-D-bicomodule.

Next, we want to construct σ. First, we construct the morphisms $\sigma_{j,M} : \Phi(M) = \Phi_j(M) \to M \circledast \Phi(C)$ for a given right C-comodule 1-morphism M; then show that these assemble to a natural transformation of functors $\Phi_j \to (- \circledast \Phi(C))_j$; then that these natural transformations assemble to a modification; and then, at last, that when Φ is a strong morphism of birepresentations, this modification is invertible.

To construct the morphism $\Phi(M) := \Phi_j(M) \to M \circledast \Phi(C)$, consider the following map, which we suggestively denote $\overline{\sigma}_{j,M} = \overline{\sigma}$:

$$\Phi(M) \xrightarrow{\Phi(\delta_{M,C})} \Phi(MC) \xrightarrow{\Phi_M} M\Phi(C)$$

By similar reasoning as before, this is a morphism of right C-comodule 1-morphisms. We want it to equalize the following maps, thereby inducing a map to the tensor product:

$$\Phi(M) \xrightarrow{\overline{\sigma}} M\Phi(C) \underset{1*\delta_{C,\Phi(C)}}{\overset{\delta_{M,C}*1}{\rightrightarrows}} MC\Phi(C)$$

We expand this to the following diagram:

$$\begin{tikzcd}
\Phi(M) \arrow[r, "\Phi(\delta_{M,C})"] \arrow[d, "\Phi(\delta_{M,C})"'] \arrow[dr, phantom, "1"] & \Phi(MC) \arrow[r, "\Phi_M"] \arrow[d, "\Phi(1*\Delta_C)"] \arrow[dr, phantom, "2"] & M\Phi(C) \arrow[d, "1*\Phi(\Delta_C)"] \\
\Phi(MC) \arrow[r, "\Phi(\delta_{M,C}*1)"] \arrow[d, "\Phi_M"'] \arrow[dr, phantom, "3"] & \Phi(MCC) \arrow[r, "\Phi_M"] \arrow[dr, "\Phi_{MC}"] \arrow[r, phantom, shift left=-6, "4"] & M\Phi(CC) \arrow[d, "1*\Phi_C"] \\
M\Phi(C) \arrow[rr, "\delta_{M,C}*1"] & & MC\Phi(C)
\end{tikzcd}$$

(1) commutes because M is a comodule 1-morphism;

(2) and (3) commute by naturality of Φ;

and (4) commutes by higher coherence for Φ.

Therefore the outer diagram commutes, that is, $\overline{\sigma}$ equalises the C-coactions of M and $\Phi(C)$, so induces a map $\sigma : \Phi(M) \to M \circledast \Phi(C)$ such that $t \circ \sigma = \overline{\sigma}$.

Next, we want to show that these assemble to a natural transformation of functors $\sigma_j : \Phi_j \to (- \circledast \Phi(C))_j$, that is, that the following diagram commutes for any morphism of right C-comodule 1-morphisms $\xi : M \to M'$:

$$\begin{tikzcd}
\Phi(M) \arrow[r, "\sigma_{j,M}"] \arrow[d, "\Phi(\xi)"'] & M \circledast \Phi(C) \arrow[d, "\xi \circledast 1"] \\
\Phi(M') \arrow[r, "\sigma_{j,M'}"] & M' \circledast \Phi(C)
\end{tikzcd}$$

Expanding this, consider the following diagram:

$$
\begin{tikzcd}
 & \Phi(MC) \arrow[d, "\Phi(\xi * 1)"] \arrow[dr, "\Phi_M"] & \\
\Phi(M) \arrow[ur, "\Phi(\delta_{M,C})"] \arrow[d, "\Phi(\xi)"] \arrow[dr, "\sigma_{j,M}"] & \Phi(M'C) \arrow[dr, "\Phi_{M'}"] & M\Phi(C) \arrow[d, "\xi * 1"] \\
\Phi(M') \arrow[ur, "\Phi(\delta_{M',C})"] \arrow[dr, "\sigma_{j,M'}"] & M \boxast \Phi(C) \arrow[ur, "t"] \arrow[d, "\xi \boxast 1"] & M'\Phi(C) \\
 & M' \boxast \Phi(C) \arrow[ur, "t"] &
\end{tikzcd}
$$

The top and bottom faces commute by definition of σ_j;

the front right face commutes by definition of $\xi \boxast 1$;

the back left face commutes because ξ is a comodule homomorphism;

the back right face commutes by naturality of Φ;

and $t : M' \boxast \Phi(C) \to M'\Phi(C)$ is monic;

so by a previous argument, the front left face commutes, which is what we wanted.

Next, we show that these natural transformations assemble to a modification, that is, that the following diagram commutes for any $X \in \mathscr{C}(j,k)$:

$$
\begin{tikzcd}
\Phi_k \circ \mathbb{M}_C(X) \arrow[r, "\sigma_k * 1"] \arrow[d, "\Phi_X"] & (- \boxast \Phi(C))_k \circ \mathbb{M}_C(X) \arrow[d, equal] \\
\mathbb{M}_D(X) \circ \Phi_j \arrow[r, "1 * \sigma_j"] & \mathbb{M}_D(X) \circ (- \boxast \Phi(C))_k
\end{tikzcd}
$$

Fixing a right C-comodule 1-morphism $M \in \mathscr{C}(i,j)$, consider the following diagram:

$$
\begin{tikzcd}
 & \Phi(XMC) \arrow[d, "\Phi_X"] \arrow[dr, "\Phi_{XM}"] & \\
\Phi(XM) \arrow[ur, "\Phi(1*\delta_{M,C})"] \arrow[d, "\Phi_X"] \arrow[dr, "\sigma_{k,XM}*1"] & X\Phi(MC) \arrow[dr, "1*\Phi_M"] & (XM)\Phi(C) \arrow[d, equal] \\
X\Phi(M) \arrow[ur, "1*\Phi(\delta_{M,C})"] \arrow[dr, "1*\sigma_{j,M}"] & (XM) \boxast \Phi(C) \arrow[ur, "t"] \arrow[d, equal] & X(M\Phi(C)) \\
 & X(M \boxast \Phi(C)) \arrow[ur, "1*t"] &
\end{tikzcd}
$$

The top and bottom faces commute by definition of σ;

the back left face commutes by naturality of Φ;

the back right face commutes by higher coherence for Φ;

the front right face commutes trivially;

and $1 * t$ is monic.

Therefore, by a previous argument, the front left face commutes, that is, σ is a modification.

Now, to show that σ is an invertible modification, it is sufficient to show that it is pointwise invertible, that is, the σ_j are all natural isomorphisms; so it is sufficient to show that the $\sigma_{j,M}$ are invertible.

Consider the following composition, which we suggestively call $\sigma_{j,M}^{-1} = \sigma^{-1}$:

$$
M \boxast \Phi(C) \xrightarrow{\quad t \quad} M\Phi(C) \xrightarrow{\quad \Phi_M^{-1} \quad} \Phi(MC) \xrightarrow{\quad \Phi(1*\varepsilon) \quad} \Phi(M)
$$

We want to show that this is, indeed, an inverse for σ.

Consider the following diagram:

$$
\begin{tikzcd}
\Phi(M) \arrow[rr, "\sigma"] \arrow[dr, "\Phi(\delta_{M,C})"] \arrow[dddrr, equal, bend right] & & M \circledast \Phi(C) \arrow[d, "t"] \\
& \Phi(MC) \arrow[r, "\Phi_M"] \arrow[dr, equal] & M\Phi(C) \arrow[d, "\Phi_M^{-1}"] \\
& & \Phi(MC) \arrow[d, "\Phi(1*\varepsilon)"] \\
& & \Phi(M)
\arrow[from=1-1, to=2-3, phantom, "1"]
\arrow[from=2-2, to=3-3, phantom, "3", shift left=3]
\arrow[from=2-2, to=4-3, phantom, "2", shift right=3]
\end{tikzcd}
$$

1 commutes by definition of σ;

2 commutes by the counitality axiom for M;

3 commutes trivially.

Therefore the outer diagram commutes, that is, σ^{-1} is a left inverse for σ.

Next, consider the following diagram:

$$
\begin{tikzcd}
M \circledast \Phi(C) \arrow[d, "t"] & & \\
M\Phi(C) \arrow[d, "\Phi_M^{-1}"] \arrow[ddddrr, equal] & & \\
\Phi(MC) \arrow[dd, "\Phi(1*\varepsilon)"] \arrow[dr, equal] & & \\
& \Phi(MC) \arrow[dr, "\Phi_M"] & \\
\Phi(M) \arrow[r, "\sigma"] \arrow[ur, "\Phi(\delta_{M,C})"] & M \circledast \Phi(C) \arrow[r, "t"] & M\Phi(C)
\arrow[from=3-1, to=4-2, phantom, "1", shift left=4]
\arrow[from=3-1, to=5-2, phantom, "2"]
\arrow[from=4-2, to=5-2, phantom, "3"]
\end{tikzcd}
$$

1 commutes trivially;

2 commutes by counitality for M;

3 commutes by definition of σ.

Therefore $t \circ \sigma \circ \sigma^{-1} = t$ as maps $M \circledast \Phi(C) \to M\Phi(C)$. Since t is monic, σ^{-1} is a right inverse for σ, so σ is invertible. So we are done.

When Φ is exact, it sends injective objects to injective objects, so $\Phi(C)$ is biinjective.

□

Proposition 1.4.11. Let N, L be right-injective C-D-bicomodules 1-morphisms in $\underline{\mathscr{C}}$, and suppose $\sigma : (- \circledast N) \to (- \circledast L)$ is a modification. Define ϕ_σ as the following composition:

$$N \xrightarrow[\delta^l_N]{} C \circledast N \xrightarrow[\sigma_{i,C}]{} C \circledast L \xrightarrow[\varepsilon^l_L]{} L$$

Then ϕ_σ is a bicomodule homomorphism, and $\sigma = - \circledast \phi_\sigma$. Moreover, this correspondence between modifications and bicomodule homomorphisms is a bijection.

Proof : First, we show that ϕ_σ is a bicomodule homomorphism. Since δ^l_N and ε^l_L are bicomodule homomorphisms, and $\sigma_{i,C}$ by definition is a right comodule homomorphism, it's sufficient to show that $\sigma_{i,C}$ is a left comodule homomorphism, that is, that the following diagram commutes:

$$\begin{tikzcd}
C \circledast N \arrow[r, "\sigma_{i,C}"] \arrow[d, "\Delta \circledast 1"] & C \circledast L \arrow[d, "\Delta \circledast 1"] \\
CC \circledast N \arrow[r, "\sigma_{i,CC}"] & CC \circledast L
\end{tikzcd}$$

But this is immediate from the fact that σ_i is a natural transformation. So ϕ_σ is a bicomodule homomorphism.

Next, we need to show that $\sigma = - \circledast \phi_\sigma$, that is, that for each $j \in \mathscr{C}$ and each $M \in \mathbb{M}_C(j)$, $\sigma_{j,M} : M \circledast N \to M \circledast L$ and $1 \circledast \phi_\sigma : M \circledast N \to M \circledast L$ are equal. This is precisely the statement that the following diagram commutes:

$$\begin{tikzcd}
M \circledast N \arrow[r, "\sigma_{j,M}"] \arrow[d, "1 \circledast \delta^l_N"] & M \circledast L \\
M \circledast C \circledast N \arrow[r, "1 \circledast \sigma_{i,C}"] & M \circledast C \circledast L \arrow[u, "1 \circledast \varepsilon^l_L"]
\end{tikzcd}$$

To see that it does, we first consider the following diagram:

$$
\begin{tikzcd}
M \circledast C \circledast N \arrow[r, shift left, "1 \circledast \sigma_{i,C}"] \arrow[r, shift right, "\sigma_{j,M \circledast C}"'] \arrow[d, "t"] & M \circledast C \circledast L \arrow[d, "t"] \\
MC \circledast N \arrow[r, shift left, "1 * \sigma_{i,C}"] \arrow[r, shift right, "\sigma_{j,MC}"'] & MC \circledast L
\end{tikzcd}
$$

We note that travelling along the top horizontal morphisms, the square commutes by definition of $1 \circledast \sigma_{i,C}$; along the bottom horizontal morphisms, this square commutes by naturality of σ_j; and the bottom morphisms are equal because σ is a modification. So the top morphisms, post-composed with t, are equal. But t is monic, so $1_M \circledast \sigma_{i,C} = \sigma_{j,M \circledast C}$.

Now, consider the following diagram:

$$
\begin{tikzcd}
M \circledast N \arrow[r, "\sigma_{j,M}"] & M \circledast L \\
M \circledast (C \circledast N) \arrow[u, shift left, "1 \circledast \varepsilon^l_N"] \arrow[u, shift right, "\varepsilon^r_M \circledast 1"'] \arrow[r, shift left, "\sigma_{j,M \circledast C}"] \arrow[r, shift right, "1 \circledast \sigma_{i,C}"'] & M \circledast (C \circledast L) \arrow[u, shift left, "\varepsilon^r_M \circledast 1"] \arrow[u, shift right, "1 \circledast \varepsilon^l_L"']
\end{tikzcd}
$$

The bottom horizontal arrows are equal by the above argument. The left vertical arrows are equal since the left and right unitors commute, and similarly the right vertical arrows are equal. The inner square commutes because σ_j is a natural transformation. Therefore the outer square commutes, which (noting that $1 \circledast \delta^l_N = (1 \circledast \varepsilon^l_N)^{-1}$) is precisely what we wanted. So $\sigma = - \circledast \phi_\sigma$.

To see that this correspondence between modifications and bicomodule homomorphisms is moreover bijective, we consider $\sigma = - \circledast \phi$, and compute ϕ_σ.

Consider the following diagram:

$$
\begin{tikzcd}
N \arrow[r, "\phi"] \arrow[d, "\delta^l_N"'] & L \\
C \circledast N \arrow[r, "\sigma_{i,C} = 1 \circledast \phi"'] & C \circledast L \arrow[u, "\varepsilon^l_L"']
\end{tikzcd}
$$

This diagram commutes by naturality of ε^l. But the top edge is ϕ, and the composition of the other edges is ϕ_σ. So $\phi_{- \circledast \phi} = \phi$, and thus our correspondence is bijective.

□

1.5 Internal cohom construction

To prove Proposition 1.4.9, we need a natural way to move from a generator of a finitary birepresentation to a coalgebra 1-morphism in $\underline{\mathscr{C}}$. This is given by the internal cohom construction.

Definition 1.5.1 (Internal cohom)**.** Let $\mathbb{M}$ be a finitary birepresentation of $\mathscr{C}$, $X \in \mathbb{M}(j), Y \in \mathbb{M}(i)$.

The functor

$$\underline{\mathscr{C}}(i,j) \to \mathscr{V}ect_{\Bbbk}, F \mapsto \mathrm{Hom}_{\underline{\mathbb{M}}(j)}(X, \underline{\mathbb{M}}(F)Y)$$

is representable, see e.g. [MMM+21] Section 4.3 for details, so there is an object

$$_{\mathbb{M}}[Y,X] \in \underline{\mathscr{C}}(i,j),$$

called the *internal cohom* of X and Y with respect to $\mathbb{M}$, and a bijection (natural in F) $\gamma^{\mathbb{M}}_{X,Y}$: $\mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}({}_{\mathbb{M}}[Y,X], F) \cong \mathrm{Hom}_{\underline{\mathbb{M}}(j)}(X, \underline{\mathbb{M}}(F)Y)$.

We can define

$$\mathrm{coev}^{\mathbb{M}}_{X,Y} := \gamma^{\mathbb{M}}_{X,Y}(1_{{}_{\mathbb{M}}[Y,X]}) : X \to \underline{\mathbb{M}}({}_{\mathbb{M}}[Y,X])Y,$$

and

$$\epsilon^{\mathbb{M}}_{X} := (\gamma^{\mathbb{M}}_{X,X})^{-1}(\underline{\mathbb{M}}^{-0}) : {}_{\mathbb{M}}[X,X] \to 1_i.$$

Then the bijection is given explicitly as follows:

$$\begin{aligned} \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}({}_{\mathbb{M}}[Y,X], F) &\cong \mathrm{Hom}_{\underline{\mathbb{M}}(j)}(X, \underline{\mathbb{M}}(F)Y) \\ \gamma^{\mathbb{M}}_{X,Y} : \alpha &\mapsto \underline{\mathbb{M}}(\alpha) \circ \mathrm{coev}^{\mathbb{M}}_{X,Y} \\ (1_F * \epsilon_Y) \circ {}_{\mathbb{M}}[Y,f] &\leftarrow\!\shortmid f : (\gamma^{\mathbb{M}}_{X,Y})^{-1} \end{aligned}$$

$\lhd$

Example 1.5.2. If M is a right C-comodule 1-morphism, then there is a bijection

$$\begin{aligned} \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}(M, F) &\cong \mathrm{Hom}_{\underline{\mathbb{M}_C}(j)}(M, FC) \\ \gamma^{\mathbb{M}_C}_{M,C} : \alpha &\mapsto (\alpha * 1) \circ \delta_{M,C} \\ (1_F * \epsilon_C) \circ \beta &\leftarrow\!\shortmid \beta : (\gamma^{\mathbb{M}_C}_{M,C})^{-1} \end{aligned}$$

so we can take ${}_{\mathbb{M}_C}[C,-] \cong 1_{\mathsf{comod}(\mathscr{C})}$, $\mathrm{coev}^{\mathbb{M}_C}_{M,C} = \delta_{M,C}$, $\epsilon_C = \epsilon_C$. $\lll$

Lemma 1.5.3. Suppose $G : j \to k$ is a 1-morphism in $\mathscr{C}$. Then ${}_{\mathbb{M}}[Y, \mathbb{M}(G)X] \cong G \circ {}_{\mathbb{M}}[Y,X]$, and $\mathrm{coev}^{\mathbb{M}}_{\mathbb{M}(G)X,Y} = \mathbb{M}^{-2} \circ (1 * \mathrm{coev}^{\mathbb{M}}_{X,Y})$. Similarly, if $G : i \to k$, then ${}_{\mathbb{M}}[\mathbb{M}(G)Y, X] \cong {}_{\mathbb{M}}[Y,X] \circ G^*$.

Proof : Consider the following sequence of bijections

$$\begin{aligned}
\mathrm{Hom}_{\underline{\mathscr{C}}(i,k)}({}_{\mathbb{M}}[Y, \mathbb{M}(G)X], F) &\cong \mathrm{Hom}_{\underline{\mathbb{M}}(k)}(\mathbb{M}(G)X, \mathbb{M}(F)Y) \\
&\cong \mathrm{Hom}_{\underline{\mathbb{M}}(j)}(X, \mathbb{M}(G^*)\mathbb{M}(F)Y) \\
&\cong \mathrm{Hom}_{\underline{\mathbb{M}}(j)}(X, \mathbb{M}(G^*F)Y) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}({}_{\mathbb{M}}[Y,X], G^*F) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}(G \circ {}_{\mathbb{M}}[Y,X], F)
\end{aligned}$$

where the first and fourth bijections are the cohom adjunction given above, and the second and fifth use the fact that (G, G^*) is an adjoint pair. Since each of these bijections is natural in F, by uniqueness of the internal cohom, we have ${}_{\mathbb{M}}[Y, \mathbb{M}(G)X] \cong G \circ {}_{\mathbb{M}}[Y,X]$, $\mathrm{coev}^{\mathbb{M}}_{\mathbb{M}(G)X,Y} = \mathbb{M}^{-2} \circ (1 * \mathrm{coev}^{\mathbb{M}}_{X,Y})$.

The other part is similar. $\square$

Using the idea of an internal cohom, we can define a nice class of coalgebra 1-morphisms and bicomodule 1-morphisms in $\mathscr{C}$.

Lemma 1.5.4. Consider the following composition of maps, which we denote τ:

$$\begin{tikzcd}
X \arrow[d, "\mathrm{coev}^{\mathbb{M}}_{X,Y}"] \\
\underline{\mathbb{M}}({}_{\mathbb{M}}[Y,X])Y \arrow[d, "1*\mathrm{coev}^{\mathbb{M}}_{Y,Z}"] \\
\underline{\mathbb{M}}({}_{\mathbb{M}}[Y,X])\underline{\mathbb{M}}({}_{\mathbb{M}}[Z,Y])Z \arrow[d, "\underline{\mathbb{M}}^{-2}"] \\
\underline{\mathbb{M}}({}_{\mathbb{M}}[Y,X]{}_{\mathbb{M}}[Z,Y])Z
\end{tikzcd}$$

Then we can define $\delta_{X,Y,Z} := \gamma^{-1}(\tau) : {}_{\mathbb{M}}[Z,X] \to {}_{\mathbb{M}}[Y,X]{}_{\mathbb{M}}[Z,Y]$. We have the following:

- $({}_{\mathbb{M}}[X,X], \delta_{X,X,X}, \epsilon_X)$ is a coalgebra 1-morphism;

- $({}_{\mathbb{M}}[Y,X], \delta_{X,X,Y}, \delta_{X,Y,Y})$ is an ${}_{\mathbb{M}}[X,X]$-${}_{\mathbb{M}}[Y,Y]$-bicomodule 1-morphism.

Proof : See Proposition 4.9 in [MMM+21]. □

This inspires a notational shorthand:

Notation 1.5.5. Let $X \in \mathbb{M}(i)$.

We write $\mathsf{C}^X_{\mathbb{M}} := {}_{\mathbb{M}}[X,X]$ as coalgebra 1-morphisms. We write the comultiplication and counit maps for $\mathsf{C}^X_{\mathbb{M}}$ as $\Delta^{\mathbb{M}}_X$ and $\epsilon^{\mathbb{M}}_X$ respectively, and define $\operatorname{coev}^{\mathbb{M}}_{\mathsf{C}^X} = \operatorname{coev}^{\mathbb{M}}_{X,X} : X \to \underline{\mathbb{M}}(\mathsf{C}^X)X$. We suppress $\mathbb{M}$ when it is clear from context, that is, we write C^X, Δ_X, and so on.

Lemma 1.5.6. Suppose $f : X \to Y$ is a morphism in $\mathbb{M}(j)$, $Z \in \mathbb{M}(i)$. Then ${}_{\mathbb{M}}[Z,f] : {}_{\mathbb{M}}[Z,X] \to {}_{\mathbb{M}}[Z,Y]$ is a right C^Z-comodule homomorphism.

Proof : We want to show that the following diagram commutes:

$$\begin{tikzcd}
{}_{\mathbb{M}}[Z,X] \arrow[r, "{}_{\mathbb{M}}[Z,f]"] \arrow[d, "\delta_{{}_{\mathbb{M}}[Z,X],\mathsf{C}^Z}"'] & {}_{\mathbb{M}}[Z,Y] \arrow[d, "\delta_{{}_{\mathbb{M}}[Z,Y],\mathsf{C}^Z}"] \\
{}_{\mathbb{M}}[Z,X]\mathsf{C}^Z \arrow[r, "{}_{\mathbb{M}}[Z,f]*1"'] & {}_{\mathbb{M}}[Z,Y]\mathsf{C}^Z
\end{tikzcd}$$

To see that it does, consider the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "\operatorname{coev}^{\mathbb{M}}_{X,Z}"] \arrow[drr, "f"] \arrow[dd, "\operatorname{coev}^{\mathbb{M}}_{X,Z}"'] & & \mathbb{M}({}_{\mathbb{M}}[Z,X])Z \arrow[rr, "\mathbb{M}({}_{\mathbb{M}}[Z,f])"] \arrow[d, phantom, "1"] & & \mathbb{M}({}_{\mathbb{M}}[Z,Y])Z \arrow[dl, "1*\operatorname{coev}_{\mathsf{C}^Z}"'] \arrow[dddd, "\mathbb{M}(\delta_{{}_{\mathbb{M}}[Z,Y],\mathsf{C}^Z})"] \\
& & Y \arrow[urr, "\operatorname{coev}^{\mathbb{M}}_{Y,Z}"] \arrow[dd, phantom, "3"] & & \\
\mathbb{M}({}_{\mathbb{M}}[Z,X])Z \arrow[uurrrr, "\mathbb{M}({}_{\mathbb{M}}[Z,f])"] \arrow[dr, "1*\operatorname{coev}_{\mathsf{C}^Z}"] \arrow[dd, "\mathbb{M}(\delta_{{}_{\mathbb{M}}[Z,X],\mathsf{C}^Z})"'] \arrow[ur, phantom, "2"] & & & & \\
& \mathbb{M}({}_{\mathbb{M}}[Z,X])\mathbb{M}(\mathsf{C}^Z)Z \arrow[rr, "\mathbb{M}({}_{\mathbb{M}}[Z,f])*1"] \arrow[dl, "\mathbb{M}^{-2}"] \arrow[l, phantom, "4"] \arrow[dr, phantom, "6"] & & \mathbb{M}({}_{\mathbb{M}}[Z,Y])\mathbb{M}(\mathsf{C}^Z)Z \arrow[dr, "\mathbb{M}^{-2}"] \arrow[ur, phantom, "5"] & \\
\mathbb{M}({}_{\mathbb{M}}[Z,X]\mathsf{C}^Z)Z \arrow[rrrr, "\mathbb{M}({}_{\mathbb{M}}[Z,f]*1)"'] & & & & \mathbb{M}({}_{\mathbb{M}}[Z,Y]\mathsf{C}^Z)Z
\end{tikzcd}$$

1 and 2 commute by definition of ${}_{\mathbb{M}}[Z,f]$;

3 commutes by the interchange law;

4 precomposed with $\operatorname{coev}^{\mathbb{M}}_{X,Z}$, and 5 precomposed with $\operatorname{coev}^{\mathbb{M}}_{Y,Z}$, commute by definition of $\delta_{{}_{\mathbb{M}}[Z,X],\mathsf{C}^Z}$ and $\delta_{{}_{\mathbb{M}}[Z,Y],\mathsf{C}^Z}$ respectively;

and 6 commutes by naturality of $\mathbb{M}^{-2}$.

Therefore the outer diagram commutes. Passing via $\gamma_{X,Z}$, this is precisely what we wanted. $\square$

2

Internalising birepresentations

Following from the work of Section 1.4, we know that essentially all data about (cyclic) birepresentations can be expressed by discussing internal coalgebra and biinjective bicomodule 1-morphisms in $\underline{\mathscr{C}}$; see Propositions 1.4.8 to 1.4.11. We make this precise in two parts: first showing that $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ and $\mathscr{C}-cfmod_{ex}^*$ (defined in Definitions 1.3.13 and 1.4.5 respectively) are biequivalent as bicategories, and then constructing an explicit form for this equivalence.

2.1 A biequivalence between $\mathscr{B}\mathscr{B}$icom$_{\underline{\mathscr{C}}}$ and $\mathscr{C}-cfmod_{ex}^*$

This subsection follows the structure of [MMM+21], whose Theorem 4.28 is the same claim as our Theorem 2.1.2.

First, we prove Proposition 1.4.9.

Proposition 2.1.1 (Restatement of Proposition 1.4.9)**.** Suppose $\mathbb{M}$ is a cyclic birepresentation of a fiat 2-category $\mathscr{C}$ with generator $Z \in \mathbb{M}(i)$. Then there is an adjoint equivalence between $\mathbb{M}$ and $\mathbb{M}_{\mathsf{C}^Z}$.

Proof : By Proposition 1.1.7, it's sufficient to show that there's a strong morphism of birepresentations $\Phi : \mathbb{M} \to \mathbb{M}_{\mathsf{C}^Z}$ that at each object $i \in \mathscr{C}$ is an equivalence in $\mathfrak{A}_{\Bbbk}^f$, that is, an equivalence of categories.

Define $\Phi_j : \mathbb{M}(j) \to \mathbb{M}_{\mathsf{C}^Z}(j)$ to be ${}_{\mathbb{M}}[Z, -]$.

We first show that this is well-defined, that is, for each $X \in \mathbb{M}(j)$, ${}_{\mathbb{M}}[Z, X]$ is an injective right C^Z-comodule 1-morphism, and for each $f : X \to Y$ in $\mathbb{M}(j)$, ${}_{\mathbb{M}}[Z, f]$ is a homomorphism of right C^Z-comodule 1-morphisms.

By Lemma 1.5.3, if $X = \mathbb{M}(F)Z$ for some $F \in \mathscr{C}(i, j)$, then ${}_{\mathbb{M}}[Z, X] \cong F\mathsf{C}^Z$. Since the internal cohom is additive, we know that if X is isomorphic to a summand of $\mathbb{M}(F)Z$ for some $F \in \mathscr{C}(i, j)$, then ${}_{\mathbb{M}}[Z, X]$ is isomorphic to a summand of $F\mathsf{C}^Z$, so by Lemma 1.3.7 is injective as a right C^Z-comodule

1-morphism. But by definition, since Z generates $\mathbb{M}$, every $X \in \mathbb{M}(j)$ is of such a form. So for each $X \in \mathbb{M}(j)$, ${}_{\mathbb{M}}[Z, X]$ is an injective right C^Z-comodule 1-morphism.

That ${}_{\mathbb{M}}[Z, f]$ is a homomorphism of right C^Z-comodule 1-morphisms is precisely Lemma 1.5.6.

Moreover, ${}_{\mathbb{M}}[Z, -]$ is clearly functorial.

Next, we show that this functor is an equivalence of categories, by showing it is essentially surjective and fully faithful.

To see it is essentially surjective, note that it sends a generator of $\mathbb{M}$ to a generator of $\mathbb{M}_{\mathsf{C}^Z}$.

To see it is fully faithful, we note that it is clearly faithful, and since by definition hom sets in $\mathbb{M}(j)$ and $\mathbb{M}_{\mathsf{C}^Z}(j)$ are finite-dimensional vector spaces, it's sufficient to show that $\dim_{\Bbbk} \mathrm{Hom}_{\mathbb{M}(j)}(X, Y) = \dim_{\Bbbk} \mathrm{Hom}_{\mathbb{M}_{\mathsf{C}^Z}(j)}({}_{\mathbb{M}}[Z, X], {}_{\mathbb{M}}[Z, Y])$ for any $X, Y \in \mathbb{M}(j)$. But we can compute, for $X = \mathbb{M}(F)Z$, $Y = \mathbb{M}(G)Z$, $F, G \in \mathscr{C}(i, j)$, that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{M}_{\mathsf{C}^Z}(j)}({}_{\mathbb{M}}[Z, X], {}_{\mathbb{M}}[Z, Y]) &\cong \mathrm{Hom}_{\mathbb{M}_{\mathsf{C}^Z}(j)}(F\mathsf{C}^Z, G\mathsf{C}^Z) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}(F\mathsf{C}^Z, G) \\
&\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,i)}(\mathsf{C}^Z, F^*G) \\
&\cong \mathrm{Hom}_{\mathbb{M}(i,i)}(Z, \mathbb{M}(F^*G)Z) \\
&\cong \mathrm{Hom}_{\mathbb{M}(i,j)}(\mathbb{M}(F)Z, \mathbb{M}(G)Z)
\end{aligned}$$

where isomorphisms are isomorphisms of $\Bbbk$-vector spaces.

The first isomorphism follows from Lemma 1.5.6. The second is the isomorphism given by

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{M}_{\mathsf{C}^Z}(j)}(F\mathsf{C}^Z, G\mathsf{C}^Z) &\cong \mathrm{Hom}_{\underline{\mathscr{C}}(i,j)}(F\mathsf{C}^Z, G) \\
f &\mapsto (1 * \epsilon_Z) \circ f \\
(g * 1) \circ (1 * \Delta_{\mathsf{C}^Z}) &\leftarrow\!\shortmid g
\end{aligned}$$

The third and fifth are the isomorphisms given by the adjunction $F \vdash F^*$, and the fourth is the adjunction that defines C^Z.

So these hom sets are isomorphic as vector spaces. Since every object is a summand of some $\mathbb{M}(F)Z$, and both the internal cohom and hom sets are additive, the generalisation to arbitrary objects X and Y follows, and in particular $\dim_{\Bbbk} \mathrm{Hom}_{\mathbb{M}(j)}(X, Y) = \dim_{\Bbbk} \mathrm{Hom}_{\mathbb{M}_{\mathsf{C}^Z}(j)}({}_{\mathbb{M}}[Z, X], {}_{\mathbb{M}}[Z, Y])$. So

${}_{\mathbb{M}}[Z, -] : \mathbb{M}(j) \to \mathbb{M}_{\mathsf{C}^Z}(j)$ is fully faithful and essentially surjective, so an equivalence of categories.

Finally, we show that these functors assemble to a morphism of birepresentations.

We can compute

$$\begin{aligned} {}_{\mathbb{M}}[Z, -] \circ \mathbb{M}(F) &= {}_{\mathbb{M}}[Z, \mathbb{M}(F) -] \\ &\cong F_{\mathbb{M}}[Z, -] \\ &= \mathbb{M}_{\mathsf{C}^Z}(F) \circ {}_{\mathbb{M}}[Z, -] \end{aligned}$$

from which our claim immediately follows. □

Theorem 2.1.2. Let $\mathscr{C}$ be a fiat 2-category. Consider the assignment of data $\iota : \mathscr{R}\mathscr{B}icom_{\underline{\mathscr{C}}} \to \mathscr{C} - cfmod^*$ given as follows:

- $C \mapsto (\mathbb{M}_C, C)$, for C a coalgebra 1-morphism;
- $N \mapsto - \circledast N : (\mathbb{M}_C, C) \to (\mathbb{M}_D, D)$, for N a right injective C-D-bicomodule 1-morphism;
- $(\phi : N \to N') \mapsto (- \circledast \phi : - \circledast N \to - \circledast N')$, for ϕ a bicomodule homomorphism;
- $\iota^2 = id$;
- $\iota^0 = \varepsilon^r_-$.

This is a pseudofunctor, and in fact a biequivalence. This restricts to a biequivalence $\iota : \mathscr{B}\mathscr{B}icom_{\underline{\mathscr{C}}} \to \mathscr{C} - cfmod^*_{ex}$

Proof :

By Proposition 1.4.8, these data are well-defined.

It is immediate that $- \circledast (N \circledast M)(L) = L \circledast (N \circledast M) \cong (L \circledast N) \circledast M = (- \circledast M) \circ (- \circledast N)(L)$, so this assignment respects composition of 1-morphisms.

By Proposition 1.3.10, $(- \circledast \psi) \circ (- \circledast \phi) = - \circledast \psi\phi$, so this assignment strictly respects composition of 2-morphisms.

It is straightforward to see that this assignment strictly respects identity 2-morphisms, and that $\varepsilon^r_- : \iota({}_C C_C) = - \circledast C \to 1_{\mathbb{M}_C}$ is an invertible modification of morphisms of representations that will satisfy the higher coherence diagrams for a pseudofunctor.

By Proposition 1.4.9, ι is essentially surjective on objects, that is, any object in $\mathscr{C} - cfmod^*$ is adjoint equivalent to some $\iota(C)$ for a coalgebra 1-morphism C. By Proposition 1.4.10 it is essentially full on 1-morphisms. By Proposition 1.4.11, it is fully faithful on 2-morphisms. Therefore by Theorem 7.4.1 in [JY20], it is an equivalence. By Proposition 1.4.10, biinjective bicomodule 1-morphisms correspond to exact morphisms of representations. □

2.2 An explicit form of the biequivalence

The previous theorem provided us with one half of the biequivalence, but we used a non-constructive result to assert the existence of a weak inverse. This subsection is dedicated to constructing such a weak inverse explicitly. We note that, in showing our constructed pseudofunctor is indeed a weak inverse for ι, we use Theorem 2.1.2, so this proof-by-parts is not superfluous.

For a morphism of (pointed) birepresentations $\Phi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$, that is, a morphism of birepresentations $\Phi : \mathbb{M} \to \mathbb{M}'$ which have specified generators $X \in \mathbb{M}(j)$, $Y \in \mathbb{M}'(i)$, we want to see ${}_{\mathbb{M}'}[Y, \Phi_j(X)]$ as a C^X-C^Y-bicomodule 1-morphism. We construct a coalgebra homomorphism $\mathsf{C}^{\Phi_j(X)} \to \mathsf{C}^X$, and then apply Lemma 1.3.15. Through this subsection, we fix the notation of Φ, $\mathbb{M}$, $\mathbb{M}'$, X and Y

Lemma 2.2.1. Let $\mathscr{C}$ be a finitary 2-category. Overloading notation, write Φ for Φ_j. Consider the composition of maps

$$\Phi(X) \xrightarrow{\Phi(\mathrm{coev}_{\mathsf{C}^X})} \Phi(\mathbb{M}(\mathsf{C}^X)X) \xrightarrow{\Phi_{\mathsf{C}^X}} \mathbb{M}'(\mathsf{C}^X)\Phi(X)$$

Since this defines a map $\Phi(X) \to \mathbb{M}'(\mathsf{C}^X)\Phi(X)$, we can pass to a map $\theta_\Phi : \mathsf{C}^{\Phi_j(X)} \to \mathsf{C}^X$. This map is a coalgebra homomorphism.

Proof : Let $\theta = \theta_\Phi$.

First, to see that θ respects comultiplication, consider the following diagram:

$$
\begin{tikzcd}
\Phi(X) \arrow[r, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[dr, "\Phi(\mathrm{coev}_{\mathsf{C}^X})"] \arrow[dd, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"'] & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[rr, "\mathbb{M}'(\theta)"] \arrow[d, phantom, "2"] & & \mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[dddd, "\mathbb{M}'(\Delta_X)"] \\
& \Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "\Phi(\mathbb{M}(\Delta_X))"] \arrow[urr, "\Phi_{\mathsf{C}^X}"] \arrow[dd, phantom, "5"] & \Phi(\mathbb{M}(\mathsf{C}^X\mathsf{C}^X)X) \arrow[dddr, "\Phi_{(\mathsf{C}^X\mathsf{C}^X)}"] \arrow[ur, phantom, "1"] & \\
\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[dr, "1*\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[dd, "\mathbb{M}'(\Delta_{\Phi(X)})"'] & & & \\
\arrow[r, phantom, "3"] & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[r, "\mathbb{M}'(\theta)*\mathbb{M}'(\theta)"] \arrow[dl, "\mathbb{M}'^{-2}"] \arrow[dr, phantom, "4"] & \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[dr, "\mathbb{M}'^{-2}"] & \\
\mathbb{M}'(\mathsf{C}^{\Phi(X)}\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[rrr, "\mathbb{M}'(\theta*\theta)"] & & & \mathbb{M}'(\mathsf{C}^X\mathsf{C}^X)\Phi(X)
\end{tikzcd}
$$

1 commutes by naturality of Φ;

2 commutes by definition of θ;

3, precomposed with $\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}$, commutes by definition of $\Delta_{\Phi(X)}$;

4 commutes by naturality of $\mathbb{M}'^{-2}$;

5 can be seen to commute by consideration of the following diagram:

$$
\begin{tikzcd}[column sep=small]
\Phi(X) \arrow[rr, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[d, dashed, "\Phi(\mathrm{coev}_{\mathsf{C}^X})"'] \arrow[drr, phantom, "6"] & & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[rr, "1*\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[d, dashed, "\mathbb{M}'(\theta)"'] \arrow[dr, "1*\Phi(\mathrm{coev}_{\mathsf{C}^X})"] \arrow[drr, phantom, "8"] & & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[d, dashed, "1*\mathbb{M}'(\theta)"] \\
\Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[rr, "\Phi_{\mathsf{C}^X}"] \arrow[dd, "\Phi(\mathbb{M}(\Delta_X))"'] \arrow[ddr, "\Phi(1*\mathrm{coev}_{\mathsf{C}^X})"] \arrow[dddr, phantom, "11"] & & \mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[dr, "1*\Phi(\mathrm{coev}_{\mathsf{C}^X})"] \arrow[r, phantom, "7"] \arrow[ddl, phantom, "10"] & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "1*\Phi_{\mathsf{C}^X}"] \arrow[d, dashed, "\mathbb{M}'(\theta)*1"] \arrow[dr, phantom, "9"] & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[d, dashed, "\mathbb{M}'(\theta)*1"] \\
& & & \mathbb{M}'(\mathsf{C}^X)\Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "1*\Phi_{\mathsf{C}^X}"] \arrow[dd, phantom, "12"] & \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[dd, "\mathbb{M}'^{-2}"] \\
& \Phi(\mathbb{M}(\mathsf{C}^X)\mathbb{M}(\mathsf{C}^X)X) \arrow[urr, "\Phi_{\mathsf{C}^X}*1"] \arrow[dl, "\Phi(\mathbb{M}^{-2})"] & & & \\
\Phi(\mathbb{M}(\mathsf{C}^X\mathsf{C}^X)X) \arrow[rrrr, "\Phi_{(\mathsf{C}^X\mathsf{C}^X)}"] & & & & \mathbb{M}'(\mathsf{C}^X\mathsf{C}^X)\Phi(X)
\end{tikzcd}
$$

(6) and (8) commute by definition of θ;

(7) and (9) commute by the interchange law;

(10) and (12) commute by naturality of Φ;

(11) commutes by definition of Δ_X;

Therefore the outer diagram commutes, that is, (5) commutes, so the outer square of the first diagram commutes. But this says, passing via $\gamma_{\Phi(X),\Phi(X)}$, that the following diagram commutes, that is, θ respects comultiplication, as required:

$$\begin{tikzcd}
\mathsf{C}^{\Phi(X)} \arrow[r, "\theta"] \arrow[d, "\Delta_{\Phi(X)}"'] & \mathsf{C}^X \arrow[d, "\Delta_X"] \\
\mathsf{C}^{\Phi(X)}\mathsf{C}^{\Phi(X)} \arrow[r, "\theta*\theta"] & \mathsf{C}^X\mathsf{C}^X
\end{tikzcd}$$

Next, we want to see that θ respects the counit. Consider the following diagram:

$$\begin{tikzcd}
\Phi(X) \arrow[rrrr, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[drrr, "\Phi(\mathrm{coev}_{\mathsf{C}^X})"] \arrow[ddrr, "\Phi(\mathbb{M}^{-0})"'] \arrow[dddrr, "\mathbb{M}'^{-\mathsf{C}}"'] & & & & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[d, "\mathbb{M}'(\theta)"] \\
& & (2) & \Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "\Phi_{\mathsf{C}^X}"'] \arrow[dl, "\Phi(\mathbb{M}(\epsilon_X))"] \arrow[ur, phantom, "(3)"] & \mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[ddll, "\mathbb{M}'(\epsilon_X)"] \\
& (1) & \Phi(\mathbb{M}(1)X) \arrow[d, "\Phi_1"] & (4) & \\
& & \mathbb{M}'(1)\Phi(X) & &
\end{tikzcd}$$

(1) and (4) commute by naturality of Φ;

(2) commutes by definition of ϵ_X;

(3) commutes by definition of θ.

Therefore, the outer triangle commutes. Passing via $\gamma_{\Phi(X),\Phi(X)}$, this says that the following diagram commutes, that is, θ respects the counit:

$$\begin{tikzcd}
\mathsf{C}^{\Phi(X)} \arrow[rr, "\theta"] \arrow[dr, "\epsilon_{\Phi(X)}"'] & & \mathsf{C}^X \arrow[dl, "\epsilon_X"] \\
& 1 &
\end{tikzcd}$$

So θ is a coalgebra homomorphism. $\square$

Definition 2.2.2 (Bicomodule 1-morphism generated by Φ)**.** Define $N^{\Phi} = {}^{\theta_\Phi}({}_{\mathbb{M}'}[Y, \Phi(X)])$. This is a C^X-C^Y-bicomodule 1-morphism. $\triangleleft$

Lemma 2.2.3. N^{Φ} is injective as a right comodule 1-morphism.

Proof : Since Y generates $\mathbb{M}'$, $\Phi(X)$ is isomorphic to a summand of $\mathbb{M}'(F)Y$ for some F in $\mathscr{C}$. But then ${}_{\mathbb{M}}[Y, \Phi(X)]$ is isomorphic to a summand of ${}_{\mathbb{M}}[Y, \mathbb{M}'(F)Y] \cong F_{\mathbb{M}}[Y, Y] = F\mathsf{C}^Y$. So ${}_{\mathbb{M}}[Y, \Phi(X)]$ is injective as a right comodule 1-morphism, so N^{Φ} is. $\square$

Notation 2.2.4. Write $\mathrm{coev}_{N^\Phi} = \mathrm{coev}^{\mathbb{M}'}_{\Phi(X),Y} : \Phi(X) \to \underline{\mathbb{M}}'(N^{\Phi})Y$, where here N^{Φ} is treated as an unadorned 1-morphism of $\underline{\mathscr{C}}$. We note that, as a morphism of pointed birepresentations, Φ inherently carries the data of X and Y, but without pointedness, the notation coev_{N^Φ} is ambiguous. If in any instances we work without pointedness, we will note this and disambiguate.

Lemma 2.2.5. If $\sigma : \Phi \to \Psi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$ is a modification of morphisms of pointed birepresentations of a finitary 2-category $\mathscr{C}$, then $\phi^{\sigma} := {}_{\mathbb{M}'}[Y, \sigma_X] : N^{\Phi} \to N^{\Psi}$ is a bicomodule homomorphism.

Proof : We need to show that ϕ^{σ} is compatible with the left C^X- and right C^Y-coactions. That it is compatible with the right C^Y-coaction follows immediately from Lemma 1.5.6.

To see that it is compatible with the left C^X-coaction, consider the following diagram:

$$
\begin{tikzcd}
\Phi(X) \arrow[r, "\mathrm{coev}_{N^\Phi}"] \arrow[d, "\mathrm{coev}_{N^\Phi}"'] \arrow[ddr, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"] \arrow[drr, "\sigma_X"] & \mathbb{M}'(N^\Phi)Y \arrow[rr, "\mathbb{M}'(\phi^\sigma)"] \arrow[drr, phantom, "\textcircled{3}"] & & \mathbb{M}'(N^\Psi)Y \arrow[ddd, "\mathbb{M}'(\delta_{\mathsf{C}^{\Psi(X)},N^\Psi})"] \\
\mathbb{M}'(N^\Phi)Y \arrow[dd, "\mathbb{M}'(\delta_{\mathsf{C}^{\Phi(X)},N^\Phi})"'] \arrow[ddr, phantom, "\textcircled{1}"] & & \Psi(X) \arrow[ur, "\mathrm{coev}_{N^\Psi}"] \arrow[d, dashed, "\mathrm{coev}_{\mathsf{C}^{\Psi(X)}}"] \arrow[ddr, phantom, "\textcircled{4}"] & \\
& \mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[d, dashed, "1*\mathrm{coev}_{N^\Phi}"] \arrow[r, phantom, "\textcircled{7}"] & \mathbb{M}'(\mathsf{C}^{\Psi(X)})\Psi(X) \arrow[d, dashed, "1*\mathrm{coev}_{N^\Psi}"] & \\
\mathbb{M}'(\mathsf{C}^{\Phi(X)}N^\Phi)Y \arrow[dd, "\mathbb{M}'(\theta_\Phi*1)"'] \arrow[ddr, phantom, "\textcircled{2}"] & \mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(N^\Phi)Y \arrow[l, "\mathbb{M}'^{-2}"'] \arrow[d, "\mathbb{M}'(\theta_\Phi)*1"'] & \mathbb{M}'(\mathsf{C}^{\Psi(X)})\mathbb{M}'(N^\Psi)Y \arrow[r, "\mathbb{M}'^{-2}"] \arrow[d, dashed, "\mathbb{M}'(\theta_\Psi)*1"] \arrow[ddr, phantom, "\textcircled{6}"] & \mathbb{M}'(\mathsf{C}^{\Psi(X)}N^\Psi)Y \\
& \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(N^\Phi)Y \arrow[r, "1*\mathbb{M}'(\phi^\sigma)"'] \arrow[dl, "\mathbb{M}'^{-2}"] \arrow[dr, phantom, "\textcircled{5}"] & \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(N^\Psi)Y \arrow[dr, "\mathbb{M}'^{-2}"] & \\
\mathbb{M}'(\mathsf{C}^XN^\Phi)Y \arrow[rrr, "\mathbb{M}'(1*\phi^\sigma)"] & & & \mathbb{M}'(\mathsf{C}^XN^\Psi)Y \arrow[uu, "\mathbb{M}'(\theta_\Psi*1)"']
\end{tikzcd}
$$

1 commutes by definition of $\delta_{\mathsf{C}^{\Phi(X)},N^\Phi}$;

2, 5 and 6 commute by naturality of $\mathbb{M}'^{-2}$;

3 commutes by definition of ϕ^σ;

4 commutes by definition of $\delta_{\mathsf{C}^{\Psi(X)},N^\Psi}$.

To see that 7 commutes, consider the following diagram:

$$
\begin{tikzcd}
\Phi(X) \arrow[rrr, "\sigma"'] \arrow[d, dashed, "\mathrm{coev}_{\mathsf{C}^{\Phi(X)}}"'] \arrow[dr, "\Phi(\mathrm{coev}_{\mathsf{C}^X})"] \arrow[drr, phantom, "\textcircled{8}"] & & & \Psi(X) \arrow[d, dashed, "\mathrm{coev}_{\mathsf{C}^{\Psi(X)}}"] \arrow[dl, "\Psi(\mathrm{coev}_{\mathsf{C}^X})"'] \\
\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X) \arrow[d, dashed, "1*\mathrm{coev}_{N^\Phi}"'] \arrow[dr, "\mathbb{M}'(\theta_\Phi)"] \arrow[r, phantom, "\textcircled{9}"] & \Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "\sigma"] \arrow[d, dashed, "\Phi_{\mathsf{C}^X}"] \arrow[dr, phantom, "\textcircled{12}"] & \Psi(\mathbb{M}(\mathsf{C}^X)X) \arrow[d, dashed, "\Psi_{\mathsf{C}^X}"] \arrow[r, phantom, "\textcircled{13}"] & \mathbb{M}'(\mathsf{C}^{\Psi(X)})\Psi(X) \arrow[dl, "\mathbb{M}'(\theta_\Psi)"'] \arrow[d, dashed, "1*\mathrm{coev}_{N^\Psi}"] \\
\mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(N^\Phi)Y \arrow[d, "\mathbb{M}'(\theta_\Phi)*1"'] \arrow[r, phantom, "\textcircled{10}"] & \mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[r, "1*\sigma"] \arrow[dl, dashed, "1*\mathrm{coev}_{N^\Phi}"] \arrow[dr, phantom, "\textcircled{11}"] & \mathbb{M}'(\mathsf{C}^X)\Psi(X) \arrow[dr, dashed, "1*\mathrm{coev}_{N^\Psi}"'] \arrow[r, phantom, "\textcircled{14}"] & \mathbb{M}'(\mathsf{C}^{\Psi(X)})\mathbb{M}'(N^\Psi)Y \arrow[d, dashed, "\mathbb{M}'(\theta_\Psi)*1"] \\
\mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(N^\Phi)Y \arrow[rrr, "1*\mathbb{M}'(\phi^\sigma)"'] & & & \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(N^\Psi)Y
\end{tikzcd}
$$

8 commutes since σ_i is a natural transform;

9 and 13 commute by definition of θ_Φ and θ_Ψ respectively;

10 and 14 commute by the interchange law;

11 commutes by definition of ϕ^σ;

and 12 commutes since σ is a modification.

Therefore, the outer diagram commutes. So (7) commutes. So the previous diagram commutes. But, passing via $\gamma_{\Phi(X),Y}$, this says that the following diagram commutes, that is, that ϕ^σ respects the left C^X-action:

$$\begin{tikzcd}
N^\Phi \arrow[r, "\phi^\sigma"] \arrow[d, "\delta_{\mathsf{C}^X,N^\Phi}"'] & N^\Psi \arrow[d, "\delta_{\mathsf{C}^X,N^\Psi}"] \\
\mathsf{C}^X N^\Phi \arrow[r, "1*\phi^\sigma"'] & \mathsf{C}^X N^\Psi
\end{tikzcd}$$

Therefore, ϕ^σ is a bicomodule homomorphism.

□

Lemma 2.2.6. Suppose $\Phi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$, $\Psi : (\mathbb{M}', Y) \to (\mathbb{M}'', Z)$ are morphisms of pointed birepresentations of a finitary 2-category $\mathscr{C}$. Consider the following composition:

$$\begin{tikzcd}
\Psi(\Phi(X)) \arrow[d, "\Psi(\mathrm{coev}_{N^\Phi})"] \\
\Psi(\mathbb{M}'(N^\Phi)(Y)) \arrow[d, "\Psi_{N^\Phi}"] \\
\mathbb{M}''(N^\Phi)\Psi(Y) \arrow[d, "1*\mathrm{coev}_{N^\Psi}"] \\
\mathbb{M}''(N^\Phi)\mathbb{M}''(N^\Psi)Z \arrow[d, "\mathbb{M}''^{-2}"] \\
\mathbb{M}''(N^\Phi N^\Psi)Z
\end{tikzcd}$$

Passing via $\gamma_{\Psi\Phi(X),Z}$, define the map $\overline{\phi}_{\Phi,\Psi} : N^{\Psi\Phi} \to N^\Phi N^\Psi$.

Then this map induces a bicomodule homomorphism

$$\phi_{\Phi,\Psi} : N^{\Psi\Phi} \to N^\Phi \underset{\mathsf{C}^Y}{\circledast} N^\Psi$$

.

Proof : First, we show that $\overline{\phi}_{\Phi,\Psi}$ is a bicomodule homomorphism, that is, that the following diagrams commute:

$$\begin{tikzcd} N^{\Psi\Phi} \arrow[r, "\overline{\phi}_{\Phi,\Psi}"] \arrow[d, "\delta_{N^{\Psi\Phi},\mathsf{C}^Z}"'] \arrow[dr, phantom, "\boxed{1}"] & N^{\Phi}N^{\Psi} \arrow[d, "\delta_{N^{\Phi}N^{\Psi},\mathsf{C}^Z}"] \\ N^{\Psi\Phi}\mathsf{C}^Z \arrow[r, "\overline{\phi}_{\Phi,\Psi}*1"'] & N^{\Phi}N^{\Psi}\mathsf{C}^Z \end{tikzcd} \qquad \begin{tikzcd} N^{\Psi\Phi} \arrow[r, "\overline{\phi}_{\Phi,\Psi}"] \arrow[d, "\delta_{\mathsf{C}^X,N^{\Psi\Phi}}"'] \arrow[dr, phantom, "\boxed{2}"] & N^{\Phi}N^{\Psi} \arrow[d, "\delta_{\mathsf{C}^X,N^{\Phi}N^{\Psi}}"] \\ \mathsf{C}^X N^{\Psi\Phi} \arrow[r, "1*\overline{\phi}_{\Phi,\Psi}"'] & \mathsf{C}^X N^{\Phi}N^{\Psi} \end{tikzcd}$$

We start with $\boxed{1}$. Consider the diagram overleaf:

$$
\begin{tikzcd}
\Psi\Phi(X) \arrow[r, "\mathrm{coev}_{N^{\Psi\Phi}}"] \arrow[dr, "\Psi(\mathrm{coev}_{N^{\Phi}})"] \arrow[dd, "\mathrm{coev}_{N^{\Psi\Phi}}"'] & \mathbb{M}''(N^{\Psi\Phi})Z \arrow[rrrr, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] & & & & \mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[ddddd, "\mathbb{M}''(1*\delta_{N^{\Psi},\mathsf{C}^Z})"] \\
& \Psi(\mathbb{M}'(N^{\Phi})Y) \arrow[rr, "\Psi_{N^{\Phi}}"] & & \mathbb{M}''(N^{\Phi})\Psi(Y) \arrow[r, "1*\mathrm{coev}_{N^{\Psi}}"] \arrow[ur, phantom, "\textcircled{1}"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[ur, "\mathbb{M}''^{-2}"] \arrow[dlll, "\mathbb{M}''^{-2}"] \arrow[ddl, "1*\mathrm{coev}_{\mathsf{C}Z}"] \arrow[dd, "1*\mathbb{M}''(\delta_{N^{\Psi},\mathsf{C}^Z})"] \arrow[ddr, phantom, "\textcircled{4}"] & \\
\mathbb{M}''(N^{\Psi\Phi})Z \arrow[r, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] \arrow[ddr, "1*\mathrm{coev}_{\mathsf{C}Z}"] \arrow[ddd, "\mathbb{M}''(\delta_{N^{\Psi\Phi},\mathsf{C}^Z})"'] \arrow[ur, phantom, "\textcircled{2}"] & \mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[ddrr, "1*\mathrm{coev}_{\mathsf{C}Z}"] \arrow[urr, phantom, "\textcircled{3}"] \arrow[dd, phantom, "\textcircled{6}"] & & & & \\
& & & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})\mathbb{M}''(\mathsf{C}^Z)Z \arrow[r, "1*\mathbb{M}''^{-2}"] \arrow[d, "\mathbb{M}''^{-2}*1"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi}\mathsf{C}^Z)Z \arrow[ddr, "\mathbb{M}''^{-2}"] \arrow[ul, phantom, "\textcircled{5}"] & \\
& \mathbb{M}''(N^{\Psi\Phi})\mathbb{M}''(\mathsf{C}^Z)Z \arrow[rr, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})*1"'] \arrow[dl, "\mathbb{M}''^{-2}"] \arrow[ul, phantom, "\textcircled{7}"] & & \mathbb{M}''(N^{\Phi}N^{\Psi})\mathbb{M}''(\mathsf{C}^Z)Z \arrow[drr, "\mathbb{M}''^{-2}"] \arrow[ur, phantom, "\textcircled{8}"] \arrow[dl, phantom, "\textcircled{9}"] & & \\
\mathbb{M}''(N^{\Psi\Phi}\mathsf{C}^Z)Z \arrow[rrrrr, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi}*1)"'] & & & & & \mathbb{M}''(N^{\Phi}N^{\Psi}\mathsf{C}^Z)Z
\end{tikzcd}
$$

1 and 2 commute by definition of $\overline{\phi}_{\Phi,\Psi}$;

3 commutes by the interchange law;

4 and 9 commute by naturality of $\mathbb{M}''^2$;

5 and 7, precomposed with $1 * \mathrm{coev}_{N^\Psi}$ and $\mathrm{coev}_{N^{\Psi\Phi}}$ respectively, commute by definition of the right coactions;

8 commutes by higher coherence for $\mathbb{M}''^2$.

Therefore the outer diagram commutes. But passing via $\gamma^{-1}_{\Psi\Phi(X),Z}$, this says precisely that $\overline{\phi}_{\Phi,\Psi}$ is a right C^Z-comodule homomorphism.

Next, we examine $\boxed{2}$. Consider the following two diagrams:

$$\begin{tikzcd}
\Psi\Phi(X) \arrow[rr, "\mathrm{coev}_{N^{\Psi\Phi}}"] \arrow[d, "\Psi(\mathrm{coev}_{\mathsf{C}^{\Phi(X)}})"'] \arrow[dr, "\Psi(\mathrm{coev}_{N^{\Phi}})"] & & \mathbb{M}''(N^{\Psi\Phi})Z \arrow[rr, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] \arrow[d, phantom, "8"] & & \mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[dd, "\mathbb{M}''(\delta_{\mathsf{C}^{\Phi(X)},N^{\Phi}}*1)"] \\
\Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X)) \arrow[d, "\Psi(1*\mathrm{coev}_{N^{\Phi}})"'] \arrow[r, phantom, "9"] & \Psi(\mathbb{M}'(N^{\Phi})Y) \arrow[r, "\Psi_{N^{\Phi}}"] \arrow[d, "\Psi(\mathbb{M}'(\delta_{\mathsf{C}^{\Phi(X)},N^{\Phi}}))"] \arrow[dr, phantom, "10"] & \mathbb{M}''(N^{\Phi})\Psi(Y) \arrow[r, "1*\mathrm{coev}_{N^{\Psi}}"] \arrow[d, "\mathbb{M}''(\delta_{\mathsf{C}^{\Phi(X)},N^{\Phi}})"] \arrow[dr, phantom, "11"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[ur, "\mathbb{M}''^{-2}"] \arrow[d, "\mathbb{M}''(\delta_{\mathsf{C}^{\Phi(X)},N^{\Phi}})*1"] \arrow[r, phantom, "12"] & {} \\
\Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(N^{\Phi})Y) \arrow[r, "\Psi(\mathbb{M}'^{-2})"'] & \Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)}N^{\Phi})Y) \arrow[r, "\Psi_{\mathsf{C}^{\Phi(X)}N^{\Phi}}"'] & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi})\Psi(Y) \arrow[r, "1*\mathrm{coev}_{N^{\Psi}}"'] & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[r, "\mathbb{M}''^{-2}"'] & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi}N^{\Psi})Z
\end{tikzcd}$$

$$\begin{tikzcd}
\Psi\Phi(X) \arrow[r, "\Psi(\mathrm{coev}_{\mathsf{C}^{\Phi(X)}})"] \arrow[d, "\mathrm{coev}_{N^{\Psi\Phi}}"'] & \Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)})\Phi(X)) \arrow[r, "\Psi(1*\mathrm{coev}_{N^{\Phi}})"] \arrow[d, "\Psi_{\mathsf{C}^{\Phi(X)}}"] \arrow[dr, phantom, "2"] & \Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)})\mathbb{M}'(N^{\Phi})Y) \arrow[r, "\Psi(\mathbb{M}'^{-2})"] \arrow[d, "\Psi_{\mathsf{C}^{\Phi(X)}}"] \arrow[dr, phantom, "3"] & \Psi(\mathbb{M}'(\mathsf{C}^{\Phi(X)}N^{\Phi})Y) \arrow[r, "\Psi_{\mathsf{C}^{\Phi(X)}N^{\Phi}}"] & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi})\Psi(Y) \arrow[dd, "\mathrm{coev}_{N^{\Psi}}"] \\
\mathbb{M}''(N^{\Psi\Phi})Z \arrow[ddd, "\mathbb{M}''(\delta_{\mathsf{C}^{\Phi(X)},N^{\Psi\Phi}})"'] \arrow[dr, phantom, "1"] & \mathbb{M}''(\mathsf{C}^{\Phi(X)})\Psi\Phi(X) \arrow[r, "1*\Psi(\mathrm{coev}_{N^{\Phi}})"] \arrow[dd, "1*\mathrm{coev}_{N^{\Psi\Phi}}"] \arrow[ddr, phantom, "4"] & \mathbb{M}''(\mathsf{C}^{\Phi(X)})\Psi(\mathbb{M}'(N^{\Phi})Y) \arrow[r, "1*\Psi_{N^{\Phi}}"] & \mathbb{M}''(\mathsf{C}^{\Phi(X)})\mathbb{M}''(N^{\Phi})\Psi(Y) \arrow[uur, "\mathbb{M}''^{-2}"] \arrow[d, "1*\mathrm{coev}_{N^{\Psi}}"] \arrow[r, phantom, "5"] & {} \\
& & & \mathbb{M}''(\mathsf{C}^{\Phi(X)})\mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[r, "\mathbb{M}''^{-2}*1"] \arrow[d, dashed, "1*\mathbb{M}''^{-2}"] \arrow[r, phantom, "6", shift right=20] & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[dd, "\mathbb{M}''^{-2}"] \\
& \mathbb{M}''(\mathsf{C}^{\Phi(X)})\mathbb{M}''(N^{\Psi\Phi})Z \arrow[rr, "1*\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] \arrow[dl, "\mathbb{M}''^{-2}"] \arrow[drr, phantom, "7"] & & \mathbb{M}''(\mathsf{C}^{\Phi(X)})\mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[dr, "\mathbb{M}''^{-2}"] & \\
\mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Psi\Phi})Z \arrow[rrrr, "\mathbb{M}''(1*\overline{\phi}_{\Phi,\Psi})"'] & & & & \mathbb{M}''(\mathsf{C}^{\Phi(X)}N^{\Phi}N^{\Psi})Z
\end{tikzcd}$$

1 commutes by definition of $\delta_{\mathsf{C}^{\Phi(X)},N^{\Psi\Phi}}$;

(2) and (10) commute by naturality of Ψ;

(3) commutes by higher coherence of Ψ;

(4) and (8) commute by definition of $\overline{\phi}_{\Phi,\Psi}$;

(5) and (11) commute by the interchange law;

(6) commutes by higher coherence for $\mathbb{M}''^2$;

(7) and (12) commute by naturality of $\mathbb{M}''^2$;

and (9) commutes by definition of $\delta_{\mathsf{C}^{\Phi(X)},N^{\Phi}}$.

Therefore, the bottom edge of the first diagram is equal to the top edge of the first diagram, which is equal to the bottom edge of the second, which is equal to the top edge of the second. Passing via $\gamma^{-1}_{\Psi\Phi(X),Z}$, this is precisely the statement that $\overline{\phi}_{\Phi,\Psi}$ is a left $\mathsf{C}^{\Phi(X)}$-comodule homomorphism, and therefore by Lemma 1.3.16, a left C^{X}-comodule homomorphism, as required.

Next, we show that $\overline{\phi}$ equalises the C^{Y}-coaction maps, and thus induces a map to the cotensor product.

Consider the following diagram:

$$
\begin{tikzcd}
\Psi\Phi(X) \arrow[rr, "\mathrm{coev}_{N^{\Psi\Phi}}"] \arrow[dr, "\Psi(\mathrm{coev}_{N^{\Phi}})"] \arrow[dddd, "\mathrm{coev}_{N^{\Psi\Phi}}"] & & \mathbb{M}''(N^{\Psi\Phi})Z \arrow[rrrr, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] & {} \arrow[d, phantom, "1"] & & & \mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[dddddd, "\mathbb{M}''(1*\delta_{\mathsf{C}^Y,N^{\Psi}})"] \\
& \Psi(\mathbb{M}'(N^{\Phi})Y) \arrow[rrr, "\Psi_{N^{\Phi}}"] \arrow[drr, "\Psi(1*\mathrm{coev}_{\mathsf{C}^Y})"] \arrow[ddr, "\Psi(\mathbb{M}'(\delta_{N^{\Phi},\mathsf{C}^Y}))"] \arrow[ddd, "\Psi_{N^{\Phi}}"] & & {} \arrow[d, phantom, "5"] & \mathbb{M}''(N^{\Phi})\Psi(Y) \arrow[r, dashed, "1*\mathrm{coev}_{N^{\Psi}}"] \arrow[d, dashed, "1*\Psi(\mathrm{coev}_{\mathsf{C}^Y})"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[ur, "\mathbb{M}''^{-2}"] \arrow[dddd, "1*\mathbb{M}''(\delta_{\mathsf{C}^Y,N^{\Psi}})"] & \\
{} \arrow[r, phantom, "2"] & {} \arrow[dr, phantom, "3"] & {} \arrow[phantom, "4"] & \Psi(\mathbb{M}'(N^{\Phi})\mathbb{M}'(\mathsf{C}^Y)Y) \arrow[r, dashed, "\Psi_{N^{\Phi}}"] \arrow[dl, "\Psi(\mathbb{M}'^{-2})"] & \mathbb{M}''(N^{\Phi})\Psi(\mathbb{M}'(\mathsf{C}^Y)Y) \arrow[dd, "\Psi_{\mathsf{C}^Y}"] & & \\
& & \Psi(\mathbb{M}'(N^{\Phi}\mathsf{C}^Y)Y) \arrow[dr, "\Psi_{N^{\Phi}\mathsf{C}^Y}"] & {} \arrow[r, phantom, "6"] & {} \arrow[r, phantom, "7"] & {} \arrow[r, phantom, "8"] & \\
\mathbb{M}''(N^{\Psi\Phi})Z \arrow[dd, "\mathbb{M}''(\overline{\phi}_{\Phi,\Psi})"] & \mathbb{M}''(N^{\Phi})\Psi(Y) \arrow[rr, "\mathbb{M}''(\delta_{N^{\Phi},\mathsf{C}^Y})"] \arrow[d, dashed, "1*\mathrm{coev}_{N^{\Psi}}"] & {} \arrow[d, phantom, "9"] & \mathbb{M}''(N^{\Phi}\mathsf{C}^Y)\Psi(Y) \arrow[d, dashed, "1*\mathrm{coev}_{N^{\Psi}}"] \arrow[dr, phantom, "10"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(\mathsf{C}^Y)\Psi(Y) \arrow[l, "\mathbb{M}''^{-2}"'] \arrow[d, dashed, "1*\mathrm{coev}_{N^{\Psi}}"] & & \\
& \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi})Z \arrow[rr, "\mathbb{M}''(\delta_{N^{\Phi},\mathsf{C}^Y})*1"] \arrow[dl, "\mathbb{M}''^{-2}"] & {} \arrow[dr, phantom, "12"] & \mathbb{M}''(N^{\Phi}\mathsf{C}^Y)\mathbb{M}''(N^{\Psi})Z \arrow[drrr, "\mathbb{M}''^{-2}"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(\mathsf{C}^Y)\mathbb{M}''(N^{\Psi})Z \arrow[l, "\mathbb{M}''^{-2}*1"'] \arrow[r, "1*\mathbb{M}''^{-2}"] \arrow[dr, phantom, "11"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(\mathsf{C}^Y N^{\Psi})Z \arrow[dr, "\mathbb{M}''^{-2}"] & \\
\mathbb{M}''(N^{\Phi}N^{\Psi})Z \arrow[rrrrrr, "\mathbb{M}''(\delta_{N^{\Phi},\mathsf{C}^Y}*1)"] & & & & & & \mathbb{M}''(N^{\Phi}\mathsf{C}^Y N^{\Psi})Z
\end{tikzcd}
$$

1 and 2 commute by definition of ϕ;

3 and 5 commute by naturality of Ψ;

4 precomposed with $\Psi(\mathrm{coev}_{\mathsf{C}^\Phi})$ commutes by definition of the right coaction on N^Φ;

6 commutes by higher coherence of Ψ;

7 commutes by definition of the left coaction on N^Ψ;

8 and 12 commute by naturality of $\mathbb{M}''^{-2}$;

9 and 10 commute by the interchange law;

11 commutes by higher coherence for $\mathbb{M}''^{-2}$.

So the outer diagram commutes. But passing via $\gamma_{\Psi\Phi(X),Z}$, this says the following commutes:

$$\begin{tikzcd}
 & N^\Phi N^\Psi \arrow[dr, "\delta_{N^\Phi,\mathsf{C}^Y}*1"] & \\
N^{\Psi\Phi} \arrow[ur, dashed, "\overline{\phi}_{\Phi,\Psi}"] \arrow[dr, dashed, "\overline{\phi}_{\Phi,\Psi}"'] & & N^\Phi \mathsf{C}^Y N^\Psi \\
 & N^\Phi N^\Psi \arrow[ur, "1*\delta_{\mathsf{C}^Y,N^\Psi}"'] &
\end{tikzcd}$$

That is, $\overline{\phi}_{\Phi,\Psi}$ equalises the C^Y-coaction maps. So it induces a bicomodule homomorphism $\phi_{\Phi,\Psi}$: $N^{\Psi\Phi} \to N^\Phi \circledast N^\Psi$ such that $t \circ \phi_{\Phi,\Psi} = \overline{\phi}_{\Phi,\Psi}$, as required.

□

Theorem 2.2.7. For any fiat 2-category $\mathscr{C}$, there is a colax functor $\mathcal{I} : \mathscr{C} - cfmod^* \to \mathscr{R}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$, defined as follows:

- A representation with generator $(\mathbb{M}, X)$ is sent to the coalgebra 1-morphism C^X;
- A morphism of birepresentations $\Phi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$ is sent to the C^X-C^Y-bicomodule 1-morphism N^Φ;
- A modification $\sigma : \Phi \to \Psi$ is sent to the bicomodule homomorphism ϕ^σ;
- $\mathcal{I}^0$ is the identity natural transform;
- $\mathcal{I}^2$ is given component-wise by $\mathcal{I}^2_{\Phi,\Psi} := \phi_{\Phi,\Psi}$.

Proof :

We've shown that $\mathcal{I}$ is well-defined on objects, 1-morphisms (by Lemma 2.2.3) and 2-morphisms (by Lemma 2.2.5).

Next, we consider $\mathcal{I}^0$. It is immediate that, when Φ is the identity natural transform, θ_Φ is the identity on C^X (since the image under the cohom adjunction is just $\mathrm{coev}_{\mathsf{C}^X}$), and thus $\mathcal{I}(1_{(\mathbb{M},X)}) = \mathsf{C}^X = 1_{\mathcal{I}(\mathbb{M},X)}$. So $\mathcal{I}^0$ is well-defined.

$\mathcal{I}^2$ is well-defined by Lemma 2.2.6. We need to show that this assembles to a natural transformation, so suppose $\sigma : \Phi \to \Psi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$, $\sigma' : \Phi' \to \Psi' : (\mathbb{M}', Y) \to (\mathbb{M}'', Z)$ are modifications. We want to show that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{I}(\Phi'\Phi) \arrow[r, "\mathcal{I}^2"] \arrow[d, "\mathcal{I}(\sigma' * \sigma)"] & \mathcal{I}(\Phi) \circledast \mathcal{I}(\Phi') \arrow[d, "\mathcal{I}(\sigma) \circledast \mathcal{I}(\sigma')"] \\
\mathcal{I}(\Psi'\Psi) \arrow[r, "\mathcal{I}^2"] & \mathcal{I}(\Psi) \circledast \mathcal{I}(\Psi')
\end{tikzcd}$$

Consider the diagram below:

$$\begin{tikzcd}
& & \mathcal{I}(\Phi)\mathcal{I}(\Phi') \arrow[ddd, "\mathcal{I}(\sigma) * \mathcal{I}(\sigma')"] \\
\mathcal{I}(\Phi'\Phi) \arrow[r, "\mathcal{I}^2"] \arrow[d, "\mathcal{I}(\sigma' * \sigma)"] \arrow[urr, "\overline{\phi}_{\Phi,\Phi'}"] & \mathcal{I}(\Phi) \circledast \mathcal{I}(\Phi') \arrow[d, "\mathcal{I}(\sigma) \circledast \mathcal{I}(\sigma')"] \arrow[ur, "t"] & \\
\mathcal{I}(\Psi'\Psi) \arrow[r, "\mathcal{I}^2"] \arrow[drr, "\overline{\phi}_{\Psi,\Psi'}"'] & \mathcal{I}(\Psi) \circledast \mathcal{I}(\Psi') \arrow[dr, "t"] & \\
& & \mathcal{I}(\Psi)\mathcal{I}(\Psi')
\end{tikzcd}$$

The top and bottom triangles commute by definition of $\mathcal{I}^2$; the right trapezoid commutes by definition of $\mathcal{I}(\sigma) \circledast \mathcal{I}(\sigma')$; and t is monic. So to show that $\mathcal{I}^2$ is natural, it is sufficient to show the outer square commutes. To see that it does, consider the following diagram:

$$
\begin{tikzcd}
\Phi'\Phi(X) \arrow[rrr, "\mathrm{coev}_{N^{\Phi'\Phi}}"] \arrow[ddr, "\Phi'(\sigma)"] \arrow[ddd, "\mathrm{coev}_{N^{\Phi'\Phi}}"'] \arrow[dddr, bend right, "\sigma'*\sigma"'] \arrow[ddr, phantom, bend right=20, "7"] \arrow[drr, "\Phi'(\mathrm{coev}_{N^{\Phi}})"] & & & \mathbb{M}''(N^{\Phi'\Phi})Z \arrow[rrr, "\mathbb{M}''(\bar{\phi}_{\Phi,\Phi'})"] \arrow[dr, phantom, "1"] & & & \mathbb{M}''(N^{\Phi}N^{\Phi'})Z \arrow[dddd, "\mathbb{M}''(\phi^{\sigma}*\phi^{\sigma'})"] \\
& & \Phi'(\mathbb{M}'(N^{\Phi})Y) \arrow[r, "\Phi'_{N^{\Phi}}"] \arrow[d, dashed, "\Phi'(\mathbb{M}'(\phi^{\sigma}))"] \arrow[dl, phantom, "2"] \arrow[dr, phantom, "3"] & \mathbb{M}''(N^{\Phi})\Phi'(Y) \arrow[rr, "1*\mathrm{coev}_{N^{\Phi'}}"] \arrow[d, dashed, "\mathbb{M}''(\phi^{\sigma})"] \arrow[dr, "1*\sigma'"] & & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Phi'})Z \arrow[ur, "\mathbb{M}''^{-2}"] \arrow[d, dashed, "1*\mathbb{M}''(\phi^{\sigma'})"] & \\
& \Phi'\Psi(X) \arrow[r, "\Phi'(\mathrm{coev}_{N^{\Psi}})"] \arrow[d, dashed, "\sigma'"] \arrow[dr, phantom, "9"] & \Phi'(\mathbb{M}'(N^{\Psi})Y) \arrow[r, "\Phi'_{N^{\Psi}}"'] \arrow[d, dashed, "\sigma'"] \arrow[dr, phantom, "10"] & \mathbb{M}''(N^{\Psi})\Phi'(Y) \arrow[dr, "1*\sigma'"] \arrow[r, phantom, "4"] & \mathbb{M}''(N^{\Phi})\Psi'(Y) \arrow[r, "1*\mathrm{coev}_{N^{\Psi'}}"] \arrow[d, dashed, "\mathbb{M}''(\phi^{\sigma})"] \arrow[u, phantom, "5"] \arrow[dr, phantom, "11"] & \mathbb{M}''(N^{\Phi})\mathbb{M}''(N^{\Psi'})Z \arrow[d, dashed, "\mathbb{M}''(\phi^{\sigma})*1"] \arrow[r, phantom, "6"] & \\
\mathbb{M}''(N^{\Phi'\Phi})Z \arrow[d, "\mathbb{M}''(\phi^{\sigma'*\sigma})"'] \arrow[u, phantom, "8"] & \Psi'\Psi(X) \arrow[r, "\Psi'(\mathrm{coev}_{N^{\Psi}})"] \arrow[dl, dashed, "\mathrm{coev}_{N^{\Psi'\Psi}}"] & \Psi'(\mathbb{M}'(N^{\Psi})Y) \arrow[rr, "\Psi'_{N^{\Psi}}"] \arrow[d, phantom, "12"] & & \mathbb{M}''(N^{\Psi})\Psi'(Y) \arrow[r, "1*\mathrm{coev}_{N^{\Psi'}}"] & \mathbb{M}''(N^{\Psi})\mathbb{M}''(N^{\Psi'})Z \arrow[dr, "\mathbb{M}''^{-2}"] & \\
\mathbb{M}''(N^{\Psi'\Psi})Z \arrow[rrrrrr, "\mathbb{M}''(\bar{\phi}_{\Psi,\Psi'})"] & & & & & & \mathbb{M}''(N^{\Psi}N^{\Psi'})Z
\end{tikzcd}
$$

1 and 12 commute by definition of $\overline{\phi}_{\Phi,\Phi'}$ and $\overline{\phi}_{\Psi,\Psi'}$;

2, 5 and 8 commute by definition of ϕ^{σ}, $\phi^{\sigma'}$ and $\phi^{\sigma'*\sigma}$ respectively;

3 commutes by naturality of Φ';

4 and 11 commute by the interchange law;

6 commutes by naturality of $\mathbb{M}''^{-2}$;

7 commutes by definition of $\sigma' * \sigma$;

9 commutes by naturality of σ';

and 10 commutes because σ' is a modification.

Therefore the outer diagram commutes. Passing via $\gamma_{\Phi'\Phi(X),Z}$, this says that the following diagram commutes:

$$\begin{tikzcd}
N^{\Phi'\Phi} \arrow[r, "\overline{\phi}_{\Phi,\Phi'}"] \arrow[d, "\phi^{\sigma'*\sigma}"'] & N^{\Phi}N^{\Phi'} \arrow[d, "\phi^{\sigma}*\phi^{\sigma'}"] \\
N^{\Psi'\Psi} \arrow[r, "\overline{\phi}_{\Psi,\Psi'}"] & N^{\Psi}N^{\Psi'}
\end{tikzcd}$$

which is what we wanted. So $\mathcal{I}^2$ is a natural transformation.

Finally, we need to show that $\mathcal{I}^0$, $\mathcal{I}^2$ are coherent, that is, that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{I}(\Xi\Psi\Phi) \arrow[r, "\mathcal{I}^2_{\Xi,\Psi\Phi}"'] \arrow[d, "\mathcal{I}^2_{\Xi\Psi,\Phi}"] \arrow[dr, phantom, "\boxed{1}"] & \mathcal{I}(\Psi\Phi) \circledast \mathcal{I}(\Xi) \arrow[d, "\mathcal{I}^2_{\Psi,\Phi}\circledast 1"'] \\
\mathcal{I}(\Phi) \circledast \mathcal{I}(\Xi\Psi) \arrow[r, "1\circledast\mathcal{I}^2_{\Xi,\Psi}"] & \mathcal{I}(\Phi) \circledast \mathcal{I}(\Psi) \circledast \mathcal{I}(\Xi)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{I}(\Phi 1_{\mathbb{M}}) \arrow[rr, "\mathcal{I}^2"'] \arrow[dr, equal] & \arrow[d, phantom, "\boxed{2}"] & \mathcal{I}(1_{\mathbb{M}}) \circledast \mathcal{I}(\Phi) \arrow[dl, "\epsilon^l_{\mathcal{I}(\Phi)}"] \\
& \mathcal{I}(\Phi) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}(1_{\mathbb{M}'}\Phi) \arrow[rr, "\mathcal{I}^2"'] \arrow[dr, equal] & \arrow[d, phantom, "\boxed{3}"] & \mathcal{I}(\Phi) \circledast \mathcal{I}(1_{\mathbb{M}'}) \arrow[dl, "\epsilon^r_{\mathcal{I}(\Phi)}"] \\
& \mathcal{I}(\Phi) &
\end{tikzcd}$$

To see that the $\boxed{1}$ commutes, consider the diagram below:

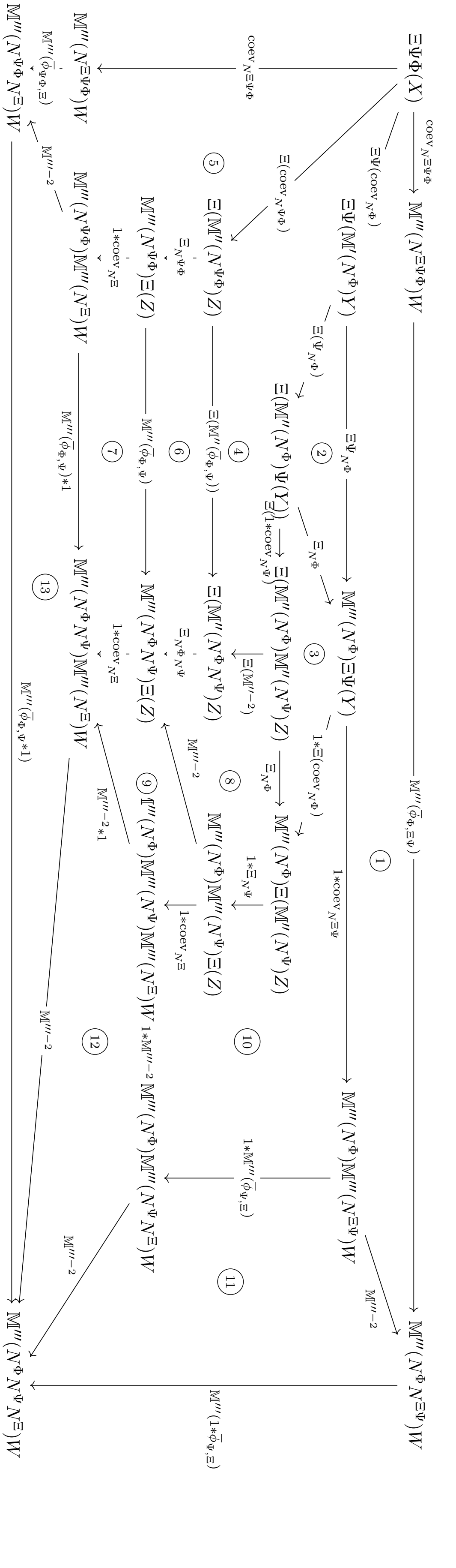
$$
\begin{tikzcd}
N^{\Xi\Psi\Phi} \arrow[rrrrrr, "\phi_{\Phi,\Xi\Psi}"] \arrow[rrrd, "\overline{\phi}_{\Phi,\Xi\Psi}"] \arrow[rrdd, "\overline{\phi}_{\Psi\Phi,\Xi}"] \arrow[dddd, "\phi_{\Psi\Phi,\Xi}"'] & & & \arrow[d, phantom, "\textcircled{1}"] & & & N^{\Phi} \circledast N^{\Xi\Psi} \arrow[llld, "t"'] \arrow[lldd, "1\circledast\overline{\phi}_{\Psi,\Xi}"'] \arrow[dddd, "1\circledast\phi_{\Psi,\Xi}"] \\
 & & \textcircled{8} & N^{\Phi}N^{\Xi\Psi} \arrow[d, "1*\overline{\phi}_{\Psi,\Xi}"] & \textcircled{6} & & \\
 & \textcircled{2} & N^{\Psi\Phi}N^{\Xi} \arrow[r, "\overline{\phi}_{\Phi,\Psi}*1"] & N^{\Phi}N^{\Psi}N^{\Xi} & N^{\Phi} \circledast N^{\Psi}N^{\Xi} \arrow[l, "t"'] & \textcircled{4} & \\
 & & \textcircled{7} & N^{\Phi}N^{\Psi} \circledast N^{\Xi} \arrow[u, "t"] \arrow[d, phantom, "\textcircled{3}"] & \textcircled{5} & & \\
N^{\Psi\Phi} \circledast N^{\Xi} \arrow[rruu, "t"] \arrow[rrru, "\overline{\phi}_{\Phi,\Psi}\circledast 1"] \arrow[rrrrrr, "\phi_{\Phi,\Psi}\circledast 1"'] & & & \, & & & N^{\Phi} \circledast N^{\Psi} \circledast N^{\Xi} \arrow[lllu, "t"] \arrow[lluu, "t"']
\end{tikzcd}
$$

1-4 commute by the definition of $\mathcal{I}^2$;

5 commutes by Lemma 1.3.9;

6 and 7 commute by definition of $1 \circledast \overline{\phi}_{\Psi,\Xi}$ and $\overline{\phi}_{\Phi,\Psi} \circledast 1$ respectively;

and $t \circ t$ is monic. So to show that the outer diagram commutes, it is sufficient to show that 8 commutes. To see that it does, consider the diagram overleaf:



(1), (4), (5) and (10) commute by definition of $\overline{\phi}$;

(2) commutes by definition of $\Xi\Psi_{N^\Phi}$;

(3) and (6) commute by naturality of Ξ;

(7) and (9) commute by the interchange law;

(8) commutes by higher coherence for Ξ;

(11) and (13) commute by naturality of $\mathbb{M}'''^{-2}$;

and (12) commutes by higher coherence of $\mathbb{M}'''^{-2}$.

So the outer square commutes. But passing via $\gamma_{\Xi\Psi\Phi(X),W}$, this is exactly what we wanted. So the first coherence diagram commutes.

Next, we examine $\boxed{2}$. We expand it to the following diagram:

$$
\begin{tikzcd}
\mathcal{I}(\Phi 1_{\mathbb{M}}) \arrow[rr, "\mathcal{I}^2"] \arrow[dr, "\overline{\phi}_{1_{\mathbb{M}},\Phi}"'] \arrow[ddrr, equal] \arrow[rr, phantom, shift right=4ex, "1"] & & \mathcal{I}(1_{\mathbb{M}}) \circledast \mathcal{I}(\Phi) \arrow[dl, "t"'] \arrow[dd, "\varepsilon^l_{\mathcal{I}(\Phi)}"] \arrow[ddl, phantom, "2"] \\
& \mathsf{C}^X\mathcal{I}(\Phi) \arrow[dr, "\varepsilon_X * 1"] \arrow[d, phantom, "3"] & \\
& & \mathcal{I}(\Phi)
\end{tikzcd}
$$

(1) commutes by definition of $\mathcal{I}^2$;

(2) commutes by definition of $\varepsilon^l_{\mathcal{I}(\Phi)}$;

So to see that the outer diagram commutes, it's sufficient to show that (3) commutes. To see that it does, consider the following diagram:

$$
\begin{tikzcd}
\Phi(X) \arrow[r, "\mathrm{coev}_{N\Phi}"] \arrow[drr, "\Phi(\mathrm{coev}_{\mathsf{C}X})"] \arrow[ddrr, "\Phi(\mathbb{M}^{-0})"'] \arrow[dddrrr, bend right, "\mathbb{M}'^{-}"'] \arrow[dddd, "\mathrm{coev}_{N\Phi}"'] & \mathbb{M}'(\mathcal{I}(\Phi 1_{\mathbb{M}}))Y \arrow[rrrr, "\mathbb{M}'(\overline{\phi}_{1_{\mathbb{M}},\Phi})"] \arrow[d, phantom, "2"] \arrow[ddd, phantom, "7"] & & \arrow[d, phantom, "1"] & & \mathbb{M}'(\mathsf{C}^X\mathcal{I}(\Phi))Y \arrow[dddd, "\mathbb{M}'(\varepsilon_X * 1)"] \\
& & \Phi(\mathbb{M}(\mathsf{C}^X)X) \arrow[r, "\Phi_{\mathsf{C}X}"] \arrow[d, dashed, "\Phi(\mathbb{M}(\varepsilon_X))"] \arrow[dr, phantom, "3"] & \mathbb{M}'(\mathsf{C}^X)\Phi(X) \arrow[r, "1*\mathrm{coev}_{N\Phi}"] \arrow[dd, "\mathbb{M}'(\varepsilon_X)"] \arrow[ddr, phantom, "4"] & \mathbb{M}'(\mathsf{C}^X)\mathbb{M}'(\mathcal{I}(\Phi))Y \arrow[ur, "\mathbb{M}'^{-2}"] \arrow[dd, "\mathbb{M}'(\varepsilon_X)*1"] \arrow[ddr, phantom, "5"] & \\
& \arrow[r, phantom, "6"] & \Phi(\mathbb{M}(1)X) \arrow[dr, "\Phi_1"] & & & \\
& & & \mathbb{M}'(1)\Phi(X) \arrow[r, "1*\mathrm{coev}_{N\Phi}"] \arrow[dr, phantom, "8"] & \mathbb{M}'(1)\mathbb{M}'(\mathcal{I}(\Phi))Y \arrow[dr, "\mathbb{M}'^{-2}"] & \\
\mathbb{M}'(\mathcal{I}(\Phi 1_{\mathbb{M}}))Y \arrow[urrrr, "\mathbb{M}'^{0}*1"] \arrow[rrrrr, equals] & & & & & \mathbb{M}'(1\mathcal{I}(\Phi))Y
\end{tikzcd}
$$

(1) commutes by definition of $\overline{\phi}$;

(2) commutes by definition of ε_X;

(3) commutes by naturality of Φ;

(4) and (7) commute by the interchange law;

(5) commutes by naturality of $\mathbb{M}'^{-2}$;

(6) commutes by higher coherence for Φ;

and (8) commutes by higher coherence for $\mathbb{M}'$.

So the outer diagram commutes. But passing via $\gamma_{\Phi(X),Y}$, this says precisely that our coherence diagram commutes.

Similarly, one can show that $\boxed{3}$ commutes precisely if

$$\begin{tikzcd}
N^{1_{\mathbb{M}'}\Phi} \arrow[r, "\overline{\phi}_{\Phi,1_{\mathbb{M}'}}"] \arrow[dr, equal] & N^{\Phi}\mathsf{C}^{Y} \arrow[d, "1*\varepsilon_Y"] \\
 & N^{\Phi}
\end{tikzcd}$$

commutes.

To see that it does, consider the following diagram:

$$\begin{tikzcd}
\Phi(X) \arrow[r, "\mathrm{coev}_{N^\Phi}"] \arrow[dd, "\mathrm{coev}_{N^\Phi}"'] & \mathbb{M}'(N^{1_{\mathbb{M}'}\Phi})Y \arrow[rr, "\mathbb{M}'(\overline{\phi}_{\Phi,1_{\mathbb{M}'}})"] \arrow[dr, "1*\mathrm{coev}_{\mathsf{C}^Y}"] \arrow[ddr, "1*\mathbb{M}^{-0}"'] \arrow[dd, phantom, "1"] & & \mathbb{M}'(N^{\Phi}\mathsf{C}^{Y})Y \arrow[ddd, "\mathbb{M}'(1*\varepsilon_Y)"] \\
 & & \mathbb{M}'(N^{\Phi})\mathbb{M}'(\mathsf{C}^{Y})Y \arrow[ur, "\mathbb{M}'^{-2}"'] \arrow[d, "1*\mathbb{M}'(\varepsilon_Y)"] \arrow[u, phantom, "2"] \arrow[dl, phantom, "3"] \arrow[dr, phantom, "4"] & \\
\mathbb{M}'(N^{1_{\mathbb{M}'}\Phi})Y \arrow[rr, "1*\mathbb{M}^{-0}"] \arrow[drrr, equal] & & \mathbb{M}'(N^{\Phi})\mathbb{M}'(1)Y \arrow[dr, "\mathbb{M}'^{-2}"] \arrow[d, phantom, "5"] & \\
 & & & \mathbb{M}'(N^{\Phi}1)Y
\end{tikzcd}$$

(1) commutes trivially;

(2) precomposed with coev_{N^Φ} commutes by definition of $\overline{\phi}$;

(3) commutes by definition of ε_Y;

(4) commutes by naturality of $\mathbb{M}'^{2}$;

and (5) commutes by higher coherence for $\mathbb{M}'$.

So the outer diagram commutes. Passing via $\gamma_{\Phi(X),Y}$, this says precisely that our coherence diagram above commutes.

So $\mathcal{I}^2$ and $\mathcal{I}^0$ are coherent. So $\mathcal{I}$ is a colax functor.

□

Now, we want to show that $\mathcal{I}$ is the other half of the equivalence $\iota : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}} \to \mathscr{C} - cfmod^*_{ex}$.

Theorem 2.2.8. The pair $(\iota, \mathcal{I})$ form a biequivalence of bicategories between $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$ and $\mathscr{C} - cfmod^*_{ex}$.

Proof : Since ι is a biequivalence of bicategories $\mathscr{R}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}} \to \mathscr{C} - cfmod^*$ which restricts to a biequivalence $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}} \to \mathscr{C} - cfmod^*_{ex}$ by Theorem 2.1.2, it is sufficient to show that $\mathcal{I} \circ \iota \cong 1_{\mathscr{R}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}}$.

One can compute that, for a coalgebra 1-morphism C, $\mathcal{I} \circ \iota(C) = \mathcal{I}(\mathbb{M}_C, C) = {}_{\mathbb{M}_C}[C, C] \cong C$.

For a C-D-bicomodule 1-morphism M, write $\Phi_M = - \circledast M$. Then $\mathcal{I} \circ \iota(M) = \mathcal{I}(\Phi_M) = N^{\Phi_M} = {}^{\theta_{\Phi_M}}{}_{\mathbb{M}_D}[D, C \circledast M]$. We claim that, as bicomodule 1-morphisms, this is isomorphic to $C \circledast M$. We note the following facts:

- ${}_{\mathbb{M}_D}[D, C \circledast M] \cong C \circledast M$ as right comodule 1-morphisms, with the left coaction $\delta_{\mathsf{C}^{C\circledast M}, {}_{\mathbb{M}_D}[D, C\circledast M]}$ mapping ${}_{\mathbb{M}}[D, C \circledast M] \to \mathsf{C}^{C\circledast M}{}_{\mathbb{M}_D}[D, C \circledast M]$;

- $\mathrm{coev}_{\mathsf{C}^C} = \Delta_C$, so $\Phi_M(\mathrm{coev}_{\mathsf{C}^C}) = \Delta_C \circledast M = \delta_{C, C\circledast M}$;

- $\mathrm{coev}^{\mathbb{M}_D}_{C\circledast M, D} = \delta_{C\circledast M, D}$;

- $\gamma_{C\circledast M, C\circledast M}(\theta_{\Phi_M}) = (\Phi_M)_C \circ \Phi_M(\mathrm{coev}_{\mathsf{C}^C}) = 1 \circ \delta_{C, C\circledast M} = \delta_{C, C\circledast M}$ by construction.

So consider the following diagram:

$$
\begin{tikzcd}
C \circledast M \arrow[rrrrr, "\mathrm{coev}^{\mathbb{M}_D}_{C \circledast M, D}"] \arrow[drrrr, "\delta_{C \circledast M, D}"] \arrow[dd, "\mathrm{coev}^{\mathbb{M}_D}_{C \circledast M, D}"] \arrow[ddr, "\delta_{C \circledast M, D}"] \arrow[ddrr, "\mathrm{coev}_{\mathsf{C}^{C \circledast M}}"'] \arrow[ddrrr, "\delta_{C, C \circledast M}"] & & & \arrow[d, phantom, "\textcircled{1}"] & & \mathbb{M}_D(C \circledast M)D \arrow[dl, equal] \arrow[dddd, "\mathbb{M}_D(\mathbb{M}[D, \delta_{C, C \circledast M}])"] \\
\arrow[r, phantom, "\textcircled{5}"] & & \textcircled{3} & \arrow[r, phantom, "\textcircled{2}"] & C \circledast MD \arrow[dd, "\delta_{C, C \circledast M} * 1"] & \\
\mathbb{M}_D(\mathbb{M}[D, C \circledast M])D \arrow[r, equal] \arrow[dd, "\mathbb{M}_D(\delta_{\mathsf{C}^{C \circledast M}, \mathbb{M}[D, C \circledast M]})"] & C \circledast MD \arrow[r, phantom, "\textcircled{4}"] & \mathsf{C}^{C \circledast M} C \circledast M \arrow[r, "\theta_{\Phi_M} * 1"] \arrow[dl, "\delta_{\mathsf{C}^{C \circledast M}, C \circledast M} * 1 * \delta_{C \circledast M, D}"'] & CC \circledast M \arrow[dr, "1 * \delta_{C \circledast M, D}"] & \arrow[r, phantom, "\textcircled{8}"] & \\
\arrow[r, phantom, "\textcircled{6}"] & \mathsf{C}^{C \circledast M} C \circledast MD \arrow[rrr, "\theta_{\Phi_M} * 1"] \arrow[dl, equal] & \arrow[r, phantom, "\textcircled{9}"] & & CC \circledast MD \arrow[dr, equal] & \\
\mathbb{M}_D(\mathsf{C}^{C \circledast M}{}_{\mathbb{M}}[D, C \circledast M])D \arrow[rrrrr, "\mathbb{M}_D(\theta_{\Phi_M} * 1)", "\textcircled{7}"'] & & & & & \mathbb{M}_D(C_{\mathbb{M}}[D, C \circledast M])D
\end{tikzcd}
$$

1 and 5 commute by definition of $\mathrm{coev}^{\mathbb{M}_D}_{C\boxast M,D}$;

2 commutes because $C \boxast M$ is a bicomodule 1-morphism, so the coactions commute;

3 commutes by definition of θ_{Φ_M};

4 commutes by definition of $\delta_{\mathsf{C}^{C\boxast M}, C\boxast M}$;

6-8 commute by definition of $\mathbb{M}_D$;

and 9 commutes by the interchange law.

So the outer diagram commutes. But passing via $\gamma_{C\boxast M,D}$, this says that N^{Φ_M} and $C \boxast M$ have the same left coaction. So they are isomorphic as bicomodule 1-morphisms.

So $\mathcal{I} \circ \iota(M) \cong C \boxast M$, and for a bicomodule homomorphism $\phi : M \to M'$, $\mathcal{I} \circ \iota(\phi) = \mathcal{I}(- \boxast \phi) = {}_{\mathbb{M}_D}[D, 1 \boxast \phi] = 1 \boxast \phi$.

Given a coalgebra 1-morphism C, we can compute $(\mathcal{I} \circ \iota)^0_C = (\mathcal{I}^0 \circ \mathcal{I}(\iota^0))(C) = \mathcal{I}(\varepsilon^r_-)(C) = {}_{\mathbb{M}_C}[C, \varepsilon^r_C] = \varepsilon^r_C$

Finally, given a C-D-bicomodule 1-morphism M and a D-E-bicomodule 1-morphism N, we can compute

$$(\mathcal{I} \circ \iota)^2_{M,N} = \mathcal{I}^2 \circ \mathcal{I}(\iota^2) = \mathcal{I}^2.$$

But note the following:

- $\mathrm{coev}_{N^{\Phi_M}} = \delta_{C\boxast M,D}$, so $\Phi_N(\mathrm{coev}_{N^{\Phi_M}}) = \delta_{C\boxast M,D} \boxast 1_N$;
- $\mathrm{coev}_{N^{\Phi_N}} = \delta_{D\boxast N,E}$.

So $\gamma_{C\boxast M\boxast N,E}(t^D_{C\boxast M,D\boxast N} \circ \mathcal{I}^2_{M,N}) = \delta_{D\boxast M,E} \circ (\delta_{C\boxast M,D} \boxast 1_N)$. So

$$\begin{aligned}
t^D_{C\boxast M,D\boxast N} \circ \mathcal{I}^2_{M,N} &= (1 * \varepsilon_E) \circ (1 * \delta_{D\boxast M,E}) \circ (\delta_{C\boxast M,D} \boxast 1_N) \\
&= \delta_{C\boxast M,D} \boxast 1_N \\
&= (t^D_{C\boxast M,D} \circ \delta^r_{C\boxast M}) \boxast 1_N \\
&= (t^D_{C\boxast M,D} \boxast 1_N) \circ (\delta^r_{C\boxast M} \boxast 1_N) \\
&= t^D_{C\boxast M,D\boxast N} \circ (\delta^r_{C\boxast M} \boxast 1_N)
\end{aligned}$$

So since t is monic, we must have $\mathcal{I}^2_{M,N} = \delta^r_{C\boxast M} \boxast 1_N$, that is, $(\mathcal{I} \circ \iota)^2 = \delta^r_{C\boxast M} \boxast 1_N$.

So define $\Gamma : \mathcal{I} \circ \iota \to 1_{\mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{C}}}$ as follows: for a coalgebra 1-morphism C, $\Gamma_C = C$ as a bicomodule 1-morphism; for a C-D-bicomodule 1-morphism M, $\Gamma_M = \varepsilon^r_{C \boxast M} : C \boxast M \boxast D \to C \boxast M$. By Proposition 1.3.11, the Γ_M assemble to a natural transformation for each C and D.

Now, we want the following diagrams to commute:

$$\begin{tikzcd}
(\mathcal{I}\circ\iota)(M \boxast N) \boxast \Gamma_E \arrow[rr, "\Gamma_{M \boxast N}"] \arrow[d, "(\mathcal{I}\circ\iota)^2 \boxast 1"'] \arrow[drr, phantom, "\boxed{1}"] & & \Gamma_C \boxast (M \boxast N) \arrow[d, equal] \\
\mathcal{I}\circ\iota(M) \boxast \mathcal{I}\circ\iota(N) \boxast \Gamma_E \arrow[r, "1 \boxast \Gamma_N"] & \mathcal{I}\circ\iota(M) \boxast \Gamma_D \boxast N \arrow[r, "\Gamma_M \boxast 1"] & \Gamma_C \boxast (M \boxast N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{I}\circ\iota(C) \boxast \Gamma_C \arrow[rr, "\Gamma_C"] \arrow[d, "(\mathcal{I}\circ\iota)^0 \boxast 1"'] \arrow[drr, phantom, "\boxed{2}"] & & \Gamma_C \boxast C \arrow[d, equal] \\
C \boxast \Gamma_C \arrow[dr, "\varepsilon^l_{\Gamma_C}"'] & & \Gamma_C \boxast C \arrow[dl, "\varepsilon^r_{\Gamma_C}"] \\
& \Gamma_C &
\end{tikzcd}$$

We first examine $\boxed{1}$, which we expand into the following diagram:

$$\begin{tikzcd}
C \boxast M \boxast N \boxast E \arrow[rr, "\varepsilon^r_{C \boxast M \boxast N} = 1_{C \boxast M} \boxast \varepsilon^r_N"] \arrow[dr, equal] \arrow[dd, "(\delta^r_{C \boxast M} \boxast 1_N) \boxast 1_E = \delta^r_{C \boxast M} \boxast 1_{l}"'] & & C \boxast M \boxast N \arrow[dd, equal] \arrow[dl, phantom, "\textcircled{2}"] \\
\arrow[r, phantom, "\textcircled{1}"] & C \boxast M \boxast N \boxast E \arrow[dr, "1_{C \boxast M} \boxast \varepsilon^r_N"] \arrow[d, phantom, "\textcircled{3}"] & \\
C \boxast M \boxast D \boxast N \boxast E \arrow[ur, "\varepsilon^r_{C \boxast M} \boxast 1_{N \boxast E}"] \arrow[r, "1_{C \boxast M} \boxast \varepsilon^r_{D \boxast N} = 1_{C \boxast M \boxast D} \boxast \varepsilon^r_N"'] & C \boxast M \boxast D \boxast N \arrow[r, "\varepsilon^r_{C \boxast M} \boxast 1_N"'] & C \boxast M \boxast N
\end{tikzcd}$$

1 commutes because $\delta^r_{C \boxast M} = (\varepsilon^r_{C \boxast M})^{-1}$;

2 commutes trivially;

and 3 commutes by the interchange law.

So the outer diagram commutes, that is, $\boxed{1}$ commutes.

Next, we consider $\boxed{2}$, which we expand into the following diagram:

$$\begin{tikzcd}
C \circledast C \circledast C \arrow[rr, "\varepsilon^r_{C \circledast C} = 1 \circledast \varepsilon^r_C"] \arrow[d, "\varepsilon^r_C \circledast 1"'] \arrow[drr, "1 \circledast \varepsilon^l_C"] \arrow[dr, phantom, "2"] & & C \circledast C \arrow[d, equal] \arrow[dl, phantom, "1"] \\
C \circledast C \arrow[rr, equal] \arrow[dr, "\varepsilon^l_C"'] & \arrow[d, phantom, "3"] & C \circledast C \arrow[dl, "\varepsilon^r_C"] \\
& C &
\end{tikzcd}$$

1 and 3 commute because $\varepsilon^l_C = \varepsilon^r_C$;

2 commutes because the left and right unitors commute in a bicategory.

Therefore the outer diagram commutes, that is, $\boxed{2}$ commutes. So Γ is a 2-natural transformation.

Now, since each Γ_C and each Γ_M is invertible, Γ defines an isomorphism $\mathcal{I} \circ \iota \cong 1_{\mathscr{R}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}}$.

So the pair $(\iota, \mathcal{I})$ is a biequivalence of bicategories.

□

As a small corollary of this result, we get an upgrade to Theorem 2.2.7: $\mathcal{I}$ is, in fact, a pseudofunctor, and in particular $\mathcal{I}^2$ is invertible.

3

Induction

Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a locally $\mathbb{k}$-linear pseudofunctor between finitary 2-categories.

The goal of this section is to define the induction functor $\mathcal{P}_{\mathscr{F}} : \mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{D}}}$. To this end, we begin with a series of lemmas, following the structure of [MMM+21] Lemma 3.11:

3.1 Local functoriality of induction

Lemma 3.1.1. The image of a coalgebra 1-morphism under $\mathscr{F}$ is a coalgebra 1-morphism.

Proof : To be specific, we send the triple (C, Δ, ϵ) to $(\mathscr{F}(C), \Delta' := \mathscr{F}^2 \circ \mathscr{F}(\Delta), \epsilon' := \mathscr{F}^0 \circ \mathscr{F}(\epsilon))$. We need to check that the latter satisfies the coalgebra axioms. First, coassociativity. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathscr{F}(\Delta)"] \arrow[d, "\mathscr{F}(\Delta)"'] \arrow[dr, phantom, "1"] & \mathscr{F}(CC) \arrow[r, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}(1*\Delta)"] \arrow[dr, phantom, "2"] & \mathscr{F}(C)\mathscr{F}(C) \arrow[d, "\mathscr{F}(1)*\mathscr{F}(\Delta)"] \\
\mathscr{F}(CC) \arrow[r, "\mathscr{F}(\Delta*1)"'] \arrow[d, "\mathscr{F}^2"'] \arrow[dr, phantom, "3"] & \mathscr{F}(CCC) \arrow[r, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}^2"'] \arrow[dr, phantom, "4"] & \mathscr{F}(C)\mathscr{F}(CC) \arrow[d, "1*\mathscr{F}^2"] \\
\mathscr{F}(C)\mathscr{F}(C) \arrow[r, "\mathscr{F}(\Delta)*\mathscr{F}(1)"'] & \mathscr{F}(CC)\mathscr{F}(C) \arrow[r, "\mathscr{F}^2*1"'] & \mathscr{F}(C)\mathscr{F}(C)\mathscr{F}(C)
\end{tikzcd}$$

1 is the image of the coassociativity diagram for C, and so commutes;

2 and 3 commute by naturality of $\mathscr{F}^2$;

4 commutes by the coassociativity condition for $\mathscr{F}$.

Therefore the outer square commutes, which is precisely the coassociativity diagram for $\mathscr{F}(C)$.

Next, counitality. Again, consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathscr{F}(\Delta)"] \arrow[d, "\mathscr{F}(\Delta)"] \arrow[dr, equal] & \mathscr{F}(CC) \arrow[rr, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}(1*\epsilon)"] \arrow[drr, phantom, "②"] & & \mathscr{F}(C)\mathscr{F}(C) \arrow[d, "1*\mathscr{F}(\epsilon)"] \\
\mathscr{F}(CC) \arrow[r, "\mathscr{F}(\epsilon*1)"] \arrow[dd, "\mathscr{F}^2"] \arrow[ddr, phantom, "③"] & \mathscr{F}(C) \arrow[r, equal] \arrow[d, equal] \arrow[ddrr, equal] \arrow[drr, phantom, "④"] & \mathscr{F}(C1) \arrow[r, "\mathscr{F}^2"'] & \mathscr{F}(C)\mathscr{F}(1) \arrow[dd, "1*\mathscr{F}^0"] \\
 & \mathscr{F}(1C) \arrow[d, "\mathscr{F}^2"] & & \\
\mathscr{F}(C)\mathscr{F}(C) \arrow[r, "\mathscr{F}(\epsilon)*1"] & \mathscr{F}(1)\mathscr{F}(C) \arrow[rr, "\mathscr{F}^0*1"'] & & \mathscr{F}(C)
\end{tikzcd}$$

1 is the image of the counitality diagram for C, and so commutes;

2 and 3 commute by naturality of $\mathscr{F}^2$;

4 commutes by counitality for $\mathscr{F}$.

Therefore the outer square (with the diagonal equality) commutes, which is precisely the counitality diagram for $\mathscr{F}(C)$.

So $\mathscr{F}(C)$ is indeed a coalgebra 1-morphism.

□

Compare this lemma also with [JS93] Proposition 5.5.

Lemma 3.1.2. The image of a bicomodule 1-morphism under $\mathscr{F}$ is a bicomodule 1-morphism.

Moreover, when $\mathscr{C}$ and $\mathscr{D}$ are fiat, $\mathscr{F}$ sends (left-, right- and) biinjective bicomodule 1-morphisms to (left-, right- and) biinjective bicomodule 1-morphisms.

Proof : Again, to be specific, we send the triple $(M, \delta_{C,M}, \delta_{M,D})$ to the triple $(\mathscr{F}(M), \delta'_{C,M} := \mathscr{F}^2 \circ \mathscr{F}(\delta_{C,M}), \delta'_{M,D} := \mathscr{F}^2 \circ \mathscr{F}(\delta_{M,D}))$. We need to check that this is indeed an $\mathscr{F}(C)$-$\mathscr{F}(D)$-bicomodule 1-morphism. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(M) \arrow[r, "\mathscr{F}(\delta_{C,M})"] \arrow[d, "\mathscr{F}(\delta_{C,M})"'] \arrow[dr, phantom, "1"] & \mathscr{F}(CM) \arrow[r, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}(\Delta * 1)"] \arrow[dr, phantom, "2"] & \mathscr{F}(C)\mathscr{F}(M) \arrow[d, "\mathscr{F}(\Delta) * 1"] \\
\mathscr{F}(CM) \arrow[r, "\mathscr{F}(1 * \delta_{C,M})"] \arrow[d, "\mathscr{F}^2"'] \arrow[dr, phantom, "3"] & \mathscr{F}(CCM) \arrow[r, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}^2"'] \arrow[dr, phantom, "4"] & \mathscr{F}(CC)\mathscr{F}(M) \arrow[d, "\mathscr{F}^2 * 1"'] \\
\mathscr{F}(C)\mathscr{F}(M) \arrow[r, "1 * \mathscr{F}(\delta_{C,M})"'] & \mathscr{F}(C)\mathscr{F}(CM) \arrow[r, "1 * \mathscr{F}^2"'] & \mathscr{F}(C)\mathscr{F}(C)\mathscr{F}(M)
\end{tikzcd}$$

1 is the image of the C-coaction condition for M, and so commutes;

2 and 3 commute by naturality of $\mathscr{F}^2$;

4 is the coassociativity diagram for $\mathscr{F}$.

Therefore the outer square commutes, which is precisely the $\mathscr{F}(C)$-coaction diagram for $\mathscr{F}(M)$.

Next, we consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(M) \arrow[r, "\mathscr{F}(\delta_{C,M})"] \arrow[dr, equal] & \mathscr{F}(CM) \arrow[r, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}(\epsilon * 1)"] \arrow[dr, phantom, "2"] \arrow[dl, phantom, "1"] & \mathscr{F}(C)\mathscr{F}(M) \arrow[d, "\mathscr{F}(\epsilon) * 1"] \\
 & \mathscr{F}(M) \arrow[r, "\mathscr{F}^2"] \arrow[dr, equal] & \mathscr{F}(1)\mathscr{F}(M) \arrow[d, "\mathscr{F}^0 * 1"] \arrow[dl, phantom, "3"] \\
 & & \mathscr{F}(M)
\end{tikzcd}$$

1 is the image of the C-counit condition for M, and so commutes;

2 commutes by naturality of $\mathscr{F}^2$;

3 is the counitality diagram for $\mathscr{F}$.

Therefore the outer triangle commutes, which is precisely the $\mathscr{F}(C)$-counit diagram for $\mathscr{F}(M)$.

So in fact $\mathscr{F}(M)$ is a left $\mathscr{F}(C)$-comodule 1-morphism.

Similarly, $\mathscr{F}(M)$ is a right $\mathscr{F}(D)$-comodule 1-morphism.

To see that $\mathscr{F}(M)$ is an $\mathscr{F}(C)$-$\mathscr{F}(D)$-bicomodule 1-morphism, consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(M) \arrow[rr, "\mathscr{F}(\delta_{C,M})"] \arrow[dd, "\mathscr{F}(\delta_{M,D})"] \arrow[ddrr, phantom, "1"] & & \mathscr{F}(CM) \arrow[rr, "\mathscr{F}^2"] \arrow[dd, "\mathscr{F}(1*\delta_{M,D})"] \arrow[ddrr, phantom, "2"] & & \mathscr{F}(C)\mathscr{F}(M) \arrow[dd, "1*\mathscr{F}(\delta_{M,D})"] \\
& & & & \\
\mathscr{F}(MD) \arrow[rr, "\mathscr{F}(\delta_{C,M}*1)"] \arrow[dd, "\mathscr{F}^2"] \arrow[ddrr, phantom, "3"] & & \mathscr{F}(CMD) \arrow[rr, "\mathscr{F}^2"] \arrow[dd, "\mathscr{F}^2"] \arrow[ddrr, phantom, "4"] & & \mathscr{F}(C)\mathscr{F}(MD) \arrow[dd, "1*\mathscr{F}^2"] \\
& & & & \\
\mathscr{F}(M)\mathscr{F}(D) \arrow[rr, "\mathscr{F}(\delta_{C,M})*1"] & & \mathscr{F}(CM)\mathscr{F}(D) \arrow[rr, "\mathscr{F}^2*1"] & & \mathscr{F}(C)\mathscr{F}(M)\mathscr{F}(D)
\end{tikzcd}$$

1 is the image of the bicomodule diagram for M, and so commutes;

2 and 3 commute by naturality of $\mathscr{F}^2$;

4 is the coassociativity diagram for $\mathscr{F}$.

Therefore the outer square commutes, which is precisely the bicomodule diagram for $\mathscr{F}(M)$. So $\mathscr{F}(M)$ is an $\mathscr{F}(C)$-$\mathscr{F}(D)$-bicomodule 1-morphism.

Finally, we need to show that left-, right- and biinjectivity are preserved when $\mathscr{C}$ and $\mathscr{D}$ are fiat. By Lemma 1.3.7, when M is injective as a left C-comodule 1-morphism, it is isomorphic to a summand of CF, for some 1-morphism $F : j \to i$. So since $\mathscr{F}$ is additive, $\mathscr{F}(M)$ is isomorphic to a summand of $\mathscr{F}(CF) \cong \mathscr{F}(C)\mathscr{F}(F)$. So $\mathscr{F}(M)$ is injective as a left $\mathscr{F}(C)$-comodule 1-morphism. Similarly, when M is right-injective, $\mathscr{F}(M)$ is injective as a right $\mathscr{F}(D)$-comodule 1-morphism. So if M is biinjective, so is $\mathscr{F}(M)$. So we are done.

□

Lemma 3.1.3. The image of a bicomodule homomorphism under $\mathscr{F}$ is a bicomodule homomorphism

Proof : We send the bicomodule homomorphism $\Phi : {}_CM_D \to {}_CN_D$ to $\mathscr{F}(\Phi) : {}_{\mathscr{F}(C)}\mathscr{F}(M)_{\mathscr{F}(D)} \to {}_{\mathscr{F}(C)}\mathscr{F}(N)_{\mathscr{F}(D)}$. We need to check that this is indeed a bicomodule homomorphism. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(M) \arrow[rr, "\mathscr{F}(\Phi)"] \arrow[d, "\mathscr{F}(\delta_{C,M})"'] \arrow[drr, phantom, "1"] & & \mathscr{F}(N) \arrow[d, "\mathscr{F}(\delta_{C,N})"] \\
\mathscr{F}(CM) \arrow[rr, "\mathscr{F}(1*\Phi)"] \arrow[d, "\mathscr{F}^2"'] \arrow[drr, phantom, "2"] & & \mathscr{F}(CN) \arrow[d, "\mathscr{F}^2"] \\
\mathscr{F}(C)\mathscr{F}(M) \arrow[rr, "1*\mathscr{F}(\Phi)"] & & \mathscr{F}(C)\mathscr{F}(N)
\end{tikzcd}$$

1 is the image of the left C-comodule homomorphism condition, and so commutes;

2 commutes by naturality of $\mathscr{F}^2$.

Therefore, the outer square commutes. But this is the left $\mathscr{F}(C)$-comodule homomorphism condition for $\mathscr{F}(\Phi)$. The right $\mathscr{F}(D)$-comodule homomorphism condition is similar, and thus $\mathscr{F}(\Phi)$ is a bicomodule homomorphism.

□

Putting all of this information together, we obtain the following result:

Proposition 3.1.4. When $\mathscr{C}$, $\mathscr{D}$ are fiat, there is an assignment of data (not necessarily defining a pseudofunctor) $\mathcal{P} := \mathcal{P}_{\mathscr{F}} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}$ as follows:

- For a coalgebra 1-morphism (C, Δ, ϵ) in $\underline{\mathscr{C}}$,

$$\mathcal{P}(C) = (\mathscr{F}(C), \Delta', \epsilon'),$$

  as defined in Lemma 3.1.1;

- For a pair of coalgebra 1-morphisms C, D, a functor

$$\mathcal{P}_{CD} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}(C, D) \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}(\mathscr{F}(C), \mathscr{F}(D))$$

  with the following data:

  – For an C-D-bicomodule 1-morphism $(M, \delta_{C,M}, \delta_{M,D})$, $\mathcal{P}_{CD}(M) = (\mathscr{F}(M), \delta'_{C,M}, \delta'_{M,D})$, as defined in Lemma 3.1.2;

– For a homomorphism of C-D-bicomodule 1-morphisms $\Phi : M \to N$, $\mathcal{P}_{CD}(\Phi) = \mathscr{F}(\Phi)$.

Proof : By Lemmas 3.1.1, 3.1.2 and 3.1.3, we know that this assignment is well-defined, sending objects to objects, 1-morphisms to 1-morphisms, and 2-morphisms to 2-morphisms with appropriate source and target. That $\mathcal{P}_{CD}$ are functors is immediate from the fact that $\mathscr{F}$ preserves vertical composition and identity for 2-morphisms. □

This gets us most of the way to defining a pseudofunctor $\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{D}}}$.

3.2 Pseudofunctoriality of induction

Next, we define $\mathcal{P}^0$ and $\mathcal{P}^2$.

Lemma 3.2.1. The following diagram commutes:

$$\begin{tikzcd} 1 \arrow[r, "I_C"] \arrow[d, equal] & \mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}(C,C) \arrow[d, "\mathcal{P}"] \\ 1 \arrow[r, "I_{\mathcal{P}(C)}"] & \mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{D}}}(\mathcal{P}(C),\mathcal{P}(C)) \end{tikzcd}$$

Proof : We know that $1_C = {}_C C_C$. So $\mathcal{P}(1_C) = {}_{\mathscr{F}(C)}\mathscr{F}(C)_{\mathscr{F}(C)} = 1_{\mathcal{P}(C)}$. □

So we can define $\mathcal{P}^0$ to be the identity.

A version of the following lemma appears as Lemma 3.9 in [MMM+23].

Lemma 3.2.2. For bicomodule 1-morphisms $M = {}_C M_D$ and $N = {}_D N_E$, there is a bicomodule homomorphism $\mathcal{P}^2_{M,N} : \mathscr{F}(M \underset{D}{\circledast} N) \to \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N)$. This collection of morphisms form a natural transformation $\mathcal{P}^2 : \mathscr{F}(- \underset{D}{\circledast} -) \to \mathscr{F}(-) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(-)$. Moreover, when $\mathscr{F}$ preserves equalizers of 1-morphisms, $\mathcal{P}^2$ is a natural isomorphism.

Proof : $(M \underset{D}{\circledast} N, t := t^D_{M,N})$ is such that, by definition, the following is an equaliser diagram:

$$\begin{tikzcd} M \underset{D}{\circledast} N \arrow[r, "t"] & MN \arrow[r, shift left, "1*\delta_{D,N}"] \arrow[r, shift right, "\delta_{M,D}*1"'] & MDN \end{tikzcd}$$

Similarly, $(\mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N), t' := t^{\mathscr{F}(D)}_{\mathscr{F}(M),\mathscr{F}(N)})$ is such that the following is an equaliser diagram:

$$\begin{tikzcd} \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \arrow[r, "t'"] & \mathscr{F}(M)\mathscr{F}(N) \arrow[r, shift left, "1*\delta_{\mathscr{F}(D),\mathscr{F}(N)}"] \arrow[r, shift right, "\delta_{\mathscr{F}(M),\mathscr{F}(D)}*1"'] & \mathscr{F}(M)\mathscr{F}(D)\mathscr{F}(N) \end{tikzcd} \tag{3.2.1}$$

We claim that the map $\mathscr{F}^2 \circ \mathscr{F}(t) : \mathscr{F}(M \underset{D}{\boxast} N) \to \mathscr{F}(M)\mathscr{F}(N)$ equalises $\delta_{\mathscr{F}(M),\mathscr{F}(D)} * 1$ and $1 * \delta_{\mathscr{F}(D),\mathscr{F}(N)}$. To prove this, we consider the following pair of diagrams:

$$\begin{tikzcd} \mathscr{F}(M \underset{D}{\boxast} N) \arrow[r, "\mathscr{F}(t)"] & \mathscr{F}(MN) \arrow[r, "\mathscr{F}(1*\delta_{D,N})"] \arrow[d, "\mathscr{F}^2"] \arrow[dr, phantom, "1"] & \mathscr{F}(MDN) \arrow[d, "\mathscr{F}^2"] & \\ & \mathscr{F}(M)\mathscr{F}(N) \arrow[r, "1*\mathscr{F}(\delta_{D,N})"] & \mathscr{F}(M)\mathscr{F}(DN) \arrow[r, "1*\mathscr{F}^2"] & \mathscr{F}(M)\mathscr{F}(D)\mathscr{F}(N) \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{F}(M \underset{D}{\boxast} N) \arrow[r, "\mathscr{F}(t)"] & \mathscr{F}(MN) \arrow[r, "\mathscr{F}(\delta_{M,D}*1)"] \arrow[d, "\mathscr{F}^2"] \arrow[dr, phantom, "2"] & \mathscr{F}(MDN) \arrow[d, "\mathscr{F}^2"] & \\ & \mathscr{F}(M)\mathscr{F}(N) \arrow[r, "\mathscr{F}(\delta_{M,D})*1"] & \mathscr{F}(MD)\mathscr{F}(N) \arrow[r, "\mathscr{F}^2*1"] & \mathscr{F}(M)\mathscr{F}(D)\mathscr{F}(N) \end{tikzcd}$$

1 and 2 commute by naturality of $\mathscr{F}^2$.

The bottom edges of both diagrams are, respectively, $(1*\delta_{\mathscr{F}(D),\mathscr{F}(N)})\circ\mathscr{F}^2\circ\mathscr{F}(t)$ and $(\delta_{\mathscr{F}(M),\mathscr{F}(D)}* 1) \circ \mathscr{F}^2 \circ \mathscr{F}(t)$. On the other hand, we can compute

$$\begin{aligned} (1 * \mathscr{F}^2) \circ \mathscr{F}^2 \circ \mathscr{F}(1 * \delta_{D,N}) \circ \mathscr{F}(t) &= (1 * \mathscr{F}^2) \circ \mathscr{F}^2 \circ \mathscr{F}((1 * \delta_{D,N}) \circ t) \\ &= (\mathscr{F}^2 * 1) \circ \mathscr{F}^2 \circ \mathscr{F}((1 * \delta_{D,N}) \circ t) \\ &= (\mathscr{F}^2 * 1) \circ \mathscr{F}^2 \circ \mathscr{F}((\delta_{M,D} * 1) \circ t) \\ &= (\mathscr{F}^2 * 1) \circ \mathscr{F}^2 \circ \mathscr{F}(\delta_{M,D} * 1) \circ \mathscr{F}(t) \end{aligned}$$

where the first and last equalities hold by functoriality of $\mathscr{F}$, the second holds by coassociativity of $\mathscr{F}$, and the third holds because t equalises $1 * \delta_{D,N}$ and $\delta_{M,D} * 1$.

So the top edges of our two diagrams are equal; so since (1) and (2) commute, the bottom edges are equal. So $\mathscr{F}^2 \circ \mathscr{F}(t)$ equalises $\delta_{\mathscr{F}(M),\mathscr{F}(D)} * 1$ and $1 * \delta_{\mathscr{F}(D),\mathscr{F}(N)}$, as required.

But then, since (3.2.1) is an equaliser, we must have a map

$$\mathcal{P}^2_{M,N} : \mathscr{F}(M \underset{D}{\boxast} N) \to \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N)$$

such that $t' \circ \mathcal{P}^2_{M,N} = \mathscr{F}^2 \circ \mathscr{F}(t)$.

To see that this collection of morphisms does, indeed, form a natural transformation, let $\Phi_M : M \to M'$ and $\Phi_N : N \to N'$ be bicomodule homomorphisms. We need the following diagram to commute:

$$\begin{tikzcd}
\mathscr{F}(M \underset{D}{\boxast} N) \arrow[r, "\mathcal{P}^2_{M,N}"] \arrow[d, "\mathscr{F}(\Phi_M \underset{D}{\boxast} \Phi_N)"'] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \arrow[d, "\mathscr{F}(\Phi_M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(\Phi_N)"] \\
\mathscr{F}(M' \underset{D}{\boxast} N') \arrow[r, "\mathcal{P}^2_{M',N'}"] & \mathscr{F}(M') \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N')
\end{tikzcd}$$

To see that it does commute, we let t and t' be as above, $\bar{t}$ and $\overline{t'}$ be the analogous maps for M' and N', and consider the following cube:

$$\begin{tikzcd}
 & & \mathscr{F}(MN) \arrow[rrd, "\mathscr{F}^2"] \arrow[d, dashed, "\mathscr{F}(\Phi_M * \Phi_N)"] & & \\
\mathscr{F}(M \underset{D}{\boxast} N) \arrow[rru, "\mathscr{F}(t)"] \arrow[d, "\mathscr{F}(\Phi_M \underset{D}{\boxast} \Phi_N)"'] \arrow[rrd, "\mathcal{P}^2_{M,N}"'] & & \mathscr{F}(M'N') \arrow[rrd, dashed, "\mathscr{F}^2"] & & \mathscr{F}(M)\mathscr{F}(N) \arrow[d, "\mathscr{F}(\Phi_M) * \mathscr{F}(\Phi_N)"] \\
\mathscr{F}(M' \underset{D}{\boxast} N') \arrow[rru, dashed, "\mathscr{F}(\bar{t})"] \arrow[rrd, "\mathcal{P}^2_{M',N'}"'] & & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \arrow[rru, "t'"] \arrow[d, "\mathscr{F}(\Phi_M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(\Phi_N)"] & & \mathscr{F}(M')\mathscr{F}(N') \\
 & & \mathscr{F}(M') \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N') \arrow[rru, "\overline{t'}"'] & &
\end{tikzcd}$$

Now, the front right face commutes by definition of $\mathscr{F}(\Phi_M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(\Phi_N)$.

The back left face is the image of the following diagram:

$$\begin{tikzcd}
M \underset{D}{\circledast} N \arrow[r, "t"] \arrow[d, "\Phi_M \underset{D}{\circledast} \Phi_N"'] & MN \arrow[d, "\Phi_M * \Phi_N"] \\
M' \underset{D}{\circledast} N' \arrow[r, "\bar{t}"] & M'N'
\end{tikzcd}$$

which, by definition of $\Phi_M \underset{D}{\circledast} \Phi_N$, commutes.

The top and bottom faces both commute by definition of $\mathcal{P}^2$.

The back right face commutes by naturality of $\mathscr{F}^2$.

$\overline{t'}$ is an equalizer, so monic, so by a previous argument, the front left face commutes.

But this is exactly what we wanted to begin with, so $\mathcal{P}^2$ is a natural transformation.

To see that $\mathcal{P}^2_{M,N}$ is a bicomodule homomorphism, we want the following diagrams to commute, where we suppress the subscripts in $\mathcal{P}^2_{M,N}$:

$$\begin{tikzcd}
\mathscr{F}(M \underset{D}{\circledast} N) \arrow[r, "\mathcal{P}^2"] \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(M \underset{D}{\circledast} N)}"'] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N) \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N)}"] \\
\mathscr{F}(C)\mathscr{F}(M \underset{D}{\circledast} N) \arrow[r, "1*\mathcal{P}^2"'] & \mathscr{F}(C)\mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{F}(M \underset{D}{\circledast} N) \arrow[r, "\mathcal{P}^2"] \arrow[d, "\delta_{\mathscr{F}(M \underset{D}{\circledast} N),\mathscr{F}(E)}"'] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N) \arrow[d, "\delta_{\mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N),\mathscr{F}(E)}"] \\
\mathscr{F}(M \underset{D}{\circledast} N)\mathscr{F}(E) \arrow[r, "\mathcal{P}^2*1"'] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N)\mathscr{F}(E)
\end{tikzcd}$$

We only consider the first diagram, as the proof that the second commutes is similar.

Consider the following diagram:

$$\begin{tikzcd}
& & \mathscr{F}(MN) \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(MN)}"] \arrow[drr, "\mathscr{F}^2"] & & \\
\mathscr{F}(M \underset{D}{\boxast} N) \arrow[urr, "\mathscr{F}(t)"] \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(M \underset{D}{\boxast} N)}"] \arrow[drr, "\mathcal{P}^2"] & & \mathscr{F}(C)\mathscr{F}(MN) \arrow[drr, "1*\mathscr{F}^2"] & & \mathscr{F}(M)\mathscr{F}(N) \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(M)}*1"] \\
\mathscr{F}(C)\mathscr{F}(M \underset{D}{\boxast} N) \arrow[urr, "1*\mathscr{F}(t)"] \arrow[drr, "1*\mathcal{P}^2"] & & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \arrow[urr, "t'"] \arrow[d, "\delta_{\mathscr{F}(C),\mathscr{F}(M)} \underset{\mathscr{F}(D)}{\boxast} 1"] & & \mathscr{F}(C)\mathscr{F}(M)\mathscr{F}(N) \\
& & \mathscr{F}(C)\mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \arrow[urr, "1*t'"] & &
\end{tikzcd}$$

Now, the top and bottom faces commute by definition of $\mathcal{P}^2$;

the front right, back left and back right faces all commute by the definitions of the respective coactions;

and $1 * t'$ is monic.

Therefore, by a previous proof, the front left face commutes, that is, $\mathcal{P}^2$ is a left $\mathscr{F}(C)$-comodule homomorphism. Similarly, it is a right $\mathscr{F}(E)$-comodule homomorphism, so an $\mathscr{F}(C)$-$\mathscr{F}(E)$-bicomodule homomorphism.

In the case that $\mathscr{F}$ preserves equalizers of 1-morphisms, the following is an equalizer diagram:

$$\begin{tikzcd}
\mathscr{F}(M \underset{D}{\boxast} N) \arrow[r, "\mathscr{F}(t)"] & \mathscr{F}(MN) \arrow[r, shift left, "\mathscr{F}(1*\delta_{D,N})"] \arrow[r, shift right, "\mathscr{F}(\delta_{M,D}*1)"'] & \mathscr{F}(MDN)
\end{tikzcd}$$

By exactly the same process as we used to obtain $\mathcal{P}^2$, we get a map $\mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \to \mathscr{F}(M \underset{D}{\boxast} N)$, which we will suggestively call $\mathcal{P}^{-2}_{M,N}$, satisfying $\mathscr{F}(t) \circ \mathcal{P}^{-2}_{M,N} = \mathscr{F}^{-2} \circ t'$. We can compute:

$$\begin{aligned}
t' \circ \mathcal{P}^2_{M,N} \circ \mathcal{P}^{-2}_{M,N} &= \mathscr{F}^2 \circ \mathscr{F}(t) \circ \mathcal{P}^{-2}_{M,N} \\
&= \mathscr{F}^2 \circ \mathscr{F}^{-2} \circ t' \\
&= t'
\end{aligned}$$

from which we deduce that, since t' is monic, we have $\mathcal{P}^2_{M,N} \circ \mathcal{P}^{-2}_{M,N} = 1$. Similarly, $\mathcal{P}^{-2}_{M,N} \circ \mathcal{P}^2_{M,N} = 1$, so these maps are indeed inverses. So $\mathcal{P}^2$ is a natural isomorphism.

□

Given all this data, we can define the following:

Theorem 3.2.3. Suppose $\mathscr{C}$, $\mathscr{D}$ are fiat, and $\mathscr{F}$ preserves equalizers of 1-morphisms. Let

$$\mathcal{P} := \mathcal{P}_{\mathscr{F}} : \mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{D}}}$$

be defined by the following data:

- On objects, 1-morphisms and 2-morphisms, $\mathcal{P}$ is defined as in Proposition 3.1.4
- Define the natural transformation $\mathcal{P}^0$ as in Lemma 3.2.1
- Define the natural transformation $\mathcal{P}^2$ as in Lemma 3.2.2.

Then $\mathcal{P}$ is a pseudofunctor.

Proof : By Proposition 3.1.4, along with Lemmas 3.2.1 and 3.2.2, our data are well-defined. Again by Proposition 3.1.4, we have everything except the coassociativity and counitality diagrams. To show counitality, we consider the following diagrams:

$$\begin{tikzcd}
\mathscr{F}(C \underset{C}{\boxast} M) \arrow[r, "\mathcal{P}^2"] \arrow[d, "\mathscr{F}(\epsilon^l_M)"'] & \mathscr{F}(C) \underset{\mathscr{F}(C)}{\boxast} \mathscr{F}(M) \arrow[r, equal, "\mathcal{P}^0 * 1"] & \mathscr{F}(C) \underset{\mathscr{F}(C)}{\boxast} \mathscr{F}(M) \arrow[d, "\epsilon^l_{\mathscr{F}(M)}"] \\
\mathscr{F}(M) \arrow[rr, equal] & & \mathscr{F}(M)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{F}(M \underset{D}{\boxast} D) \arrow[r, "\mathcal{P}^2"] \arrow[d, "\mathscr{F}(\epsilon^r_M)"'] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(D) \arrow[r, equal, "1 * \mathcal{P}^0"] & \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(D) \arrow[d, "\epsilon^r_{\mathscr{F}(M)}"] \\
\mathscr{F}(M) \arrow[rr, equal] & & \mathscr{F}(M)
\end{tikzcd}$$

We consider only the first diagram, as the proof the second commutes is completely analogous.

Writing $t := t^{\mathscr{F}(C)}_{\mathscr{F}(C),\mathscr{F}(M)}$, by definition,

$$\epsilon^l_{\mathscr{F}(M)} = (\epsilon_{\mathscr{F}(C)} * 1) \circ t = ((\mathscr{F}^0 \circ \mathscr{F}(\epsilon_C)) * 1) \circ t.$$

So expanding our previous diagram, we have

$$\begin{tikzcd}
\mathscr{F}(C \underset{C}{\boxast} M) \arrow[rr, "\mathcal{P}^2"] \arrow[d, "\mathscr{F}(t^C_{C,M})"] \arrow[drr, phantom, "1"] & & \mathscr{F}(C) \underset{\mathscr{F}(C)}{\boxast} \mathscr{F}(M) \arrow[d, "t"] \\
\mathscr{F}(CM) \arrow[rr, "\mathscr{F}^2"] \arrow[d, "\mathscr{F}(\epsilon_C * 1)"] \arrow[drr, phantom, "2"] & & \mathscr{F}(C)\mathscr{F}(M) \arrow[d, "\mathscr{F}(\epsilon_C)*1"] \\
\mathscr{F}(M) \arrow[rr, "\mathscr{F}^2"] \arrow[dr, equal] & \arrow[d, phantom, "3"] & \mathscr{F}(1)\mathscr{F}(M) \arrow[dl, "\mathscr{F}^0 * 1"] \\
& \mathscr{F}(M) &
\end{tikzcd}$$

1 commutes by definition of $\mathcal{P}^2$;

2 commutes by naturality of $\mathscr{F}^2$;

3 commutes by counitality for $\mathscr{F}$.

Moreover, the outer edge is exactly the counitality diagram for $\mathcal{P}$.

Finally, to show coassociativity, we need the following diagram to commute:

$$\begin{tikzcd}
\mathscr{F}(M \boxast N \boxast L) \arrow[r, "\mathcal{P}^2"'] \arrow[d, "\mathcal{P}^2"] & \mathscr{F}(M \boxast N) \boxast \mathscr{F}(L) \arrow[d, "\mathcal{P}^2 \boxast 1"'] \\
\mathscr{F}(M) \boxast \mathscr{F}(N \boxast L) \arrow[r, "1 \boxast \mathcal{P}^2"] & \mathscr{F}(M) \boxast \mathscr{F}(N) \boxast \mathscr{F}(L)
\end{tikzcd}$$

To see that it does, we first consider the following diagram:

$$
\begin{tikzcd}
 & & \mathscr{F}(MNL) \arrow[d, dashed, "\mathscr{F}^2"] \arrow[drr, "\mathscr{F}^2"] & & \\
\mathscr{F}(M \boxast NL) \arrow[urr, "\mathscr{F}(t)"] \arrow[d, "\mathcal{P}^2"] \arrow[drr, "\mathscr{F}^2"] & & \mathscr{F}(M)\mathscr{F}(NL) \arrow[drr, dashed, "1 * \mathscr{F}^2"] & & \mathscr{F}(MN)\mathscr{F}(L) \arrow[d, "\mathscr{F}^2 * 1"] \\
\mathscr{F}(M) \boxast \mathscr{F}(NL) \arrow[urr, dashed, "t"] \arrow[drr, "1 \boxast \mathscr{F}^2"] & & \mathscr{F}(M \boxast N)\mathscr{F}(L) \arrow[urr, "\mathscr{F}(t) * 1"] \arrow[d, "\mathcal{P}^2 * 1"] & & \mathscr{F}(M)\mathscr{F}(N)\mathscr{F}(L) \\
 & & \mathscr{F}(M) \boxast \mathscr{F}(N)\mathscr{F}(L) \arrow[urr, "t * 1"] & &
\end{tikzcd}
$$

In this diagram, we see that the top face commutes by naturality of $\mathscr{F}^2$;

The bottom face commutes by definition of $1 \boxast \mathscr{F}^2$;

The back left face commutes by definition of $\mathcal{P}^2$;

The back right face commutes by coherence of $\mathscr{F}^2$;

The front right face commutes by definition of $\mathcal{P}^2$;

And $t * 1$ is monic.

Therefore, the front left face commutes by a previous argument.

Next, consider the following diagram:

$$
\begin{tikzcd}
 & & \mathscr{F}(M \boxast NL) \arrow[d, dashed, "\mathcal{P}^2"] \arrow[drr, "\mathscr{F}^2"] & & \\
\mathscr{F}(M \boxast N \boxast L) \arrow[urr, "\mathscr{F}(1 \boxast t)"] \arrow[d, "\mathcal{P}^2"] \arrow[drr, "\mathcal{P}^2"] & & \mathscr{F}(M) \boxast \mathscr{F}(NL) \arrow[drr, dashed, "1 \boxast \mathscr{F}^2"] & & \mathscr{F}(M \boxast N)\mathscr{F}(L) \arrow[d, "\mathcal{P}^2 * 1"] \\
\mathscr{F}(M) \boxast \mathscr{F}(N \boxast L) \arrow[urr, dashed, "1 \boxast \mathscr{F}(t)"] \arrow[drr, "1 \boxast \mathcal{P}^2"] & & \mathscr{F}(M \boxast N) \boxast \mathscr{F}(L) \arrow[urr, "t"] \arrow[d, "\mathcal{P}^2 \boxast 1"] & & \mathscr{F}(M) \boxast \mathscr{F}(N)\mathscr{F}(L) \\
 & & \mathscr{F}(M) \boxast \mathscr{F}(N) \boxast \mathscr{F}(L) \arrow[urr, "1 \boxast t"] & &
\end{tikzcd}
$$

Here, we see that the front left face commutes by definition of the cotensor $\mathcal{P}^2 \boxast 1$;

The back right face commutes by the argument for the previous diagram;

The back left face commutes by naturality of $\mathcal{P}^2$;

The top and bottom faces commute by definition of $\mathcal{P}^2$;

and $1 \circledast t$ is monic.

Therefore, by the same argument as before, the front left face must commute, that is, the diagram we want commutes.

Hence, we've shown that $\mathcal{P}$ is a colax functor.

It is immediate from Lemma 3.2.2 that when $\mathscr{F}$ preserves equalizers of 1-morphisms, $\mathcal{P}$ is a pseudofunctor.

□

4

Restriction

Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a $\Bbbk$-linear pseudofunctor of fiat 2-categories. In the setting of birepresentations of $\mathscr{D}$, defining restriction along $\mathscr{F}$ is simple. We start in this context.

However, for the purposes of constructing an adjunction, we need $\mathcal{P}$ and $\mathcal{R}$ to pass between the same 2-categories. For this reason, we use Theorem 2.1.2 alongside some technical considerations to obtain a pseudofunctor in the correct setting.

4.1 Restriction of birepresentations

Definition 4.1.1 (Restriction)**.** We define *restriction along* $\mathscr{F}$ as the strict 2-functor

$$\mathcal{R} = \mathcal{R}_{\mathscr{F}} : \mathscr{D} - afmod \to \mathscr{C} - afmod$$

given by pre-composition by $\mathscr{F}$, that is, by the following data:

- For $\mathbb{M}$ a birepresentation of $\mathscr{D}$, define $\mathcal{R}(\mathbb{M}) = \mathbb{M} \circ \mathscr{F}$;
- For $\Phi : \mathbb{M} \to \mathbb{M}'$ a morphism of birepresentations of $\mathscr{D}$, define $\mathcal{R}(\Phi) = \Phi_{\mathscr{F}}$;
- For $\sigma : \Phi \to \Psi : \mathbb{M} \to \mathbb{M}'$ a modification of morphisms of birepresentations of $\mathscr{D}$, define $\mathcal{R}(\sigma) = \sigma_{\mathscr{F}}$;

$\triangleleft$

Theorem 4.1.2. $\mathcal{R}$ is, indeed, a strict 2-functor. When $\mathscr{F}$ is a strict 2-functor, $\mathcal{R}$ sends 2-representations to 2-representations.

Proof : First, we show that $\mathcal{R}$ is well-defined.

Since the composition of pseudofunctors is a pseudofunctor, it is immediate that $\mathcal{R}$ is well-defined on objects. The claim that for $\mathscr{F}$ a strict 2-functor, $\mathcal{R}$ sends 2-representations to 2-representations, is similarly immediate.

Now, suppose $\Phi : \mathbb{M} \to \mathbb{M}'$ is a strong transform. $\Phi_{\mathscr{F}(j)}$ clearly gives a morphism $\mathbb{M}(j) \to \mathbb{M}'(j)$, and similarly $\Phi_{\mathscr{F}(F)} : \Phi_{\mathscr{F}(k)} \circ \mathbb{M}(\mathscr{F}(F)) \to \mathbb{M}'(\mathscr{F}(F)) \circ \Phi_{\mathscr{F}(j)}$, which is natural in F as Φ is. For $\Phi_{\mathscr{F}}$ to be a 2-natural transform, we need the following diagram to commute:

$$\begin{tikzcd}
\Phi_l \circ \mathbb{M}(\mathscr{F}(GF)) \arrow[rr, "\Phi_{\mathscr{F}(GF)}"] \arrow[d, "1*(\mathcal{R}(\mathbb{M}))^2"'] & & \mathbb{M}'(\mathscr{F}(GF)) \circ \Phi_j \arrow[d, "\mathcal{R}(\mathbb{M}')^2*1"] \\
\Phi_l \circ \mathbb{M}(\mathscr{F}(G))\mathbb{M}(\mathscr{F}(F)) \arrow[r, "\Phi_{\mathscr{F}(G)}*1"'] & \mathbb{M}'(\mathscr{F}(G)) \circ \Phi_k \circ \mathbb{M}(\mathscr{F}(F)) \arrow[r, "\Phi_{\mathscr{F}(F)}"'] & \mathbb{M}'(\mathscr{F}(G))\mathbb{M}'(\mathscr{F}(F)) \circ \Phi_j
\end{tikzcd}$$

To see that it does, we consider the expanded diagram below:

$$\begin{tikzcd}
\Phi_l \circ \mathbb{M}(\mathscr{F}(GF)) \arrow[rr, "\Phi_{\mathscr{F}(GF)}"] \arrow[d, "1*\mathbb{M}(\mathscr{F}^2)"'] & & \mathbb{M}'(\mathscr{F}(GF)) \circ \Phi_j \arrow[d, "\mathbb{M}'(\mathscr{F}^2)*1"] \\
\Phi_l \circ \mathbb{M}(\mathscr{F}(G)\mathscr{F}(F)) \arrow[rr, "\Phi_{\mathscr{F}(G)\mathscr{F}(F)}"] \arrow[d, "1*\mathbb{M}^2"'] & 1 & \mathbb{M}'(\mathscr{F}(G)\mathscr{F}(F)) \circ \Phi_j \arrow[d, "\mathbb{M}'^2*1"] \\
\Phi_l \circ \mathbb{M}(\mathscr{F}(G))\mathbb{M}(\mathscr{F}(F)) \arrow[r, "\Phi_{\mathscr{F}(G)}*1"'] & \mathbb{M}'(\mathscr{F}(G)) \circ \Phi_k \circ \mathbb{M}(\mathscr{F}(F)) \arrow[u, phantom, "2"] \arrow[r, "1*\Phi_{\mathscr{F}(F)}"'] & \mathbb{M}'(\mathscr{F}(G))\mathbb{M}'(\mathscr{F}(F)) \circ \Phi_j
\end{tikzcd}$$

Now, 1 commutes by naturality of Φ;

2 commutes by the higher naturality condition for Φ.

Therefore, the outer square commutes as required.

Moreover, we need the following diagram to commute:

$$\begin{tikzcd}
\Phi_{\mathscr{F}(j)} \circ \mathbb{M}(\mathscr{F}(1_j)) \arrow[rr, "\Phi_{\mathscr{F}(1_j)}"] \arrow[dr, "1*\mathcal{R}(\mathbb{M})^0"'] & & \mathbb{M}'(\mathscr{F}(1_j)) \circ \Phi_{\mathscr{F}(j)} \arrow[dl, "\mathcal{R}(\mathbb{M}')^0*1"] \\
& \Phi_{\mathscr{F}(j)} &
\end{tikzcd}$$

To see that it does, consider the following diagram:

$$\begin{tikzcd}
\Phi_{\mathscr{F}(j)} \circ \mathbb{M}(\mathscr{F}(1_j)) \arrow[rr, "\Phi_{\mathscr{F}(1_j)}"] \arrow[d, "1 * \mathbb{M}(\mathscr{F}^0)"] \arrow[drr, phantom, "1"] & & \mathbb{M}'(\mathscr{F}(1_j)) \circ \Phi_{\mathscr{F}(j)} \arrow[d, "\mathbb{M}'(\mathscr{F}^0) * 1"] \\
\Phi_{\mathscr{F}(j)} \circ \mathbb{M}(1_{\mathscr{F}(j)}) \arrow[rr, "\Phi_{1_{\mathscr{F}(j)}}"'] \arrow[dr, "1 * \mathbb{M}^0"'] & \arrow[d, phantom, "2"] & \mathbb{M}'(1_{\mathscr{F}(j)}) \circ \Phi_{\mathscr{F}(j)} \arrow[dl, "\mathbb{M}'^0 * 1"] \\
& \Phi_{\mathscr{F}(j)} &
\end{tikzcd}$$

1 commutes by naturality of Φ;

2 commutes by the naturality condition for Φ.

Therefore, the outer diagram commutes as required.

$\Phi_{\mathscr{F}}$ is locally invertible since Φ is, so $\Phi_{\mathscr{F}}$ is indeed a strong transform.

That $\mathcal{R}$ sends strict transformations to strict transformations, and modifications to modifications is immediate.

Finally, we can see that $\mathcal{R}(\Psi\Phi) = (\Psi\Phi)_{\mathscr{F}} = \Psi_{\mathscr{F}}\Phi_{\mathscr{F}} = \mathcal{R}(\Psi)\mathcal{R}(\Phi)$, and $\mathcal{R}(1_{\mathbb{M}}) = (1_{\mathbb{M}})_{\mathscr{F}} = 1_{\mathbb{M}\circ\mathscr{F}}$, so that $\mathcal{R}$ is indeed a strict 2-functor.

□

If $\mathcal{R} : \mathscr{D} - afmod \to \mathscr{C} - afmod$ restricted to a functor $\mathscr{D} - cfmod^*_{ex} \to \mathscr{C} - cfmod^*_{ex}$, we could use Theorem 2.1.2 to obtain a functor $\mathscr{BB}$icom$_{\underline{\mathscr{D}}} \to \mathscr{BB}$icom$_{\underline{\mathscr{C}}}$, as required, but unfortunately this is not necessarily the case: the restriction of a cyclic birepresentation isn't necessarily cyclic, and even when this is the case, there is not always an obvious choice of generator for $\mathcal{R}(\mathbb{M})$.

We solve this problem in two steps.

Proposition 4.1.3. Given a finitary 2-category $\mathscr{C}$, there is a multifinitary 2-category $\mathscr{C}^{\oplus}$ with the following properties:

- $\mathscr{C} - afmod$ is biequivalent to $\mathscr{C}^{\oplus} - afmod$;
- Every birepresentation of $\mathscr{C}^{\oplus}$ is cyclic;
- When $\mathscr{C}$ is a fiat 2-category, $\mathscr{C}^{\oplus}$ is multifiat.

- This construction is natural in $\mathscr{C}$.

Proof : The construction of $\mathscr{C}^{\oplus}$ is given in full in [MMM+21, Section 2.4].

In that paper, Lemma 2.25 states that $\mathscr{C}^{\oplus}$ is multifinitary when $\mathscr{C}$ is finitary.

Proposition 2.27 states that $\mathscr{C} - afmod$ is biequivalent to $\mathscr{C}^{\oplus} - afmod$.

Remark 4.11 notes that every birepresentation of $\mathscr{C}^{\oplus}$ is cyclic.

Lemma 2.26 states that $\mathscr{C}^{\oplus}$ is multifiab when $\mathscr{C}$ is fiab. These are weakenings of multifiat and fiat respectively, and this result clearly restricts to the strict case (that is, $\mathscr{C}^{\oplus}$ is multifiat when $\mathscr{C}$ is fiat).

Naturality in $\mathscr{C}$ is immediate from the construction, although this claim isn't made in [MMM+21].

□

This construction is called the additive closure of $\mathscr{C}$, and in a particular sense, is the bicategory obtained by taking the the sum of all objects in $\mathscr{C}$.

As a consequence, we can assume without loss of generality that we work in a 2-category where all birepresentations are cyclic, solving our first issue.

Remark. $\mathscr{C}^{\oplus}$ has only one object. This is not a peculiarity of the construction: in fact, any 2-category with only cyclic representations will necessarily be biequivalent to a bicategory with one object. We assume, going forward, that our 2-category $\mathscr{C}$ has one object, which we denote $*$.

Finding a canonical generator for $\mathcal{R}\mathbb{M}$ is slightly subtle. First, note that in this new context, $\mathcal{R}\mathbb{P}_*^{\mathscr{D}}$ is cyclic, and so must have a generator.

Lemma 4.1.4. Let $\mathscr{C}$, $\mathscr{D}$ be one-object finitary 2-categories for which every representation is cyclic. Let $\mathbb{M}$ be a birepresentation of $\mathscr{D}$ with generator $X \in \mathbb{M}(*)$, and pick a generator $G \in \mathcal{R}\mathbb{P}_*^{\mathscr{D}}(*)$ for $\mathcal{R}\mathbb{P}_i^{\mathscr{D}}$. Then $\mathbb{M}(G)X \in \mathcal{R}\mathbb{M}(*)$ is a generator for $\mathcal{R}\mathbb{M}$.

Proof :

Suppose $Y \in \mathcal{R}\mathbb{M}(*)$. Since X generates $\mathbb{M}$, Y is isomorphic to a summand of $\mathbb{M}(F)X$ for some $F \in \mathscr{D}(*, \mathscr{F}(*))$. Since G generates $\mathcal{R}\mathbb{P}_*^{\mathscr{D}} = \mathscr{D}(*, \mathscr{F}(-))$, there is some $H \in \mathscr{C}(*, *)$ such that F is isomorphic to a summand of $\mathcal{R}\mathbb{P}_*(H)G = \mathscr{F}(H)G$. Then $\mathbb{M}(F)X$ is isomorphic to a summand of $\mathbb{M}(\mathscr{F}(H)G)X \cong \mathcal{R}\mathbb{M}(H)(\mathbb{M}(G)X)$, so Y is isomorphic to a summand of $\mathcal{R}\mathbb{M}(H)(\mathbb{M}(G)X)$ for some $H \in \mathscr{C}(*, *)$. So the inclusion $(\mathcal{R}\mathbb{M} \cdot \mathbb{M}(G)X)(*) \hookrightarrow \mathcal{R}\mathbb{M}(*)$ is essentially surjective (and by definition fully faithful), so an equivalence of categories. By Proposition 1.4.4, $\mathbb{M}(G)X \in \mathcal{R}\mathbb{M}(*)$

generates $\mathcal{R}\mathbb{M}$, as required. $\square$

This lets us define $\mathcal{R} : \mathscr{D} - cfmod_{ex}^{*} \to \mathscr{C} - cfmod_{ex}^{*}$.

Proposition 4.1.5. Let $\mathscr{C}$, $\mathscr{D}$ be one-object multifiat 2-categories whose birepresentations are all cyclic. Pick a generator $G \in \mathcal{R}\mathbb{P}_{*}^{\mathscr{D}}(*)$. Define $\mathcal{R} : \mathscr{D} - cfmod_{ex}^{*} \to \mathscr{C} - cfmod_{ex}^{*}$ as follows:

- For $(\mathbb{M}, X) \in \mathscr{D} - cfmod_{ex}^{*}$, define $\mathcal{R}(\mathbb{M}, X) = (\mathbb{M} \circ \mathscr{F}, \mathbb{M}(G)X)$;
- For $\Phi : (\mathbb{M}, X) \to (\mathbb{M}', Y)$, define $\mathcal{R}(\Phi) = \Phi_{\mathscr{F}}$;
- For $\sigma : \Phi \to \Psi$, define $\mathcal{R}(\sigma) = \sigma_{\mathscr{F}}$.

Then this is a strict 2-functor. $\square$

Note that this definition is independent of our choice of the G (up to natural isomorphism of 2-functors), since if X, Y generate $\mathbb{M}$, then $1_{\mathbb{M}} : (\mathbb{M}, X) \to (\mathbb{M}, Y)$ is an isomorphism.

4.2 Restriction of coalgebra and bicomodule 1-morphisms

This finally gives us all the tools we need to define $\mathcal{R} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}} \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$.

Theorem 4.2.1. Let $\mathscr{C}$, $\mathscr{D}$ be one-object multifiat categories for which every birepresentation is cyclic. Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a pseudofunctor. Pick a generator $G \in \mathcal{R}\mathbb{P}_{*}^{\mathscr{D}}(*)$.

Overloading notation, $\mathcal{R}_{\mathscr{F}}^{G} = \mathcal{R} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}} \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$ is given by the following data:

- For a coalgebra 1-morphism C of $\underline{\mathscr{D}}$, $\mathcal{R}(C) = {}_{\mathbb{M}_C \circ \mathscr{F}}[GC, GC]$;
- For a C-D-bicomodule 1-morphism N of $\underline{\mathscr{D}}$, $\mathcal{R}(N) = {}_{\mathbb{M}_D \circ \mathscr{F}}[GD, GN]$, viewed as a $\mathcal{R}(C)$-$\mathcal{R}(D)$-bicomodule 1-morphism;
- For a bicomodule homomorphism ϕ, $\mathcal{R}(\phi) = {}_{\mathbb{M}_D \circ \mathscr{F}}[GD, 1 * \phi]$;
- $\mathcal{R}^2$ is defined in the lemma below;
- $\mathcal{R}^0 = 1_{\mathcal{R}(C)}$.

Then this is a pseudofunctor, and the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathscr{D} - cfmod^*_{ex} \arrow[r, "\mathcal{R}"] \arrow[d, "\simeq"] & \mathscr{C} - cfmod^*_{ex} \arrow[d, "\simeq"] \\ \mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{D}} \arrow[r, "\mathcal{R}"] & \mathscr{B}\mathscr{B}\mathrm{icom}_{\mathscr{C}} \end{tikzcd}$$

□

This theorem requires the following lemma:

Lemma 4.2.2. Note that the following composition of maps defines a map $G(M \circledast N) \to \mathcal{R}\mathbb{M}_E(\mathcal{R}(M)\mathcal{R}(N))GE$:

$$\begin{tikzcd} G(M \circledast N) \arrow[d, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{GM,GD} \circledast 1"] \\ \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GM])GD \circledast N \arrow[d, equal] \\ \mathscr{F}\mathcal{R}(M)GD \circledast N \arrow[d, "1 * \varepsilon^l_N"] \\ \mathscr{F}\mathcal{R}(M)GN \arrow[d, "1 * \mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GN,GE}"] \\ \mathscr{F}\mathcal{R}(M)\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[d, equal] \\ \mathscr{F}\mathcal{R}(M)\mathscr{F}\mathcal{R}(N)GE \arrow[d, "\mathscr{F}^{-2} * 1"] \\ \mathscr{F}(\mathcal{R}(M)\mathcal{R}(N))GE \arrow[d, equal] \\ \mathcal{R}\mathbb{M}_E(\mathcal{R}(M)\mathcal{R}(N))GE \end{tikzcd}$$

and thus induces a map $\mathcal{R}(M \circledast N) \to \mathcal{R}(M)\mathcal{R}(N)$. This map is a homomorphism of bicomodule 1-morphisms. Moreover, this map equalises the $\mathcal{R}(D)$ coactions, so induces a map $\mathcal{R}^2 : \mathcal{R}(M \circledast N) \to \mathcal{R}(M) \circledast \mathcal{R}(N)$.

Proof : Identical to the proof of Lemma 2.2.6. □

With this, we can show that $\mathcal{R}$ is a pseudofunctor.

Proof : (of Theorem 4.2.1)

First, we can compute the composition $\mathcal{R}' = \mathcal{I} \circ \mathcal{R} \circ \iota$:

- For a coalgebra 1-morphism C of $\underline{\mathscr{D}}$, $\mathcal{R}'(C) = \mathcal{R}(C)$.
- For a C-D-comodule 1-morphism N of $\underline{\mathscr{D}}$, $\mathcal{R}'(N) = {}_{\mathbb{M}_D \circ \mathcal{F}}[GD, G(C \circledast N)]$
- For a bicomodule homomorphism ϕ of $\underline{\mathscr{D}}$, $\mathcal{R}'(\phi) = {}_{\mathbb{M}_D \circ \mathcal{F}}[GD, 1 \circledast \phi]$.

In particular, since $\mathcal{R}'(N) \cong \mathcal{R}(N)$ for any biinjective bicomodule 1-morphism N, we know that $\mathcal{R}(N)$ is a biinjective bicomodule 1-morphism, so the data of $\mathcal{R}$ is well-defined.

We want to use Proposition 1.1.8. To this end, we define $\Gamma : \mathcal{R}' \to \mathcal{R}$ as follows:

- At a coalgebra 1-morphism C of $\underline{\mathscr{D}}$, $\Gamma_C = 1 : \mathcal{R}'(C) \to \mathcal{R}(C)$;
- For a C-D-bicomodule 1-morphism M of $\underline{\mathscr{D}}$, $\Gamma_M = {}_{\mathbb{M}_D \circ \mathcal{F}}[GD, 1 * \varepsilon^l_{-}] : \mathcal{R}'(M) \to \mathcal{R}(M)$

Since ε^l_{-} is a natural isomorphism by Proposition 1.3.11, we know that Γ satisfies the epimorphism and $\boxed{1}$ conditions of Proposition 1.1.8.

We want to show that Γ satisfies the $\boxed{2}$ conditions of the proposition. To see that it does, consider the following diagram:

$$
\begin{tikzcd}[column sep=small]
GC \underset{C}{\boxtimes} M \underset{D}{\boxtimes} N \arrow[rrrr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GC \underset{C}{\boxtimes} M \underset{D}{\boxtimes} N, GE}"] \arrow[dddd, "\varepsilon^l_M \boxtimes 1"'] \arrow[dr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{GC \boxtimes M, GD} \boxtimes 1"'] & & & & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GC \underset{C}{\boxtimes} M \underset{D}{\boxtimes} N])GE \arrow[rrr, "\mathcal{R}\mathbb{M}_E(\mathcal{R}'^2) * 1"] \arrow[d, phantom, "6"] & & & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M] \underset{\mathcal{R}(D)}{\boxtimes} {}_{\mathcal{R}\mathbb{M}_E}[GE, GD \underset{D}{\boxtimes} N])GE \arrow[dl, "\mathcal{R}\mathbb{M}_E(t) * 1"'] \arrow[dddd, "\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M] \boxtimes {}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]) * 1"] \\
 & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M])GD \underset{D}{\boxtimes} N \arrow[rr, "1 * \mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GD \boxtimes N, GE}"'] \arrow[dd, "\mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M]) * 1"] \arrow[dr, "1 * \varepsilon^l_N"] \arrow[dr, phantom, "3"' near end, shift left=4ex] & & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M])\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GD \underset{D}{\boxtimes} N])GE \arrow[r, equal] \arrow[d, "1 * \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]) * 1"] & \mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M])\mathscr{F}({}_{\mathcal{R}\mathbb{M}_E}[GE, GD \underset{D}{\boxtimes} N])GE \arrow[r, "\mathscr{F}^2 * 1"'] \arrow[dd, "\mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M]) * \mathscr{F}({}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]) * 1"] & \mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M]_{\mathcal{R}\mathbb{M}_E}[GE, GD \underset{D}{\boxtimes} N])GE \arrow[r, equal] \arrow[dd, "\mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M] * {}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]) * 1"] & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M]_{\mathcal{R}\mathbb{M}_E}[GE, GD \underset{D}{\boxtimes} N])GE \arrow[dd, "\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M] * {}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]) * 1"] & \\
 \arrow[r, phantom, "1"] & \arrow[dr, phantom, "2"] & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M])GN \arrow[r, "1 * \mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GN, GE}"'] \arrow[d, "\mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M]) * 1"] \arrow[dr, phantom, "4"] & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GC \underset{C}{\boxtimes} M])\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[d, "\mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M]) * 1"] \arrow[r, phantom, "5"] & \arrow[r, phantom, "8"] & \arrow[r, phantom, "9"] & \arrow[r, phantom, "10"] & \\
 & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GM])GD \underset{D}{\boxtimes} N \arrow[r, "1 * \varepsilon^l_N"'] & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GM])GN \arrow[r, "1 * \mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GN, GE}"'] & \mathcal{R}\mathbb{M}_D({}_{\mathcal{R}\mathbb{M}_D}[GD, GM])\mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[r, equal] & \mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, GM])\mathscr{F}({}_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[r, "\mathscr{F}^2 * 1"] \arrow[d, phantom, "7"] & \mathscr{F}({}_{\mathcal{R}\mathbb{M}_D}[GD, GM]_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[r, equal] & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, GM]_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE & \\
GM \underset{D}{\boxtimes} N \arrow[ur, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{GM, GD} \boxtimes 1"] \arrow[rrrr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_E}_{GM \underset{D}{\boxtimes} N, GE}"] & & & & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_E}[GE, GC \underset{C}{\boxtimes} M \underset{D}{\boxtimes} N])GE \arrow[rrr, "\mathcal{R}\mathbb{M}_E(\mathcal{R}^2) * 1"] & & & \mathcal{R}\mathbb{M}_E({}_{\mathcal{R}\mathbb{M}_D}[GD, GM] \underset{\mathcal{R}(D)}{\boxtimes} {}_{\mathcal{R}\mathbb{M}_E}[GE, GN])GE \arrow[ul, "\mathcal{R}\mathbb{M}_E(t) * 1"]
\end{tikzcd}
$$

1 commutes by definition of ${}_{\mathcal{R}\mathbb{M}_D}[GD, \varepsilon^l_M]$;

2 and 4 commute by the interchange law;

3 commutes by definition of ${}_{\mathcal{R}\mathbb{M}_E}[GE, \varepsilon^l_N]$;

5 and 9 commute trivially;

6 commutes by definition of $\mathcal{R}'^2$;

7 commutes by definition of $\mathcal{R}^2$;

8 commutes by naturality of $\mathscr{F}^2$;

9 commutes by definition of ;

10 commutes by definition of the cotensor product;

and $\mathcal{R}\mathbb{M}_E(t) * 1$ is monic.

So the outer diagram commutes. Passing via $\gamma^{\mathcal{R}\mathbb{M}_E}_{GC \boxtimes M \boxtimes N, GE}$, this says precisely that $\boxed{2}$ of the proposition commutes in this case.

$\boxed{3}$ of the proposition commutes trivially.

So we immediately get that $\mathcal{R}$ is a pseudofunctor, and $\Gamma : \mathcal{R}' \to \mathcal{R}$ is a 2-natural isomorphism. □

We prove one final lemma:

Lemma 4.2.3. Suppose $\mathscr{F}$ is essentially 1-surjective. Then $1_{\mathscr{F}(i)}$ generates $\mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}$ whenever $\mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}$ is cyclic.

Proof : Notice first that $1_{\mathscr{F}(i)} \in \mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}(i)$. Now, $\mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}$ must have a generator, say $G_i \in \mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}(j)$. So if $F \in \mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}(k)$, then there is some $H \in \mathscr{C}(j,k)$ such that F is isomorphic to a summand of $\mathcal{R}\mathbb{P}_i(H)G_i = \mathscr{F}(H)G_i$. Since $\mathscr{F}$ is essentially 1-surjective, $\mathscr{F}(H)G_i$ is isomorphic to $\mathscr{F}(H')$ for some $H' : i \to k$. But then F is isomorphic to a summand of $\mathscr{F}(H')1_{\mathscr{F}(i)} = \mathcal{R}\mathbb{P}_i(H')1_{\mathscr{F}(i)}$. So $1_{\mathscr{F}(i)}$ generates $\mathcal{R}\mathbb{P}^{\mathscr{D}}_{\mathscr{F}(i)}$ as required. □

It may be possible to work without this lemma, but certain constructions rapidly become very unwieldy.

Since we made a choice of G in defining $\mathcal{R}$, we note that this actually defines a collection of pseudofunctors, one for each possible choice. By a previous remark, these are all naturally isomorphic, so the choice is unimportant. Going forward, we fix G, and use the pseudofunctor $\mathcal{R} := \mathcal{R}^G_{\mathcal{F}}$. We often assume, in order to use the above lemma, that $\mathscr{F}$ is essentially 1-surjective and $G = 1$.

5

Unit of the adjunction

Throughout this chapter, we assume $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an essentially 1-surjective locally $\Bbbk$-linear pseudofunctor between one-object multifiat 2-categories whose birepresentations are all cyclic.

In this section, we must construct a 2-natural transform $\eta : 1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}} \to \mathcal{RP}$, the unit for our adjunction.

First, we explicitly compute the form of $\mathcal{RP} : \mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}$.

Proposition 5.0.1.

- For a coalgebra 1-morphism C of $\underline{\mathscr{C}}$, $\mathcal{RP}(C) = \mathsf{C}^{\mathscr{F}(C)}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}$;
- for a C-D-bicomodule 1-morphism M of $\underline{\mathscr{C}}$, $\mathcal{RP}(M) = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(M)]$;
- for a bicomodule 1-morphism $\phi : M \to M'$ of $\underline{\mathscr{C}}$, $\mathcal{RP}(\phi) = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(\phi)]$.

Proof : First, recall that by definition, $\mathcal{RP}(C) = \mathcal{R}(\mathscr{F}(C)) = {}_{\mathbb{M}_{\mathscr{F}(C)} \circ \mathscr{F}}[G\mathscr{F}(C), G\mathscr{F}(C)]$. But since $\mathscr{F}$ is essentially 1-surjective, by Lemma 4.2.3 we can assume, without loss of generality, that G is 1, so $\mathcal{RP}(C) = {}_{\mathbb{M}_{\mathscr{F}(C)} \circ \mathscr{F}}[\mathscr{F}(C), \mathscr{F}(C)] = \mathsf{C}^{\mathscr{F}(C)}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}$.

The other two parts can be computed similarly. □

So what do we need for the 2-natural transform $\eta : 1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}} \to \mathcal{RP}$?

First, for each object of $\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}$ (that is, for each coalgebra 1-morphism C of $\underline{\mathscr{C}}$), we need a 1-morphism $\eta_C : C \to \mathcal{RP}(C)$ in $\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}$ (that is, a biinjective C-$\mathcal{RP}(C)$-bicomodule 1-morphism in $\underline{\mathscr{C}}$).

Second, for each 1-morphism in $\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}(C, D)$ (that is, each biinjective C-D-bicomodule 1-morphism of $\underline{\mathscr{C}}$), we need a 2-morphism (that is, bicomodule homomorphism) $\eta_M : M \circledast \eta_D \to \eta_C \circledast \mathcal{RP}(M)$.

These must be natural in M and satisfy the two coherence diagrams.

We use our explicit form of $\mathcal{RP}$ to construct this data.

5.1 Unit at coalgebra 1-morphisms

Proposition 5.1.1. Let C be a coalgebra 1-morphism in $\underline{\mathscr{C}}$, and Y be an object of $\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(j)$ for some j. ${}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \mathscr{F}(C)]$ is a C-C^Y-bicomodule 1-morphism.

Proof : Write $M = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \mathscr{F}(C)]$, $\delta_{C,M} = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \Delta_{\mathscr{F}(C)}]$ (noting that ${}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \mathscr{F}(C)\mathscr{F}(C)] = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(C)\mathscr{F}(C)] \cong C_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[Y, \mathscr{F}(C)]$ as 1-morphisms), and $\mathrm{coev}_M{=}\mathrm{coev}^{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}_{\mathscr{F}(C),Y}$. Clearly, M is a right C^Y-comodule 1-morphism. To see that it is a left C-comodule 1-morphism, we need the following diagrams to commute:

$$\begin{tikzcd}
M \arrow[r, "\delta_{C,M}"] \arrow[dr, equal] \arrow[r, phantom, shift right=4ex, "\boxed{1}"] & CM \arrow[d, "\varepsilon_C * 1"] \\
& M
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "\delta_{C,M}"] \arrow[d, "\delta_{C,M}"'] \arrow[dr, phantom, "\boxed{2}"] & CM \arrow[d, "1*\delta_{C,M}"] \\
CM \arrow[r, "\Delta_C * 1"'] & CCM
\end{tikzcd}$$

To see that $\boxed{1}$ commutes, consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathrm{coev}_M"] \arrow[dr, "\Delta_{\mathscr{F}(C)}"] \arrow[dddr, equal] \arrow[ddddd, "\mathrm{coev}_M"'] & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[rrr, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{C,M})"] & & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CM)Y \arrow[dl, equal] \arrow[ddddd, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\varepsilon_C * 1)"] \\
& \mathscr{F}(C)\mathscr{F}(C) \arrow[r, "1*\mathrm{coev}_M"] \arrow[d, "\mathscr{F}(\varepsilon_C)*1"] & \mathscr{F}(C)\mathscr{F}(M)Y \arrow[r, "\mathscr{F}^{-2}"] \arrow[dd, "\mathscr{F}(\varepsilon_C)*1"] & \mathscr{F}(CM)Y \arrow[ddd, "\mathscr{F}(\varepsilon_C * 1)"] & \\
& \mathscr{F}(1)\mathscr{F}(C) \arrow[d, dashed, "\mathscr{F}^0 * 1"] \arrow[dr, "1*\mathrm{coev}_M"] & & & \\
& \mathscr{F}(C) \arrow[d, "\mathrm{coev}_M"'] & \mathscr{F}(1)\mathscr{F}(M)Y \arrow[dl, "\mathscr{F}^0 * 1"] \arrow[dr, "\mathscr{F}^{-2}"] & & \\
& \mathscr{F}(M)Y \arrow[rr, equal] \arrow[dl, equal] & & \mathscr{F}(M)Y \arrow[dr, equal] & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[rrrr, equal] & & & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y
\end{tikzcd}$$

1 commutes by definition of $\delta_{C,M}$;

2 commutes because $\mathscr{F}(C)$ is a coalgebra 1-morphism;

3 and 4 commute by the interchange law;

5 and 8 commute trivially;

(6) commutes by naturality of $\mathscr{F}^2$;

and (7) commutes by higher coherence for $\mathscr{F}$.

So the outer diagram commutes. Passing via $\gamma^{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}_{\mathscr{F}(C),Y}$, this says that $\boxed{1}$ commutes.

To see that $\boxed{2}$ commutes, consider the following diagram:

$$
\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathrm{coev}_M"] \arrow[ddd, "\mathrm{coev}_M"'] \arrow[ddrr, "\Delta_{\mathscr{F}(C)}"] \arrow[drrr, "\Delta_{\mathscr{F}(C)}"] & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[rrrrr, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{C,M})"] & & & & \arrow[d, phantom, "2"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CM)Y \arrow[dl, equals] \arrow[dddddd, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(1*\delta_{C,M})"] \\
& & \arrow[d, phantom, "1"] & \mathscr{F}(C)\mathscr{F}(C) \arrow[r, "1*\mathrm{coev}_M"] \arrow[d, dashed, "1*\Delta_{\mathscr{F}(C)}"] & \mathscr{F}(C)\mathscr{F}(M)Y \arrow[r, "\mathscr{F}^{-2}"] \arrow[dd, "1*\mathscr{F}(\delta_{C,M})"] & \mathscr{F}(CM)Y \arrow[dddd, "\mathscr{F}(1*\delta_{C,M})"] \arrow[dddd, phantom, "10"', near start] & & \\
& & \mathscr{F}(C)\mathscr{F}(C) \arrow[r, "\Delta_{\mathscr{F}(C)}*1"] \arrow[d, dashed, "1*\mathrm{coev}_M"] \arrow[dr, phantom, "5"] & \mathscr{F}(C)\mathscr{F}(C)\mathscr{F}(C) \arrow[d, dashed, "1*\mathrm{coev}_M"] \arrow[r, phantom, "4"] & \arrow[dr, phantom, "6"] & & & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[ddd, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{C,M})"] \arrow[ddr, phantom, "3"] & & \mathscr{F}(C)\mathscr{F}(M)Y \arrow[r, "\Delta_{\mathscr{F}(C)}*1"] \arrow[dr, "\mathscr{F}(\Delta_C"'] \arrow[dd, "\mathscr{F}^{-2}"] \arrow[dr, phantom, "9"] & \mathscr{F}(C)\mathscr{F}(C)\mathscr{F}(M)Y \arrow[r, "1*\mathscr{F}^{-2}"] \arrow[d, "\mathscr{F}^{-2}*1"] & \mathscr{F}(C)\mathscr{F}(CM)Y \arrow[ddr, "\mathscr{F}^{-2}"] \arrow[d, phantom, "8"] & & & \\
& & & \mathscr{F}(CC)\mathscr{F}(M)Y \arrow[ddrr, "\mathscr{F}^{-2}"] \arrow[d, phantom, "7"] & & & & \\
& & \mathscr{F}(CM)Y \arrow[rrr, "(\Delta_C*1)"] \arrow[dll, equals] \arrow[drr, phantom, "11"] & & & \mathscr{F}(CCM)Y \arrow[drr, equals] & & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CM)Y \arrow[rrrrrrr, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\Delta_C*1)"'] & & & & & & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CCM)Y
\end{tikzcd}
$$

1 commutes as $\mathscr{F}(C)$ is coassociative;

2, 3 and 4 commute by definition of $\delta_{C,M}$;

5 commutes by the interchange law;

6 and 7 commute by naturality of $\mathscr{F}^{-2}$;

8 commutes by higher coherence for $\mathscr{F}$;

9 commutes by definition of $\Delta_{\mathscr{F}(C)}$;

and 10 and 11 commute by definition of $\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}$.

So the outer diagram commutes, which, passing via $\gamma^{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}_{\mathscr{F}(C),Y}$, says that $\boxed{2}$ commutes. So M is a left C-comodule 1-morphism.

Finally, we need to show that the coactions commute, that is, that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\delta_{M,\mathcal{RP}(C)}"] \arrow[d, "\delta_{C,M}"'] \arrow[dr, phantom, "\boxed{3}"] & M\mathcal{RP}(C) \arrow[d, "\delta_{C,M}*1"] \\
CM \arrow[r, "1*\delta_{M,\mathcal{RP}(C)}"'] & CM\mathcal{RP}(C)
\end{tikzcd}$$

To see that it does, consider the following diagram:

$$
\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\mathrm{coev}_M"] \arrow[ddr, "\Delta_{\mathscr{F}(C)}"] \arrow[dddd, swap, "\mathrm{coev}_M"] & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[rrrr, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{M,\mathsf{C}^Y})"] \arrow[dr, equal] & & \arrow[d, phantom, "8"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M\mathsf{C}^Y)Y \arrow[dl, equal] \arrow[dddddd, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{C,M}*1)"] \\
& \arrow[r, phantom, "2"] & \mathscr{F}(M)Y \arrow[rr, "\mathscr{F}(\delta_{M,\mathsf{C}^Y})"] \arrow[d, "\mathscr{F}(\delta_{C,M})"] \arrow[dr, "1*\mathrm{coev}_{\mathsf{C}^Y}"] & \arrow[d, phantom, "5"] & \mathscr{F}(M\mathsf{C}^Y)Y \arrow[dddd, "\mathscr{F}(\delta_{C,M}*1)"] & \\
& \mathscr{F}(C)\mathscr{F}(C) \arrow[d, "1*\mathrm{coev}_M"] & \mathscr{F}(CM)Y \arrow[r, phantom, "4"] \arrow[d, phantom, "3"] \arrow[dr, "1*\mathrm{coev}_{\mathsf{C}^Y}"] & \mathscr{F}(M)\mathscr{F}(\mathsf{C}^Y)Y \arrow[ur, "\mathscr{F}^{-2}"] \arrow[d, "\mathscr{F}(\delta_{C,M})*1"] & & \\
 \arrow[r, phantom, "1"] & \mathscr{F}(C)\mathscr{F}(M)Y \arrow[ur, "\mathscr{F}^{-2}"] \arrow[r, "1*1*\mathrm{coev}_{\mathsf{C}^Y}"] \arrow[dr, "1*\mathscr{F}(\delta_{M,\mathsf{C}^Y})"] \arrow[dr, phantom, "6"] \arrow[ddr, "\mathscr{F}^{-2}"] & \mathscr{F}(C)\mathscr{F}(M)\mathscr{F}(\mathsf{C}^Y)Y \arrow[r, swap, "\mathscr{F}^{-2}*1"] \arrow[d, "1*\mathscr{F}^{-2}"] & \mathscr{F}(CM)\mathscr{F}(\mathsf{C}^Y)Y \arrow[r, phantom, "12"] \arrow[d, phantom, "13"] \arrow[ddr, "\mathscr{F}^{-2}"] & \arrow[r, phantom, "10"] & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(M)Y \arrow[r, equal] \arrow[dd, swap, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(\delta_{C,M})"] & \mathscr{F}(M)Y \arrow[dr, swap, "\mathscr{F}(\delta_{C,M})"] \arrow[dd, phantom, "7"] & \mathscr{F}(C)\mathscr{F}(M\mathsf{C}^Y)Y \arrow[drr, "\mathscr{F}^{-2}"] & \arrow[d, phantom, "11"] & & \\
& & \mathscr{F}(CM)Y \arrow[rr, swap, "\mathscr{F}(1*\delta_{M,\mathsf{C}^Y})"] \arrow[dll, equal] & \arrow[d, phantom, "9"] & \mathscr{F}(CM\mathsf{C}^Y)Y \arrow[dr, equal] & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CM)Y \arrow[rrrrr, swap, "\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(1*\delta_{M,\mathsf{C}^Y})"] & & & & & \mathcal{R}\mathbb{M}_{\mathscr{F}(C)}(CM\mathsf{C}^Y)Y
\end{tikzcd}
$$

1 and 2 commute by definition of $\delta_{C,M}$;

3 and 4 commute by the interchange law;

5 and 6 commute by definition of δ_{M,C^Y};

7-10 commute by definition of $\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}$;

11 and 12 commute by naturality of $\mathscr{F}^2$;

13 commutes by higher coherence for $\mathscr{F}$.

So the outer diagram commutes. But, passing via $\gamma^{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}_{\mathscr{F}(C),Y}$, this says that $\boxed{3}$ commutes. So the left and right coactions of M commute, so M is a bicomodule 1-morphism.

□

Now, setting $Y = \mathscr{F}(C)$, this tells us that ${}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C), \mathscr{F}(C)]$ can be viewed as a C-$\mathsf{C}^{\mathscr{F}(C)}$-bicomodule 1-morphism. But $\mathsf{C}^{\mathscr{F}(C)} = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C), \mathscr{F}(C)] = \mathcal{RP}(C)$. So we can make the following definition:

Definition 5.1.2 (Unit at coalgebra 1-morphisms)**.** $\eta_C = \mathcal{RP}(C) = {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C), \mathscr{F}(C)]$, viewed as a C-$\mathcal{RP}(C)$-bicomodule 1-morphism. ◁

5.2 Unit at bicomodule 1-morphisms

Next, we construct a map $\eta_M : M \underset{D}{\circledast} \eta_D \to \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M)$.

Lemma 5.2.1. Suppose M is an injective right C-comodule 1-morphism in $\underline{\mathscr{C}}$, N is a $\mathscr{F}(C)$-$\mathscr{F}(D)$-bicomodule 1-morphism in $\underline{\mathscr{D}}$, L is a right $\mathscr{F}(D)$-comodule 1-morphism in $\underline{\mathscr{D}}$. Then $M \underset{C}{\circledast} {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L, N] \cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L, \mathscr{F}(M) \underset{\mathscr{F}(C)}{\circledast} N]$. Moreover, this isomorphism is natural in M.

Proof : Since $\mathscr{C}$ is fiat, by Lemma 1.3.7 (and since the cotensor and internal cohom are additive), we can assume, without loss of generality, that $M = FC$ for some $F \in \mathscr{C}(i, j)$. So

$$\begin{aligned} M \underset{C}{\circledast} {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L,N] &= FC \underset{C}{\circledast} {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L,N] \\ &\cong F_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L,N] \\ &\cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L, \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}(F)N] \\ &\cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L, \mathscr{F}(F)\mathscr{F}(C) \underset{\mathscr{F}(C)}{\circledast} N] \\ &\cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[L, \mathscr{F}(M) \underset{\mathscr{F}(C)}{\circledast} N] \end{aligned}$$

Each of these isomorphisms is natural in F, so altogether this is natural in M, as required. □

Lemma 5.2.2. Let M be a biinjective C-D-bicomodule 1-morphism. There is an invertible bicomodule homomorphism $\eta_M : M \underset{D}{\circledast} \eta_D \to \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M)$, natural in M.

Proof : First, note that $\eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M) = \mathcal{RP}(C) \underset{\mathcal{RP}(C)}{\circledast} {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(M)] \cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(M)]$.

Moreover, note that $M \underset{D}{\circledast} \eta_D = M \underset{D}{\circledast} {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(D)]$. Then by Lemma 5.2.1, $M \underset{D}{\circledast} \eta_D \cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(M) \circledast \mathscr{F}(D)] \cong {}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}}[\mathscr{F}(D), \mathscr{F}(M)]$. Moreover, both of these are natural in M. So we have an isomorphism $\eta_M : M \underset{D}{\circledast} \eta_D \to \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M)$ natural in M, as required. □

5.3 2-naturality

Finally, we put this data together to obtain the following result:

Proposition 5.3.1. The η_C, η_M assemble to a 2-natural transformation.

Proof : By Lemma 5.1.1, for a coalgebra 1-morphism C of $\mathscr{C}$, the bicomodule 1-morphism η_C is indeed a 1-morphism $\eta_C : C \to \mathcal{RP}(C)$, as needed.

By Lemma 5.2.2, for a biinjective C-D-bicomodule 1-morphism M of $\mathscr{C}$, the bicomodule isomorphisms $\eta_M : M \underset{D}{\circledast} \eta_D \to \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M)$ are natural in M.

For η to be a 2-natural transformation, we further need the following diagrams to commute for any biinjective C-D-bicomodule 1-morphism M and D-E-bicomodule 1-morphism N:

$$\begin{tikzcd}
(M \underset{D}{\circledast} N) \underset{E}{\circledast} \eta_E \arrow[rr, "\eta_{M \circledast N}"'] \arrow[d, equal] \arrow[drr, phantom, "\boxed{1}"] & & \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M \underset{D}{\circledast} N) \arrow[d, "1 \circledast (\mathcal{RP})^2"'] \\
M \underset{D}{\circledast} N \underset{E}{\circledast} \eta_E \arrow[r, "1 \circledast \eta_N"] & M \underset{D}{\circledast} \eta_D \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[r, "\eta_M \circledast 1"] & \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(M) \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N)
\end{tikzcd}$$

$$\begin{tikzcd}
C \underset{C}{\circledast} \eta_C \arrow[rr, "\eta_C" description] \arrow[dr, "\varepsilon^r_{\eta_C}" description] & \boxed{2} & \eta_C \underset{\mathcal{RP}(C)}{\circledast} \mathcal{RP}(C) \arrow[dl, "\varepsilon^l_{\eta_C}" description] \\
& \eta_C &
\end{tikzcd}$$

To see that $\boxed{1}$ commutes, consider the following diagrams:

$$
\begin{tikzcd}
\eta_C \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M \underset{D}{\boxast} N) \arrow[r, equal] \arrow[dddddd, "1\boxast(\mathcal{RP})^2"'] \arrow[ddr, phantom, "1"] & \mathcal{RP}(C) \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M \underset{D}{\boxast} N) \arrow[r, "\cong"] \arrow[d, "1\boxast(\mathcal{RP})^2"] \arrow[dr, phantom, "2"] & \mathcal{RP}(M \underset{D}{\boxast} N) \arrow[d, "(\mathcal{RP})^2"] \\
& \mathcal{RP}(C) \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M) \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[r, "\cong"] \arrow[ddddl, equal] \arrow[dddd, phantom, "3"] & \mathcal{RP}(M) \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[d, equal] \\
& & \mathcal{RM}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(M)] \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[d, "\cong"] \\
& & \mathcal{RM}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(D)] \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[d, "\cong"] \\
& & M \underset{D}{\boxast} \mathcal{RM}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(D)] \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[d, equal] \\
& & \\
\eta_C \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M) \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[rr, "\eta_M^{-1}\boxast 1"'] & & M \underset{D}{\boxast} \eta_D \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N)
\end{tikzcd}
$$

$$\begin{tikzcd}
\eta_C \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M \underset{D}{\boxast} N) \arrow[rrr, "\eta^{-1}_{M \boxast N}"] \arrow[d, equal] & & & (M \underset{D}{\boxast} N) \underset{E}{\boxast} \eta_E \arrow[dl, equal] \arrow[dddddddd, equal] \\
\mathcal{RP}(C) \underset{\mathcal{RP}(C)}{\boxast} \mathcal{RP}(M \underset{D}{\boxast} N) \arrow[d, "\cong"] \arrow[r, phantom, "\textcircled{4}"] & {} & & \\
\mathcal{RP}(M \underset{D}{\boxast} N) \arrow[d, equal] & & & \\
\mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M \underset{D}{\boxast} N)] \arrow[r, "\cong"] \arrow[d, "\mathcal{R}(\mathcal{P}^2)"] \arrow[dr, phantom, "\textcircled{5}"] & \mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M \underset{D}{\boxast} N) \underset{\mathscr{F}(E)}{\boxast} \mathscr{F}(E)] \arrow[dr, "\cong"] \arrow[d, "\mathcal{R}(\mathcal{P}^2 \boxast 1)"] & & \\
\mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N)] \arrow[r, "\cong"] \arrow[d, dashed, "\cong"] \arrow[dr, phantom, "\textcircled{6}"] & \mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M) \underset{\mathscr{F}(D)}{\boxast} \mathscr{F}(N) \underset{\mathscr{F}(E)}{\boxast} \mathscr{F}(E)] \arrow[d, "\cong"] \arrow[r, phantom, "\textcircled{8}"] & M \underset{D}{\boxast} N \underset{E}{\boxast} \mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(E)] \arrow[r, phantom, "\textcircled{9}"] & {} \\
M \underset{D}{\boxast} \mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(N)] \arrow[r, "\cong"] \arrow[d, equal] & M \underset{D}{\boxast} \mathcal{RM}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(N) \underset{\mathscr{F}(E)}{\boxast} \mathscr{F}(E)] \arrow[ur, "\cong"] & & \\
M \underset{D}{\boxast} \mathcal{RP}(N) \arrow[r, phantom, "\textcircled{7}"] & {} & & \\
M \underset{D}{\boxast} \mathcal{RP}(D) \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[u, "\cong"] \arrow[d, equal] & & & \\
M \underset{D}{\boxast} \eta_D \underset{\mathcal{RP}(D)}{\boxast} \mathcal{RP}(N) \arrow[rrr, "1 \boxast \eta_N^{-1}"'] & & & M \underset{D}{\boxast} N \underset{E}{\boxast} \eta_E \arrow[uuuul, equal]
\end{tikzcd}$$

1 and 9 commute trivially;

2, 5 and 6 commute by naturality of the unitors;

and 3, 4 and 7 commute by definition of η_M.

to see that 8 commutes, consider the following cube:

$$\begin{tikzcd}
 & & KM \underset{C}{\boxast} \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,N] \arrow[d, "\cong"] \arrow[drr, "\cong"] & & \\
K \underset{B}{\boxast} M \underset{C}{\boxast} \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,N] \arrow[urr, "t"] \arrow[d, "\cong"'] \arrow[drr, "\cong"] & & K\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(M) \underset{\mathscr{F}(C)}{\boxast} N] \arrow[drr, "\cong"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(KM) \underset{\mathscr{F}(C)}{\boxast} N] \arrow[d, "{[L,\mathscr{F}^2 \boxast 1]}"] \\
K \underset{B}{\boxast} \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(M) \underset{\mathscr{F}(C)}{\boxast} N] \arrow[urr, "t"] \arrow[drr, "\cong"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(K \underset{B}{\boxast} M) \underset{\mathscr{F}(C)}{\boxast} N] \arrow[urr, "{[L,\mathscr{F}(t)\boxast N]}"] \arrow[d, "{[L,\mathcal{P}^2 \boxast 1]}"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(K)\mathscr{F}(M) \underset{\mathscr{F}(C)}{\boxast} N] \\
 & & \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[L,\mathscr{F}(K) \underset{\mathscr{F}(B)}{\boxast} \mathscr{F}(M) \underset{\mathscr{F}(C)}{\boxast} N] \arrow[urr, "{[L,t\boxast N]}"'] & &
\end{tikzcd}$$

The back left face commutes by definition of the cotensor product;

the back right face commutes by Lemma 1.5.3;

the top and bottom faces commute by naturality of the isomorphism;

the front right face commutes by definition of $\mathcal{P}^2$;

and $[L, t \boxast N]$ is monic.

So by a previous argument, the front left face commutes, that is, 8 commutes.

So we can reduce $\boxed{1}$ to the commutativity of the following diagram:

$$
\begin{tikzcd}
\mathcal{RP}(M \underset{D}{\circledast} N) \arrow[rr, equal] \arrow[d, "(\mathcal{RP})^2"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M \underset{D}{\circledast} N)] \arrow[dd, "\mathcal{R}(\mathcal{P}^2)"] \\
\mathcal{RP}(M) \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[d, equal] & 9 & \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(M)] \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[d, "\cong"] & & \mathcal{R}\mathbb{M}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(N)] \arrow[ll, "\mathcal{R}^2"'] \arrow[dd, "\cong"] \\
\mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(M) \underset{\mathscr{F}(D)}{\circledast} \mathscr{F}(D)] \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[d, "\cong"] & & \\
M \underset{D}{\circledast} \mathcal{R}\mathbb{M}_{\mathscr{F}(D)}[\mathscr{F}(D), \mathscr{F}(D)] \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[d, equal] & 10 & M \underset{D}{\circledast} \mathcal{R}\mathbb{M}_{\mathscr{F}(E)}[\mathscr{F}(E), \mathscr{F}(N)] \arrow[dd, equal] \\
M \underset{D}{\circledast} \eta_D \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[d, equal] & & \\
M \underset{D}{\circledast} \mathcal{RP}(D) \underset{\mathcal{RP}(D)}{\circledast} \mathcal{RP}(N) \arrow[rr, "\cong"] & & M \underset{D}{\circledast} \mathcal{RP}(N)
\end{tikzcd}
$$

9 commutes by definition of $(\mathcal{RP})^2$;

and 10 commutes by definition of $\mathcal{R}^2$.

So $\boxed{1}$ commutes.

That $\boxed{2}$ commutes is trivial.

□

6

Counit of the adjunction

Throughout this chapter, we assume $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an essentially 1-surjective locally $\Bbbk$-linear pseudofunctor between one-object multifiat 2-categories whose birepresentations are all cyclic.

In this section, we must construct a 2-natural transform $\epsilon : \mathcal{PR} \to 1_{\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}}$, the counit for our adjunction.

What do we need for the 2-natural transform $\epsilon : \mathcal{PR} \to 1_{\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}}$?

First, for each object of $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}$ (that is, for each coalgebra 1-morphism C of $\underline{\mathscr{D}}$), we need a 1-morphism $\epsilon_C : \mathcal{PR}(C) \to C$ in $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}$ (that is, a biinjective $\mathcal{PR}(C)$-C-bicomodule 1-morphism in $\underline{\mathscr{D}}$).

Second, for each 1-morphism in $\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}(C, D)$ (that is, each biinjective C-D-bicomodule 1-morphism of $\underline{\mathscr{D}}$), we need a 2-morphism (that is, bicomodule homomorphism) $\epsilon_M : \mathcal{PR}(M) \circledast \eta_D \to \eta_C \circledast M$.

These must be natural in M and satisfy the two coherence diagrams.

6.1 Counit at coalgebra 1-morphisms

We start by constructing ϵ_C.

Proposition 6.1.1. Let C be a coalgebra 1-morphism in $\underline{\mathscr{D}}$. Then $(C, \delta_{\mathcal{PR}(C),C} := \mathrm{coev}_{C,C}^{\mathcal{R}\mathbb{M}_C}, \Delta_C)$ is a $\mathcal{PR}(C)$-C-bicomodule 1-morphism.

Proof : Note first that $\mathcal{PR}(C)C = \mathscr{F}({}_{\mathcal{R}\mathbb{M}_C}[C, C])C = \mathcal{R}\mathbb{M}_C({}_{\mathcal{R}\mathbb{M}_C}[C, C])C$ as objects in some $\mathcal{R}\mathbb{M}_C(*)$, so as objects in $\underline{\mathscr{D}}$, and that $\mathrm{coev}_{C,C}^{\mathcal{R}\mathbb{M}_C}$ is a morphism in $\mathcal{R}\mathbb{M}_C(*)$ so a 2-morphism in $\underline{\mathscr{D}}$. Now, C is clearly a right C-comodule 1-morphism. To see that it is a left $\mathcal{PR}(C)$-comodule 1-morphism, we need the following diagram to commute:

$$\begin{tikzcd}
C \arrow[r, "\delta_{\mathcal{PR}(C),C}"] \arrow[d, "\delta_{\mathcal{PR}(C),C}"'] & \mathcal{PR}(C)C \arrow[d, "\Delta_{\mathcal{PR}(C)}*1"] \\
\mathcal{PR}(C)C \arrow[r, "1*\delta_{\mathcal{PR}(C),C}"'] & \mathcal{PR}(C)\mathcal{PR}(C)C
\end{tikzcd}$$

We first consider the following diagram:

$$\begin{tikzcd}
C \arrow[rr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] \arrow[ddd, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] & & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[dl, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] \arrow[dd, "\mathcal{R}\mathbb{M}_C(\Delta_{\mathcal{R}(C)})"] \\
& \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))\mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[dr, "\mathcal{R}\mathbb{M}^2_C"] \arrow[r, phantom, "1"] & {} \\
& 2 & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C)\mathcal{R}(C))C \arrow[d, "\mathcal{R}\mathbb{M}^{-2}_C"] \\
\mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[rr, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] & & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))\mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C
\end{tikzcd}$$

1 commutes trivially, and 2 precomposed with $\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}$ commutes by definition of $\Delta_{\mathcal{R}(C)}$. So the outer diagram commutes. Next, consider the following diagram:

$$
\begin{tikzcd}
C \arrow[rrrr, "\delta_{\mathcal{PR}(C),C}"] \arrow[drr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] \arrow[dddr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] \arrow[dddd, "\delta_{\mathcal{PR}(C),C}"'] & & & & \mathcal{PR}(C)C \arrow[dl, "\mathscr{F}(\Delta_{\mathcal{R}(C)})*1"'] \arrow[dddd, "\Delta_{\mathcal{PR}(C)}*1"] \arrow[dddd, phantom, "7"', shift right=6] \\
& & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[urr, equal] \arrow[d, dashed, "\mathcal{R}\mathbb{M}_C(\Delta_{\mathcal{R}(C)})"] \arrow[u, phantom, "2"] \arrow[dr, phantom, "6"] & & \\
& \arrow[r, phantom, "1"] & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C)\mathcal{R}(C))C \arrow[r, equal] \arrow[d, dashed, "\mathcal{R}\mathbb{M}_C^{-2}"] & \mathcal{P}(\mathcal{R}(C)\mathcal{R}(C))C \arrow[ddr, "\mathscr{F}^2"] \arrow[d, phantom, "5"] & \\
\arrow[r, phantom, "3"] & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[r, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] \arrow[dl, equal] & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))\mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[drr, equal] \arrow[d, phantom, "4"] & & \\
\mathcal{PR}(C)C \arrow[rrrr, "1*\delta_{\mathcal{PR}(C),G_iC}"'] & & & & \mathcal{PR}(C)\mathcal{PR}(C)C
\end{tikzcd}
$$

1 commutes by the argument above;

2-6 commute by definition of $\mathcal{R}\mathbb{M}_C$;

and 7 commutes by definition of $\Delta_{\mathcal{P}(D)}$.

So the outer diagram commutes, which is what we wanted.

Next, we need C to be counital for the left coaction, that is, for the following diagram to commute:

$$\begin{tikzcd}
C \arrow[r, "\delta_{\mathcal{PR}(C),C}"] \arrow[dr, equal] & \mathcal{PR}(C)C \arrow[d, "\epsilon_{\mathcal{PR}(C)}*1"] \\
& C
\end{tikzcd}$$

We first consider the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] \arrow[dr, "\mathcal{R}\mathbb{M}_C^{-0}"] \arrow[ddr, equal] \arrow[r, phantom, shift right=6, "1"] & \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[d, "\mathcal{R}\mathbb{M}_C(\epsilon_{\mathcal{R}(C)})"] \\
& \mathcal{R}\mathbb{M}_C(1)C \arrow[d, "\mathcal{R}\mathbb{M}^0_C"] \arrow[l, phantom, shift right=4, "2"] \\
& C
\end{tikzcd}$$

1 commutes by definition of $\epsilon_{\mathcal{R}(C)}$;

2 commutes trivially.

So the outer diagram commutes.

Next, we consider the following diagram:

$$\begin{tikzcd}
C \arrow[rrr, "\delta_{\mathcal{PR}(C),C}"] \arrow[dr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{G_iC,G_iC}"] \arrow[dddr, equal] & & & \mathcal{PR}(C)C \arrow[ddl, "\mathscr{F}(\epsilon_{\mathcal{R}(C)})"] \arrow[dddll, "\epsilon_{\mathcal{PR}(C)}*1"] \\
& \mathcal{R}\mathbb{M}_C(\mathcal{R}(C))C \arrow[urr, equal] \arrow[d, "\mathcal{R}\mathbb{M}_C(\epsilon_{\mathcal{R}(C)})"] \arrow[ur, phantom, "2"] \arrow[r, phantom, "3"] & & \\
& \mathcal{R}\mathbb{M}_C(1)C \arrow[r, equal] \arrow[d, "\mathcal{R}\mathbb{M}^0_C"] \arrow[l, phantom, "1"] & \mathscr{F}(1)C \arrow[dl, "\mathscr{F}^0"] \arrow[r, phantom, "5"] & \\
& C \arrow[ur, phantom, "4"] & &
\end{tikzcd}$$

1 commutes by the argument above;

2-4 commute by definition of $\mathcal{R}\mathbb{M}_C$;

and 5 commutes by definition of $\Delta_{\mathcal{P}(D)}$.

So the outer diagram commutes, which is what we wanted.

So C is a left $\mathcal{PR}(C)$ comodule 1-morphism.

□

So we can define:

Definition 6.1.2 (Counit at coalgebra 1-morphisms)**.** $\epsilon_C := C$, viewed as a $\mathcal{PR}(C)$-C-bicomodule 1-morphism. ◁

6.2 Counit at bicomodule 1-morphisms

Next, we want to construct $\epsilon_M : \epsilon_C \underset{C}{\circledast} M \to \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\circledast} \epsilon_D$.

To begin with, we state the following definition:

Definition 6.2.1 (Adjoint of maps from internal cohoms)**.** Let C be a coalgebra 1-morphism in $\underline{\mathscr{D}}$, F a 1-morphism in $\mathscr{C}$, $M \in \mathcal{R}\mathbb{M}_C(j)$. Consider the following sequence of maps, where the horizontal maps are the respective cohom adjunctions:

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{\mathscr{C}}}({}_{\mathcal{R}\mathbb{M}_C}[C,M],F) & \cong & \mathrm{Hom}_{\mathcal{R}\mathbb{M}_C(j)}(M,\mathcal{R}\mathbb{M}_C(F)C) \\ & & \| \\ \mathrm{Hom}_{\underline{\mathscr{D}}}({}_{\mathbb{M}_C}[C,M],\mathscr{F}(F)) & \cong & \mathrm{Hom}_{\mathbb{M}_C(\mathscr{F}(j))}(M,\mathscr{F}(F)C) \end{array}$$

This gives us a bijection between maps $f : {}_{\mathcal{R}\mathbb{M}_C}[C,M] \to F$ in $\underline{\mathscr{C}}$, and maps $f_{M,C} : M \to \mathscr{F}(F)$ in $\underline{\mathscr{D}}$. Explicitly, we can compute

$$f_{M,C} = (\mathscr{F}(f) * \epsilon_C) \circ \mathrm{coev}_{M,C}^{\mathcal{R}\mathbb{M}_C}.$$

In particular, we have a 2-morphism $\psi_C := (1_{\mathcal{R}(C)})_{C,C} = (1 * \epsilon_C) \circ \mathrm{coev}_{C,C}^{\mathcal{R}\mathbb{M}_C} : C \to \mathcal{PR}(C)$ for each coalgebra 1-morphism C of $\underline{\mathscr{D}}$, and a 2-morphism $\Psi_M := (1_{\mathcal{R}(M)})_{M,D} = (1 * \epsilon_D) \circ \mathrm{coev}_{M,D}^{\mathcal{R}\mathbb{M}_D}$: $M \to \mathcal{PR}(M)$ for each C-D-bicomodule 1-morphism M.

◁

Lemma 6.2.2. Let C be a coalgebra 1-morphism in $\underline{\mathscr{D}}$. Then

$$\psi_C : C \to \mathcal{PR}(C)$$

is a monic coalgebra homomorphism.

Proof : Write $\psi := \psi_C$. We need to show that the following diagrams commute:

$$\begin{tikzcd}
C \arrow[r, "\psi"] \arrow[d, "\Delta_C"] & \mathcal{PR}(C) \arrow[d, "\Delta_{\mathcal{PR}(C)}"] \\
CC \arrow[r, "\psi * \psi"] & \mathcal{PR}(C)\mathcal{PR}(C)
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[rr, "\psi"] \arrow[dr, "\epsilon_C"'] & & \mathcal{PR}(C) \arrow[dl, "\epsilon_{\mathcal{PR}(C)}"] \\
& 1 &
\end{tikzcd}$$

We consider the left diagram first. Expanding definitions, we get the following diagram:

$$\begin{tikzcd}
C \arrow[r, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] \arrow[dd, "\Delta_C"] \arrow[dr, phantom, "1"] & \mathcal{PR}(C)C \arrow[r, "1*\epsilon_C"] \arrow[d, bend right, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"'] \arrow[d, bend left, "\Delta_{\mathcal{PR}(C)}*1"] \arrow[d, phantom, "2"] & \mathcal{PR}(C) \arrow[dd, "\Delta_{\mathcal{PR}(C)}"] \arrow[dl, phantom, "3"] \\
& \mathcal{PR}(C)\mathcal{PR}(C)C \arrow[dr, "1*1*\epsilon_C"] \arrow[dr, phantom, "4"'] & \\
CC \arrow[r, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C} * \mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] & \mathcal{PR}(C)C\mathcal{PR}(C)C \arrow[u, "1*\epsilon_C*1"] \arrow[r, "1*\epsilon_C*1*\epsilon_C"] & \mathcal{PR}(C)\mathcal{PR}(C)
\end{tikzcd}$$

(1) commutes because $\mathcal{PR}(C)$ is a coalgebra 1-morphism;

(2), precomposed with $\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}$, commutes by definition of $\Delta_{\mathcal{PR}(C)}$;

(3) commutes by the interchange law;

and (4) commutes trivially.

Therefore the outer diagram commutes, that is, ψ respects comultiplication.

Next, we consider the counit diagram. Consider the following expanded version:

$$\begin{tikzcd}
C \arrow[r, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}"] \arrow[dr, "\mathscr{F}^{-0}*1"'] \arrow[ddr, "\epsilon_C"'] \arrow[dr, phantom, bend left=15, "1"] \arrow[ddr, phantom, bend left=10, "3"] & \mathcal{PR}(C)C \arrow[rr, "1*\epsilon_C"] \arrow[d, "\mathcal{P}(\epsilon_{\mathcal{R}(C)})"] \arrow[dr, "\epsilon_{\mathcal{PR}(C)}*1"] \arrow[dr, phantom, bend right=20, "2"] & & \mathcal{PR}(C) \arrow[ddll, "\epsilon_{\mathcal{PR}(C)}"] \arrow[dll, phantom, "4"] \\
& \mathscr{F}(1)C \arrow[r, dashed, "\mathscr{F}^0*1"] & C \arrow[dl, "\epsilon_C"] & \\
& 1 & &
\end{tikzcd}$$

1 commutes by definition of $\epsilon_{\mathcal{R}(C)}$;

2 commutes by definition of $\epsilon_{\mathcal{PR}(C)}$;

3 commutes trivially;

4 commutes by the interchange law.

Therefore the outer diagram commutes, that is, ψ respects the counit.

So ψ is a coalgebra homomorphism, as required.

It is monic since $(1 * \epsilon_C)$ and $\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}$ are.

□

One can see immediately that $\epsilon_C = {}^{\psi_C}C$, and we use these definitions interchangeably

Lemma 6.2.3. Let M be a C-D-bicomodule 1-morphism. When viewed as a morphism $\Psi_M : {}^{\psi_C}M \to \mathcal{PR}(M)$, Ψ_M is a left comodule homomorphism natural in M.

Proof : Write $\Psi := \Psi_M$. We need Ψ to respect the left $\mathcal{PR}(C)$- and right $\mathcal{PR}(D)$-coactions. We first examine the left coaction. We want the following diagram to commute:

$$\begin{tikzcd}
M \arrow[r, "\Psi_M"] \arrow[d, "\delta_{\mathcal{PR}(C),M}"] & \mathcal{PR}(M) \arrow[d, "\delta_{\mathcal{PR}(C),\mathcal{PR}(M)}"] \\
\mathcal{PR}(C)M \arrow[r, "1*\Psi_M"] & \mathcal{PR}(C)\mathcal{PR}(M)
\end{tikzcd}$$

Consider the expanded diagram:

$$
\begin{tikzcd}
M \arrow[rrrr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{M,D} *1"] \arrow[drr, "\delta_{\mathcal{P}\mathcal{R}(C),M}"] \arrow[dd, "\delta_{C,M}"'] & & & & \mathcal{P}\mathcal{R}(M)D \arrow[r, "1*\epsilon_D"] \arrow[ddl, "\mathscr{F}(\delta_{\mathcal{R}(C),\mathcal{R}(M)})*1"'] \arrow[dd, "\delta_{\mathcal{P}\mathcal{R}(C),\mathcal{P}\mathcal{R}(M)}*1"] \arrow[dd, phantom, shift right=6ex, "3"] & \mathcal{P}\mathcal{R}(M) \arrow[ddd, "\delta_{\mathcal{P}\mathcal{R}(C),\mathcal{P}\mathcal{R}(M)}"] \arrow[ddd, phantom, shift right=6ex, "4"] \\
 & \arrow[r, phantom, "2"] & \mathcal{P}\mathcal{R}(C)M \arrow[r, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{M,D} *1", "1"'{yshift=6ex}] \arrow[ddll, equal, bend left] & \mathcal{P}\mathcal{R}(C)\mathcal{P}\mathcal{R}(M)D \arrow[d, dashed, "\mathscr{F}^{-2}"] & & \\
CM \arrow[r, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C} *1"] \arrow[d, "\psi_C *1"'] \arrow[dr, phantom, "6"] & \mathcal{P}\mathcal{R}(C)CM \arrow[dl, "1*\epsilon_C *1"] & & \mathcal{P}(\mathcal{R}(C)\mathcal{R}(M))D \arrow[r, "\mathscr{F}^2"] \arrow[d, phantom, "5"] & \mathcal{P}\mathcal{R}(C)\mathcal{P}\mathcal{R}(M)D \arrow[dr, "1*1*\epsilon_D"] & \\
\mathcal{P}\mathcal{R}(C)M \arrow[rrrrr, "1*\Psi_M"'] & & & & & \mathcal{P}\mathcal{R}(C)\mathcal{P}\mathcal{R}(M)
\end{tikzcd}
$$

(1) commutes by definition of $\delta_{\mathcal{R}(C),\mathcal{R}(M)}$;

(2) commutes by definition of $\delta_{\mathcal{PR}(C),M}$;

(3) commutes by definition of $\delta_{\mathcal{PR}(C),\mathcal{PR}(M)}$;

(4) commutes by the interchange law;

(5) commutes by definition of Ψ;

and (6) commutes by definition of ψ.

Therefore the outer diagram commutes, that is, Ψ is a left $\mathcal{PR}(C)$-comodule homomorphism. Since $\mathrm{coev}^{\mathcal{R}\mathbb{M}_D}_{M,D}$ is natural in M, this is in fact a natural transformation of bicomodule 1-morphisms.

□

Note that $\epsilon_C = {}^{\psi_C}C$. Note also that since ψ_D is monic, by Lemma 1.3.17, ${}^{\psi_C}M \underset{D}{\circledast} D = {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\circledast} {}^{\psi_D}D$. So we can use these data to construct ϵ_M as follows:

Definition 6.2.4 (Counit at comodule 1-morphisms)**.** ϵ_M is the following composition:

$$
\begin{array}{c}
\mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\circledast} \epsilon_D \\
\Big\| \\
\mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\circledast} {}^{\phi_D}D \\
\Big\downarrow {\scriptstyle \Psi_M^{-1} \underset{\mathcal{PR}(D)}{\circledast} 1} \\
{}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\circledast} {}^{\psi_D}D \\
\Big\| \\
{}^{\psi_C}M \underset{D}{\circledast} D \\
\Big\downarrow {\scriptstyle \varepsilon^r_M} \\
{}^{\psi_C}M \\
\Big\downarrow {\scriptstyle (\varepsilon^l_M)^{-1}} \\
{}^{\psi_C}C \underset{C}{\circledast} M \\
\Big\| \\
\epsilon_C \underset{C}{\circledast} M
\end{array}
$$

◁

6.3 2-naturality

Finally, we put this data together to obtain the following result:

Proposition 6.3.1. ϵ_C, ϵ_M assemble to a 2-natural transform. $\epsilon : \mathcal{PR} \to 1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\mathscr{D}}}$

Proof :

By Lemma 6.2.2, the bicomodule 1-morphism ${}^{\psi_C}C$ is indeed a 1-morphism $\epsilon_C : \mathcal{PR}(C) \to C$, as needed.

We recall that $(\psi_D) \upharpoonright^{(M,D)}$ is a bicomodule homomorphism natural in M

$$(\psi_D) \upharpoonright^{(M,D)} : {}^{\psi_C}M \underset{D}{\boxast} D \to {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}D$$

by Lemma 1.3.18; ϵ^l_M and ϵ^r_M are bicomodule isomorphisms natural in M by Lemma 1.3.11; and Φ is a bicomodule homomorphism natural in M by Lemma 6.2.3. So ϵ_M is a bicomodule homomorphism natural in M, as required.

We further need the following diagrams to commute:

$$\begin{tikzcd}
\mathcal{PR}(M \underset{D}{\boxast} N) \underset{\mathcal{PR}(E)}{\boxast} \epsilon_E \arrow[rr, "\epsilon_{M \underset{D}{\boxast} N}"] \arrow[d, "\mathcal{PR}^2 \underset{\mathcal{PR}(E)}{\boxast} 1"'] \arrow[drr, phantom, "\boxed{1}"] & & \epsilon_C \underset{C}{\boxast} M \underset{D}{\boxast} N \arrow[d, equal] \\
\mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \mathcal{PR}(N) \underset{\mathcal{PR}(E)}{\boxast} \epsilon_E \arrow[r, "1 \underset{\mathcal{PR}(D)}{\boxast} \epsilon_N"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \epsilon_D \underset{D}{\boxast} N \arrow[r, "\epsilon_M \underset{D}{\boxast} 1"'] & \epsilon_C \underset{C}{\boxast} M \underset{D}{\boxast} N
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{PR}(C) \underset{\mathcal{PR}(C)}{\boxast} \epsilon_C \arrow[rr, "\epsilon_C"] \arrow[dr, "\varepsilon^r_{\epsilon_C}"'] & \boxed{2} & \epsilon_C \underset{C}{\boxast} C \arrow[dl, "\varepsilon^l_{\epsilon_C}"] \\
& \epsilon_C &
\end{tikzcd}$$

To see that $\boxed{1}$ commutes, consider the following diagrams:

$$
\begin{tikzcd}
{}^{\psi_C}M \underset{D}{\boxast} N \arrow[r, "(\varepsilon^r_{M\boxast N})^{-1}"] \arrow[d, "(\varepsilon^r_M)^{-1}\boxast 1"'] \arrow[dr, equal] \arrow[dr, phantom, "(7)" near start] & {}^{\psi_C}M \underset{D}{\boxast} N \underset{E}{\boxast} E \arrow[rr, equal] \arrow[dr, equal] \arrow[dr, phantom, "(11)" near start] & & {}^{\psi_C}M \underset{D}{\boxast} N^{\psi_E} \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[r, "\Psi_{M\boxast N}\boxast 1"] \arrow[dd, equal] \arrow[ddr, phantom, "(16)"] & \mathcal{PR}(M \underset{D}{\boxast} N) \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[ddd, "\mathcal{PR}^2\boxast 1"] \\
{}^{\psi_C}M \underset{D}{\boxast} D \underset{D}{\boxast} N \arrow[r, "1\boxast\varepsilon^l_N"'] \arrow[d, equal] \arrow[dr, phantom, "(9)"] & {}^{\psi_C}M \underset{D}{\boxast} N \arrow[r, "1\boxast(\varepsilon^r_N)^{-1}"'] \arrow[d, equal] \arrow[dr, phantom, "(12)"] & {}^{\psi_C}M \underset{D}{\boxast} N \underset{E}{\boxast} E \arrow[d, equal] \arrow[dr, phantom, "(14)"] & & \\
{}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}D \underset{D}{\boxast} N \arrow[r, "1\boxast\varepsilon^l_N"'] \arrow[d, "\Psi_M\boxast 1"'] \arrow[dr, phantom, "(10)"] & {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \arrow[r, "1\boxast(\varepsilon^r_N)^{-1}"'] \arrow[d, "\Psi_M\boxast 1"] \arrow[dr, phantom, "(13)"] & {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \underset{E}{\boxast} E \arrow[r, equal] \arrow[d, "\Psi_M\boxast 1"] \arrow[dr, phantom, "(15)"] & {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N^{\psi_E} \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[d, "\Psi_M\boxast 1"] & \\
\mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}D \underset{D}{\boxast} N \arrow[r, "1\boxast\varepsilon^l_N"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \arrow[r, "1\boxast(\varepsilon^r_N)^{-1}"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \underset{E}{\boxast} E \arrow[r, equal] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N^{\psi_E} \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[r, "1\boxast\Psi_N\boxast 1"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \mathcal{PR}(N) \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E
\end{tikzcd}
$$

$$
\begin{tikzcd}
\epsilon_C \underset{C}{\boxast} M \underset{D}{\boxast} N \arrow[rrrrrrr, "\epsilon^{-1}_{M\boxast N}"] \arrow[ddddd, equal] \arrow[ddr, equal] & & & & \arrow[d, phantom, "(3)"] & & & \mathcal{PR}(M \underset{D}{\boxast} N) \underset{\mathcal{PR}(E)}{\boxast} \epsilon_E \arrow[ddddd, "\mathcal{PR}^2\boxast 1"] \\
& & {}^{\psi_C}M \underset{D}{\boxast} N \underset{E}{\boxast} E \arrow[r, equal] & {}^{\psi_C}M \underset{D}{\boxast} N^{\psi_E} \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[rrr, "\Psi_{M\boxast N}\boxast 1"] & & & \mathcal{PR}(M \underset{D}{\boxast} N) \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[ur, equal] \arrow[ddd, "\mathcal{PR}^2\boxast 1"] \arrow[ddd, phantom, "(6)", shift left=8ex] & \\
\arrow[r, phantom, "(1)"] & {}^{\psi_C}C \underset{C}{\boxast} M \underset{D}{\boxast} N \arrow[r, "\varepsilon^l_{M\boxast N}"] \arrow[dddl, equal] & {}^{\psi_C}M \underset{D}{\boxast} N \arrow[u, dashed, "(\varepsilon^r_{M\boxast N})^{-1}"] \arrow[d, "(\varepsilon^r_M)^{-1}\boxast 1"'] & & \arrow[d, phantom, "(4)"] & & & \\
& \arrow[d, phantom, "(2)"] & {}^{\psi_C}M \underset{D}{\boxast} D \underset{D}{\boxast} N \arrow[d, equal] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \arrow[r, "1\boxast(\varepsilon^r_N)^{-1}"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N \underset{E}{\boxast} E \arrow[d, equal] & & & \\
& & {}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}D \underset{D}{\boxast} N \arrow[r, "\Psi_M\boxast 1"'] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}D \underset{D}{\boxast} N \arrow[u, "1\boxast\varepsilon^l_N"] \arrow[d, equal] & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} {}^{\psi_D}N^{\psi_E} \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[rr, "1\boxast\Psi_N\boxast 1"'] \arrow[rr, phantom, "(5)", shift right=4ex] & & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \mathcal{PR}(N) \underset{\mathcal{PR}(E)}{\boxast} {}^{\psi_E}E \arrow[dr, equal] & \\
\epsilon_C \underset{C}{\boxast} M \underset{D}{\boxast} N \arrow[rrr, "\epsilon^{-1}_M\boxast 1"'] & & & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \epsilon_D \underset{D}{\boxast} N \arrow[rrrr, "1\boxast\epsilon^{-1}_N"'] & & & & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxast} \mathcal{PR}(N) \underset{\mathcal{PR}(E)}{\boxast} \epsilon_E
\end{tikzcd}
$$

1 and 6 commute trivially;

and 2, 3 and 5 commute by definition of ϵ_M.

4 is the outer edge of the right-hand diagram;

7 commutes since left and right unitors commute;

9, 11, 12, 14 and 15 commute trivially;

10 and 13 commute by the interchange law;

so it suffices to consider 16.

$$
\begin{tikzcd}
{}^{\psi_C}M \underset{D}{\boxtimes} N^{\psi_E} \arrow[rrrr, "\Psi_{M\boxtimes N}"] \arrow[rrrd, "\mathrm{coev}^{\mathcal{R}M_E}_{M\boxtimes N,E}"] \arrow[rrdd, "\mathrm{coev}^{\mathcal{R}M_D}_{M,D}\boxtimes 1"] \arrow[rdd, "t"] \arrow[dd, equal] & & & & \mathcal{PR}(M \underset{D}{\boxtimes} N) \arrow[dddddd, "\mathcal{PR}^2"] \\
 & & & \mathcal{PR}(M \underset{D}{\boxtimes} N)E \arrow[ur, "1*\epsilon_E"] \arrow[dddd, "\mathcal{PR}^2*1"] \arrow[u, phantom, "24"] \arrow[ddr, phantom, "26"] & \\
{}^{\psi_C}M^{\psi_D} \underset{\mathcal{PR}(D)}{\boxtimes} {}^{\psi_D}N^{\psi_E} \arrow[r, "t"] \arrow[dddd, "\Psi_M\boxtimes 1"] \arrow[ur, phantom, "17"] & {}^{\psi_C}MN^{\psi_E} \arrow[d, "\mathrm{coev}^{\mathcal{R}M_D}_{M,D}*1"] \arrow[dd, bend right, "\Psi_M"'] \arrow[u, phantom, "20"] & \mathcal{PR}(M)D \underset{D}{\boxtimes} N^{\psi_E} \arrow[dl, "t"] \arrow[ddl, bend left, "1*\epsilon^l_N"] \arrow[dr, phantom, "23"] & & \\
 & \mathcal{PR}(M)DN^{\psi_E} \arrow[d, "1*\epsilon_D*1"] \arrow[u, phantom, "19"] \arrow[r, phantom, "21"] & & & \\
 & \mathcal{PR}(M)N^{\psi_E} \arrow[r, "1*\mathrm{coev}^{\mathcal{R}M_E}_{N,E}"'] \arrow[dr, phantom, "22"] & \mathcal{PR}(M)\mathcal{PR}(N)E & & \\
 & & & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxtimes} \mathcal{PR}(N)E \arrow[ul, "t*1"] \arrow[dr, "1*\epsilon_E"] \arrow[d, phantom, "25"] & \\
\mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxtimes} {}^{\psi_D}N^{\psi_E} \arrow[uur, "t"] \arrow[urrr, "1\boxtimes \mathrm{coev}^{\mathcal{R}M_E}_{N,E}"] \arrow[rrrr, "1\boxtimes\Psi_N"'] \arrow[uur, phantom, "18"'] & & & & \mathcal{PR}(M) \underset{\mathcal{PR}(D)}{\boxtimes} \mathcal{PR}(N)
\end{tikzcd}
$$

17 commutes trivially;

18, 20 and 22 commute by definition of the cotensor product;

19, 24 and 25 commute by definition of Ψ_M;

21 commutes by definition of ϵ^l_N;

23 commutes by definition of $\mathcal{R}^2$;

26 commutes by the interchange law;

and t is monic.

Note that here, unlike previous diagrams, we are considering equality of 2-morphisms in $\underline{\mathscr{D}}$, but since being equal as 2-morphisms implies being equal as bicomodule homomorphisms, this causes no issues.

So 16 commutes, so 4 commutes, so $\boxed{1}$ commutes.

That $\boxed{2}$ commutes is trivial.

So ϵ is a 2-natural transform.

□

7

Triangulators and the swallowtail diagrams

Throughout this chapter, we assume $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ is an essentially 1-surjective locally $\mathbb{k}$-linear pseudofunctor between one-object multifiat 2-categories whose birepresentations are all cyclic.

We next need a pair of modifications, called *triangulators*, as in the following diagrams:

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathcal{P}(\eta)"] \arrow[dr, equal] & \mathcal{PRP} \arrow[d, "\epsilon_{\mathcal{P}}"] \arrow[dl, phantom, "\Uparrow\sigma"] \\ & \mathcal{P} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R} \arrow[r, "\eta_{\mathcal{R}}"] \arrow[dr, equal] & \mathcal{RPR} \arrow[d, "\mathcal{R}(\epsilon)"] \arrow[dl, phantom, "\Downarrow\tau"] \\ & \mathcal{R} \end{tikzcd}$$

7.1 Construction of σ

We start with the left diagram. Fix a coalgebra 1-morphism C in $\underline{\mathscr{C}}$, and note that

$$(\epsilon_{\mathcal{P}} \circ \mathcal{P}(\eta))(C) = \mathcal{P}(\eta_C) \underset{\mathcal{PRP}(C)}{\circledast} \epsilon_{\mathcal{P}(C)}.$$

Recalling that $\eta_D = {}^*\mathcal{RP}(D)$ (with the * highlighting the special left coaction), and $\epsilon_D = {}^{\psi_D}D$, we find that this composite takes the form

$${}^*\mathcal{PRP}(C) \underset{\mathcal{PRP}(C)}{\circledast} {}^{\psi_{\mathcal{P}(C)}}\mathcal{P}(C).$$

Now, we'd like to use $\varepsilon^l_{\epsilon_{\mathcal{P}(C)}}$ to go from this object to $\mathcal{P}(C)$. This is coherent as a 2-morphism of 1-morphisms, and respects the right coactions. We need to understand how it interacts with the left coactions. This leads us to the following lemma:

Lemma 7.1.1. Let D be a coalgebra 1-morphism in a finitary 2-category $\mathscr{C}$. Suppose that D has another left coaction $\delta_{D',D}$, so can be viewed as a D'-D-bicomodule 1-morphism ${}^*D$. Moreover, let M be a left D-comodule 1-morphism, with coaction $\delta_{D,M}$.

Then there's a left D'-comodule 1-morphism ${}^{\dagger}M$, with underlying 1-morphism M and left coaction given by the composite

$$M \xrightarrow{\delta_{D,M}} DM \xrightarrow{\delta_{D',D}*1} D'DM \xrightarrow{1*\epsilon_D*1} D'M \ ,$$

and $\varepsilon^l_M : {}^*D \underset{D}{\boxast} M \to {}^{\dagger}M$ is an isomorphism of left comodule 1-morphisms.

Proof : By Proposition 1.3.11, it's sufficient to check that ε^l_M is a homomorphism of left comodule 1-morphisms. But this is immediate from the following diagram:

$$\begin{tikzcd}
{}^*C \boxast M \arrow[r, "t"] \arrow[ddd, "\delta_{C',C}\boxast 1"'] & {}^*CM \arrow[r, "\epsilon_C*1"] \arrow[dr, equal] \arrow[ddr, phantom, "1"'] & {}^{\dagger}M \arrow[d, "\delta_{C,M}"] \arrow[dl, phantom, "2"] \\
& & {}^*CM \arrow[d, "\delta_{C',C}*1"] \\
& & C'{}^*CM \arrow[d, "1*\epsilon_C*1"] \arrow[dl, equal] \\
C'{}^*C \boxast M \arrow[r, "1*t"'] & C'{}^*CM \arrow[r, "1*\epsilon_C*1"'] \arrow[ur, phantom, "3"] & C'{}^{\dagger}M
\end{tikzcd}$$

1 commutes by definition of $\delta_{C',C} \boxast 1$;

2 and 3 commute trivially.

So the outer diagram commutes, that is, ε^l_M is a homomorphism of left comodule 1-morphisms. □

So with the correct left coactions, we have an isomorphism

$$\varepsilon^l_{\epsilon_{\mathcal{P}(C)}} : {}^*\mathcal{PRP}(C) \underset{\mathcal{PRP}(C)}{\boxast} {}^{\psi_{\mathcal{P}(C)}}\mathcal{P}(C) \to {}^{\dagger}\mathcal{P}(C).$$

We compute the left coaction needed for $\mathcal{P}(C)$ in the following lemma:

Lemma 7.1.2. The composition

$$\mathcal{P}(C) \xrightarrow{\delta_{\mathcal{PRP}(C),\mathcal{P}(C)}} \mathcal{PRP}(C)\mathcal{P}(C) \xrightarrow{\delta_{\mathcal{P}(C),\mathcal{PRP}(C)}*1} \mathcal{P}(C)\mathcal{PRP}(C)\mathcal{P}(C) \xrightarrow{1*\epsilon_{\mathcal{PRP}(C)}*1} \mathcal{P}(C)\mathcal{P}(C) \ ,$$

is equal to $\Delta_{\mathcal{P}(C)}$.

Proof :

Consider the following diagram:

$$
\begin{tikzcd}
\mathcal{P}(C) \arrow[r, "\delta_{\mathcal{P}\mathcal{R}\mathcal{P}(C),\mathcal{P}(C)}"] \arrow[dddd, "\Delta_{\mathcal{P}(C)}"'] \arrow[ddr, phantom, "1"] & \mathcal{P}\mathcal{R}\mathcal{P}(C)\mathcal{P}(C) \arrow[rr, "\delta_{\mathcal{P}(C),\mathcal{P}\mathcal{R}\mathcal{P}(C)}*1"] \arrow[drr, equal] \arrow[ddr, phantom, "3"] & & \mathcal{P}(C)\mathcal{P}\mathcal{R}\mathcal{P}(C)\mathcal{P}(C) \arrow[rr, "1*\epsilon_{\mathcal{P}\mathcal{R}\mathcal{P}(C)}*1"] \arrow[dr, phantom, "5"] & & \mathcal{P}(C)\mathcal{P}(C) \arrow[dddd, equal] \\
 & & & \mathcal{P}({}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C),\mathscr{F}(C)])\mathcal{P}(C) \arrow[dd, "\mathcal{P}({}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C),\Delta_{\mathscr{F}(C)}])*1"] & \mathcal{P}(C_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C),\mathscr{F}(C)])\mathcal{P}(C) \arrow[ul, "\mathscr{F}^2*1"'] \arrow[ur, "\mathcal{P}(1*\epsilon_{\mathcal{R}\mathcal{P}(C)})*1"] \arrow[ur, phantom, "7"'] & \\
 & \mathcal{P}\mathcal{R}\mathcal{P}(C)\mathcal{P}(C)\mathcal{P}(C) \arrow[uu, "1*\epsilon_{\mathcal{P}(C)}*1"] & \mathcal{P}(C) \arrow[ur, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_{\mathcal{P}(C)}}_{\mathcal{P}(C),\mathcal{P}(C)}"] & & & \\
 & & & \mathcal{P}({}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C),\mathscr{F}(C)\mathscr{F}(C)])\mathcal{P}(C) \arrow[uur, "\cong"] & \mathcal{P}(C)\mathcal{P}({}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C),\mathscr{F}(C)])\mathcal{P}(C) \arrow[uu, "\mathscr{F}^{-2}*1"] \arrow[uuur, "1*\mathcal{P}(\epsilon_{\mathcal{R}\mathcal{P}(C)})*1"'] \arrow[uuur, phantom, "8"] \arrow[dr, phantom, "9"] & \\
\mathcal{P}(C)\mathcal{P}(C) \arrow[uuuur, "\psi_{\mathcal{P}(C)}*1"] \arrow[uur, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_{\mathcal{P}(C)}}_{\mathcal{P}(C),\mathcal{P}(C)}*1"'] \arrow[uur, phantom, "2"] \arrow[uurr, "\epsilon_{\mathcal{P}(C)}*1"'] \arrow[uurr, phantom, "4"'] \arrow[urrr, "\mathrm{coev}^{\mathcal{R}\mathbb{M}_{\mathcal{P}(C)}}_{\mathcal{P}(C)\mathcal{P}(C),\mathcal{P}(C)}"'] \arrow[rrrrr, equal] \arrow[rrrrr, phantom, "6"] & & & & & \mathcal{P}(C)\mathcal{P}(C) \arrow[ul, "1*\mathrm{coev}^{\mathcal{R}\mathbb{M}_{\mathcal{P}(C)}}_{\mathcal{P}(C),\mathcal{P}(C)}"]
\end{tikzcd}
$$

1 commutes by definition of $\delta_{\mathcal{PRP}(C),\mathcal{P}(C)}$;

2 commutes by definition of $\psi_{\mathcal{P}(C)}$;

3 commutes by the interchange law;

4 commutes by definition of ${}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(C)}}[\mathscr{F}(C), \Delta_{\mathscr{F}(C)}]$, noting that $(\epsilon_D * 1) \circ \Delta_D = 1$;

5 commutes by definition of $\delta_{\mathcal{P}(C),\mathcal{PRP}(C)}$;

6 commutes by Lemma 1.5.3;

7 commutes by naturality of $\mathscr{F}^2$ and definition of $\epsilon_{\mathcal{PRP}(C)}$;

8 commutes by naturality of $\mathscr{F}^2$;

and 9 commutes by definition of $\epsilon_{\mathcal{RP}(C)}$.

Therefore the outer diagram commutes, which is precisely what we wanted. □

So we have a bicomodule isomorphism

$$\varepsilon^l_{\epsilon_{\mathcal{P}(C)}} : \mathcal{P}(\eta_C) \underset{\mathcal{PRP}(C)}{\circledast} \epsilon_{\mathcal{P}(C)} \to \mathcal{P}(C).$$

So we define $\sigma_C := (\varepsilon^l_{\epsilon_{\mathcal{P}(C)}})^{-1}$.

7.2 Construction of τ

Next, we look at the second triangulator diagram:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\eta_{\mathcal{R}}"] \arrow[dr, equal] & \mathcal{RPR} \arrow[d, "\mathcal{R}(\epsilon)"] \arrow[dl, phantom, "\swarrow \tau"] \\ & \mathcal{R} \end{tikzcd}$$

Fixing a coalgebra 1-morphism C of $\underline{\mathscr{C}}$, we know that

$$(\mathcal{R}(\epsilon) \circ \eta_{\mathcal{R}})(C) = {}^*{}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(\mathcal{R}(C))}}[\mathscr{F}(\mathcal{R}(C)), \mathscr{F}(\mathcal{R}(C))] \underset{\mathcal{RPR}(C)}{\circledast} \mathcal{R}({}^{\psi_C}C),$$

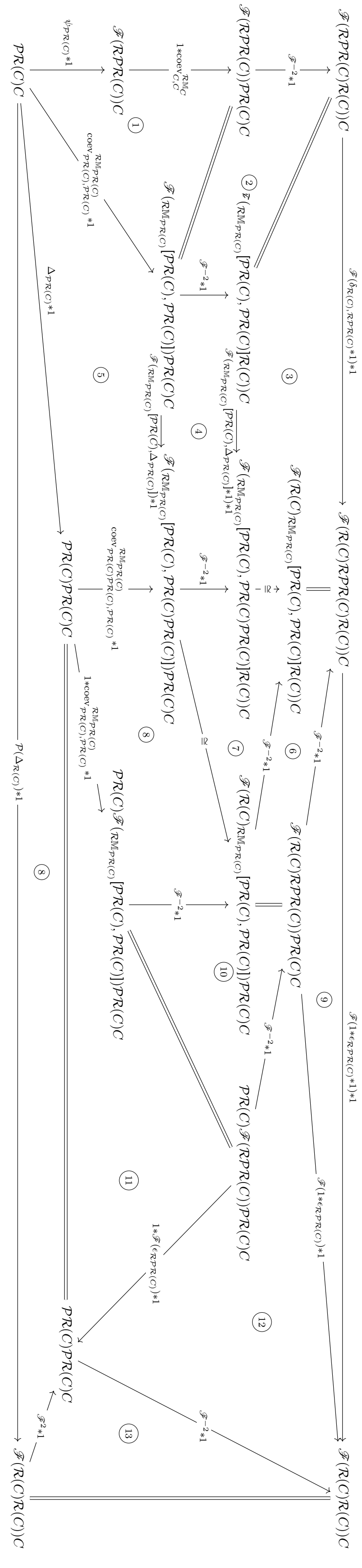
and note that as a right comodule 1-morphism, ${}_{\mathcal{R}\mathbb{M}_{\mathscr{F}(\mathcal{R}(C))}}[\mathscr{F}(\mathcal{R}(C)), \mathscr{F}(\mathcal{R}(C))] = \mathcal{RPR}(C)$. We want to again invoke Lemma 7.1.1, which will give us an isomorphism between this cotensor product and $\mathcal{R}(C)$. So we compute the left coaction of ${}^{\dagger}\mathcal{R}({}^{\psi_C}C)$.

Lemma 7.2.1. The composition

$$\mathcal{R}(C) \xrightarrow{\delta_{\mathcal{RPR}(C),\mathcal{R}(C)}} \mathcal{RPR}(C)\mathcal{R}(C) \xrightarrow{\delta_{\mathcal{R}(C),\mathcal{RPR}(C)}*1} \mathcal{R}(C)\mathcal{RPR}(C)\mathcal{R}(C) \xrightarrow{1*\varepsilon_{\mathcal{RPR}(C)}*1} \mathcal{R}(C)\mathcal{R}(C)$$

is equal to $\Delta_{\mathcal{R}(C)}$.

Proof : We do this in two parts. First, consider the following diagram:



1 commutes by definition of $\psi_{\mathcal{PR}(C)}$;

2, 4, 7, 9 and 12 commute by naturality of $\mathscr{F}^{-2}$;

3 commutes by definition of $\delta_{\mathcal{R}(C),\mathcal{RPR}(C)}$;

5 commutes by definition of ${}_{\mathcal{R}\mathbb{M}_{\mathcal{PR}(C)}}[\mathcal{PR}(C), \Delta_{\mathcal{PR}(C)}]$;

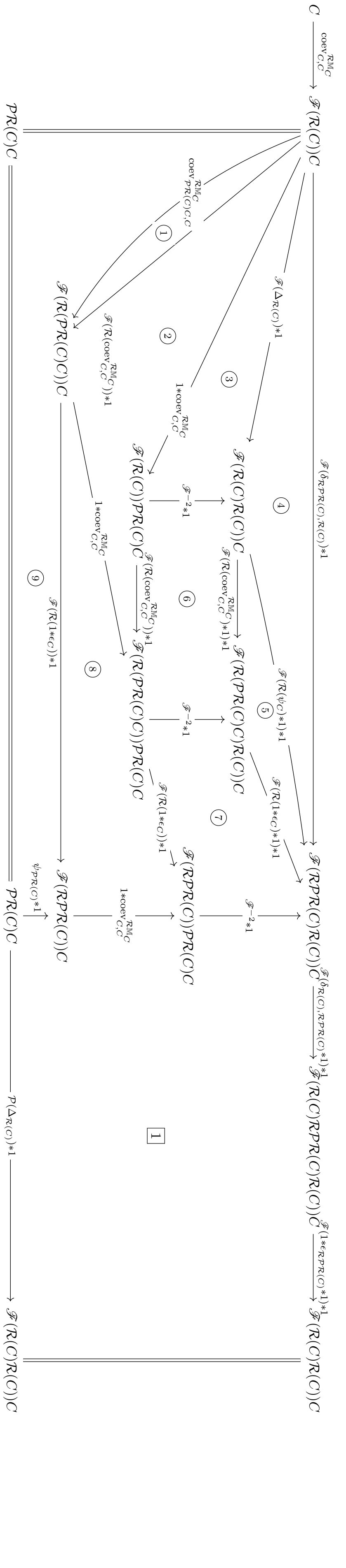
6, 10 and 13 commute trivially;

8 commutes by Lemma 1.5.3;

and 11 commutes by definition of $\varepsilon_{\mathcal{RPR}(C)}$.

So the outer diagram commutes. Call this outer diagram $\boxed{1}$.

Now, consider the following diagram:



1, precomposed with $\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C}$, commutes by definition of $\mathcal{R}(\mathrm{coev}^{\mathcal{R}\mathbb{M}_C}_{C,C})$;

2 and 8 commute by the interchange law;

3 commutes by definition of $\Delta_{\mathcal{R}(C)}$;

4 commutes by definition of $\delta_{\mathcal{RPR}(C),\mathcal{R}(C)}$;

5 commutes by definition of ψ_C;

6 and 7 commute by naturality of $\mathscr{F}^{-2}$;

9 commutes by definition of $\psi_{\mathcal{PR}(C)}$;

and we have already established that $\boxed{1}$ commutes.

So the outer diagram commutes. But passing via $\gamma^{\mathcal{R}\mathbb{M}_C}_{C,C}$, this says precisely that the composition

$$\mathcal{R}(C) \xrightarrow{\delta_{\mathcal{RPR}(C),\mathcal{R}(C)}} \mathcal{RPR}(C)\mathcal{R}(C) \xrightarrow{\delta_{\mathcal{R}(C),\mathcal{RPR}(C)}*1} \mathcal{R}(C)\mathcal{RPR}(C)\mathcal{R}(C) \xrightarrow{1*\varepsilon_{\mathcal{RPR}(C)}*1} \mathcal{R}(C)\mathcal{R}(C)$$

is equal to $\Delta_{\mathcal{R}(C)}$, which is what we wanted.

$\square$

So we have a bicomodule isomorphism

$$\varepsilon^l_{\mathcal{R}(\epsilon_C)} : \eta_{\mathcal{R}(C)} \circledast \mathcal{R}(\epsilon_C) \to \mathcal{R}(C)$$

and thus define $\tau_C = \varepsilon^l_{\mathcal{R}(\epsilon_C)}$

7.3 Swallowtail diagrams

Finally, for all this data to assemble to a 2-natural transformation, we need the swallowtail diagrams to commute. We recall the swallowtail diagrams from Definition 1.1.6, as they appear in our context:

$$\begin{tikzcd}
1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}} \arrow[r, "\eta"] \arrow[d, "\eta"'] & \mathcal{RP} \arrow[dl, Rightarrow, "\eta_\eta"'] \arrow[d, "\mathcal{RP}(\eta)"] \arrow[ddr, equal, bend left] & \\
\mathcal{RP} \arrow[r, "\eta_{\mathcal{RP}}"] \arrow[drr, equal, bend right] & \mathcal{RPRP} \arrow[dr, "\mathcal{R}(\epsilon_{\mathcal{P}})"] \arrow[r, phantom, "\Leftarrow \mathcal{R}(\sigma)"] \arrow[d, phantom, "\Downarrow \tau_{\mathcal{P}}"] & {} \\
& {} & \mathcal{RP}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{C}}}} \arrow[dr, "\eta"] & \\
& \mathcal{RP}
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathcal{PR} \arrow[dr, "\mathcal{P}(\eta_{\mathcal{R}})"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] \arrow[ddr, phantom, "\sigma_{\mathcal{R}} \Rightarrow"] \arrow[drr, phantom, "\Uparrow \mathcal{P}(\tau)"] & & \\
& \mathcal{PRPR} \arrow[r, "\mathcal{PR}(\epsilon)"] \arrow[d, "\epsilon_{\mathcal{PR}}"] & \mathcal{PR} \arrow[d, "\epsilon"] \\
& \mathcal{PR} \arrow[r, "\epsilon"] \arrow[ur, Rightarrow, "\epsilon_{\epsilon}"] & 1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{D}}}}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{PR} \arrow[dr, "\epsilon"] & \\
& 1_{\mathscr{B}\mathscr{B}\mathsf{icom}_{\underline{\mathscr{D}}}}
\end{tikzcd}
$$

It suffices to show that these equalities hold for each coalgebra 1-morphism C of $\underline{\mathscr{C}}$ (respectively, $\underline{\mathscr{D}}$). Recall that composition of 1-morphisms in bicomodule categories is given by the cotensor product; that the identity 1-morphism at a coalgebra 1-morphism C is the C-C-bicomodule 1-morphism C; and that, since η_C (respectively, ϵ_C) is a bicomodule 1-morphism for any suitable coalgebra 1-morphism C, η_{η_C} (respectively, ϵ_{ϵ_C}) is a bicomodule homomorphism as defined in Lemma 5.2.2 (respectively, Lemma 6.2.4).

We now recall specific definitions.

Lemma 7.3.1. In the first swallowtail diagram, for C a coalgebra 1-morphism in $\underline{\mathscr{C}}$:

- η_{η_C} is the bicomodule homomorphism $\mathcal{R}((\epsilon^r_{\mathcal{P}(\eta_C)})^{-1}) \circ \epsilon^l_{\mathcal{RP}(\eta_C)}$;

- $\mathcal{R}(\sigma_C) = \mathcal{R}((\varepsilon^l_{\epsilon_{\mathcal{P}(C)}})^{-1})$;

- $\tau_{\mathcal{P}(C)} = \epsilon^l_{\mathcal{R}(\epsilon_{\mathcal{P}(C)})}$.

So recalling that left and right unitors commute, we immediately see that the first swallowtail diagram identity holds.

In the second swallowtail diagram, for C a coalgebra 1-morphism in $\underline{\mathscr{D}}$:

- $\epsilon_{\epsilon_C} = (\varepsilon^r_{\epsilon_C})^{-1} \circ \varepsilon^l_{\eta_C}$

- $\sigma_{\mathcal{R}(C)} = (\varepsilon^l_{\epsilon_{\mathcal{PR}(C)}})^{-1}$

- $\mathcal{P}(\tau) = \mathcal{P}(\epsilon^l_{\mathcal{R}(\epsilon_C)})$.

So similarly the second swallowtail diagram identity holds.

7.4 Conclusion

Finally, we can put everything together to obtain the central result of our paper:

Theorem 7.4.1 (Frobenius reciprocity for fiat 2-representations)**.** Let $\mathscr{C}$, $\mathscr{D}$ be multifiat 2-categories for which every birepresentation is cyclic. Let $\mathscr{F} : \mathscr{C} \to \mathscr{D}$ be a locally $\Bbbk$-linear essentially 1-surjective pseudofunctor. Then induction along $\mathscr{F}$ is left biadjoint to restriction along $\mathscr{F}$.

Proof : Recall the definition of a biadjunction from Definition 1.1.6.

By Theorem 3.2.3, $\mathcal{P} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}} \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}$ is a pseudofunctor. By Theorem 4.2.1, $\mathcal{R} : \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}} \to \mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}$ is a pseudofunctor.

By Proposition 5.3.1, $\eta : 1_{\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{C}}}} \to \mathcal{R}\mathcal{P}$ is a 2-natural transform. By Proposition 6.3.1, $\epsilon : \mathcal{P}\mathcal{R} \to 1_{\mathscr{B}\mathscr{B}\text{icom}_{\underline{\mathscr{D}}}}$ is a 2-natural transform.

By Section 7.1, σ in the following diagram is a modification:

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\mathcal{P}(\eta)"] \arrow[dr, equal] & \mathcal{P}\mathcal{R}\mathcal{P} \arrow[d, "\epsilon_{\mathcal{P}}"] \\ & \mathcal{P} \arrow[u, phantom, ""] \end{tikzcd} \quad \text{with } \overset{\sigma}{\Rrightarrow}$$

By Section 7.2, τ in the following diagram is a modification:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\eta_{\mathcal{R}}"] \arrow[dr, equal] & \mathcal{R}\mathcal{P}\mathcal{R} \arrow[d, "\mathcal{R}(\epsilon)"] \\ & \mathcal{R} \end{tikzcd}$$

And by Lemma 7.3.1, the swallowtail diagram identities hold.

□

8

Final remarks

Some natural questions and conjectures arise from this thesis.

First, and most straightforward:

Conjecture 8.0.1. Theorem 7.4.1 still holds for $\mathscr{C}$, $\mathscr{D}$ (quasi) multifiab bicategories.

where the definition of multifiab bicategories can be found in Definition 2.5 of [MMM+21].

In theory, the result for multifiab bicategories might be considered a corollary of Theorem 7.4.1. By [MP85], any bicategory is biequivalent to a strict 2-category, and by [Cam19] this 'strictification' respects 2-adjunctions. The strictification of a multifiab bicategory is a multifiat 2-category, so the result follows. However, strictification loses important information about a bicategory, and the form of the adjunction may not translate straightforwardly. For quasi multifiab bicategories, the strict equivalent are weak multifiat 2-categories, and many of results we have used are still known to hold (courtesy of [MMM+21]).

Of more importance is the following conjecture:

Conjecture 8.0.2. The assumption of essential 1-surjectivity in Theorem 7.4.1 is unnecessary.

The main roadblock to this conjecture is in constructing the unit. Our choice of bicomodule 1-morphism η_C relies on essential 1-surjectivity unavoidably: the left C-coaction defined in Proposition 5.1.1 would otherwise be blocked by the appearance of some 1-morphism G. By contrast, the counit construction requires only slight modifications. By parallel with the representation theory of algebras, there is seemingly no a priori reason for the assumption of essential 1-surjectivity to be necessary.

The next conjecture arises from a detail in section 1.3. We can observe that $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ naturally has additional structure: that of the coalgebra homomorphisms. This additional structure turns $\mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}}$ into a (pseudo) double category, $\mathscr{B}\mathscr{B}\mathrm{icom}^{+}_{\underline{\mathscr{C}}}$ (see [GP99] for the basics of double categories). We might then reasonably speculate:

Conjecture 8.0.3. Restriction and induction can be extended to pseudo-double functors between $\mathscr{B}\mathscr{B}\mathrm{icom}^+_{\underline{\mathscr{C}}}$ and $\mathscr{B}\mathscr{B}\mathrm{icom}^+_{\underline{\mathscr{D}}}$. These pseudo-double functors are double adjoint.

Extending induction is straightforward: the image of a coalgebra homomorphism under a pseudofunctor is a coalgebra homomorphism, so there's little to check. If we pick a suitably rich class of vertical 1-morphisms when turning the 2-category $\mathscr{C} - afmod$ into a double category, then the inclusion $\iota : \mathscr{B}\mathscr{B}\mathrm{icom}_{\underline{\mathscr{C}}} \to \mathscr{C} - afmod$ is also straightforward to extend: a coalgebra homomorphism $\phi : C \to D$ naturally gives a morphism of representations $\mathbb{M}_C \to \mathbb{M}_D$. While ι is still essentially 0-surjective, vertically and horizontally 1-surjective, and fully faithful on 2-morphisms, it's not immediately clear if the internal cohom construction can be used to complete the construction of a pseudo double inverse of ι, and attempting to define the restriction of a coalgebra homomorphism directly is similarly not straightforward.

Finally, while this thesis focused on a single pseudofunctor $\mathscr{F} : \mathscr{C} \to \mathscr{D}$, we can ask how this adjunction interacts with the 3-category of fiat 2-categories. We make a series of bold conjectures.

Conjecture 8.0.4. If $\mathscr{F} : \mathscr{C} \to \mathscr{D}$, $\mathscr{G} : \mathscr{D} \to \mathscr{E}$ are (locally $\Bbbk$-linear, essentially 1-surjective) pseudofunctors between multifiat 2-categories, then

- $\mathcal{P}_{\mathscr{G}\mathscr{F}} \simeq \mathcal{P}_{\mathscr{G}} \circ \mathcal{P}_{\mathscr{F}}$;
- $\mathcal{R}_{\mathscr{G}\mathscr{F}} \simeq \mathcal{R}_{\mathscr{F}} \circ \mathcal{R}_{\mathscr{G}}$;
- $\eta^{\mathscr{G}\mathscr{F}} \simeq \mathcal{R}(\eta^{\mathscr{G}}_{\mathcal{P}}) \circ \eta^{\mathscr{F}}$;
- $\epsilon^{\mathscr{G}\mathscr{F}} \simeq \epsilon^{\mathscr{F}} \circ \mathcal{P}(\epsilon^{\mathscr{G}}_{\mathcal{R}})$;

and these congruences are coherent.

The first two of the above claims are relatively straightforward to check. The latter require more thought.

Conjecture 8.0.5. If $\Gamma : \mathscr{F} \to \mathscr{G} : \mathscr{C} \to \mathscr{D}$ is a strong transform of (locally $\Bbbk$-linear, essentially 1-surjective) pseudofunctors between multifiat 2-categories, then there exists:

- a 2-natural transform $\mathcal{P}_\Gamma : \mathcal{P}_{\mathscr{G}} \to \mathcal{P}_{\mathscr{F}}$;
- a 2-natural transform $\mathcal{R}_\Gamma : \mathcal{R}_{\mathscr{G}} \to \mathcal{R}_{\mathscr{F}}$;

These are such that the following diagrams commute, up to suitable modification:

$$\begin{tikzcd}
 & \mathcal{R}_{\mathscr{G}}\mathcal{P}_{\mathscr{G}} \arrow[dd, "\mathcal{R}_\Gamma * \mathcal{P}_\Gamma"] \\
1 \arrow[ur, "\eta^{\mathscr{G}}"] \arrow[dr, "\eta^{\mathscr{F}}"'] & \\
 & \mathcal{R}_{\mathscr{F}}\mathcal{P}_{\mathscr{F}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}_{\mathscr{G}}\mathcal{R}_{\mathscr{G}} \arrow[dd, "\mathcal{P}_\Gamma * \mathcal{R}_\Gamma"'] \arrow[dr, "\epsilon^{\mathscr{G}}"] & \\
 & 1 \\
\mathcal{P}_{\mathscr{F}}\mathcal{R}_{\mathscr{F}} \arrow[ur, "\epsilon^{\mathscr{F}}"'] &
\end{tikzcd}$$

For a coalgebra 1-morphism $C : i \to i$ of $\underline{\mathscr{C}}$, we note that $\Gamma_i\mathscr{F}(C)$ is a $\mathscr{G}(C)$-$\mathscr{F}(C)$-bicomodule 1-morphism in a straightforward way, which motivated this last conjecture.

Bibliography

[AF92] Frank Anderson and Kent Fuller. *Rings and Categories of Modules*. Springer-Verlag, 1992.

[Bur04] William Burnside. On groups of order $p^\alpha q^\beta$. *Proceedings of the London Mathematical Society*, s2-1:388–392, 1904.

[Cam19] Alexander Campbell. How strict is strictification? *Journal of Pure and Applied Algebra*, 223(7):2948–2976, July 2019.

[CR04] Joseph Chuang and Raphael Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Annals of Mathematics*, 167(1):245–298, Jul 2004.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor Categories*. AMS Mathematical Surveys and Monographs 205. American Mathematical Society, 2015.

[FH91] William Fulton and Joe Harris. *Representation theory. A first course*. Springer-Verlag, 1991.

[Fin22] Jessica Fintzen. Representations of p-adic groups, 2022.

[Fre64] Peter Freyd. *Abelian Categories: An Introduction to the Theory of Functors*. Harper & Row, 1964.

[Fro96] Ferdinand G. Frobenius. Uber gruppencharaktere. *Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften zu Berlin*, page 985–1021, 1896.

[Gor79] Daniel Gorenstein. The classification of finite simple groups. i. simple groups and local analysis. *Bulletin of the American Mathematical Society, New Series*, 1:43–199, 1979.

[GP99] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40:162–220, 1999.

[JS93] André Joyal and Ross Street. Braided tensor categories. *Advances in Mathematics*, 102(1):20–78, 1993.

[JY20] Niles Johnson and Donald Yau. *2-Dimensional Categories*. Oxford University Press, 2020.

[Lei98] Tom Leinster. *Basic Bicategories*. arXiv, 1998.

[MM11] Volodymyr Mazorchuk and Vanessa Miemietz. Cell 2-representations of finitary 2-categories. *Compositio Mathematica*, 147(5):1519–1545, July 2011.

[MM16a] Volodymyr Mazorchuk and Vanessa Miemietz. Endomorphisms of cell 2-representations. *International Mathematics Research Notices*, 2016(24):7471–7498, Dec 2016.

[MM16b] Volodymyr Mazorchuk and Vanessa Miemietz. Transitive 2-representations of finitary 2-categories. *Transactions of the American Mathematical Society*, 368(11):7623–7627, Nov 2016.

[MMM+21] Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Daniel Tubbenhauer, and Xiaoting Zhang. Finitary birepresentations of finitary bicategories. *Forum Mathematicum*, 33(5):1261–1320, Sep 2021.

[MMM+23] Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, Daniel Tubbenhauer, and Xiaoting Zhang. Simple transitive 2-representations of soergel bimodules for finite coxeter types. *Proceedings of the London Mathematical Society*, 126(5):1585–1655, feb 2023.

[MMMT19] Marco Mackaay, Volodymyr Mazorchuk, Vanessa Miemietz, and Daniel Tubbenhauer. Simple transitive 2-representations via (co)algebra 1-morphisms. *Indiana University Mathematics Journal*, 68(1):1–33, 2019.

[MP85] Saunders Maclane and Robert Paré. Coherence for bicategories and indexed categories. *Journal of Pure and Applied Algebra*, 37:59–80, 1985.

[Rou08] Raphael Rouquier. 2-kac-moody algebras. *arXiv preprint*, 2008.

[Sch05] Issai Schur. Neue begründung der theorie der gruppencharaktere. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pages 406–432, 1905.

[Tak77] Mitsuhiro Takeuchi. Morita theorems for categories of comodules. *Transactions of the American Mathematical Society*, 3(11):629–644, 1977.

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