The Derived l-Modular Unipotent Block of p-adic GL_n

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Abstract

Representations of p-adic groups have deep applications to number theoretic questions via the conjectured Langlands correspondence. While the complex representation theory is well-understood in a wide variety of cases, the l-modular theory for $l \neq p$ is still largely unsolved. For the case of GL_n, a block decomposition is known, as is a description of the irreducible representations in each block, but the full structure of the blocks remains open. Recent developments in the categorification of the Langlands correspondence have suggested that it is in fact the study of the derived category that is of central interest.

We obtain, for the derived unipotent l-modular block $D^b_{fg}(\mathsf{H}_1(G))$ of $G = \mathrm{GL}_n(F)$ for a p-adic field F, an explicit classical generator V. In the process, we also obtain an analogous result in the case of $G = \mathrm{GL}_n(k)$ for k a finite field. The proof proceeds in two parts. Firstly, we show that another representation Q, which plays a key role in the underived l-modular representation theory, is a classical generator. This requires establishing various finiteness properties for the unipotent block $\mathcal{B}_1(G)$, namely that it is Noetherian and possesses a certain subcategory $\mathcal{B}'_1(G)$ of finite global dimension. Secondly, we relate the two classical generators using the theory of irreducible l-modular representations of $\mathrm{GL}_n(k)$.

Using this classical generator, we give a (triangulated, linear) equivalence from $D^b_{fg}(\mathsf{H}_1(G))$ to the perfect complexes over a dg Schur algebra. This is a derived l-modular analogue for the result in the complex setting that the unipotent block is equivalent to modules over the Iwahori-Hecke algebra. We conclude with a composition formula for the dg Schur algebra.

Contents

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Introduction

Synopsis

We summarise the history of the study of p-adic groups, from classical problems to the Langlands program, and establish the state of the art of both the complex and l-modular representation theory, as well as modern categorical and derived approaches. We then summarise the key results of this thesis, and lay out the structure of how we will go about presenting our argument.

1.1 Background

1.1.1 Representation Theory and the Langlands Program

The ideas of reciprocity and class field theory date back hundreds of years (see Cox [2022] for a modern summary). The former was used to find prime solutions to integer equations, and provided an equivalence between the existence of nth roots modulo different primes, which could be computed effectively. Class field theory extended this to the problem of determining when certain prime ideals in a number field split over an abelian extension, and in this setting reciprocity was generalised to an isomorphism between the class group and the galois group of an (unramified) abelian extension.

In Langlands [1970], Langlands proposed a series of conjectures massively generalising reciprocity and class field theory to the non-abelian case. In particular, he proposed a reciprocity between automorphic forms of reductive groups over number fields (which generalise class groups) and representations over a hypothetical object called the Weil group, which he proposed to be closely related to the Galois group of the number field. These would allow for the solution of equations which relate to non-abelian extensions, and hence would have far-reaching applications throughout number theory.

One insightful method is to localise the conjectured correspondence for a single prime p, that is, to work over a p-adic field. This provides numerous advantages, most notably, that the Weil group in this setting has an explicit definition, and that the automorphic forms become ordinary irreducible (smooth) representations of the reductive p-adic groups, allowing for the introduction of techniques from representation theory. We refer to Kaletha [2023] for an overview.

Group representation theory has been extensively studied since the 19th century, with much of the theory focusing on finite groups of lie type and real lie groups. Both of these classes of groups provide techniques that have analogues for p-adic groups, allowing for the automorphic side of the local Langlands correspondence to be described in detail.

Historically, the theory for p-adic groups drew from the theory of real lie groups, but more recently purely algebraic methods have been developed which connect to the theory of finite groups of lie type. In particular, the representation theory of reductive finite groups forms in many ways a special case of that of p-adic groups, and proofs of results for p-adic groups often proceed by reducing to the case of finite groups.

1.1.2 Complex Representations of Finite and p-adic Groups

Let F be a non-archimedean local field, with residue field k of characteristic p and cardinality q, and let G (respectively G_f) be the F-points (respectively k-points) of a reductive algebraic group over F (respectively k).

The theory of representations of both G and G_f with complex coefficients is well-understood. The latter was classified in general by Lusztig in Lusztig [1984, 1988], and this was recently translated into a Langlands-compatible language in Imai and Jr [2025], Imai [2025]. Work is still underway to provide a p-adic analogue for the geometric methods used in the finite case (Chan and Ivanov

[2021] provides such a construction for inner forms of GL_n).

This classification can be formulated in the language of supercuspidal support (also known as Harish-Chandra series) and Hecke algebras. Namely, there is a partition of the irreducible representations of G_f according to their supercuspidal support. The summand generated by an equivalence class of such representations is then equivalent to modules over a finite Hecke algebra of some explicit Coxeter group. A key ingredient in the proof of this result is showing that the summand corresponding to a given supercuspidal support is equivalent to a summand arising from a unipotent supercuspidal support of some other group. In this form it is known to be generalisable to the p-adic setting.

By Bernstein [1984], the category of smooth complex representations of G is known to decompose into blocks, which are parameterised by inertial supercuspidal support. Furthermore, in many cases the supercuspidal representations, and hence the blocks, can be parameterised via the theory of types. This was done first for GL_n in Bushnell and Kutzko [1999], and has since been shown for inner forms of GL_n in Sécherre and Stevens [2012], for tamely ramified groups in Fintzen [2021b,a] (when p does not divide the order of the Weyl group), for classical groups in Miyauchi and Stevens [2014] (when p is odd), or for depth zero representations in Moy and Prasad [1996].

In these cases, the types give an explicit progenerator for the block, whose endomorphisms are a twisted affine Hecke algebra, and furthermore, the Hecke algebra often has an explicit description in terms of generators and relations (for example Adler et al. [2024b,a] for tamely ramified groups, Morris [1993] for blocks of depth zero, and Miyauchi and Stevens [2014] for certain cases of classical groups), and a well-understood representation theory.

These blocks are often equivalent to each other. For example, there is a reduction to depth zero blocks for tamely ramified groups in Adler et al. [2024a]. In particular, for GL_n, there is a single unipotent block $\mathcal{B}_1(G)$, namely the block containing the trivial representation $\mathbb{1}$, which has progenerator $P = \mathrm{ind}_I^G \mathbb{1}$ (I an Iwahori subgroup) and Hecke algebra $\mathsf{H}_R(n)$ extended affine of type A_{n-1}. It was shown, first in Bushnell and Kutzko [1999] and expanded on in Dat [2017], that each block of $\mathrm{GL}_n(F)$ is equivalent to some $\mathcal{B}_1(H)$, where H is a finite product of general linear groups over finite extensions of F.

1.1.3 l-modular Representation Theory

While complex representations already contain a great deal of information, they are in a sense the simplest kind of representations. Algebraically, they are split (irreducible representations are absolutely irreducible) and for finite groups they are semisimple (indecomposable representations are irreducible). Furthermore, they are amenable to geometric and analytic methods. A more complete picture of the representation theory can be obtained by generalising to other coefficient rings, but the techniques that may be used become more restrictive, and the theory becomes more complicated.

The next natural case to consider is l-modular representation theory, that is, representations over algebraically closed fields of characteristic $l \neq 0$. Representations in this setting remain split, but additional complexity can arise from l no longer being invertible (for finite groups, for example, semisimplicity can fail). The case where $l = p$ displays a very different structure to the complex case, and is beyond the scope of discussion for this thesis. We shall henceforth focus on the case where $l \neq p$.

The simplest case is when l is banal, which means that it does not divide the index of any compact open subgroup of G or G_f in any larger compact open subgroup. For G_f, this case remains semisimple, and so the representation theory is the same as the complex case. The representation theory is expected to be the same as the complex case for G also. Indeed, Bernstein's decomposition still holds (Dat et al. [2024b]). It is likely that the equivalence of the blocks with modules over Hecke algebras also holds via the arguments from the complex case (in the cases where the types are known), but the author does not know anywhere this has been written down.

When l is not necessarily banal, the situation becomes different. The construction of types in the known cases still holds and gives every supercuspidal representation (Fintzen [2022], Kurinczuk and Stevens [2020], Mínguez and Sécherre [2014]). However, while the subcategories of Bernstein's decomposition are still well-defined, in general they are not direct summands, such as for SL_n (Cui [2022]), or even disjoint, such as for Sp_8 (Dat [2018b]). Blocks are instead expected to be unions of Bernstein subcategories. These unions have been found explicitly in some cases, such as for inner forms of classical groups when $p \neq 2$ (Helm et al. [2024]).

Surprisingly, it is shown in Vignéras [1998] that, for $G = \mathrm{GL}_n(F)$, Bernstein's block decomposition still holds. Indeed, this is also known to be true for inner forms of GL_n also (Sécherre and Stevens [2016]), though we shall not consider that case here. The decomposition for G_f is also known explicitly (Fong and Srinivasan [1982]), though, like in the complex case, it is finer than the Bernstein decomposition. It thus makes sense to ask, for $\mathrm{G} = \mathrm{GL}_n$, if these blocks are related to the modules over the Hecke algebra associated to their type, and if they are equivalent to a unipotent block.

We address the latter question first. For $G = \mathrm{GL}_n(F)$, there is still a single unipotent block $\mathcal{B}_1(G)$, which contains $\mathbb{1}$, and the same equivalence of an arbitrary block to $\mathcal{B}_1(G)$ for some product of $\mathrm{GL}_m(E)$. To the author's knowledge, this has not been recorded anywhere. As such, we interrupt the introduction to provide a proof, at least over $\bar{\mathbb{F}}_l$.

Theorem 1.1.1. *Let F be a p-adic field, B a block of the category of representations of $\mathrm{GL}_n(F)$ with coefficients in $\bar{\mathbb{F}}_l$. Then B is equivalent to the principal block (that is, the block containing the trivial representation) of representations of some $\prod_i \mathrm{GL}_{m_i}(E_i)$, where E_i are degree d_i extensions of F such that $\sum_i m_i d_i = n$.*

Proof. By Chinello [2018], we may reduce to the case where B has depth zero. We wish to conclude by Dat [2018a]; however, Dat works over $\bar{\mathbb{Z}}_l$. We may think of any representation over $\bar{\mathbb{F}}_l$ as a representation over $\bar{\mathbb{Z}}_l$ via inflation, that is, taking the maximal ideal $\mathfrak{m}$ of $\bar{\mathbb{Z}}_l$ to act as zero. Furthermore, by Helm [2016], each block over $\bar{\mathbb{F}}_l$ is exactly the subcategory of a block of $\bar{\mathbb{Z}}_l$ consisting of those representations where $\mathfrak{m}$ acts as zero. Thus it suffices to show that Dat's equivalence restricts to an equivalence of the corresponding blocks over $\bar{\mathbb{F}}_l$.

To this end, it suffices to show that Dat's equivalence is $\bar{\mathbb{Z}}_l$-linear. To see this suffices, observe that, given a representation V over $\bar{\mathbb{Z}}_l$, by considering scalar multiplication by elements of $\bar{\mathbb{Z}}_l$ as endomorphisms of V, the equivalence being $\bar{\mathbb{Z}}_l$-linear would imply that $\mathfrak{m}$ acts as zero on one side of the equivalence if and only if it acts as zero on the other side. Hence the equivalence would then restrict to the blocks over $\bar{\mathbb{F}}_l$.

To see that Dat's equivalence is linear, we inspect its construction. In Theorem 4.2.2 of Dat [2018a] Dat first gives an equivalence between a depth zero block and a category of certain modules over a multi-object Hecke algebra, sending a representation V to, at each object, a space of invariant vectors. This is manifestly $\bar{\mathbb{Z}}_l$-linear. Then in Theorems 4.3.8 and 4.3.9 of the same paper Dat gives a Morita equivalence between two such categories over multi-object Hecke algebras, where one is equivalent to a principal block via the previous equivalence. But Morita equivalences are always linear. □

It seems likely to the author that one could also show directly that Dat's proof works over more general coefficient rings.

We return to the former question of whether the blocks are equivalent to modules over their Hecke algebra: the answer is negative. While we can still define the representation $P = \mathrm{ind}_I^G \mathbb{1}$ and the Hecke algebra $\mathsf{H}_R(n) = \mathrm{End}_G(P)$ from the type as in the complex case, the equivalence of $\mathcal{B}_1(G)$ with the category of modules over $\mathsf{H}_R(n)$ fails. Specifically, P fails to be a generator (and when l divides $q-1$ it also fails to be projective).

For example, consider l odd and dividing $q+1$, and $G_f = \mathrm{GL}_2(k)$. Then $\mathcal{B}_1(G)$ contains two irreducible representations, but the Hecke algebra only has one irreducible module (James [1986]). A similar argument on the subcategories of fixed central character (Mínguez and Sécherre [2014]) shows that the equivalence also fails for $G = \mathrm{GL}_2(F)$ in this case.

Despite this, for GL_n, a great deal is still known. Most notably, all the irreducible representations of G_f have been classified by Richard Dipper and Gordon James (Dipper [1985], James [1986], Dipper and James [1986]), extending the classification of Lusztig in the complex case, and a similar method describes the irreducible representations of G (Mínguez and Sécherre [2014]).

A description of the full category of representations, instead of just the irreducible representations, has however proven elusive. The state of the art can be found in Vignéras [2003], building on the result of Takeuchi [1996] for G_f. Vignéras defines for G a subcategory $\mathcal{B}'_1(G)$ of $\mathcal{B}_1(G)$ given by the representations annihilated by $\mathcal{I} = \mathrm{Ann}_{\mathsf{H}(G)} P$, where $\mathsf{H}(G)$ is the global Hecke algebra of G.

She then shows two things. Firstly, that some power of $\mathcal{I}$ annihilates $\mathcal{B}_1(G)$. This implies that $\mathcal{B}'_1(G)$ generates $\mathcal{B}_1(G)$ under extensions, and so in particular contains all irreducible representations. Secondly, she shows that $\mathcal{B}'_1(G)$ is equivalent to the category of modules over the Schur algebra $\mathsf{S}_R(n)$, an algebra closely related to $\mathsf{H}_R(n)$ but with a richer l-modular structure.

Vignéras achieves this result by giving an explicit progenerator Q of $\mathcal{B}'_1(G)$, whose endomorphisms give $\mathsf{S}_R(n)$. However, she also observes that $\mathsf{S}_R(n)$ is the endomorphisms of a much simpler representation V, whose annihilator is $\mathcal{I}$, but which is not projective or a generator in $\mathcal{B}'_1(G)$ in general. Note that, as ever, all of this holds analogously for G_f as well as for G. Thus, while we know a great deal, we do not have a complete description of $\mathcal{B}_1(G)$.

1.1.4 Derived Representation Theory

One may recall that the local Langlands correspondence concerned only irreducible representations, so at first it might not seem obvious why an understanding of the full category would be useful. However, beyond independent interest, the (local) Langlands correspondence has been categorified in Fargues and Scholze [2024] into an equivalence between certain derived categories of sheaves on stacks corresponding to p-adic groups and their Langlands parameters. In particular, for a given quasi-split G, the automorphic side of the categorical local Langlands correspondence has a semi-orthogonal decomposition into the derived categories of representations of the inner forms of the Levi subgroups of G.

Thus, for applications to the Langlands program, it is the derived category of representations that is of primary interest. In the complex case, the detailed understanding of the underived theory naturally extends to the derived setting. However, in the l-modular setting, it may be possible to provide a complete description of the derived setting using only the partial structure results known in the underived setting. This is the goal of this thesis.

1.2 Results

Recall that we consider representations over an algebraically closed field of characteristic different from p. Write $D^b_{fg}(\mathsf{H}_1(G))$ for the bounded derived category of finitely generated unipotent representations, and for a triangulated category $\mathcal{T}$ with object G write $\langle G\rangle_{\mathcal{T}}$ for the subcategory classically generated by G. The main result of this thesis is the following theorem:

Theorem 1.2.1. *For $G = \mathrm{GL}_n(F)$, and for V and Q as in the previous subsection, we have that $D^b_{fg}(\mathsf{H}_1(G)) = \langle Q\rangle_{D^b_{fg}(\mathsf{H}_1(G))} = \langle V\rangle_{D^b_{fg}(\mathsf{H}_1(G))}$.*

This allows us to describe $D^b_{fg}(\mathsf{H}_1(G))$ via the theory of dg algebras. Let $V^\bullet$ be a projective resolution of V in $\mathcal{B}_1(G)$, and write dg-End for the dg endomorphism algebra of a complex and per for the perfect complexes over a dg algebra.

Corollary 1.2.2. *There is a triangulated equivalence $D^b_{fg}(\mathsf{H}_1(G)) \simeq \mathrm{per}(\text{dg-End}_G(V^\bullet))$.*

Now, $\text{dg-End}_G(V^\bullet)$ has zeroth cohomology $\mathsf{S}_R(n)$, and so can be seen as a dg enhancement of the latter. Due to the relative simplicity of V, we can describe the composition law of $\text{dg-End}_G(V^\bullet)$ explicitly in terms of resolutions on the finite group, analogously to the composition law for $\mathsf{S}_R(n)$.

To establish that Q classically generates $D^b_{fg}(\mathsf{H}_1(G))$, we use the results of Dat [2009] to show that the categories under consideration are Noetherian, and extend a result of Cui [2015] to show a further key finiteness property, which is implicit in the literature but spelled out explicitly for the first time herein:

Lemma 1.2.3. *For any n, the Schur algebra $\mathsf{S}_R(n)$ has finite global dimension.*

To show an equivalence between the categories generated by Q and V, we use the unipotent block $\mathcal{B}_1(G_f)$ of the finite reductive quotient $G_f = \mathrm{GL}_n(k)$. It contains certain finite analogues P_f, $\mathcal{I}_f$, Q_f and V_f of P, $\mathcal{I}$, Q and V. This allows us to use the structure theory of G_f found in James [1986] to describe the composition factors of V_f and Q_f. Thus we may show a finite version of our main theorem:

Lemma 1.2.4. $D^b_{fg}(\mathsf{H}_1(G_f)) = \langle Q_f\rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f\rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$.

We then attempt to lift this equivalence into the p-adic setting. However, a barrier occurs, as Q is not a priori parahorically induced from Q_f. We prove equality by showing inclusion in each direction. To show one direction, we prove the following property of $\mathcal{I}_f$:

Lemma 1.2.5. *$\mathcal{I}_f \subseteq \mathcal{I}$.*

We show this by giving an abstract argument that works for any reductive group.

To show the other direction, we use the structure theory of $\mathcal{I}_f$ given in Takeuchi [1996], Dipper and James [1989] to show the following:

Theorem 1.2.6. *Q_f is a direct sum of submodules of P_f.*

We also give many examples of these theorems, such as in the case of cyclic defect group, where the results are more readily apparent. It would be interesting to see if a direct proof of the theorem could be achieved via the classification of Mínguez and Sécherre [2014].

1.3 Organisation of the Thesis

Section 2 of this thesis recalls the relevant background theory of finite and p-adic groups and their representations. In particular, we recall notions of parabolic and parahoric subgroups and induction and restriction along them, as well as blocks, projective covers, affine cellular algebras, dg algebras, and derived categories.

Section 3 features our arguments for the finite group G_f. We recall the structure theory of Dipper and James, and use it to prove our results about V_f and Q_f. Section 4 recalls the results of Vignéras, and then uses the work of Dat and Cui to show that the category of smooth representations is Noetherian and that the Schur algebra has finite global dimension.

In section 5, we combine the results of the previous sections, together with our result on the relationship between the annihilators, to show our main theorem. Finally, we investigate the structure of the dg algebra $\text{dg-End}_G(V^\bullet)$, and provide a composition formula using results on double cosets of extended affine Weyl groups.

2

Groups, Algebras, and their Representation Theory

Synopsis

We review the general theory of representations of algebras, including the relations between projective representations, idempotents, and blocks, as well as the operations of induction and restriction. We also explore certain particular types of algebras, namely the group algebras of reductive, topological, and p-adic groups, which have the further operations of parabolic and parahoric induction. Finally, we review the theory of complexes and dg algebras, as well as the definition of an affine cellular algebra.

2.1 Notation and Conventions

We shall throughout take R to be a commutative ring, and A a (not necessarily commutative or unital, but associative) R-algebra. In particular, we do not assume that subalgebras have the same unit, even when they are unital.

All modules and representations will be left modules and left representations. We will assume that all A-modules M are nondegenerate, that is, that $AM = M$. We write $\mathrm{Mod}(A)$ for the category of (nondegenerate) modules over A. When $A = R[G]$ is a group algebra, we will just write G instead of $R[G]$ when it appears as a decoration, such as $\mathrm{Mod}(G)$ and Hom_G instead of $\mathrm{Mod}(R[G])$ and $\mathrm{Hom}_{R[G]}$ respectively.

Whenever G is a topological group, we will assume that all representations of G are smooth (see Section 2.7 for the definition). In particular, in this case $\mathrm{Mod}(G)$ shall denote the category of smooth representations.

We write the trivial representation of a group G over R as $\mathbb{1}_G$, or simply $\mathbb{1}$ when G is clear from context.

We write the indicator function of $S \subseteq X$, that is, the function $X \to R$ which is 1 on S and 0 on $X \backslash S$, as 1_S.

We use the term p-adic field to denote any non-archimedean local field F with residual characteristic P, without assuming F has characteristic zero. We endow p-adic fields with the topology arising from their local field structure.

2.2 Projective Representations, Blocks, Idempotents, and Endomorphisms

Suppose that that A is idempotent (that is, that $A^2 = A$).

Definition 2.2.1. We say that two indecomposable A-modules are in the same block if they are equivalent under the equivalence relation generated by $M \sim N$ if $\mathrm{Hom}_A(M, N) \neq 0$.

The blocks of $\mathrm{Mod}(A)$ are the closure under direct sums of the above equivalence classes.

$\mathrm{Mod}(A)$ has a block decomposition if, writing C_i for the block of $\mathrm{Mod}(A)$, we have that each $M \in \mathrm{Mod}(A)$ decomposes as $M = \bigoplus_i M_i$ for $M_i \in C_i$.

Observe in particular that if M and N are in different blocks then $\mathrm{Hom}_A(M, N) = 0$. When a block decomposition exists, we may then decompose A as a direct sum of left ideals $\bigoplus_i A_i$ for $A_i \in C_i$. We call the ideals A_i the block algebras.

For $a \in A$, we have module maps $A_i \to A = \bigoplus_i A_i$ given by $a' \mapsto a'a$. By composing with the projection maps and using that $\mathrm{Hom}_A(A_i, A_j) = 0$ for $i \neq j$, we can see that this map must have image contained in A_i. Thus the A_i are also right ideals, and so in particular the A_i are (non-unital) algebras themselves, with $A_iA_j = 0$ for $i \neq j$. Thus, by taking the summands of A^2 and A in each C_i and using that $A^2 = A$, we must also have $A_i^2 = A_i$, that is, that the A_i are idempotent.

Given an A-module M, write $M = \bigoplus_i M_i$ for $M_i \in C_i$. Then for $m \in M_i$, the map $A_j \to M_i$ given by $a \mapsto am$ must be zero for $i \neq j$. Thus $A_jM_i = 0$ for $i \neq j$. Thus, by taking summands of AM and M in each C_i and using that $AM = M$, we must also have that $A_iM_i = M_i$, and hence also that $A_iM = M_i$.

Thus, the C_i are exactly the full subcategories of modules M such that $A_iM = M$, and $C_i \cong \mathrm{Mod}(A_i)$ by restricting the A-action on such M to an A_i-action.

We now restrict to the case where R is an algebraically closed field and A is unital and finite dimensional over R, and follow Assem et al. [2006].

In this case, we can decompose the identity element $1 \in A$ as a sum $1 = \sum_i e_i$ for $e_i \in A_i$. Then e_i is the identity element of A_i, and the e_i form a set of primitive orthogonal central idempotents, such that $e_iA = A_i$ and $e_iM = M_i$. We call e_i the central idempotent of the block C_i.

A has a module decomposition into indecomposable modules $\bigoplus_{k=1}^n P_k$, and any two such module decompositions must contain the same modules up to isomorphism with the same multiplicities. In particular, as indecomposable modules must lie in a single block, we have $A_i = \bigoplus_{P_k \in \mathrm{Mod}(A_i)} P_k$. Furthermore, the P_k are projective, and every projective indecomposable module is isomorphic to some P_k. There is a unique idempotent e_k for each k such that $P_k = Ae_k$, and conversely any set of primitive orthogonal idempotents e_k with $\sum_k e_k = 1$ gives a decomposition $A = \bigoplus_{k=1}^n Ae_k$ with the $P_k = Ae_k$ projective and indecomposable. If P_k is in the block C_i then $e_k \in A_i$, and conversely. We call e_k the idempotent generating P_k.

By considering the left action of $1 = \sum_{l=1}^n e_l$ on $A = \bigoplus_{k=1}^n Ae_k$, we may further decompose $A = \bigoplus_{k,l=1}^n e_lAe_k$. We have $\phi : e_lAe_k \xrightarrow{\sim} \mathrm{Hom}_A(Ae_l, Ae_k)$ by sending an element to its right multiplication action. Thus, we may evaluate the left action of $a \in A$ on $p \in P_j$ by as $\sum_{k,l,m=1}^n \phi^{-1}(\phi(a_{kl})\phi(p_m))$ where $a = \sum_{k,l=1}^n a_{kl}$ for $a_{kl} \in e_lAe_k$ and $p = \sum_{m=1}^n p_m$ for $p_m \in e_mAe_j$.

P_k possesses a unique maximal submodule, and hence a unique simple quotient L_k. All simple modules are the quotient of some P_k, and P_k is the projective cover of L_k.

2.3 l-Modular Reduction

Here we follow Linckelmann [2018].

Definition 2.3.1. An l-modular system is a complete discrete valuation ring $\mathcal{O}_K$ with residue field R algebraically closed of characteristic l and fraction field K of characteristic zero.

Suppose that A is the extension $R \otimes_{\mathcal{O}_K} \mathcal{O}_B$ of some unital $\mathcal{O}_K$-algebra $\mathcal{O}_B$ that is finite free as an $\mathcal{O}_K$-module. Let $B = K \otimes \mathcal{O}_B$, and suppose that B is split semisimple, that is, that every simple B-module M is projective and has $\mathrm{End}_B(M) = K$.

For a simple B-module M, there exists a $\mathcal{O}_B$-module $\mathcal{O}_M$ with $M = K \otimes_{\mathcal{O}_K} \mathcal{O}_M$. Given such an $\mathcal{O}_M$, we call $\bar{M} := R \otimes_{\mathcal{O}_K} \mathcal{O}_M$ an l-modular reduction of M. Note that l-modular reductions are not in general unique. Given also a simple A-module N, call the decomposition number d_{MN} of N in M the multiplicity of N in some (hence any) composition series for $\bar{M}$. This is independent of the choice of $\mathcal{O}_M$.

The decomposition matrix of $\mathcal{O}_B$ is the matrix with rows indexed by simple B-modules M, columns by simple A-modules N, and (M, N)-entry d_{MN}. Where $\mathcal{O}_B$ is clear, we shall often abuse notation and simply speak of the decomposition matrix of A.

Conversely, given such M and N, there exists a primitive idempotent e of $\mathcal{O}_B$ such that Ae is the projective cover of N. Then M has multiplicity d_{MN} in any composition series of Be. Thus, N' occurs in any composition series for Ae with multiplicity $\sum_M d_{MN'} d_{MN}$.

2.4 Induction and Restriction

We again follow Linckelmann [2018]. Suppose A and A' are both R-algebras with $f : A \to A'$ a R-algebra homomorphism.

Definition 2.4.1. Restriction of scalars $\mathrm{res}_A^{A'} : \mathrm{Mod}(A') \to \mathrm{Mod}(A)$ is the functor sending $M \in \mathrm{Mod}(A')$ to the A-module with the same underlying R-module as M and A-action $(a, m) \mapsto f(a)m$ where the right-hand side is the A'-action of M.

Induction $\mathrm{ind}_A^{A'} : \mathrm{Mod}(A) \to \mathrm{Mod}(A')$ is the functor $A' \otimes_A -$. Note that in the case of topological groups, we shall instead use this notation for compact induction (see Section 2.7).

Both functors are R-linear and transitive, that is, functorial in f up to coherent natural isomorphism.

Now suppose further that A and A' are unital.

By Frobenius Reciprocity, $\mathrm{ind}_A^{A'}$ is left adjoint to $\mathrm{res}_A^{A'}$. Furthermore, $\mathrm{res}_A^{A'}$ is exact.

Proposition 2.4.2. *Suppose* f *has a right adjoint* g*. If* g *is exact then* f *preserves projective modules.*

Hence, $\mathrm{ind}_A^{A'}$ always preserves projective modules.

We adopt the standard convention that ind_H^G and res_H^G are reserved for when $f : R[H] \to R[G]$ is induced by an inclusion of groups $H \hookrightarrow G$, and that we instead call them co-inv_H^G and infl_H^G respectively when $f : R[G] \to R[H]$ is induced by a quotient of groups $G \twoheadrightarrow H$ with kernel K.

In the case of an $H \hookrightarrow G$, as $R[G]$ is free over $R[H]$, it follows that ind_H^G is also exact. In the particular case that H is a finite index subgroup of G, Frobenius reciprocity also gives that ind_H^G is right-adjoint to res_H^G, and so res_H^G preserves projective modules.

We may also identify $\mathrm{ind}_H^G(M)$ with the H-equivariant functions $G \to M$ whose support is a finite union of left-H-cosets, where we give G a left-H-action via $(h, x) \mapsto xh^{-1}$ and a right-G-action via $(x, g) \mapsto g^{-1}x$, and the left-G-action on $\mathrm{ind}_H^G(M)$ is induced by the right-G-action on G. The element $g \otimes m \in \mathrm{ind}_H^G(M)$ is identified with the unique map with support gH such that $g \mapsto m$. In particular, $\mathrm{ind}_H^G(\mathbb{1})$ has basis 1_{xH} for $x \in G/H$, where $g \in G$ sends 1_{xH} to 1_{gxH}.

If K and H are both subgroups of G, then the Mackey Decomposition gives an isomorphism

$$\mathrm{res}^G_H \mathrm{ind}^G_K M \cong \bigoplus_{g \in HgK} \mathrm{ind}^H_{H \cap gKg^{-1}} \mathrm{res}^{gKg^{-1}}_{H \cap gKG^{-1}} M^g$$

where M^g is the representation of gKg^{-1} with the same underlying R-module as M and action $(x, m) \mapsto (g^{-1}xg)m$, where the right-hand side is the K-action of M.

In the case of a quotient $G \twoheadrightarrow H$ with kernel K, we may identify $\text{co-inv}^G_H(M)$ with the quotient M/KM, where KM denotes the R-module spanned by km for $k \in K$ and $m \in M$. In particular, $gK \otimes m$ is identified with $gm + KM$. If $|K|$ is finite and invertible in R, then M/KM can be identified with the R-submodule of K-invariant elements of M via $m + KM \mapsto \frac{1}{|K|}\sum_{k \in K} km$. Thus, in this case, co-inv^G_H is also right adjoint to infl^G_H, and hence is exact, and so infl^G_H preserves projective modules. In particular, if $|G|$ invertible in R, then by inflating from the trivial group we get that $\mathbb{1}$ is projective.

By the third isomorphism theorem, if G has normal subgroup K, and L is a subgroup of G containing K, then there is a natural isomorphism $\mathrm{ind}^H_L \mathrm{infl}^L_{L/K} \cong \mathrm{infl}^H_{H/K} \mathrm{ind}^{H/K}_{L/K}$.

2.5 Reductive Groups

We follow Milne [2017]. Let F be a field, and G be an affine algebraic group over F, that is, a group object in the category of affine schemes. For a field extension E of F, write the extension of scalars of G to E as G_E. Write F^{alg} for the algebraic closure of F.

Definition 2.5.1. We say that G is unipotent if may be obtained by a finite number of extensions from subgroups of the additive algebraic group F over F.

The unipotent radical of G is the largest smooth connected unipotent normal algebraic subgroup over F. This is well-defined.

We say that G is reductive if it is smooth, connected, and the unipotent radical of $\mathrm{G}_{F^{\mathsf{alg}}}$ is trivial.

Henceforth, we assume that G is reductive.

Definition 2.5.2. A split torus in G is any algebraic subgroup S isomorphic to $\mathrm{F}^{\times n}$, where $\mathrm{F}^{\times}$ is the multiplicative algebraic group over F.

Let $N_{\mathrm{G}}(-)$ and $Z_{\mathrm{G}}(-)$ denote respectively the normaliser and the centraliser in G.

Let S be a maximal split torus in G. Then the (finite) Weyl group of G is $W_f := N_{\mathrm{G}}(\mathrm{S})/Z_{\mathrm{G}}(\mathrm{S})$. This does not depend on S.

A parabolic subgroup of G is a smooth algebraic subgroup C over F such that the quotient G/C (which is always a quasi-projective scheme over F) is projective.

The quotient of C by its unipotent radical U is a reductive group over F. We call any splitting M of this quotient a Levi subgroup of C. Every parabolic subgroup has a Levi subgroup.

The Levi subgroups of parabolic subgroups C of G are precisely the subgroups of the form $Z_{\mathrm{G}}(\mathrm{S})$ for S a split torus in G. Furthermore, S is maximal exactly when C is minimal.

A parabolic subgroup C of G containing a maximal split torus S has a unique Levi subgroup containing S, which is $Z_{\mathrm{G}}(\mathrm{S}_0)$ for some split torus $\mathrm{S}_0 \subseteq \mathrm{S}$.

Given a parabolic C with Levi M, there exists a unique parabolic $\bar{\mathrm{C}}$ with Levi M whose intersection with C is exactly M. We call $\bar{\mathrm{C}}$ the opposite parabolic of C with respect to M.

Let C be a minimal parabolic subgroup of G with Levi subgroup $\mathrm{M} = Z_{\mathrm{G}}(\mathrm{S})$ for some maximal split torus S in G. The Bruhat decomposition says that the map $W_f = N_{\mathrm{G}}(\mathrm{S})/Z_{\mathrm{G}}(\mathrm{S}) \to C\backslash G/C$ given by $w \mapsto CwC$ is a bijection.

Definition 2.5.3. Let S be a maximal split torus in G. We write $X(\mathrm{S})$ for the set of morphisms $\mathrm{S} \to \mathrm{F}^{\times}$ of algebraic groups over F.

Write F^{sep} for the separable closure of F.

Suppose G is split, that is, that we can choose S so that $\mathrm{S}_{F^{\mathsf{sep}}}$ is a maximal split torus in $\mathrm{G}_{F^{\mathsf{sep}}}$. For $\alpha \in X(\mathrm{S})$, a root group U_{α} of G with respect to S is an algebraic subgroup of G over F such that

- S normalises U_α,
- there is an isomorphism of algebraic groups $u_\alpha : \mathrm{U}_\alpha \to \mathrm{F}$, and
- under u_α the conjugation action of $s \in \mathrm{S}$ is sent to multiplication by $\alpha(s)$.

Note that U_α is unique when it exists, but there can be many possible choices for u_α. We call any α which has a root group a root.

Suppose now G is not necessarily split. Then for any maximal split torus S, there exists an algebraic subgroup T of G containing S such that $\mathrm{T}_{F^{\mathsf{sep}}}$ is a maximal split torus in $\mathrm{G}_{F^{\mathsf{sep}}}$. Fix such a S and T. A root of G with respect to S is any nonzero $\alpha \in X(\mathrm{S})$ that is the restriction to S of a root β of $\mathrm{G}_{F^{\mathsf{sep}}}$ with respect to $\mathrm{T}_{F^{\mathsf{sep}}}$.

For a root α, the group generated by the U_β for all β that restrict to a α is the extension of scalars to F^{sep} of a unique algebraic subgroup U_α of G. We call U_α the root group of α.

Neither the roots nor the root groups depend on the choice of T, and they agree with the previous definition in the case that G is split.

A base for the roots is a set Δ of roots such that

1. if $\alpha, \beta \in \Delta$ then $\alpha + \beta \notin \Delta$, and
2. every root is a $\mathbb{Z}$-linear combination of roots in Δ with all coefficients of the same sign.

We call roots in Δ simple with respect to Δ. We also call a root positive with respect to Δ if the coefficients in (2) above are all non-negative.

Let C be a minimal parabolic subgroup of G with Levi subgroup $\mathrm{M} = Z_{\mathrm{G}}(\mathrm{S})$. Then there exists a unique base Δ for the roots with respect to S such that the unipotent radical U of C is generated by root groups U_α for α positive with respect to Δ. Conversely, the positive roots with respect to some base Δ for the roots with respect to S will generate the U of some unique minimal parabolic subgroup of G with Levi subgroup $Z_{\mathrm{G}}(\mathrm{S})$.

2.6 Parabolic Induction for Finite Groups

We now follow Dipper and Fleischmann [1992] and Hiss [1993]. We now suppose that F is a finite field, and continue to let G be an affine algebraic group over F.

Fix a parabolic subgroup C with Levi subgroup M. Write $G = \mathrm{G}(F)$ for the F-points of G, and similarly for C, M, and U.

Definition 2.6.1. Parabolic induction from M to G along C is the functor $\mathrm{i}_{M,C}^{G} = \mathrm{ind}_{C}^{G}\mathrm{infl}_{M}^{C} : \mathrm{Mod}_R(M) \to \mathrm{Mod}(G)$.

Parabolic restriction from G to M along C is the functor $\mathrm{r}_{M,C}^{G} = \text{co-inv}_{M}^{C}\mathrm{res}_{C}^{G} : \mathrm{Mod}(G) \to \mathrm{Mod}_R(M)$.

$\mathrm{i}_{M,C}^{G}$ is exact. As C has finite index in G, we have that $\mathrm{r}_{M,C}^{G}$ is left adjoint to $\mathrm{i}_{M,C}^{G}$, so $\mathrm{r}_{M,C}^{G}$ preserves projective modules.

Furthermore, if M has index in C which is invertible in R, then $\mathrm{r}_{M,C}^{G}$ is also right adjoint to $\mathrm{i}_{M,C}^{G}$, so $\mathrm{r}_{M,C}^{G}$ is exact and $\mathrm{i}_{M,C}^{G}$ preserves projective modules. As the index of M in C is the order of U, which is a power of the characteristic p of F, this holds precisely when p is invertible in R.

Definition 2.6.2. We call an irreducible G-representation V cuspidal if $\mathrm{Hom}_G(\mathrm{i}_{M,C}^{G}W, V) = 0$ for all irreducible M-representations W.

The cuspidal support of an irreducible G-representation V is a Levi subgroup M and an irreducible cuspidal M-representation W such that $\mathrm{Hom}_G(\mathrm{i}_{M,C}^{G}W, V) \neq 0$.

Every irreducible G-representation V has a cuspidal support (L, W), and furthermore it is unique up to G-conjugacy.

2.7 Topological Groups and Smooth Representations

We follow Vignéras [1996]. We now let G be a (Hausdorff) topological group, and furthermore assume that it is locally compact and totally disconnected, that is, that it has a neighbourhood basis of the identity given by compact open subgroups.

Definition 2.7.1. A representation V of G is smooth if every element $v \in V$ is fixed by an open subgroup of G.

A Haar measure on G is a nonzero R-valued map μ on the compact open subgroups of G that is

- finitely additive, that is, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and
- left-G-invariant, that is, $\mu(gA) = \mu(A)$ for all $g \in G$.

A Haar measure always exists in the case where we can take the neighbourhood base of the identity to have pro-order invertible in R, and in this case any compact open subgroup of invertible pro-order will have nonzero Haar measure. Haar measures are unique up to multiplication by elements of $R^\times$.

Definition 2.7.2. Fix a Haar measure μ on G. The global Hecke algebra $\mathsf{H}(G)$ of G is the (non-unital) algebra whose underlying R-module is the space of compactly supported maps $f : G \to R$ with that are right-and-left-U-invariant for some compact open subgroup U. Composition is given by convolution:

$$(ff')(x) = \mu(U) \sum_{g \in \mathrm{supp}(f)/U} f(g)f'(g^{-1}x).$$

where U is any compact open subgroup fixing f and f' on the right and left respectively. This does not depend on the choice of U.

$\mathsf{H}(G)$ is idempotent. Note that when G is discrete, $\mathsf{H}(G)$ is isomorphic to $R[G]$ via $1_g \mapsto \mu(1)g$.

Let $V \in \mathrm{Mod}(G)$, let $v \in V$ and $f \in \mathsf{H}(G)$, and let U be a compact open subgroup of K^1 fixing v and fixing f on the right. We define an action of $\mathsf{H}(G)$ on V by

$$fv = \mu(U) \sum_{k \in \mathrm{supp}(f)/U} f(k)kv.$$

Then $\mathrm{Mod}(G)$ is isomorphic to $\mathrm{Mod}(\mathsf{H}(G))$ by mapping $V \in \mathrm{Mod}(G)$ to the same underlying R-module equipped with the above action.

Definition 2.7.3. Let I be a compact open subgroup of G. The Hecke algebra of G with respect to I is the R-algebra $\mathsf{H}_R(G, I) = \mathrm{End}(\mathrm{ind}_I^G \mathbb{1})^{\mathsf{op}}$.

By Frobenius reciprocity, $\mathsf{H}_R(G,I)$ is isomorphic to the algebra of the left-and-right-I-invariant functions $G \to R$ with multiplication given by convolution:

$$(f \star f')(g) = \sum_{x \in G/I} f(x) f'(x^{-1}g).$$

The indicator functions of I-double cosets give a R-basis of $\mathsf{H}_R(G,I)$, and it is unital with identity 1_I.

If I is normal, then $\mathsf{H}_R(G,I)$ is isomorphic to $R[G/I]$ via $1_{gI} \mapsto g$. When $\mu(I) \neq 0$, $\mathsf{H}_R(G,I)$ is isomorphic to the subalgebra $1_I\mathsf{H}(G)1_I$ of $\mathsf{H}(G)$ via $f \mapsto \frac{1}{I}f$.

A closed subgroup H of G is also locally compact and totally disconnected. The functor res_H^G sends smooth representations to smooth representations. If H is open, ind_H^G also sends smooth representations to smooth representations.

Definition 2.7.4. The smooth part of a representation V of G is the subset of all $v \in V$ that is fixed by an open subgroup of G. It is a smooth subrepresentation.

For H a closed subgroup of a locally compact and totally disconnected topological group G, compact induction is the functor $\mathrm{ind}_H^G : \mathrm{Mod}(H) \to \mathrm{Mod}(G)$ sending M to the smooth part of the space of H-equivariant maps $f : G \to M$ whose support is contained in KH for K compact.

Compact induction is exact and preserves smooth modules, and agrees with induction whenever H is open in G. Furthermore, if G/H is compact, then compact induction is right adjoint to res_H^G, and so res_H^G preserves projective modules.

When H is an open subgroup of G, we can pick a Haar measure on H that is the restriction of the Haar measure on G. We then have an inclusion of global Hecke algebras $\mathsf{H}(H) \to \mathsf{H}(G)$ by considering $\mathsf{H}(H)$ as the subset of functions whose support is contained in H. In this case, $\mathrm{res}_{\mathsf{H}(H)}^{\mathsf{H}(G)} = \mathrm{res}_H^G$ and $\mathrm{ind}_{\mathsf{H}(H)}^{\mathsf{H}(G)} = \mathrm{ind}_H^G$.

Similarly, if K is a closed normal subgroup, the quotient $H = G/K$ is locally compact and totally disconnected. The functors infl_H^G and co-inv_H^G both send smooth representations to smooth representations. If K is open and compact with $\mu(K) \neq 0$, then $\text{co-inv}_H^G M$ can be identified with the R-submodule of K-invariant elements of M via $m + KM \mapsto \frac{1}{|K:K'|}\sum_{k \in K/K'} km$, where K' is an open compact subgroup of K fixing m. Thus, in this case, co-inv_H^G is also right adjoint to infl_H^G, and hence is exact, and so infl_H^G preserves projective modules.

If K is open, compact, and normal in G with $\mu(K) \neq 0$, μ also gives a Haar measure on K. Then $\mathsf{H}(H) \cong R[H] \cong \mathsf{H}_R(G, K) \cong 1_K \mathsf{H}(G) 1_K$ via $f \mapsto \mu(K) f \mapsto \mu(K) f \mapsto f$. In particular, we have a map $\mathsf{H}(G) \mapsto \mathsf{H}(H)$ given by $f \mapsto \frac{1}{\mu(K)^2} 1_K f 1_K$, and then $\text{ind}_{\mathsf{H}(G)}^{\mathsf{H}(H)} = \text{co-inv}_H^G$ and $\text{res}_{\mathsf{H}(G)}^{\mathsf{H}(H)} = \text{infl}_H^G$.

2.8 Parahoric Subgroups

We follow Kaletha and Prasad [2023]. We now take $G = \mathrm{G}(F)$, for F a p-adic field with valuation ν and residue field k, and G a reductive group over F. Let S be the F-points of a maximal split torus S of G.

Suppose that G is split, that is, that $\mathrm{S}_{F^{\mathsf{sep}}}$ is a maximal split torus in $\mathrm{G}_{F^{\mathsf{sep}}}$. In particular, this is true for GL_n. This assumption is only for simplicity, and there is a definition for the compact torus and parahoric subgroups for any reductive group.

Definition 2.8.1. The compact torus I_0 as the intersection of the kernels of all $\nu\lambda$ for $\lambda \in X(\mathrm{S})$. Note that this is an abstract subgroup of S, not an algebraic group over F.

Write $Y(\mathrm{S})$ for the morphisms $\mathrm{F}^\times \to \mathrm{S}$ of algebraic groups over F.

There is a perfect pairing $X(\mathrm{S}) \times Y(\mathrm{S}) \to \mathbb{Z}$ given by $\langle \lambda, \phi \rangle = n$ where $\lambda\phi = (-)^n : \mathrm{F}^\times \to \mathrm{F}^\times$.

Let $v \in \mathbb{R} \otimes_{\mathbb{Z}} Y(\mathrm{S})$. Then the perfect pairing $X(\mathrm{S}) \times Y(\mathrm{S}) \to \mathbb{Z}$ extends to a pairing $X(\mathrm{S}) \times \mathbb{R} \otimes_{\mathbb{Z}} Y(\mathrm{S}) \to \mathbb{R}$. Thus we may define $\alpha(v) \in \mathbb{R}$ for any root α of G with respect to S.

Given such a root α, there exist unique $u', u'' \in U_{-\alpha}$ such that $s_\alpha := u' u_\alpha^{-1}(1) u'' \in N_{\mathrm{G}}(\mathrm{S})(F)$. For all roots α and β, we have that $s_\alpha u_\beta^{-1}(1) s_\alpha^{-1} \in U_\gamma$ for some root γ. A weak Chevalley system is a choice for the u_α such that the above element is equal to $u_\gamma^{-1}(\pm 1)$ for some choice of sign. Note that for a weak Chevalley system, it is in fact the case that $u' = u'' = u_{-\alpha}^{-1}(\pm 1)$ for some choice of sign.

Definition 2.8.2. Let $v \in \mathbb{R} \otimes_{\mathbb{Z}} Y(S)$ and let (u_α) be a weak Chevalley system. Define the subgroup $U_{\alpha,0}$ of U_α as the preimage of $[-\alpha(v), \infty]$ under νu_α. Again, note that this is an abstract subgroup of U_α, not an algebraic subgroup over F.

A parahoric subgroup of G is the group J generated by I_0 and the $U_{\alpha,0}$ for all roots α, for some choice of v and (u_α).

A minimal parahoric subgroup is called an Iwahori subgroup.

Parahoric subgroups J are open and compact. Furthermore, there exists a unique group scheme $\mathcal{J}$ over $\mathcal{O}$ such that $\mathcal{J}_F = \mathrm{G}$ and $\mathcal{J}(\mathcal{O}) = J$, and which retains both of these properties over any unramified extension of F. Then $\mathcal{J}_k$ is a smooth connected algebraic group over k. We define the pro-p radical J^1 of J to be the preimage under $J = \mathcal{J}(\mathcal{O}) \to \mathcal{J}(k) = \mathcal{J}_k(k)$ of the k-points of the unipotent radical of $\mathcal{J}_k$. Then J^1 is the maximal normal open pro-p subgroup of J. The quotient of $\mathcal{J}_k$ by its unipotent radical is reductive, and we call it the reductive quotient of J. Its k-points are in bijection with J/J^1 in the obvious way.

All split groups are unramified, that is, they are the extension of scalars to F of a (not necessarily unique) reductive group scheme over $\mathcal{O}$. If we fix G some such choice of reductive group scheme over $\mathcal{O}$, then G is in fact the group scheme of a maximal parahoric of G.

Definition 2.8.3. The Iwahori-Weyl group of G is the group $W = N_{\mathrm{G}}(\mathrm{S})(F)/I_0$.

If I is an Iwahori subgroup of G containing I_0, the Iwahori decomposition says that the map $W \to I \backslash G/I$ given by $w \to IwI$ is a bijection.

2.9 Parabolic and Parahoric Induction for p-adic Groups

We continue to follow Vignéras [1996]. We now take $G = \mathrm{G}(F)$ for F a p-adic field and G a reductive group over F. Then G is locally compact and totally disconnected, and has a neighbourhood basis of the identity of pro-p subgroups.

Fix a parabolic subgroup C with Levi subgroup M. Write $G = \mathrm{G}(F)$ for the F-points of G, and similarly for C, M, and U. Then C, U, and M are closed subgroups of G.

Definition 2.9.1. Parabolic induction from M to G along C is the functor $\mathrm{i}^G_{M,C} = \mathrm{ind}^G_C \mathrm{infl}^C_M : \mathrm{Mod}_R(M) \to \mathrm{Mod}(G)$. Note that we are using compact induction.

Parabolic restriction from G to M along C is the functor $\mathrm{r}^G_{M,C} = \text{co-inv}^C_M \mathrm{res}^G_C : \mathrm{Mod}(G) \to \mathrm{Mod}_R(M)$.

Both functors respect smooth representations, and $\mathrm{i}^G_{M,C}$ is exact. Furthermore, G/C is compact, and so $\mathrm{r}^G_{M,C}$ is left adjoint to $\mathrm{i}^G_{M,C}$, and $\mathrm{r}^G_{M,C}$ preserves projectives.

Now let K be a parahoric subgroup of G with pro-p radical K^1, and write $G_f = K/K^1$. Recall that both K and K^1 are open and compact.

Definition 2.9.2. Parahoric induction from G_f to G along K is the functor $\mathrm{I}^G_{G_f,K} = \mathrm{ind}^G_K \mathrm{infl}^K_{G_f}$.

Parahoric restriction from G to G_f along K is the functor $\mathrm{R}^G_{G_f,K} = \text{co-inv}^K_{G_f} \mathrm{res}^G_K$.

Both functors preserve smooth representations. $\mathrm{I}^G_{G_f,K}$ is exact and left adjoint to $\mathrm{R}^G_{G_f,K}$.

When p is invertible in R we have that G has a Haar measure μ such that $\mu(K^1) \neq 0$. In this case $\mathrm{R}^G_{G_f,K}$ is exact, and so $\mathrm{I}^G_{G_f,K}$ preserves projectives.

Viewing smooth representations of G_f, K and G as modules over $\mathsf{H}(G_f)$, $\mathsf{H}(K)$ and $\mathsf{H}(G)$ respectively, and choosing Haar measures as in the previous section, we have $\mathrm{I}^G_{G_f,K} = \mathsf{H}(G) \otimes_{\mathsf{H}(K)} -$ and $\mathrm{R}^G_{G_f,K} = \mathsf{H}(G_f) \otimes_{\mathsf{H}(K)} -$.

2.10 Complexes, Derived Categories, and dg Algebras

Here we follow Keller [2006].

Definition 2.10.1. A complex over A is a $\mathbb{Z}$-graded A-module $M^\bullet$ together with a graded module homomorphism d' of degree 1, the differential, such that $d'^2 = 0$. The shift operator $[1]$ on complexes $(M^\bullet, d')$ is given by $M[1]^n = M^{n+1}$ and $d'[1] = -d'$ (note the $-$ sign). The cohomology of $M^\bullet$ is the graded A-module $H^\bullet(M^\bullet) = \ker(d')/\operatorname{im}(d')$.

Let $(M^\bullet, d')$ and $(N^\bullet, d'')$ be A-complexes. The R-complex of dg morphisms $\text{dg-Hom}_A(M^\bullet, N^\bullet)$ has in degree n the graded A-module homomorphisms $f : M^\bullet \to N^\bullet$ of degree n, with differential

$$df := d''f - (-1)^n f d'$$

for all $f : M^\bullet \to N^\bullet$ of degree n.

A morphism of complexes is a dg morphism f of degree 0 that lies in $\ker(d)$, that is, such that $d''f = fd'$.

A homotopy of morphisms of complexes $f, g : M^\bullet \to N^\bullet$ is a dg morphism h of degree -1 such that $dh = f - g$, that is, such that $f - g = d''h - hd'$. If such an h exists we say f and g are homotopic. This gives an equivalence relation on morphisms of complexes.

Write $K(A)(M^\bullet, N^\bullet)$ for the homotopy equivalence classes of morphisms of complexes $f : M \to N$. We have that $H^n(\text{dg-Hom}_A(M^\bullet, N^\bullet)) = K(A)(M^\bullet, N^\bullet[n])$.

A dg algebra over R is an R-complex (B, d) whose underlying graded R-module is a graded algebra over R, satisfying the graded Leibniz rule

$$d(fg) = d(f)g + (-1)^n f d(g)$$

for all $f \in B$ of degree n and $g \in B$.

We think of an ordinary (ie non-dg) algebra as a dg algebra with all elements having degree 0.

The dg endomorphism algebra $\text{dg-End}_A(M^\bullet)$ is the dg algebra whose complex is $\text{dg-Hom}_A(M^\bullet, M^\bullet)$, with multiplication given by componentwise composition.

A dg module over B is an R-complex $(M^\bullet, d')$ with a graded module action of B of degree 0, extending the R-action, such that

$$d'(fv) = (df)v + (-1)^n f(d'v)$$

for all $f \in B$ of degree n and $v \in M^\bullet$.

Let $(M^\bullet, d')$ and $(N^\bullet, d'')$ be dg modules. The complex of dg morphisms of dg modules $\text{dg-Hom}_B(M^\bullet, N^\bullet)$ is the subcomplex of $\text{dg-Hom}_R(M^\bullet, N^\bullet)$ consisting of those dg morphisms that are also B-module homomorphisms. When $B = A$ is an ordinary algebra, the two notions of $\text{dg-Hom}_A(M^\bullet, N^\bullet)$ agree.

Morphisms (respectively homotopies of morphisms) of dg modules are the dg morphisms of dg modules that are also morphisms (respectively homotopies of morphisms) of complexes.

Write $K(B)$ for the category whose objects are dg modules over B and whose morphisms are homotopy equivalence classes of morphisms of dg modules.

A morphism of dg modules $M^\bullet \to N^\bullet$ is a quasi-isomorphism if the induced morphism of graded modules $H^\bullet(M^\bullet) \to H^\bullet(N^\bullet)$ is an isomorphism.

The derived category of dg modules over B, written $D(B)$, is the localisation of $K(B)$ with respect to the quasi-isomorphisms.

Let $0 \to L^\bullet \xrightarrow{f} M^\bullet \xrightarrow{g} N^\bullet \to 0$ be an exact sequence of morphisms of dg modules that is split as a sequence of morphisms of graded B-modules, with splitting $0 \to N^\bullet \xrightarrow{i} M^\bullet \xrightarrow{p} L^\bullet \to 0$. Then $h := pd'i$ is a morphism of dg modules $N^\bullet \to L^\bullet[1]$, and we take as our distinguished triangle $(L^\bullet, M^\bullet, N^\bullet, f, g, h)$. With these distinguished triangles, $K(B)$ is a triangulated category. This induces a triangulated structure on $D(B)$.

For any short exact sequence of morphisms of dg modules $0 \to L^\bullet \xrightarrow{f} M^\bullet \xrightarrow{g} N^\bullet \to 0$ there exists a morphism of dg modules $h : N^\bullet \to L^\bullet[1]$ such that $(L^\bullet, M^\bullet, N^\bullet, f, g, h)$ is a distinguished triangle in $D(B)$.

In particular, if two dg modules in a short exact sequence of morphisms of dg modules both live in a full triangulated subcategory of $D(B)$, then so does the third, and hence such categories are closed under kernels, cokernels, and extensions in the category of dg modules.

Definition 2.10.2. A resolution of a module M is a complex $M^\bullet$ such that $M^i = 0$ for $i \geq 0$ and

$$H^i(M^\bullet) = \begin{cases} M, & i = 0 \\ 0, & i \neq 0. \end{cases}$$

A resolution is finitely generated (respectively, projective) if it is a complex of finitely generated modules (respectively, projective modules). The length of a complex is $i_{\mathsf{max}} - i_{\mathsf{min}} - 1$, where i_{max} and i_{min} are respectively the largest and smallest indices i such that $M^i \neq 0$.

Let $M^\bullet$ be a projective resolution of M in $\mathrm{Mod}(A)$. Then

$$H^n(\text{dg-End}_A(M^\bullet)) \cong \mathrm{Hom}_{D(A)}(M, M[n]).$$

Definition 2.10.3. We say a set of objects G of a triangulated category T classically generates a triangulated subcategory T' of T if T' is the smallest full triangulated subcategory of T closed under isomorphisms and direct summands and containing G. We also write $T' = \langle G \rangle_T$.

The triangulated category $\mathrm{per}(B)$ of perfect objects in $D(B)$ is $\langle B \rangle_{D(B)}$.

Observe that, in the case that B is an ordinary algebra A, then $\mathrm{per}(A)$ is the full subcategory of $D(A)$ consisting of objects isomorphic to finite length complexes of finitely generated projective A-modules.

We write $D^b_{fg}(A)$ for the subcategory of $D(A)$ consisting of objects isomorphic to finite length complexes of finitely generated A-modules. This is a triangulated subcategory of $D(A)$ that is closed under direct summands in $D(A)$.

Theorem 2.10.4. *Let $\mathcal{T}$ be a full triangulated subcategory of $D(A)$ that is closed under direct summands, let M be an object in both $\mathrm{Mod}(A)$ and $\mathcal{T}$, such that $\langle M \rangle_{\mathcal{T}} = \mathcal{T}$, and let $M^\bullet$ be a projective resolution of M in $\mathrm{Mod}(A)$. Then there is a triangulated equivalence*

$$\mathcal{T} \simeq \mathrm{per}(\text{dg-End}_A(M^\bullet)).$$

Proof. We seek to apply Theorem 3.8(b) of Keller [2006], so we must check that all the conditions of said theorem hold. Section 3.6 of the same establishes that $D(B)$ is algebraic, and hence so is $\mathcal{T}$, as it is a triangulated subcategory. Furthermore, Section 3.5 of the same establishes that $D(B)$ is idempotent-complete (as it has arbitrary coproducts), and hence, as $\mathcal{T}$ is closed under direct summands, $\mathcal{T}$ is also idempotent-complete. Finally, as $H^n(\text{dg-End}_A(M^\bullet)) \cong \text{Hom}_{D^b_{fg}(A)}(M, M[n])$ and $\mathcal{T}$ is full, we get $H^n(\text{dg-End}_A(M^\bullet)) \cong \text{Hom}_{\mathcal{T}}(M, M[n])$. Thus all the conditions of Theorem 3.8(b) hold. □

Definition 2.10.5. An object M in $\text{Mod}(A)$ is a generator if every object in $\text{Mod}(A)$ is the quotient of a direct sum of copies of M.

A finitely generated projective generator is called a progenerator.

Proposition 2.10.6. *Suppose M is a progenerator of $\text{Mod}(A)$. Then M classically generates $\text{per}(A)$.*

Proof. As finitely generated projective modules are precisely the direct summands of finite direct sums of A, we have $\text{per}(A) = \langle A \rangle_{\text{per}(A)}$. Furthermore, as a progenerator is finitely generated and projective, we have $M \in \text{per}(A)$. It thus suffices to show that $A \in \langle M \rangle_{\text{per}(A)}$. But as M is a generator, A is the quotient of a direct sum of copies of M. As A is finitely generated, this direct sum may be taken to be finite, and as A is projective, the quotient splits, so A is a direct summand of a finite direct sum of copies of M. □

We shall also need the following general homological observations.

Proposition 2.10.7. *If A is noetherian and of finite global dimension, then every finitely generated A-module M has a finite length finitely generated projective resolution.*

Proof. As A is noetherian and M is finitely generated we know by Rotman [2009] Lemma 7.19 that M has a finitely generated projective resolution. But as A has finite global dimension, say n, replacing the n-th term with the $(n-1)$th syzygy gives, by Rotman [2009] Proposition 8.6, a finitely generated projective resolution of length n. □

Proposition 2.10.8. *If $M^\bullet$ is a finite length complex of A-modules, and each M^i has a finite length finitely generated projective resolution, then $M^\bullet$ is quasi-isomorphic to a finite length complex of finitely generated projective modules.*

Proof. For each i, write $P^{i\bullet}$ for a choice of finite length finitely generated projective resolution of M^i.

By Gelfand and Manin [2003] Lemma III.7.12, $M^\bullet$ is quasi-isomorphic to the complex $T^\bullet$ whose terms are $T^k = \oplus_{i+j=k} P^{ij}$. As $M^\bullet$ has finite length, each T^k is a finite direct sum of finitely generated projective modules, and hence is finitely generated and projective. Furthermore, as $M^\bullet$ and all of the $P^{i\bullet}$ have finite length, $T^\bullet$ also has finite length. □

2.11 Affine Cellular Algebras

We follow Koenig and Xi [2012]. In this section, R will be a noetherian domain and A will be unital and have an involution i (that is, an R-linear anti-automorphism).

Definition 2.11.1. A 2-sided ideal J in A such that $i(J) = J$ is called an affine cell ideal if there are

- a free R-module V of finite rank,
- a finitely generated commutative R-algebra B with involution σ,
- and a left A-module structure on $\Delta = V \otimes_R B$ that commutes with the regular right B-module structure,

such that, if we define a right A-module structure on $\Delta' = B \otimes_R V$ by $xa = \tau^{-1}(i(a)\tau(x))$ where $\tau : \Delta' \to \Delta$, $b \otimes v \mapsto v \otimes b$, there is an isomorphism of A-A-bimodules $\alpha : J \to \Delta \otimes_B \Delta' = V \otimes_R B \otimes_R V$ making the following diagram commute:

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & V \otimes_R B \otimes_R V \arrow[d, "v\otimes b\otimes v' \mapsto v'\otimes\sigma(b)\otimes v"] \\ J \arrow[r, "\alpha"'] & V \otimes_R B \otimes_R V \end{tikzcd}$$

A is said to be affine cellular if there is an R-module decomposition $A = \bigoplus_{k=1}^{K} J'_k$ such that, for all k, we have that $i(J'_k) = J'_k$, and furthermore that $J_k = \bigoplus_{k'=1}^{k} J'_{k'}$ is a 2-sided ideal in A such that $J'_k = J_k/J_{k-1}$ is an affine cell ideal in A/J_{k-1}.

Write V_k and B_k for the V and B as above that give an affine cell ideal structure for J'_k.

An affine cellular algebra is said to be idempotent affine cellular if, for all k, we have that J'_k is generated as a 2-sided ideal in A/J_{k-1} by a nonzero idempotent.

Let A be affine cellular with notation as above. If, as a 2-sided ideal in A/J_{k-1}, we have that $J'^2_k = J'_k$ and J'_k contains a nonzero idempotent e, then e generates J'_k as a 2-sided ideal in A/J_{k-1}.

Proposition 2.11.2. *Suppose e is an idempotent in A.*

1. *If $i(e) = e$, and if A is affine cellular, with notation as above, then so is eAe, with the same B_k, and J'_k replaced with eJ'_ke.*
2. *If $AeA = A$, then restriction of scalars gives a Morita equivalence from A to eAe.*
3. *If $j \in eAe$ generates a 2-sided ideal J in A, then it generates eJe in eAe.*

Proof. The first claim is Yang [2014], Lemma 3.3, and the second is Proposition 2.4 from the same paper.

The third claim is a quick direct calculation:

$$\begin{aligned} eJe &= eAjAe \\ &= eAejeAe \end{aligned} \tag{2.11.1}$$

where the last line follows as $j \in eAe$. □

Let R' be a noetherian domain that is an R-algebra, A an (idempotent) affine cellular R-algebra. Then $R' \otimes_R A$ is an (idempotent) affine cellular algebra with affine cellular structure induced from that on A by taking the tensor product throughout by R'.

Definition 2.11.3. The global dimension of A, written $\operatorname{gl}\dim(A)$, is the smallest $m \in \mathbb{N} \cup \{\infty\}$ such that any A-module M has a projective resolution of length at most m. If $m \in \mathbb{N}$ then we say A has *finite global dimension*.

Theorem 2.11.4. *Suppose A is idempotent affine cellular, with notation as above, and suppose $\operatorname{rad}(B_k) = 0$ and $\operatorname{gl}\dim(B_k) < \infty$ for all k. Then $\operatorname{gl}\dim(A) < \infty$.*

Proof. This is Theorem 4.4(b) of Koenig and Xi [2012]. □

3

l-Modular Unipotent Representations of Finite Reductive Groups

Synopsis

We prove the finite version of our main theorem: that $D^b_{fg}(\mathsf{H}_1(G_f))$ for $G_f = \mathrm{GL}_n(k)$ is classically generated by the two representations Q_f and V_f. We also prove that Q_f is a direct sum of subrepresentations of the representation P_f, which will allow us to lift this theorem to the p-adic setting in Chapter 5. We proceed by first defining all the relevant objects, then using the theory of l-modular representations of $\mathrm{GL}_n(k)$ to describe the composition factors of Q_f and V_f. We then explore various special cases, where the results and reasoning can be seen more explicitly, and which provide intuition for the general case.

3.1 Definitions and Notation

Let R be an algebraically closed field of characteristic l, let k be a finite field of characteristic $p \neq l$ and cardinality q, and let G be a (connected) reductive algebraic group over k. Write $G_f = \mathrm{G}(k)$ for the k-points of G.

Fix in G_f a minimal parabolic subgroup I_f. For GL_n, we may without loss of generality take I_f to be the upper triangular matrices. Similarly, we write I_f^1 for the unipotent radical of I_f, which is then the unipotent upper triangular matrices. Fix also a maximal split torus T_f in I_f, which for GL_n we again take without loss of generality to be the diagonal matrices. Let $\bar{I}_f^1$ be the unipotent radical of the opposite parabolic of I_f with respect to T_f, which is then the unipotent lower triangular matrices. Furthermore, for GL_n the Weyl group $W_f = N(T_f)/T_f$ is isomorphic to $\mathfrak{S}_n$, the symmetric group on $\{1, \ldots, n\}$, and has a canonical splitting sending each permutation in $\mathfrak{S}_n$ to the corresponding permutation matrix in $N(T_f) \subseteq G_f$.

Definition 3.1.1. Write $\mathsf{H}(G_f)$ for the group algebra of G_f over R.

Write $\mathrm{Mod}(G_f)$ for the category of G_f-representations over R, that is, modules over $\mathsf{H}(G_f)$.

Write $P_f = \mathrm{ind}_{I_f}^{G_f} \mathbb{1}$.

Let $\mathcal{B}_1(G_f)$ be the full subcategory of $\mathrm{Mod}(G_f)$ consisting of all representations all of whose irreducible subquotients are subquotients of P_f. Note that this is a direct summand of $\mathrm{Mod}(G_f)$ (see eg Vignéras [2003] D12), and hence a direct sum of blocks. We call the blocks in this summand, as well as the representations in the summand, unipotent. Note this is is the correct definition for GL_n, but does not agree with the usual definition of unipotent for other groups. Write $\mathcal{B}_{\neq 1}(G_f)$ for the direct sum of all non-unipotent blocks.

Denote the corresponding direct sums of block algebras of $\mathsf{H}(G_f)$ by $\mathsf{H}_1(G_f)$ and $\mathsf{H}_{\neq 1}(G_f)$ respectively.

We call the block containing the trivial representation $\mathbb{1}$ the principal block.

Observe that the principal block is unipotent, and that both P_f and $\mathbb{1}$ are unipotent and finitely generated.

Let $\mathcal{J}_f$ be the set of parabolic subgroups of G_f containing I_f. Elements of $\mathcal{J}_f$ are called standard parabolic subgroups. For $J_f \in \mathcal{J}_f$, let M_{J_f} be the Levi subgroup of J_f containing T_f. Let U_{J_f} be the unipotent radical of a minimal parabolic subgroup of M_{J_f}. Then U_{J_f} corresponds to a base for the roots of M_{J_f} with respect to T_f. Let X_{J_f} be the set characters of U_{J_f} that are nontrivial on all simple root groups in U_{J_f} but trivial on all other positive root groups.

Definition 3.1.2. Write

$$\Gamma_f = \bigoplus_{J_f \in \mathcal{J}_f} \bigoplus_{\chi_{J_f} \in X_{J_f}} \mathrm{i}^{G_f}_{M_{J_f}, J_f} \mathrm{ind}^{M_{J_f}}_{U_{J_f}} \chi_{J_f}.$$

Let $\mathcal{I}_f$ be the annihilator of P_f in $\mathsf{H}(G_f)$. Then put $Q_f = \Gamma_f / \mathcal{I}_f \Gamma_f$, and put $\mathsf{H}'_1(G_f) = \mathsf{H}(G_f)/\mathcal{I}_f$.

Put $\mathcal{B}'_1(G_f)$ the full subcategory of $\mathrm{Mod}(G_f)$ consisting of representations M with $\mathcal{I}_f M = 0$.

Observe that, as P_f is unipotent, $\mathcal{I}_f$ contains $\mathsf{H}_{\neq 1}(G_f)$, and so Q_f is also unipotent, and $\mathsf{H}'_1(G_f)$ (resp $\mathcal{B}'_1(G_f)$) is a quotient (respectively subcategory) of $\mathsf{H}_1(G_f)$ (resp $\mathcal{B}_1(G_f)$).

Definition 3.1.3. Write

$$V_f = \bigoplus_{J_f \in \mathcal{J}_f} \mathrm{ind}^{G_f}_{J_f} \mathbb{1}.$$

Observe that V_f is a finite direct sum of submodules of P_f, and so is unipotent and finitely generated.

We seek to establish two facts which shall enable us to describe the p-adic setting:

Theorem 3.1.4. *For* $\mathrm{G} = \mathrm{GL}_n$*,*

1. *Q_f is a direct sum of subrepresentations of P_f*
2. $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = D^b_{fg}(\mathsf{H}_1(G_f))$

Note that the latter result is of independent interest, and provides an analogue for finite groups of the main theorem of this thesis.

3.2 Classical Generators for Finite GL_n

Henceforth, we shall assume that $\mathrm{G} = \mathrm{GL}_n$. Note that in this case, for fixed J_f, all $\chi_{J_f} \in X_{J_f}$ are conjugate, and so all $\mathrm{ind}^{M_{J_f}}_{U_{J_f}} \chi_{J_f}$ are isomorphic (see Vignéras [2003], Section 5.6).

Proposition 3.2.1. *The unipotent part of Γ_f is a progenerator for $\mathcal{B}_1(G_f)$.*

Proof. The proof mirrors Theorem 5.13(1) and 5.10 of Vignéras [2003] (see also Takeuchi [1996] for another proof). Γ_f is by construction finitely generated and projective, and for any unipotent irreducible representation we may apply Property H1 of Vignéras [2003] to show that it has a nonzero vector invariant under a certain unipotent subgroup, which implies that it is a quotient of Γ_f by 5.4(3) of the same source. Hence we conclude by said source's Corollary 3.7. □

Corollary 3.2.2. *Q_f is a progenerator of $\mathcal{B}'_1(G_f)$.*

Our proof proceeds by describing Q_f using the work of Dipper and James (James [1986], Dipper and James [1989]). Recall that a partition λ of a nonnegative integer n is a non-increasing tuple (λ_i) of positive integers with sum n. The dominance order on partitions is the partial order where $\lambda \geq \mu$ precisely when $\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$ for all i. We associate to each partition λ a standard parabolic $J_f(\lambda)$ of G_f, namely the upper block triangular matrices with the ith block having size λ_i.

In Theorem 8.1 of James [1986], it is shown that there is a bijection from partitions λ of n to unipotent irreducible representations $D(\lambda)$. As this claim holds for any choice of R algebraically closed of characteristic $l \neq p$, it is also true for an algebraically closed field K of characteristic 0.

We thus fix an l-modular system $\mathcal{O}_K$ with fraction field K and residue field R, such that the group algebra $K[G]$ is split semisimple. Theorem 8.1 of James [1986] gives a canonical choice $S(\lambda)$ for an l-modular reduction of the unipotent irreducible representation over K corresponding to λ.

Lemma 3.2.3. *$S(\lambda)$ and $D(\lambda)$ are objects in $\mathcal{B}'_1(G_f)$.*

Proof. It is shown in Theorem 8.1 of James [1986] that each $S(\lambda)$ is a submodule of $\mathrm{ind}_{J_f(\lambda)}^{G_f} \mathbb{1}$, and that $D(\lambda)$ is a quotient of $S(\lambda)$. Hence $S(\lambda)$ and $D(\lambda)$ are subquotients of P_f. □

We now introduce the finite Schur algebra, whose decomposition matrix is deeply entwined with that of G_f. In later chapters, we shall see a p-adic analogue, which we shall simply call the Schur algebra, hence the use of the qualifier 'finite' for this version (perhaps it would be better to call the p-adic version the 'Iwahori-Schur' algebra, but we have not seen this convention anywhere).

Definition 3.2.4. The finite Schur algebra $\mathsf{S}_R(n)_f$ is the endomorphism algebra $\mathrm{End}_{\mathsf{H}(G_f)}(V_f)$.

Observe that, while we have defined this over R, by considering endomorphisms of induced representations over a general ring, this definition would make sense over K and $\mathcal{O}_K$. Indeed, by Theorem 2.24 and Note 2.18(ii) of Dipper and James [1989], the finite Schur algebras over K and R are the extensions of scalars of the finite Schur algebra over $\mathcal{O}_K$, and by Dipper and James [1991] the finite Schur algebra over $\mathcal{O}_K$ is free, so we may speak of l-modular reduction of modules over the finite Schur algebra. The surprising property of $\mathsf{S}_R(n)_f$ that makes it relevant for us is the following:

Proposition 3.2.5. $\mathrm{End}_{\mathsf{H}(G_f)}(Q_f)$ *is Morita equivalent to* $\mathsf{S}_R(n)_f$.

Proof. This is part (a) of the theorem in the introduction of Takeuchi [1996] (see also Theorem 5.8 of Vignéras [2003]). □

With this we may now show the first part of Theorem 3.1.4. Write $P(\mu)$ for the projective cover of $D(\mu)$ in $\mathcal{B}_1(G_f)$.

Theorem 3.2.6. Q_f *is a direct sum of subrepresentations of* P_f.

Proof. Let $(d_{\lambda\mu})$ denote the submatrix of the decomposition matrix of $\mathsf{H}(G_f)$ corresponding to the $S(\lambda)$ and $D(\mu)$.

As Q_f is a progenerator of $\mathcal{B}'_1(G_f)$, there is an induced equivalence of categories between $\mathcal{B}'_1(G_f)$ and the category of modules over $\mathsf{S}_R(n)_f$. By parts (b) and (c) of the theorem in the introduction of Takeuchi [1996], this equivalence identifies the $S(\lambda)$ with the l-modular reduction of the simple modules of $\mathsf{S}_K(n)_f$, and the $D(\mu)$ with the simple modules of $\mathsf{S}_R(n)_f$. Thus $(d_{\lambda\mu})$ is the full decomposition matrix of $\mathsf{S}_R(n)_f$.

Write $P'(\mu)$ for the projective cover of $D(\mu)$ in $\mathcal{B}'_1(G_f)$. Then $P'(\mu) = P(\mu)/\mathcal{I}_f P(\mu)$. Furthermore, as $\mathcal{B}'_1(G_f)$ is equivalent to modules over $\mathsf{S}_R(n)_f$, and the latter has decomposition matrix $(d_{\lambda\mu})$, we have that $D(\nu)$ occurs in any composition series for $P'(\mu)$ with multiplicity $\sum_\lambda d_{\lambda\nu}d_{\lambda\mu}$.

In Theorem 3.8 of Dipper and James [1989] they construct a quotient of $P(\mu)$, which we shall call $Y(\mu)$, which is a submodule of P_f, and such that $D(\nu)$ occurs in any composition series for $Y(\mu)$ with multiplicity $\sum_\lambda d_{\lambda\nu}d_{\lambda\mu}$. The former property implies that $\mathcal{I}_f Y(\mu) = 0$, and hence that the quotient $P(\mu) \to Y(\mu)$ factors through $P'(\mu)$. But the latter property says that $P'(\mu)$ and $Y(\mu)$ have the same composition factors. Hence they must in fact be isomorphic.

Thus every projective indecomposable representation in $\mathcal{B}_1'(G_f)$ is a subrepresentation of P_f. But Q_f is projective in $\mathcal{B}_1'(G_f)$, and so Q_f must be a direct sum of the $Y(\mu)$. □

To show that $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = D^b_{fg}(\mathsf{H}_1(G_f))$, we proceed by showing that the first two categories contain every unipotent irreducible representation. This in fact suffices, as the next lemma shows. Let $\mathscr{D}$ be the set of all unipotent irreducible representations, that is, the set of all $D(\lambda)$.

Lemma 3.2.7. $D^b_{fg}(\mathsf{H}_1(G_f)) = \langle \mathscr{D} \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$.

Proof. As the $D(\lambda)$ are finitely generated and unipotent, we know that $\langle \mathscr{D} \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} \subseteq D^b_{fg}(\mathsf{H}_1(G_f))$. But all finitely generated representations of G_f have finite length, and so all objects of $D^b_{fg}(\mathsf{H}_1(G_f))$ arise from objects in $\mathscr{D}$ via finitely many distinguished triangles. □

Thus it is enough to show that $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ and $\langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contain $\mathscr{D}$. We first consider V_f, for which we make use of the explicit structure theory of the $D(\lambda)$.

Lemma 3.2.8. $\mathscr{D} \subseteq \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$.

Proof. We show $D(\kappa) \in \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ by decreasing induction along the dominance order for κ. First, observe that $\mathrm{ind}_{J_f(\kappa)}^{G_f} \mathbb{1}$ is a summand of V_f, and so $\mathrm{ind}_{J_f(\kappa)}^{G_f} \mathbb{1} \in \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$.

Next, by Theorem 7.19(iii) of James [1986], $\mathrm{ind}_{J_f(\kappa)}^{G_f} \mathbb{1}$ has a composition series with all factors of the form $S(\lambda)$ with $\lambda \geq \kappa$, in which $S(\kappa)$ occurs with multiplicity 1. But by Theorem 8.1 of James [1986], $S(\lambda)$ itself has a composition series with all factors of the form $D(\mu)$ with $\mu \geq \lambda$, in which $D(\lambda)$ occurs with multiplicity 1. Thus $\mathrm{ind}_{J_f(\kappa)}^{G_f} \mathbb{1}$ has a composition series with all factors of the form $D(\mu)$ with $\mu \geq \kappa$, in which $D(\kappa)$ occurs with multiplicity 1.

But by the inductive hypothesis, all $D(\mu)$ with $\mu > \kappa$ are in $\langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$. Thus, by considering the sequence of distinguished triangles giving the composition series $\mathrm{ind}^{G_f}_{J_f(\kappa)} \mathbb{1}$ in terms of $D(\mu)$, we see that $D(\kappa) \in \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$. $\square$

To show the same for Q_f, we make use of the following property, which comes from deep results about $\mathsf{S}_R(n)_f$.

Lemma 3.2.9. $\mathsf{S}_R(n)_f$ *has finite global dimension.*

Proof. By Theorem 3.7.2 of Cline et al. [1990] (see also the main theorem of Du et al. [1998]), a family of algebras $\mathsf{S}_R(N, n)_f$ (written $S_q(N, n, R)$ in their notation) are quasi-hereditary. By Theorem 3.6(a) of Cline et al. [1990] any quasi-hereditary algebra over a field has finite global dimension. But by Theorem 2.24 of Dipper and James [1989] and Lemma 1.3 of Dipper and James [1991] $\mathsf{S}_R(N, n)_f$ and $\mathsf{S}_R(n)_f$ are Morita equivalent whenever $N \geq n$. $\square$

This allows us to conclude by a purely formal argument.

Lemma 3.2.10. $\mathscr{D} \subseteq \langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$.

Proof. Q_f is a progenerator of $\mathcal{B}'_1(G_f)$, so by Proposition 2.10.6 we have that $\langle Q_f \rangle_{D^b_{fg}(\mathcal{B}'_1(G_f))} = \mathrm{per}(\mathcal{B}'_1(G_f))$. But, as $\mathcal{B}'_1(G_f)$ is equivalent to modules over $\mathsf{S}_R(n)_f$, and the latter has finite global dimension, and is furthermore Noetherian (as it is a finite dimensional algebra over a field), we have by Proposition 2.10.7 that $\mathrm{per}(\mathcal{B}'_1(G_f)) = D^b_{fg}(\mathcal{B}'_1(G_f))$. Thus $\mathscr{D} \subseteq D^b_{fg}(\mathcal{B}'_1(G_f)) = \mathrm{per}(\mathcal{B}'_1(G_f)) = \langle Q_f \rangle_{D^b_{fg}(\mathcal{B}'_1(G_f))}$. $\square$

Thus we have the second part of Theorem 3.1.4.

Theorem 3.2.11. $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = D^b_{fg}(\mathsf{H}_1(G_f))$.

While the above proof works in general, the next sections provide further insight into the structure of various special cases, where the main theorem can be shown more directly.

3.3 Finite GL_n for $l \mid q-1$

For this section, we shall assume that the characteristic l of R divides $q-1$, and that $l > n$. This case behaves somewhat differently to the other cases, as the Schur algebra is semisimple, and so it merits special consideration.

Lemma 3.3.1. *The only unipotent block is the principal block.*

Proof. By Theorem 7.11 of Dipper and James [1989], each block contains exactly those $D(\lambda)$ which have the same fixed 1-core. But all λ have the same 1-core, namely the empty partition. ☐

Lemma 3.3.2. *P_f is a direct sum of the $D(\mu)$.*

Proof. This is Ackermann [2006], Proposition 4.22, and its proof. The point is that in this case the endomorphisms of P_f are just the group algebra of $\mathfrak{S}_n$, and as $l > n$ this is semisimple. Hence P_f is a direct sum of irreducible representations. But any such representation must be unipotent, and hence is some $D(\mu)$. ☐

Corollary 3.3.3. *Every $D(\mu)$ is a summand of P_f.*

Proof. The $D(\mu)$ are by definition the irreducible subquotients of P_f. But P_f is a direct sum of irreducible representations, so every $D(\mu)$ must occur as a summand of P_f. ☐

Corollary 3.3.4. $\mathcal{I}_f = \mathsf{H}_{\neq 1}(G_f) \oplus \mathrm{rad}(\mathsf{H}_1(G_f))$.

Proof. As P_f is unipotent, every element of $\mathsf{H}_{\neq 1}(G_f)$ annihilates it. It remains to see which elements of $\mathsf{H}_1(G_f)$ annihilate P_f.

Now, $\mathrm{rad}(\mathsf{H}_1(G_f))$ is the intersection of the annihilators of all irreducible $\mathsf{H}_1(G_f)$-modules, that is, all $D(\mu)$. But each $D(\mu)$ is a summand of P_f, and P_f is a direct sum of $D(\mu)$. Thus its annihilator in $\mathsf{H}_1(G_f)$ is exactly $\mathrm{rad}(\mathsf{H}_1(G_f))$. ☐

Corollary 3.3.5. *Q_f is a direct sum of the $D(\mu)$. Every $D(\mu)$ is a summand of Q_f.*

Proof. As $\mathcal{I}_f$ contains $\mathsf{H}_{\neq 1}(G_f)$, the summand of $\mathcal{I}_f\Gamma_f$ in $\mathcal{B}_{\neq 1}(G_f)$ is the same as the summand of Γ_f in $\mathcal{B}_{\neq 1}(G_f)$. As the unipotent part of Γ_f is a progenerator of $\mathcal{B}_1(G_f)$, it will be a direct sum of $P(\mu)$, and every $P(\mu)$ will be a summand of it. As $P(\mu)$ is finitely generated and $\mathsf{H}_1(G_f)$ is Artinian, we have that $\mathrm{rad}(\mathsf{H}_1(G_f))P(\mu) = \mathrm{rad}(P(\mu))$, and so $P(\mu)/\mathrm{rad}(\mathsf{H}_1(G_f))P(\mu) \cong D(\mu)$. Thus the quotient Q_f will be a direct sum of $D(\mu)$, and all $D(\mu)$ will be a summand of Q_f. □

In particular, as both Q_f and P_f are direct sums of the $D(\mu)$, and all $D(\mu)$ are summands of both Q_f and P_f, it is clear that Q_f is a direct sum of submodules of P_f, and that $\langle Q_f\rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle P_f\rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f\rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = D^b_{fg}(\mathsf{H}_1(G_f))$.

3.3.1 Example: $n = 1$

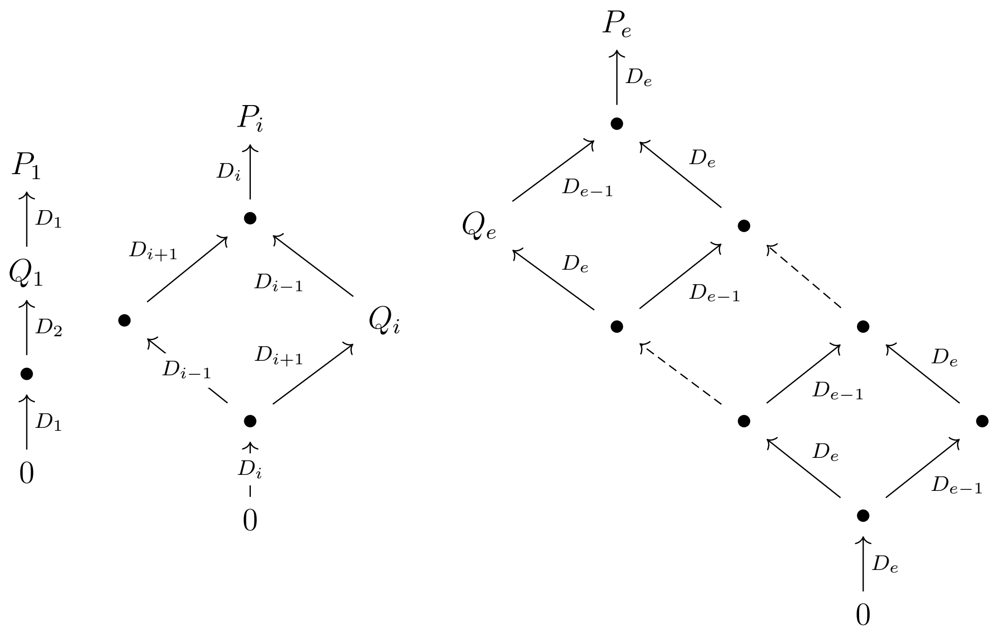
We give the case $n = 1$ explicitly, as it is instructive for the case of general n. In this setting, $I_f = G_f$ and $I_f^1 = 1$. Thus, $P_f = V_f = \mathbb{1}$ and $\Gamma_f = \mathrm{ind}_1^{G_f}\mathbb{1} = \mathsf{H}(G_f)$. Hence we can explicitly calculate $\mathcal{I}_f = \left\{\sum_{g\in G_f} r_g g \middle| \sum_{g\in G_f} r_g = 0\right\}$, and so $Q_f = \mathbb{1}$.

Thus, for $n = 1$, we see directly that the only unipotent block is the principal block, containing a single irreducible representation $\mathbb{1}$. Furthermore, we can see that P_f and Q_f are not just direct sums of $D(1) = \mathbb{1}$, but are in fact both exactly $\mathbb{1}$. In particular, Theorem 3.2.6 and Theorem 3.2.11 both tautologically hold in this case: $\mathbb{1}$ is a direct sum of submodules of itself, and generates the same derived category as itself.

3.4 Finite GL_n for $l \mid q^e - 1$, $e > \frac{n}{2}$

For this section, we shall take $n \geq 2$, and we shall further assume that the characteristic l of R divides $q^e - 1$ for some $e > \frac{n}{2}$, but not $q^{e'} - 1$ for any $e' < e$. This is exactly the case of cyclic defect, and in this case the full structure of the blocks is well-understood.

Lemma 3.4.1. *The principal block has e simple representations $L_1, \dots, L_e$. Write their respective projective covers as $P_1 \dots, P_e$. Then the P_i have the* *following subrepresentation lattices:*



Furthermore, all other blocks are either of this form, in which case we say they are blocks of cyclic defect, or are semisimple (that is, contain a unique projective indecomposable representation, which is also irreducible).

Notice that we have named a certain subrepresentation Q_i of each P_i, and that $P_i/Q_i \cong Q_{i-1}$ for $2 \leq i \leq n$.

Proof. Theorem 4.2 of Ackermann [2006] says that any block with cyclic defect group is of this form. The initial remarks from Section 4.2 of the same source establish that all blocks have cyclic or trivial defect group for the n and e we consider. □

Now observe that, as l does not divide q or $q-1$, we have that P_f is projective. We also have by definition that $P_f \in \mathcal{B}_1(G_f)$. Hence P_f is a direct sum of the $P(\mu)$.

Recall from Section 2.6 the notions of cuspidal representation and cuspidal support.

Proposition 3.4.2. *All unipotent representations have one of the following* *two cuspidal supports: either $(T_f, \mathbb{1}_{T_f})$, or $(M, C) = (\mathrm{GL}_e(k) \times \mathrm{GL}_{n-e}(k), \sigma \times \mathbb{1}_{\mathrm{GL}_{n-e}(k)})$, where σ is the unique cuspidal unipotent representation of $\mathrm{GL}_e(k)$.*

Proof. This is Proposition 2.2 of Ackermann [2006]. □

Proposition 3.4.3. *There is a bijection between unipotent blocks with cyclic defect group and unipotent representations with cuspidal support (M, C), given by sending a block with irreducible representations $D_1, \dots, D_e$ as above to D_e. We may parameterise these blocks by partitions ν of $n - e$, and in each block we have that $D_i = D(\mu_i)$ for $\mu_1, \dots, \mu_e$ the unique partitions of n with e-core ν such that $\mu_1, \dots, \mu_e$ is in decreasing lexicographical order.*

Proof. By Proposition 2.2 of Ackermann [2006], with $r = 1$ and $s = n - e$, the irreducible representations $D(\rho)$ with cuspidal support (M, C) are exactly those for which ρ ends with at least e copies of 1. To get the e-core of such a partition we can remove the final e copies of 1: this is indeed the e-core as the resulting partition has size $n - e < e$. Conversely, given an arbitrary partition ν of size $n - e$, we can form a partition of size n by adding e copies of 1 to the end. The above maps are mutually inverse and so give a bijection between e-cores ν and representations $D(\rho)$ with cuspidal support (M, C).

Meanwhile, the same proposition with $r = 0$ and $s = n$ gives that the irreducible representations $D(\mu)$ with cuspidal support $(T_f, \mathbb{1})$ come exactly from the μ that are e-regular. Thus, either μ has e-core μ, which is thus not equal to the e-core of any other partition, or it has e-core ν of size $n - e$, and there is a unique ρ with e-core μ such that $D(\rho)$ has cuspidal support (M, C). Furthermore, as we can always form an e-regular μ from a partition ν of size $n - e$ by adding e to the first entry, each $D(\rho)$ with cuspidal support (M, C) shares an e-core ν with at least one $D(\mu)$ with cuspidal support $(T_f, \mathbb{1})$.

Now, by Proposition 4.1 and Theorem 4.2 of Ackermann [2006] (recall also the previously-mentioned Theorem 7.11 of Dipper and James [1989]), two partitions are in the same block precisely when they have the same e-core. Furthermore, the same source also gives that the blocks are either semisimple or have e irreducible representations, and in the latter case the order is given by decreasing lexicographical order on the partitions. Thus, either the block contains a single $D(\mu)$ with cuspidal support $(T_f, \mathbb{1})$ and e-core μ, and is semisimple, or it contains a single $D(\rho)$ with cuspidal support (M, C), as well as at least one $D(\mu)$ with cuspidal support $(T_f, \mathbb{1})$, both with e-core ν. In the latter case the block must hence contain exactly e irreducible representations, and the partition of minimal lexicographical order is the one corresponding to the $D(\rho)$ with cuspidal support (M, C), so this $D(\rho)$ must be D_e. $\square$

Corollary 3.4.4. *Every $P(\mu)$, except for the P_e of each unipotent block with cyclic defect, is a summand of P_f.*

Proof. Every irreducible $D(\mu)$ in a unipotent block, except the D_e of a block with cyclic defect, has cuspidal support $(T_f, \mathbb{1}_{T_f})$, meaning exactly that it is a quotient of P_f. But as P_f is projective, the corresponding $P(\mu)$ must hence be a summand of P_f. Thus, every $P(\mu)$ except the P_e must be a summand. $\square$

As in Section 2.2, we may decompose $\mathsf{H}_1(G_f)$ as a direct sum (with multiplicities) of the $P(\mu)$, and furthermore, by identifying an element of $\mathsf{H}_1(G_f)$ with its right multiplication action on the $P(\mu)$, this sum then further decomposes into a direct sum of all homomorphisms between the $P(\mu)$ in the first sum. This identifies each copy of $P(\mu)$ in the first sum with the space of all morphisms $P(\nu) \to P(\mu)$ running over all $P(\nu)$ in the first sum. We may thus consider the left action of $h \in \mathsf{H}(G_f)$ on $p \in P(\mu)$ as precomposing the sum of morphisms $P(\nu) \to P(\mu)$ corresponding to p with the sum of morphisms $P(\kappa) \to P(\lambda)$ corresponding to h.

Let $\mathsf{H}_{1,e}(G_f)$ be the linear span of all $h \in \mathsf{H}_1(G_f)$ corresponding to morphisms whose domain and codomain are both isomorphic to the P_e of some unipotent block of cyclic defect. This does not depend on the choice of decomposition.

Corollary 3.4.5. *$\mathcal{I}_f = \mathsf{H}_{\neq 1}(G_f) \oplus \mathrm{rad}(\mathsf{H}_{1,e}(G_f))$.*

Proof. As P_f is unipotent, its annihilator contains all of $\mathsf{H}_{\neq 1}(G_f)$. It thus remains to see which elements of $\mathsf{H}_1(G_f)$ annihilate P_f. We may consider the summand in each unipotent block separately.

Consider first a block of cyclic defect. Then P_f is a direct sum of the P_i, and so we may consider its elements as morphisms of the P_i, on which $\mathsf{H}(G_f)$, also thought of as morphisms of UPIRs, acts by precomposition. From Lemma 3.4.1 we can see that the only morphisms of UPIRs that annihilate via precomposition every P_i apart from P_e are the morphisms with domain and codomain isomorphic to P_e, and whose image is contained in Q_e. This is exactly the nilpotent morphisms with domain and codomain isomorphic to P_e, which is $\mathrm{rad}(\mathsf{H}_{1,e}(G_f))$.

The case of a semisimple block is much simpler, as here no nonzero morphism annihilates the unique $P(\mu) = D(\mu)$ in the block, and so the part of $\mathcal{I}_f$ in the block is zero. □

Lemma 3.4.6. *Q_f is a direct sum of $P(\mu)$ not isomorphic to the P_e of some unipotent block of cyclic defect, and representations isomorphic to the P_e/Q_e of some unipotent block of cyclic defect. All said representations occur in the sum.*

Proof. As $\mathcal{I}_f$ contains $\mathsf{H}_{\neq 1}(G_f)$, the summand of $\mathcal{I}_f\Gamma_f$ in $\mathcal{B}_{\neq 1}(G_f)$ is the same as the summand of Γ_f in $\mathcal{B}_{\neq 1}(G_f)$. As the unipotent part of Γ_f is a progenerator of $\mathcal{B}_1(G_f)$, it is a direct sum of the $P(\mu)$, and every $P(\mu)$ is a summand of it. The unipotent part of $\mathcal{I}_f$ is $\mathrm{rad}(\mathsf{H}_{1,e}(G_f))$. Considering these as morphisms of UPIRs as before, and considering each unipotent block separately, we can see that $\mathrm{rad}(\mathsf{H}_{1,e}(G_f))P(\mu) = 0$ for all $P(\mu)$ apart from the P_e of a block of cyclic defect, where $\mathrm{rad}(\mathsf{H}_{1,e}(G_f))P_e = Q_e$. Thus the quotient Q_f will be a direct sum of $P(\mu)$ except for the P_e, and of representations isomorphic to P_e/Q_e, and all such representations will occur in the sum. □

Note that $P_e/Q_e \cong Q_{e-1}$. Thus we can see directly the first part of the theorem.

Corollary 3.4.7. *Q_f is a direct sum of subrepresentations of P_f.*

Proof. Every summand of Q_f is either a $P(\mu)$ that isn't a P_e, which is a summand of P_f, or Q_{e-1}, which is a subrepresentation of P_{e-1} and hence a subrepresentation of P_f. □

We now turn to the second part of the theorem. Let ν be a partition of $n - e$. To ν we associate the partition κ of n given by adding e to the first element of ν. Then κ has e-core ν, and is lexicographically the largest κ with this property. Hence $D(\kappa)$ is the D_1 of the block of cyclic defect associated to ν. Because of this, we can expand the reasoning used in the proof of Theorem 3.2.11.

Lemma 3.4.8. *The D_1 of the block associated to ν is a summand of* $\mathrm{ind}_{J_f(\kappa)}^{G_f} \mathbb{1}$.

Proof. We know that D_1 is $D(\kappa)$. By Theorem 8.1 of James [1986], this is the unique simple quotient of an indecomposable module $S(\kappa)$, all of whose other irreducible subquotients are $D(\mu)$ for $\mu > \kappa$. But since κ is the greatest partition in its block with respect to the lexicographical order, and hence also for the dominance order, D_1 cannot have any nontrivial extensions by such a representation. Thus in fact $D_1 = S(\kappa)$.

Now, again by Theorem 8.1 of James [1986], $S(\kappa)$ is in turn a submodule of $\mathrm{ind}_{J_f(\kappa)}^{G_f}\mathbb{1}$, and then applying Theorem 7.19(iii) of the same gives that $\mathrm{ind}_{J_f(\kappa)}^{G_f}\mathbb{1}$ has a composition series by $S(\lambda)$ with multiplicities zero unless $\lambda \geq \kappa$, in which $S(\kappa) = D(\kappa)$ occurs with multiplicity one. But all the irreducible subquotients of $S(\lambda)$ for $\lambda > \kappa$ are $D(\mu)$ for $\mu \geq \lambda > \kappa$, and hence lie in different blocks to $D(\kappa)$, so they cannot have nontrivial extension with $D(\kappa)$. Thus, $D_1 = D(\kappa)$ is a summand of $\mathrm{ind}_{J_f(\kappa)}^{G_f}\mathbb{1}$. $\square$

Thus we can give a much more explicit proof of Theorem 3.2.11.

Theorem 3.4.9. $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = D^b_{fg}(\mathsf{H}_1(G_f))$.

Proof. We shall show that both $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ and $\langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contain every $D(\mu)$.

Both Q_f and V_f contain as summands $P(\mu)$ apart from the P_e of the blocks of cyclic defect. In particular, they both contain every $D(\mu)$ in a semisimple block. It thus remains to consider a block of cyclic defect.

Observe that, if some triangulated and idempotent-complete subcategory of $D^b_{fg}(\mathsf{H}_1(G_f))$ contains D_i and P_i, then it must contain D_{i-1} and D_{i+1}. This is because taking the quotient of the inclusion $D_i \to P_i$ and then the kernel of the quotient $P_i/D_i \to D_i$ gives $D_{i-1} \oplus D_{i+1}$.

Now, V_f has summands isomorphic to D_1 and to $P_1, \ldots, P_{e-1}$. Hence by the above claim $\langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contains all irreducible representations $D_1, \ldots, D_e$.

Next, observe that the kernel of the inclusion $Q_i \to P_i$ is Q_{i-1}, so any triangulated category containing P_i and Q_i must contain Q_{i-1}.

Hence, as Q_f has summands isomorphic to Q_{e-1} and $P_1, \ldots, P_{e-1}$, by the above claim $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contains Q_1. But the quotient of the inclusion of

Q_1 in P_1 is D_1, so $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contains D_1, and so by the previous case $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))}$ contains all irreducible representations $D_1, \ldots, D_e$. □

3.4.1 Example: $n = 2$

We explore in more depth the simplest case, namely when $n = 2$. Hence $l \mid q^2 - 1$ but $l \nmid q - 1$ and $l \nmid q$, or in other words l is odd and $l \mid q + 1$.

Here, Lemma 3.4.1 and its corollaries tell us that the principal block contains two irreducible representations $D_1 = \mathbb{1}$ and D_2 cuspidal, with projective covers P_1 and P_2 respectively, such that P_1 a summand of P_f but P_2 is not.

In fact, by Frobenius reciprocity, we get $\mathrm{Hom}_{G_f}(P_f, \mathbb{1}) = \mathrm{Hom}_{I_f}(\mathbb{1}, \mathbb{1})$, and so P_f has a trivial quotient. Thus we can see directly that P_f must contain P_1 as a summand.

Furthermore, by Frobenius reciprocity again, we have that $\mathrm{Hom}_{G_f}(P_f, P_f) = \mathrm{Hom}_{I_f}(\mathbb{1}, \mathrm{res}^{G_f}_{I_f} P_f)$. Applying the Mackey decomposition and using the Bruhat decomposition $G_f = I_f \sqcup I_f \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) I_f$ we hence obtain $\mathrm{Hom}_{G_f}(P_f, P_f) = \mathrm{Hom}_{I_f}(\mathbb{1}, \mathbb{1}) \oplus \mathrm{Hom}_{I_f}(\mathbb{1}, \mathrm{ind}^{I_f}_{T_f} \mathbb{1}) = \mathrm{Hom}_{I_f}(\mathbb{1}, \mathbb{1}) \oplus \mathrm{Hom}_{T_f}(\mathbb{1}, \mathbb{1})$. Hence $\mathrm{End}(P_f)$ is two-dimensional.

But, by Lemma 3.4.1, $\mathrm{End}(P_1)$ is already two-dimensional, so we in fact must have equality $P_f = P_1$. In particular, this also directly shows that P_2 is not a summand of P_f. Now, as the unipotent blocks are those whose irreducible representations are subquotients of P_f, and the only subquotients of $P_f = P_1$ are D_1 and D_2, we can see directly also that the principal block is the only unipotent block. We could also have seen this by noting that (2) and $(1, 1)$ are the only two partitions of 2, and both have empty 2-core, so there is indeed only one unipotent block, containing two irreducible representations.

Furthermore, as $P_1 = P_f = \mathrm{ind}^{G_f}_{I_f} \mathbb{1}$, we may thus describe D_2 explicitly: D_2 is the quotient of $\mathrm{rad}(P_f) = \left\{ \sum_{g \in G_f/I_f} r_g g \,\middle|\, \sum_{g \in G_f/I_f} r_g = 0 \right\}$ by the relation $\sim$ such that $\sum_{g \in G_f/I_f} r_g g \sim 0$ if and only if $r_g = r_{g'}$ for all g, g'.

Now, in this case, we have the explicit description $Q_f = P_1 \oplus Q_1 = P_f \oplus \mathrm{rad}(P_f)$, which is not equal to $V_f = P_f \oplus \mathbb{1}$. Nonetheless, Q_f is manifestly a direct sum of submodules of P_f, and we can see that $\langle Q_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} = \langle V_f \rangle_{D^b_{fg}(\mathsf{H}_1(G_f))} =$

$D^b_{fg}(\mathsf{H}_1(G_f))$ via the short exact sequences

$$0 \longrightarrow \mathrm{rad}(P_1) \longrightarrow P_1 \longrightarrow \mathbb{1} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{1} \longrightarrow \mathrm{rad}(P_1) \longrightarrow D_2 \longrightarrow 0.$$

We can also describe the structure of $\mathcal{I}_f = \mathsf{H}_{\neq 1}(G_f) \oplus \mathrm{rad}(\mathsf{H}_{1,e}(G_f))$ explicitly in this case. Write $\mathsf{H}_1(G_f)$ as a direct sum of $P(\mu)$, and fix a copy of P_2 in this sum. Henceforth, P_2 shall be considered as a subrepresentation of $\mathsf{H}_1(G_f)$ via the inclusion of the summand we have fixed. We can see from Lemma 3.4.1 that $\mathrm{rad}(\mathsf{H}_{1,e}(G_f))$ is generated by any element γ corresponding to a morphism $P_2 \to P_2$ with image Q_2. It is possible to give an explicit description of such a γ. For simplicity of exposition, we assume $p \neq 2$.

Write e for the central idempotent of the unipotent block. Note that $e_1 = \frac{1}{|I_f|}\sum_{g \in I_f} g$ is idempotent and has $\mathsf{H}(G_f)\frac{1}{|I_f|}\sum_{g\in I_f} g = P_f$. Thus $P_1 = P_f$ is generated by e_1. Write e_2 for the primitive idempotent generating P_2.

Consider now the quadratic extension k' of k gained by adjoining a square root of a nonsquare element ϵ, and let $x + \sqrt{\epsilon}y$ for $x, y \in k$ generate the l-torsion in $k'^{\times}$. Note that, as there is no l-torsion in k (by our assumption that l does not divide $q-1$), but there is nontrivial l-torsion in $k'^{\times}$ (as l does divide $q+1$ and hence q^2-1) we have that x and y are both nonzero: if not, then $(x+\sqrt{\epsilon}y)^2$ lies in k, and hence $x+\sqrt{\epsilon}y$ has order dividing $2(q-1)$, which has no l-torsion, a contradiction. Write l^r for the order of $x + \sqrt{\epsilon}y$ in $k'^{\times}$, noting that $r > 0$ by the previous discussion.

We shall also make use of the following two conjugacy classes in G_f: the conjugacy class in G_f of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ will be denoted C_1, and the conjugacy class of $\left(\begin{smallmatrix} x & y \\ \epsilon y & x \end{smallmatrix}\right)$ will be denoted C_2. Also write $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for the identity matrix in G_f.

We make the initial observation that

$$C_1 = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \middle| a \in k^{\times} \right\} \cup \left\{ \begin{pmatrix} 1-ab & a \\ -ab^2 & 1+ab \end{pmatrix} \middle| a \in k^{\times}, b \in k \right\}$$

and

$$C_2 = \left\{ \begin{pmatrix} b & \frac{\epsilon y^2 - (x-b)^2}{a} \\ a & 2x - b \end{pmatrix} \middle| a \in k^{\times}, b \in k \right\}.$$

To see this, note that, for two-by-two matrices, any noncentral matrices sharing a characteristic polynomial must lie in the same conjugacy class. Hence, as the matrices listed have characteristic polynomials $(X-1)^2$ and $(X-x)^2 - \epsilon y^2$ respectively, and are not central, they do indeed lie in C_1 and C_2 respectively. But it is known (see for example Digne and Michel [1991], Chapter 15 Table 1) that the sizes of C_1 and C_2 are q^2-1 and $q(q-1)$ respectively, so we have in fact obtained the entire class.

We finally write $Z_\lambda = -\sum_{g\in C_1} g + \sum_{g\in C_2} g + \lambda 1$.

Lemma 3.4.10. *There exists some (necessarily unique) $\lambda \in R$ such that we can take γ to be $e_2 Z_\lambda$.*

Proof. By Paige [2014], Proposition 2.6, P_2 is the extension of scalars to R of a representation $\tilde{P}_2$ over $\mathcal{O}_K$, which, when extended to K, decomposes as a direct sum of irreducible representations: the Steinberg representation π_0 and each of the supercuspidal lifts π_i of D_2, for i in the range $0 < i \leq \frac{l^r-1}{2}$.

From the final remarks of Section 2 on page 363 and the opening remarks of Section 4 on page 368 of Paige [2014], there is an injective homomorphism $\psi : \mathrm{End}(\tilde{P}_2) \hookrightarrow \mathbb{K}^{\oplus \frac{l^r+1}{2}}$, given by sending the endomorphism υ to a tuple (u_0, u_i) where u_0 and u_i are the induced actions of υ on π_0 and π_i respectively (necessarily scalar as π_0 and π_i are irreducible).

Write $\mathsf{H}_{\mathcal{O}_K}(G_f)$ for the group algebra of G_f over $\mathcal{O}_K$. Then, by those same remarks, the natural map $r : Z(\mathsf{H}_{\mathcal{O}_K}(G_f)) \to \mathrm{End}(\tilde{P}_2)$, sending an element of the centre of the group algebra to its action by multiplication, is surjective.

Then, by Theorem 4.11 and Remark 1 in section 4 of Paige [2014], we see that $\mathrm{End}(\tilde{P}_2)$ is generated as an algebra by $r_{\tilde{Y}}$ for some $\tilde{Y} \in Z(\mathsf{H}_{\mathcal{O}_K}(G_f))$, and that $\psi(r_{\tilde{Y}}) = (2, \zeta^i + \zeta^{iq}) = (2, \zeta^i + \zeta^{-i})$, for ζ a primitive l^rth root of unity.

To find an explicit description for a $\tilde{Y}$ with this tuple, we make repeated use of Lemma 4.1 from Paige [2014], which says that, if C is a conjugacy class in G_f, then $\psi(r_{\sum_{g\in C} g}) = |C|(\frac{\mathrm{Tr}(\pi_0(C))}{\dim \pi_0}, \frac{\mathrm{Tr}(\pi_i(C))}{\dim \pi_i})$. We combine this with the character table information from Digne and Michel [1991] Chapter 15 Table 1, which says that

Cong. class	1	C_1	C_2
Size	1	$(q+1)(q-1)$	$q(q-1)$
$\mathrm{Tr}(\pi_0)$	q	0	-1
$\mathrm{Tr}(\pi_i)$	$q-1$	-1	$-\zeta^i - \zeta^{iq}$

Using this, and that $\mathrm{Tr}(\pi(1)) = \dim \pi$ for any representation π over K, we get that $\psi(r_{\sum_{g \in C_1} g}) = (0, -(q+1))$, that $\psi(r_{\sum_{g \in C_2} g}) = (-(q-1), -q(\zeta^i + \zeta^{iq}))$, and that $\psi(r_1) = (1, 1)$.

Thus we find that a valid choice is $\tilde{Y} = \frac{1}{q}\sum_{g \in C_1} g - \frac{1}{q}\sum_{g \in C_2} g + \frac{q+1}{q}1$, as then $\psi(r_{\tilde{Y}}) = (2, \zeta^i + \zeta^{iq})$, and as $\tilde{Y}$ is a sum of sums over conjugacy classes it is central. Now, as $\tilde{Y}$ is central, we have that the left action of $\tilde{Y}$ and the right action of $e_2\tilde{Y}e_2 = e_2\tilde{Y}$ on $\tilde{P}_2 = \mathsf{H}_{\mathcal{O}_K}(G_f)e_2$ define the same endomorphism of $\tilde{P}_2$, and hence the right action of $e_2\tilde{Y}$ also generates $\mathrm{End}(\tilde{P}_2)$.

Recall that $\mathrm{End}(\tilde{P}_2) \cong e_2\mathsf{H}_{\mathcal{O}_K}(G_f)^{\mathsf{op}}e_2$ and $\mathrm{End}(P_2) \cong e_2\mathsf{H}(G_f)^{\mathsf{op}}e_2$. Thus we get a surjective map $\mathrm{End}(\tilde{P}_2) \cong e_2\mathsf{H}_{\mathcal{O}_K}(G_f)^{\mathsf{op}}e_2 \to e_2\mathsf{H}(G_f)^{\mathsf{op}}e_2 \cong \mathrm{End}(P_2)$ induced by the quotient $\mathsf{H}_{\mathcal{O}_K}(G_f) \to \mathsf{H}(G_f)$. Hence the right action of $e_2\tilde{Y} \in e_2\mathsf{H}(G_f)e_2$ generates $\mathrm{End}(P_2)$.

Define $Y = -\sum_{g \in C_1} g + \sum_{g \in C_2} g$. As $q + 1 = 0$ and q is invertible in R, and since the right action of $e_2\tilde{Y}$ generates $\mathrm{End}(P_2)$, the right action of e_2Y also generates $\mathrm{End}(P_2)$. Observe that Y is also central.

Fix some $\gamma' \in \mathrm{End}(P_2)$ with image Q_2. Then $\mathrm{End}(P_2) \cong e_2\mathsf{H}(G_f)e_2$ is spanned by e_2 and the positive powers of γ'. Thus, there exists some unique λ such that $e_2Z_\lambda = e_2(Y + \lambda 1)$ is a linear combination of strictly positive powers of γ'. But if the image of e_2Z_λ is a strict subset of Q_2, then it must in fact be a linear combinations of $\gamma' k$ for $k \geq 2$, and so $e_2Y = -\lambda 1 + \sum_{k \geq 2} r_k \gamma'^k$ cannot generate the endomorphism γ', a contradiction. Thus e_2Z_λ must have image Q_2, and so is a valid choice for γ. $\square$

Lemma 3.4.11. *We can write* $Z_{q-1} = \sum_i \mu_i c_i x_i z_i + \sum_j \nu_j d_j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_j$ *for some* $c_i, d_j \in I_f^1$*, some* $x_i, y_j \in T_f$*, some* $z_i \in \bar{I}_f^1$*, and some* $\mu_i, \nu_j \in R$*. Furthermore, we may choose this sum such that it satisfies* $\sum_i \mu_i c_i = 0$ *and* $\sum_j \nu_j d_j = 0$.

Proof. We proceed by factorising each g in $C_1 \cup C_2$ into one of the forms μcxz or $\nu d \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y$ for $c, d \in I_f^1$, $x, y \in T_f$, $z \in \bar{I}_f^1$, and $\mu, \nu \in R$. We shall make repeated use of the following formula:

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & xz^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w - xyz^{-1} & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ yz^{-1} & 1 \end{pmatrix} & \text{if } z \neq 0 \\ \begin{pmatrix} 1 & wy^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} & \text{if } z = 0,\ y \neq 0 \end{cases}$$

observing that these are respectively of the two above forms.

We start with $g \in C_1$. First, note that $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \in \bar{I}_f^1$ for all $a \in k^\times$, and so summing over the $q-1$ terms in C_1 of this form we get $\sum_{a \in k^\times} z_{1,0,a}$ for $z_{1,0,a} = \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \in \bar{I}_f^1$.

Next, let $a \in k^\times$, $b \in k$ be such that $1 + ab \neq 0$. Then

$$\begin{pmatrix} 1-ab & a \\ -ab^2 & 1+ab \end{pmatrix} \in \begin{pmatrix} 1 & a(1+ab)^{-1} \\ 0 & 1 \end{pmatrix} T_f \bar{I}_f^1.$$

Now, observe that $a(1+ab)^{-1} = (a^{-1}+b)^{-1}$, and so we have two cases:

If $b = 0$, then in fact we just have

$$\begin{pmatrix} 1-ab & a \\ -ab^2 & 1+ab \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

Hence summing the $q-1$ terms we get by varying a over $k^\times$ gives $\sum_{a \in k^\times} \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$.

If $b \neq 0$, then $1 + ab = 0$ precisely when $a = -b^{-1}$, so fixing b and varying a over $k^\times \backslash \{-b^{-1}\}$ means $(a^{-1}+b)^{-1}$ takes every value a' in k a single time (by setting $a = (a'-b)^{-1}$) except 0 (which would need $a = -b^{-1}$) and b^{-1} (which would need $a = 0$). Hence summing these $q-2$ terms gives $\sum_{a' \in k^\times \backslash \{b^{-1}\}} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) x_{1,b,a'} z_{1,b,a'}$ for some $x_{1,b,a'} \in T_f$ and $z_{1,b,a'} \in \bar{I}_f^1$.

Meanwhile, if $1 + ab = 0$, we observe that necessarily $b \neq 0$, and that if we fix such a b then there is exactly one a satisfying this relation, namely $a = -b^{-1}$. Now, we have that

$$\begin{pmatrix} 1-ab & a \\ -ab^2 & 1+ab \end{pmatrix} \in \begin{pmatrix} 1 & (1-ab)(-ab^2)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{I}_f^1.$$

Simplifying gives $(1-ab)(-ab^2)^{-1} = 2b^{-1}$, and varying b over $k^\times$ means $2b^{-1}$ takes every value $a' \in k^\times$ a single time (by setting $b = 2a'^{-1}$). Hence summing these $q-1$ terms gives $\sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_{1,a'}$ for some $y_{1,a'} \in T_f$.

Now we consider $g \in C_2$. Let $a \in k^\times$ and $b \in k$, and suppose first that $2x - b \neq 0$. Then

$$\begin{pmatrix} b & \frac{\epsilon y^2 - (x-b)^2}{a} \\ a & 2x - b \end{pmatrix} \in \begin{pmatrix} 1 & \frac{\epsilon y^2 - (x-b)^2}{a(2x-b)} \\ 0 & 1 \end{pmatrix} T_f \bar{I}_f^1.$$

Now, note that $\epsilon y^2 - (x-b)^2 \neq 0$ as ϵ is by assumption not a square. Hence fixing b and varying a over $k^\times$ means $\frac{\epsilon y^2 - (x-b)^2}{a(2x-b)}$ takes every value $a' \in k^\times$ a single time (by setting $a = \frac{\epsilon y^2 - (x-b)^2}{a'(2x-b)}$). Hence summing these $q-1$ terms gives $\sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) x_{2,b,a'} z_{2,b,a'}$ for some $x_{2,b,a'} \in T_f$ and $z_{2,b,a'} \in \bar{I}_f^1$.

The remaining case is when $2x - b = 0$, noting there is exactly one such b, namely $2x$, and that it is not zero, since x cannot be zero and $l \neq 2$. Then

$$\begin{pmatrix} b & \frac{\epsilon y^2 - (x-b)^2}{a} \\ a & 2x - b \end{pmatrix} \in \begin{pmatrix} 1 & ba^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T_f$$

and so varying a over $k^\times$ means ba^{-1} takes every value $a' \in k^\times$ a single time (by setting $a = ba'^{-1}$). Hence summing these $q-1$ terms gives $\sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_{2,a'}$ for some $y_{2,a'} \in T_f$.

Now, we recall that we have $Z_{q-1} = -\sum_{g \in C_1} g + \sum_{g \in c_2} g + (q-1)1$. Hence, putting all our results together, we have that

$$\begin{aligned} Z_{q-1} = &-\sum_{a \in k^\times} z_{1,0,a} - \sum_{a \in k^\times} \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) - \sum_{b \in k^\times} \sum_{a' \in k^\times \setminus \{b^{-1}\}} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) x_{1,b,a'} z_{1,b,a'} \\ &- \sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_{1,a'} + \sum_{b \in k \setminus \{2x\}} \sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) x_{2,b,a'} z_{2,b,a'} \\ &+ \sum_{a' \in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_{2,a'} + (q-1)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{aligned}$$

which is of the form $\sum_i \mu_i c_i x_i z_i + \sum_j \nu_j d_j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_j$ for some $c_i, d_j \in I_f^1$, some $x_i, y_j \in T_f$, some $z_i \in \bar{I}_f^1$, and some $\mu_i, \nu_j \in R$. It remains to check that $\sum_i \mu_i c_i = 0$ and $\sum_j \nu_j d_j = 0$.

Considering first $\sum_j \nu_j d_j$, we can see that we get

$$-\sum_{a'\in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a'\in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) = 0.$$

Now we consider $\sum_i \mu_i c_i$. This is

$$\begin{aligned}
&-\sum_{a\in k^\times} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) - \sum_{a\in k^\times} \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) - \sum_{b\in k^\times} \sum_{a'\in k^\times\setminus\{b^{-1}\}} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) \\
&+ \sum_{b\in k\setminus\{2x\}} \sum_{a'\in k^\times} \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) + (q-1) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\
&= (-(q-1)+(q-1)) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a'\in k^\times} (-1-(q-2)+(q-1)) \left(\begin{smallmatrix} 1 & a' \\ 0 & 1 \end{smallmatrix}\right) \\
&= 0.
\end{aligned}$$

□

The above calculation allows us to find the value of λ such that $\gamma = e_2 Z_\lambda$.

Lemma 3.4.12. *$(e - e_1)Z_{q-1} \in \mathcal{I}_f$.*

Proof. We shall directly calculate $(e - e_1)Z_{q-1}P_1$ and observe that it is zero.

Now, $eP_1 = P_1$, so it in fact suffices to show that $(1 - e_1)Z_{q-1}P_1 = 0$. Thus, we need to show for any $g \in G_f$ that $(1 - e_1)Z_{q-1}g \sum_{b\in I_f} b = 0$, that is, that $Z_{q-1}g \sum_{b\in I_f} b$ is left-I_f-invariant.

By the Bruhat decomposition, we may without loss of generality take $g = iw$ for $i \in I_f$ and w either 1 or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. But i commutes with Z_{q-1} as the latter is central, and so $Z_{q-1}g \sum_{b\in I_f} b = iZ_{q-1}w \sum_{b\in I_f} b$. Thus we may without loss of generality take $i = 1$, and show that $Z_{q-1}w \sum_{b\in I_f} b$ is left-I_f-invariant.

Now, we can write $I_f = I_f^1 T_f$. Observe first that T_f commutes with w, and also with Z_{q-1} as Z_{q-1} is central. Thus, if $i \in T_f$, then $iZ_{q-1}w \sum_{b\in I_f} b = Z_{q-1}wi \sum_{b\in I_f} b = Z_{q-1}w \sum_{b\in I_f} b$, so $Z_{q-1}w \sum_{b\in I_f} b$ is left-T_f-invariant. Thus, it only remains to prove that $Z_{q-1}w \sum_{b\in I_f} b$ is left I_f^1-invariant.

We shall divide this into two cases depending on the value of w, respectively 1 and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Consider the first case. Then $Z_{q-1}w\sum_{b\in I_f} b = Z_{q-1}\sum_{b\in I_f} b$. As Z_{q-1} is central, I_f^1 commutes with Z_{q-1}, so if $i \in I_f^1$ then $iZ_{q-1}\sum_{b\in I_f} b = Z_{q-1}i\sum_{b\in I_f} b = Z_{q-1}\sum_{b\in I_f} b$, so we have left-$I_f^1$-invariance.

For the second case, we use that, by Lemma 3.4.11, we have $Z_{q-1} = \sum_i \mu_i c_i x_i z_i + \sum_j \nu_j d_j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_j$ for some $c_i, d_j \in I_f^1$, some $x_i, y_j \in T_f$, some $z_i \in \bar{I}_f^1$, and some $\mu_i, \nu_i \in R$, such that $\sum_i \mu_i c_i = 0$ and $\sum_j \nu_j d_j = 0$. Now, we have that $\bar{I}_f^1 w = wI_f^1 \subseteq wI_f$, so $z_i w \in wI_f$. Additionally, $T_f w = wT_f \subseteq wI_f$, so $x_i w, y_j w \in wI_f$. Thus,

$$\begin{aligned} Z_{q-1}w\sum_{b\in I_f} b &= \sum_i \mu_i c_i x_i z_i w \sum_{b\in I_f} b + \sum_j \nu_j d_j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) y_j w \sum_{b\in I_f} b \\ &= \sum_i \mu_i c_i w \sum_{b\in I_f} b + \sum_j \nu_j d_j \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) w \sum_{b\in I_f} b \\ &= 0 \end{aligned}$$

where the last equality follows as $\sum_i \mu_i c_i = 0$ and $\sum_j \nu_j d_j = 0$. As in fact $Z_{q-1}w\sum_{b\in I_f} b = 0$ in this case, it is certainly in particular left-I_f^1-invariant. □

Corollary 3.4.13. $\gamma = e_2 Z_{q-1}$.

Proof. We have that $(e - e_1)Z_{q-1} \in \mathcal{I}_f$, and hence $e_2 Z_{q-1} = e_2(e - e_1)Z_{q-1} \in \mathcal{I}_f$. But then for all $\lambda \neq q-1$ we have that $e_2 Z_\lambda P_1 = e_2(\lambda - (q-1))P_1$ is one-dimensional and spanned by β, so $e_2 Z_\lambda \notin \mathcal{I}_f$. Thus as $\gamma \in \mathcal{I}_f$ we must have $\gamma = e_2 Z_{q-1}$. □

Henceforth we shall simply write Z for Z_{q-1}.

Lemma 3.4.14. $(e - e_1)Z$ *generates the unipotent part of* $\mathcal{I}_f$.

Proof. We know that $(e - e_1)Z \in \mathcal{I}_f$. Furthermore, we have that $\gamma = e_2 Z = e_2(e - e_1)Z$ and that γ generates the unipotent part of $\mathcal{I}_f$. Hence so does $(e - e_1)Z$. □

We thus have an explicit description of $\mathcal{I}_f$, namely that it is generated by the non-unipotent elements plus a single unipotent element $(e - e_1)Z$, where $Z = -\sum_{g\in C_1} g + \sum_{g\in C_2} g + (q-1)1$ for certain conjugacy classes C_1 and C_2 in G_f, whose elements we have given explicitly. These results generalise to the case $n = e > 2$ by considering a degree n extension instead of a quadratic extension, with appropriate choices of C_1, C_2 and λ.

4

l-Modular Unipotent Representations of p-adic Groups

Synopsis

We establish the first part of the main theorem of this thesis, namely that Q classically generates $D^b_{fg}(\mathsf{H}_1(G))$. This essentially amounts to showing that various finiteness and generation results from the finite setting continue to hold in the p-adic setting. After defining relevant objects, we present Vignéras's p-adic version of Takeuchi's results on generators, as well as two key finiteness properties: that $\mathrm{Mod}(G)$ is Noetherian, and that $\mathsf{S}_R(n)$ has finite global dimension. The ingredients for both already exist in the literature, but they nonetheless require some care to rigorously prove. In particular, the former involves the property of second adjunction, and the latter employs the theory of affine cellular algebras. We then use these to give the first part of the main theorem.

4.1 Definitions and Notation

Let F denote a p-adic field, with ring of integers $\mathcal{O}$, uniformiser ϖ, and residue field k. Let G be a (connected) reductive algebraic group over F, and write $G = \mathrm{G}(F)$ for its F-points, which we consider as a topological group via the topology on F. For simplicity of exposition we will assume that G is unramified, that is, it is the extension of scalars to F of a reductive group scheme over $\mathcal{O}$.

Moreover, we will fix such a reductive group scheme over $\mathcal{O}$, and also write it as G. For GL_n we simply take $\mathrm{G} = \mathrm{GL}_n$ over $\mathcal{O}$.

Let $K = \mathrm{G}(\mathcal{O})$. Recall that this is a maximal parahoric subgroup. Let K^1 be its pro-p radical. For GL_n, we therefore have that $K^1 = 1 + \varpi M_{n,n}(\mathcal{O})$. Write G_f for the reductive quotient of K, that is, $\mathrm{G}(k)$. We retain the notation from the previous chapter for the various subgroups and representations of G_f.

Inside K, fix an Iwahori subgroup I. We may without loss of generality take I to be the preimage of I_f under the quotient $K \to G_f$. We fix a split torus S, which we can take without loss of generality for GL_n to be the diagonal matrices. We write I_0 for the compact part of S, which is the intersection of S and I. For GL_n, the Iwahori-Weyl group $W = N(S)/I_0$ is isomorphic to $\mathbb{Z}^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ acts on $\mathbb{Z}^n$ by permuting the entries. Furthermore, W has a canonical splitting by sending $\mathfrak{S}_n$ to the permutation matrices and $(i_1, \ldots, i_n) \in \mathbb{Z}^n$ to the diagonal matrix with (j, j)th entry ϖ^{i_j}.

Let R be a commutative ring in which p is invertible. Recall from Section 2.7 that we write $\mathrm{Mod}(G)$ for the category of smooth representations of G over R, and that $\mathrm{Mod}(G)$ is isomorphic to the category of nondegenerate modules over the global Hecke algebra $\mathsf{H}(G)$. We fix a Haar measure on G with $\mu(K^1) = 1$.

4.2 The Unipotent Block of p-adic GL_n

For this section only, R shall be an algebraically closed field of characteristic different from p, and G shall be GL_n. Many of the properties of G_f used in the previous chapter to establish the finite version of the main theorem can be proven in an analogous manner for G. We first recall Vignéras's generalisation of Takeuchi [1996].

Definition 4.2.1. We call the block of $\mathrm{Mod}(G)$ containing the trivial representation $\mathbb{1}$ the unipotent block. Write $\mathcal{B}_{\neq 1}(G)$ for the direct sum of all non-unipotent blocks.

Write $\mathsf{H}_1(G)$ and $\mathsf{H}_{\neq 1}(G)$ for the corresponding direct sums of block algebras of $\mathsf{H}(G)$.

For a representation M, write M_1 for its summand in the unipotent block.

Let $P = \mathrm{I}^G_{G_f,K} P_f = \mathrm{ind}^G_I \mathbb{1}$.

Let $\mathcal{I}$ be the annihilator in $\mathsf{H}(G)$ of P.

Let $\mathsf{H}'_1(G) = \mathsf{H}(G)/\mathcal{I}$, and let $\mathcal{B}'_1(G)$ be the category of $\mathsf{H}(G)$-modules annihilated by $\mathcal{I}$, that is, the category of modules over $\mathsf{H}'_1(G)$.

Let $\mathcal{J}$ be the set of all parahoric subgroups containing I and contained in K, that is, the preimages of $J_f \in \mathcal{J}_f$ under the quotient $K \to G_f$. We call those parahoric subgroups in $\mathcal{J}$ standard.

Let $\Gamma = \mathrm{I}^G_{G_f,K}\Gamma_f = \bigoplus_{J\in\mathcal{J}} \bigoplus_{\chi_{J_f}\in X_{J_f}} \mathrm{ind}^G_J \mathrm{infl}^J_{M_{J_f}} \mathrm{ind}^{M_{J_f}}_{U_{J_f}} \chi_{J_f}$, and let $Q = \Gamma/\mathcal{I}\Gamma$.

Remark. In Vignéras [2003], the definition of $\mathcal{J}$ is larger: it is the set of all J containing I, not just those contained in K. Hence, her definition of Γ contains more summands of the form $\Gamma_J = \mathrm{ind}^G_J \mathrm{infl}^J_{M_{J_f}} \mathrm{ind}^{M_{J_f}}_{U_{J_f}} \chi_{J_f}$. However, for GL_n, all J containing I are conjugate to some J' containing I and contained in K, and hence each summand Γ_J of her Γ is isomorphic to some summand $\Gamma_{J'}$ of our Γ. Hence, her notions of Γ_1 and Q are progenerators if and only if ours are, and the two notions have Morita equivalent endomorphism algebras and classically generate the same category.

By Section 5.12 of Vignéras [2003], we have that $P \in \mathcal{B}_1(G)$. Hence the quotient $\mathsf{H}(G) \to \mathsf{H}'_1(G)$ factors through $\mathsf{H}_1(G)$, and so $\mathcal{B}'_1(G)$ is a subcategory of $\mathcal{B}_1(G)$. We can now state Vignéras's result.

Proposition 4.2.2. *There exists some positive integer N such that $\mathcal{I}^N\mathcal{B}_1(G) = 0$. Furthermore, $\mathcal{B}'_1(G)$ has progenerator Q, and Γ has progenerator Γ_1.*

Proof. These are respectively Theorem 5.13 (3), Proposition 5.10, and Theorem 5.13 (1) of Vignéras [2003]. □

4.3 The Noetherian Property and Second Adjunction

We return to the case of R a general commutative ring (in which p is invertible) and G a general reductive group. To prove the first part of our main theorem, we need analogues for G of two finiteness properties that are known for G_f. The first is that $\mathrm{Mod}(G)$ is Noetherian. For $\mathrm{Mod}(G_f)$ this is immediate as $\mathsf{H}(G_f)$ is a finite dimensional algebra over a field. For $\mathrm{Mod}(G)$ this is essentially already known. We shall nonetheless be fully explicit in our exposition.

Definition 4.3.1. Let $\delta_G : G \to R$ be the map $\delta_G(g) = \mu(gK^1g^{-1})$.

We say that G has second adjunction over R if, for all parabolic subgroups C of G and all Levi subgroups M of C, we have that parabolic induction $\mathrm{i}^G_{M,C}$ is left adjoint to $\delta_C \mathrm{r}^G_{M,\bar{C}}$, where recall that $\bar{\mathrm{C}}$ is the opposite parabolic of C with respect to M.

Proposition 4.3.2. *Suppose R is Noetherian, and suppose that G has second adjunction over R. Then $\mathrm{Mod}(G)$ is Noetherian.*

Proof. This is Dat [2009] Theorem 1.3. □

Theorem 1.5 of Dat [2009] then gives that GL_n, alongside several other families of groups, have second adjunction. Since then, the result has been proven in full generality for any reductive group over F.

Proposition 4.3.3. G *has second adjunction over R.*

Proof. This is Dat et al. [2024a], Corollary 1.3. □

Remark. In fact, Dat et al. [2024a] establishes second adjunction via even stronger finiteness conditions on various Hecke algebras associated to G, which are proven in full generality in Dat et al. [2024b].

Corollary 4.3.4. *Suppose R is Noetherian. Then $\mathrm{Mod}(G)$ is Noetherian.*

In particular, when R is an algebraically closed field of characteristic not p, $\mathcal{B}_1'(G)$ and $\mathcal{B}_1(G)$ are Noetherian. The former is equivalent to the category of modules over $\mathsf{S}_R(n)$ by Proposition 4.2.2, so $\mathsf{S}_R(n)$ is Noetherian. Similarly, the latter is equivalent to the category of modules over $\mathrm{End}(\Gamma_1)$, and so $\mathrm{End}(\Gamma_1)$ is Noetherian. But being Noetherian is preserved by Morita equivalences of non-unital rings with enough idempotents (Ánh and Márki [1987], Proposition 3.3), and so $\mathsf{H}_1'(G)$ and $\mathsf{H}_1(G)$ are Noetherian. Thus, the ideal $\mathcal{I}_1$ in $\mathsf{H}_1(G)$ is finitely generated.

4.4 The Iwahori-Hecke and Schur Algebras

The other finiteness property we need to prove the first part of the main theorem is that a p-adic version of $\mathsf{S}_R(n)_f$ has finite global dimension. To define this algebra, we first need to introduce the Iwahori-Hecke algebra.

Definition 4.4.1. The Iwahori-Hecke algebra of G over R is the Hecke algebra $\mathsf{H}_R(G, I)$ of G with respect to I, as defined in Section 2.7. Thus, it is the algebra $\mathrm{End}_G(P)^{\mathsf{op}}$. For GL_n we shall also write the Iwahori-Hecke algebra as $\mathsf{H}_R(n)$.

Recall again from Section 2.7 that $\mathsf{H}_R(G, I)$ is isomorphic to the R-algebra spanned by indicator functions 1_{IgI}, with multiplication given by convolution. Thus, by the Iwahori decomposition, it has R-basis $\{T_w \mid w \in W\}$, where $T_w = 1_{IwI}$.

The structure of W can be used to give explicit generators and relations for $\mathsf{H}_R(G, I)$. Write l for the length function of W, and $\mathtt{S}$ for the simple affine reflections in W. For simplicity, we only recall their definitions for GL_n (see for example Morris [1993] for the definition in full generality). Write X_i for the ith standard basis element of $\mathbb{Z}^n$, and write elements of $\mathfrak{S}_n$ using cycle notation. Let $\sigma_i = (i \ \ i+1)$ for $i \in \{1, \ldots, n-1\}$, let $\sigma_0 = \sigma_{n-1} \ldots \sigma_2\sigma_1\sigma_2 \ldots \sigma_{n-1}(X_1X_n^{-1})$, and let $\tau = s_{n-1} \ldots s_1X_1$. Then $\mathtt{S} = \{\sigma_i | 0 \le i \le n-1\}$, and the length $l(w)$ of $w \in W$ is the minimal number of σ_i required to express w as a product of σ_i and τ.

Using this, we can give explicitly the relations defining $\mathsf{H}_R(G, I)$:

Proposition 4.4.2. *$\mathsf{H}_R(G, I)$ is generated by the T_w for $w \in W$ subject to the relations*

$$T_wT_w' = T_{ww'} \text{ for } w, w' \in W \text{ with } l(ww') = l(w) + l(w')$$
$$T_s^2 = (q-1)T_s + q \text{ for } s \in \mathtt{S}$$

Proof. This is Theorem 2.1 of Vignéras [2016]. □

Definition 4.4.3. Write $\mathbb{Z}[\mathrm{q}]$ for polynomials in q. Write $\mathsf{H}_{\mathrm{q}}(G, I)$ for the $\mathbb{Z}[\mathrm{q}]$-algebra with the generators and relations of Proposition 4.4.2, where q is replaced with q. We call this the Iwahori-Hecke algebra of G over $\mathbb{Z}[\mathrm{q}]$. For GL_n we shall also write it as $\mathsf{H}_{\mathrm{q}}(n)$.

Note that, by Remark 2.4(1) of Vignéras [2003], $\mathsf{H}_R(G,I) \cong R \otimes_{\mathbb{Z}[\mathrm{q}]} \mathsf{H}_{\mathrm{q}}(G,I)$, where we make R a $\mathbb{Z}[\mathrm{q}]$-algebra by sending $\mathrm{q} \mapsto q$.

Recall that W_f denotes the Weyl group of G_f. As G is unramified, this is the same as the (finite) Weyl group $W_f \subseteq W$ of G. Recall that for GL_n this is $\mathfrak{S}_n \subseteq \mathbb{Z}^n \rtimes \mathfrak{S}_n$. Write $\mathtt{S}_f$ for the simple reflections in W_f. For GL_n these are the transpositions σ_i for $1 \leq i \leq n-1$. For each subset $\mathtt{P}$ of $\mathtt{S}_f$, let $W_\mathtt{P}$ denote the subgroup of W_f generated by $\mathtt{P}$, and let $x_\mathtt{P} = \sum_{w \in W_\mathtt{P}} T_w$.

Definition 4.4.4. The Schur algebra of G over R is the R-algebra $\mathsf{S}_R(G,I)$ defined by

$$\mathsf{S}_R(G,I) = \mathrm{End}_{\mathsf{H}_R(G,I)}\left(\bigoplus_{\mathtt{P} \subseteq \mathtt{S}_f} x_\mathtt{P}\mathsf{H}_R(G,I)\right)$$

where the endomorphisms are of right $\mathsf{H}_R(G,I)$-modules.

The Schur algebra $\mathsf{S}_{\mathrm{q}}(G,I)$ of G over $\mathbb{Z}[\mathrm{q}]$ is similarly defined by replacing $\mathsf{H}_R(G,I)$ with $\mathsf{H}_{\mathrm{q}}(G,I)$ in the above definition.

For GL_n, we also denote the Schur algebras over R and $\mathbb{Z}[\mathrm{q}]$ as $\mathsf{S}_R(n)$ and $\mathsf{S}_{\mathrm{q}}(n)$ respectively.

Again by Remark 2.4(1) of Vignéras [2003], we have $\mathsf{S}_R(G,I) = R \otimes_{\mathbb{Z}[\mathrm{q}]} \mathsf{S}_{\mathrm{q}}(G,I)$. Furthermore, as $x_\varnothing = 1$, the endomorphisms of the $x_\varnothing\mathsf{H}_R(n)$ summand of $\mathsf{S}_R(G,I)$ (respectively the $x_\varnothing\mathsf{H}_{\mathrm{q}}(n)$ summand of $\mathsf{S}_{\mathrm{q}}(G,I)$) are canonically isomorphic to $\mathsf{H}_R(G,I)$ (respectively $\mathsf{H}_{\mathrm{q}}(G,I)$) by identifying an element of the latter with its action by left scalar multiplication on the former. Thus, $\mathsf{H}_R(G,I)$ (respectively $\mathsf{H}_{\mathrm{q}}(G,I)$) is an idempotent subalgebra of $\mathsf{S}_R(G,I)$ (respectively, $\mathsf{S}_{\mathrm{q}}(G,I)$).

As in the finite case, the importance of $\mathsf{S}_R(n)$ comes from the following surprising property:

Proposition 4.4.5. *Let* $\mathrm{G} = \mathrm{GL}_n$. *Then* $\mathrm{End}_G(Q)$ *is Morita equivalent to* $\mathsf{S}_R(n)$.

Proof. This is Proposition 5.8 of Vignéras [2003]. □

For GL_n, we can describe the subsets of $\mathtt{S}_f$ in a more explicit way.

Definition 4.4.6. A composition of n is a tuple $\lambda = (\lambda_1, \ldots, \lambda_M)$ of positive integers that sums to n.

Given a composition λ, define $\mathtt{P}(\lambda)$ to be the subset of $\mathtt{S}_f$ of all $(i\ i+1)$ with $\sum_{m'=1}^{m} \lambda_{m'} \le i < \sum_{m'=1}^{m+1} \lambda_{m'}$ for some m. This gives a bijection between compositions of n and subsets of $\mathtt{S}_f$.

4.5 The Global Dimension of the Schur Algebra

We now assume that $\mathrm{G} = \mathrm{GL}_n$. We also assume, without changing our notation, that $\mathsf{H}_{\mathrm{q}}(n)$ and $\mathsf{S}_{\mathrm{q}}(n)$ are defined over $\mathscr{Z} = \mathbb{Z}[\mathrm{q}^{\pm\frac{1}{2}}]$, and that R is a $\mathscr{Z}$-algebra (with $\mathrm{q} \mapsto q$ as before). Note that this requires a choice of $q^{\frac{1}{2}} \in R$.

Remark. This mild additional assumption is in order to draw on results from the literature. It would be interesting to investigate what happens without this assumption. For example, Du et al. [1998] show that $\mathsf{S}_R(n)_f$ is quasi-hereditary and has finite global dimension without this assumption, and it seems likely similar methods could be used for $\mathsf{S}_R(n)$.

Recall that $\mathsf{S}_R(n)_f$ was shown to have finite global dimension by giving the Morita equivalent algebra $\mathsf{S}_R(N, n)_f$ the structure of a quasi-hereditary algebra. We wish to show that $\mathsf{S}_R(n)$ has finite global dimension. Hence we proceed analogously to the finite case by finding a Morita equivalent algebra that has a well-behaved affine cellular structure.

Firstly, we give another presentation for the Hecke algebra: the Bernstein presentation. This is taken as the definition of the Hecke algebra in much of the literature on Schur algebras, in particular in McGerty [2003].

Proposition 4.5.1. $\mathsf{H}_{\mathrm{q}}(n)$ *has a presentation with generators* $T_i, 1 \le i \le n-1$, *and $X_i^{\pm 1}, 1 \leq i \leq n$, and relations*

$$\begin{aligned} T_i^2 &= (\mathrm{q}-1)T_i + \mathrm{q} \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \\ T_iT_j &= T_jT_i \text{ if } |i-j| > 1 \\ X_iX_i^{-1} &= 1 = X_i^{-1}X_i \\ X_iX_j &= X_jX_i \\ T_iX_iT_i &= \mathrm{q}X_{i+1} \\ T_iX_j &= X_jT_i \text{ if } j \notin \{i, i+1\}. \end{aligned}$$

Furthermore, $T_i = T_{s_i}$ for $1 \leq i \leq n-1$.

Proof. This is Theorem 1.4 of Vignéras [2006], which also contains a statement of the Bernstein decomposition for a general G, and a definition of the X_i in terms of the T_w. □

The proof in the finite case used the family of algebras $\mathsf{S}_R(N,n)_f$. These have a p-adic analogue.

Definition 4.5.2. Fix some $N > n$. An N-composition of n is an N-tuple of nonnegative integers that sum to n. Write the set of all N-compositions of n as $\Lambda(N,n)$.

For $\lambda \in \Lambda(N,n)$ we define $\mathtt{P}(\lambda)$ to be $\mathtt{P}(\lambda')$ where λ' is the composition of n given by deleting all the zero entries from λ.

Definition 4.5.3.

$$\mathsf{S}_{\mathrm{q}}(N,n) = \mathrm{End}_{\mathsf{H}_{\mathrm{q}}(n)}\left(\bigoplus_{\lambda \in \Lambda(N,n)} x_{\mathtt{P}(\lambda)}\mathsf{H}_{\mathrm{q}}(n)\right).$$

Let $\Lambda_0(N,n)$ denote the elements of $\Lambda(N,n)$ where all zero entries occur after every nonzero entry. There is thus a bijection between $\Lambda_0(N,n)$ and compositions of n. Write $\lambda(\mathtt{P})$ for the element of $\Lambda_0(N,n)$ corresponding to the composition λ with $\mathtt{P}(\lambda) = \mathtt{P}$. Then $\mathtt{P}(\lambda(\mathtt{P})) = \mathtt{P}$. Thus, by gluing the identity maps $x_{\mathtt{P}}\mathsf{H}_{\mathrm{q}}(n) \rightarrow x_{\mathtt{P}(\lambda(\mathtt{P}))}\mathsf{H}_{\mathrm{q}}(n)$, we obtain an inclusion

$\bigoplus_{\mathrm{P}\subseteq \mathrm{S}_f} x_{\mathrm{P}}\mathsf{H}_{\mathrm{q}}(n) \subseteq \bigoplus_{\lambda\in\Lambda(N,n)} x_{\mathrm{P}(\lambda)}\mathsf{H}_{\mathrm{q}}(n)$, and hence an inclusion $\mathsf{S}_{\mathrm{q}}(n) \subseteq \mathsf{S}_{\mathrm{q}}(N,n)$.

By the theory of affine cellular algebras, which we will expand on in the next section, $\mathsf{S}_{\mathrm{q}}(N,n)$ is known to have finite global dimension, and this is preserved under reasonable extensions of scalars.

Proposition 4.5.4. *Let R be a Noetherian domain of finite global dimension. Then $\mathsf{S}_{\mathrm{q}}(N,n) \otimes_{\mathscr{Z}} R$ has finite global dimension.*

Proof. In Section 1.10 of Lusztig [1999], Lusztig defines an algebra $\hat{\mathsf{S}}_{\mathrm{q}}(N,n)$ (written $\mathfrak{U}_{n,N,N;\mathscr{Z}}$ in his notation). In Cui [2015], Theorem 4.7 (see also Nakajima [2015]), $\hat{\mathsf{S}}_{\mathrm{q}}(N,n) \otimes_{\mathscr{Z}} R$ is shown to have finite global dimension. Note that they only claim that $\hat{\mathsf{S}}_{\mathrm{q}}(N,n) \otimes_{\mathbb{Z}} R$ has finite global dimension, but their proof in fact holds for $\hat{\mathsf{S}}_{\mathrm{q}}(N,n) \otimes_{\mathscr{Z}} R$, as their proof that $\hat{\mathsf{S}}_{\mathrm{q}}(N,n)$ is affine cellular over $\mathbb{Z}$ in fact shows that it is affine cellular over $\mathscr{Z}$. But $\hat{\mathsf{S}}_{\mathrm{q}}(N,n)$ is isomorphic to $\mathsf{S}_{\mathrm{q}}(N,n)$ by Pages 2 and 3 of McGerty [2003]. □

As in the finite case, we show that $\mathsf{S}_{\mathrm{q}}(N,n) \otimes_{\mathscr{Z}} R$ and $\mathsf{S}_R(n)$ are Morita equivalent. Let e denote the identity of $\mathsf{S}_{\mathrm{q}}(n)$, that is, the sum of the identities on each $x_{\mathrm{P}(\lambda(\mathrm{P}))}\mathsf{H}_{\mathrm{q}}(n)$. It is an idempotent in $\mathsf{S}_{\mathrm{q}}(N,n)$, and we have $\mathsf{S}_{\mathrm{q}}(n) = e\mathsf{S}_{\mathrm{q}}(N,n)e$.

Lemma 4.5.5. $\mathsf{S}_{\mathrm{q}}(N,n)e\mathsf{S}_{\mathrm{q}}(N,n) = \mathsf{S}_{\mathrm{q}}(N,n)$.

Proof. The identity map on $x_{\mathrm{P}(\lambda)}\mathsf{H}_{\mathrm{q}}(n)$ is gef, where f and g are the identity maps $x_{\mathrm{P}(\lambda)}\mathsf{H}_{\mathrm{q}}(n) \underset{g}{\overset{f}{\rightleftarrows}} x_{\mathrm{P}(\lambda(\mathrm{P}(\lambda)))}\mathsf{H}_{\mathrm{q}}(n)$. Thus $\mathsf{S}_{\mathrm{q}}(N,n)e\mathsf{S}_{\mathrm{q}}(N,n)$ contains the identity of $\mathsf{S}_{\mathrm{q}}(N,n)$, and thus is all of $\mathsf{S}_{\mathrm{q}}(N,n)$. □

Theorem 4.5.6. *Let R be a Noetherian domain of finite global dimension. Then $\mathsf{S}_R(n)$ has finite global dimension.*

Proof. $\mathsf{S}_{\mathrm{q}}(n)$ is Morita equivalent to $\mathsf{S}_{\mathrm{q}}(N,n)$ by Proposition 2.11.2, and hence $\mathsf{S}_R(n)$ is Morita equivalent to $\mathsf{S}_{\mathrm{q}}(N,n) \otimes_{\mathscr{Z}} R$. But the latter has finite global dimension. □

4.6 Aside: Affine Cellular Structures for the Schur and Hecke Algebras

We retain the assumptions of the previous section. We have so far treated the finite global dimension of $\mathsf{S}_{\mathrm{q}}(N,n)\otimes R$ as a black box. However, analogously to the finite case, this property is a single consequence of a much richer structure theory of $\mathsf{S}_{\mathrm{q}}(N,n)$: it is an affine cellular algebra. In this aside we will transfer this affine cellular structure to $\mathsf{S}_{\mathrm{q}}(n)$ and $\mathsf{H}_{\mathrm{q}}(n)$.

Remark. While the affine cellular structures we give are transferred from $\mathsf{S}_{\mathrm{q}}(N,n)$, it should in principle be possible to attempt to construct them directly on $\mathsf{S}_{\mathrm{q}}(n)$ or $\mathsf{H}_{\mathrm{q}}(n)$ using analogous methods to the construction for $\mathsf{S}_{\mathrm{q}}(N,n)$. It would be interesting to know whether this affine cellular structure, if it exists, is the same one that we construct here.

We first expand on the affine cellular structure of $\mathsf{S}_{\mathrm{q}}(N,n)$.

Proposition 4.6.1. $\mathsf{S}_{\mathrm{q}}(N,n)$ *is idempotent affine cellular. Using the notation of Definition 2.11.1, we furthermore have that:*

1. $\mathsf{S}_{\mathrm{q}}(N,n)$ *has a basis* e_A*, where* A *are certain* $\mathbb{Z}$*-by-*$\mathbb{Z}$ *matrices.*
2. *The involution* i *on* $\mathsf{S}_{\mathrm{q}}(N,n)$ *is given by* $i(e_A)=e_{A^t}$*.*
3. *For all* $\lambda\in\Lambda(N,n)$*, we have that* $e_{\mathrm{diag}(\lambda)}$ *is the identity map on* $x_{\mathrm{P}(\lambda)}\mathsf{H}_{\mathrm{q}}(n)$*, where we write* $\mathrm{diag}(\lambda)$ *for the* $\mathbb{Z}$*-by-*$\mathbb{Z}$ *matrix which is* N*-by-*N*-block-diagonal, with each diagonal block itself diagonal with* m*th entry* λ_m*.*
4. *For each* k*, the module* J_k' *contains the element* $l_k=e_{\mathrm{diag}(\lambda^{(k)})}$*, where* $\lambda^{(k)}\in\Lambda(N,n)$ *has decreasing entries.*
5. $l_ke_A=e_A$ *for any* $A=(a_{ij})$ *with* $\mathrm{row}(A)=\lambda^{(k)}$*, where* $\mathrm{row}(A)=\left(\sum_{j\in\mathbb{Z}}a_{ij}\right)_{1\le i\le N}$.
6. *For each* k*,* B_k *is of the form* $\mathscr{X}[X_1,\dots,X_{m_k},X_{i_1}^{-1},\dots,X_{i_{n_k}}^{-1}]$ *for some* $\{i_1,\dots,i_{n_k}\}\subseteq\{1,\dots,m_k\}$*.*

Proof. By Cui [2015], Theorem 4.7 (see also Nakajima [2015]), the algebra $\hat{\mathsf{S}}_{\mathrm{q}}(N,n)$ defined in Section 1.10 of Lusztig [1999] is idempotent affine cellular.

Again, they only claim that it is affine cellular over $\mathbb{Z}$, but in fact their proof shows it is affine cellular over $\mathscr{Z}$. But as we observed in the last section, $\hat{\mathsf{S}}_{\mathrm{q}}(N, n)$ is isomorphic to $\mathsf{S}_{\mathrm{q}}(N, n)$ by Pages 2 and 3 of McGerty [2003].

The details of this structure are collated in Sections 3 and 4 of Deng and Yang [2016]:

The first claim is their Definition 3.1.

The second claim is their Equation 4.6 (they call the involution τ).

The third claim is their Equation 8.4.

Their Proposition 4.3, meanwhile, defines certain sets $\mathfrak{c}_\lambda$ for every $\lambda \in \Lambda(N, n)$ with decreasing entries. Then, by their Equation 4.7 and the preceding paragraphs on Page 443, the J'_k are spanned by certain $\mathfrak{c}_{\lambda^{(k)}}$. (In their notation, the modules J'_k are written as C'_i).

From their Definition 3.1, for any $\lambda \in \Lambda(N, n)$, there is the element $l_\lambda = e_{\mathrm{diag}(\lambda)}$. Their Proposition 4.1 says that there is some other element, written $\{l_\lambda\}$, equal to l_λ. Then their Equation 4.3 says that $\{l_\lambda\} \in \mathfrak{c}_\lambda$ when the entries of λ are decreasing. Thus we get the fourth claim.

The fifth claim is their Equation 3.6.

In their Equation 4.8, they define, for every $\lambda \in \Lambda(N, n)$ with decreasing entries, rings $B_\lambda = \mathscr{Z}[X_1, \ldots, X_{\lambda_1}, X_{i_1}^{-1}, \ldots, X_{i_N}^{-1}]$ where $i_j = \lambda_1 - \lambda_{j+1}$ and we set $\lambda_{N+1} = 0$. Their Proposition 4.4 then gives a generalised matrix algebra structure on the J'_k, with coefficients in $B_{\lambda^{(k)}}$. But their Remark 2.2 says that this is exactly an affine cellular structure, with $B_k = B_{\lambda^{(k)}}$. Thus we have the sixth claim. □

To reduce to $\mathsf{S}_{\mathrm{q}}(n)$, we need some further properties of the idempotents.

Lemma 4.6.2. *For all k, we have that l_k lies in $\mathsf{S}_{\mathrm{q}}(n)$, and l_k is an idempotent generating the affine cell ideal J'_k.*

Proof. Now we observe that every decreasing tuple in $\Lambda(N, n)$ is an element of $\Lambda_0(N, n)$, and that by Claim (4) of Proposition 4.6.1 $l_k = e_{\mathrm{diag}(\lambda^{(k)})}$ and $\lambda^{(k)}$ is decreasing. Thus $l_k \in \mathsf{S}_{\mathrm{q}}(n)$.

Finally, by Claim (5) of Proposition 4.6.1 we have that l_k is idempotent, and as $\mathsf{S}_q(N,n)$ is idempotent affine cellular the J'_k are idempotent, so l_k generates J'_k. □

Recall from the previous section that e is the identity element of $\mathsf{S}_q(n)$.

Lemma 4.6.3. *Both e and the $e_{\mathrm{diag}(\lambda)}$ are fixed under the involution.*

Proof. By definition, e is the sum of the identity maps on each of the summands in Definition 4.4.4. But, by claim (3) of Proposition 4.6.1, these are precisely the $e_{\mathrm{diag}(\lambda)}$ for $\lambda \in \Lambda_0(N,n)$. It thus suffices to show that the $e_{\mathrm{diag}(\lambda)}$ are preserved under i. But $\mathrm{diag}(\lambda)$ is a diagonal matrix, and by Claim (2) of Proposition 4.6.1 we have that i sends e_A to e_{A^t}. □

Using these we can reduce to our Schur Algebra.

Theorem 4.6.4. *$\mathsf{S}_q(n)$ is idempotent affine cellular. For each k, we have that B_k is of the form $\mathscr{Z}[X_1,\dots,X_{m_k},X_{i_1}^{-1},\dots,X_{i_{n_k}}^{-1}]$ for some $\{i_1,\dots,i_{n_k}\} \subseteq \{1,\dots,m_k\}$. The ideals eJ'_ke are generated by l_k.*

Proof. We know $\mathsf{S}_q(N,n)$ is idempotent affine cellular with the B_k of the stated form by Proposition 4.6.1. We want to apply Claims (1) and (3) of Proposition 2.11.2. But, by Lemma 4.6.3 and Lemma 4.6.2, the conditions of Proposition 2.11.2 hold. □

To use this structure to show finite global dimension, we need some properties of the B_k.

Lemma 4.6.5. *Let R be a commutative ring. Fix some $m \in \mathbb{N}$ and some set $\{i_1,\dots,i_n\} \subseteq \{1,\dots,m\}$.*

1. *If R is a domain, then $\mathrm{rad}(R[X_1,\dots,X_m,X_{i_1}^{-1},X_{i_n}^{-1}]) = 0$.*
2. *If $\mathrm{gl\,dim}(R) < \infty$, then $\mathrm{gl\,dim}(R[X_1,\dots,X_m,X_{i_1}^{-1},X_{i_n}^{-1}]) < \infty$.*

Proof. We show the first result by proving that if A is a domain then $A[x]$ and $A[x,x^{-1}]$ are domains with zero Jacobson radical. That $A[x]$ is a domain follows as the leading coefficients of any nonzero polynomials must multiply to give a nonzero leading coefficient for the product. To see that $\mathrm{rad}(A[x]) = 0$, note that if $p(x) \in \mathrm{rad}(A[x])$ then $1 + xp(x)$ must be a unit, so $p(x) = 0$. The logic for $A[x, x^{-1}]$ is identical except that we now must consider $1 + x^k p(x)$ for k large enough that $x^k p(x)$ only has terms of strictly positive degree.

The second follows from the Hilbert Syzygy Theorem, which gives the polynomial case (see Rotman [2009] Theorem 8.37). The case for Laurent series follows as localisation is exact and preserves projective modules. Alternatively, it is a special case of McConnell and Robson [2001], Theorem 7.5.3 (iii) and (iv). □

With this we can give an alternate proof of Theorem 4.5.6.

Theorem 4.6.6. *Let R be a Noetherian domain such that* $\mathrm{gl}\dim(R) < \infty$. *Then* $\mathrm{gl}\dim(\mathsf{S}_R(n)) < \infty$.

Proof. By Theorem 4.6.4, $\mathsf{S}_\mathrm{q}(n)$ is idempotent affine cellular, and hence so is $\mathsf{S}_R(n)$. But by Lemma 4.6.5 we have $\mathrm{rad}(B_k) = 0$ and $\mathrm{gl}\dim(B_k) < \infty$. Thus we may apply Theorem 2.11.4. □

We now use the affine cellular structure we have obtained on the Schur algebra to obtain an affine cellular structure on the Hecke algebra.

Lemma 4.6.7. $\mathsf{H}_\mathrm{q}(n)$ *is of the form* $l\mathsf{S}_\mathrm{q}(n)l$ *(equivalently,* $l\mathsf{S}_\mathrm{q}(N, n)l$*) for some idempotent* $l \in \mathsf{S}_\mathrm{q}(n)$ *such that* $i(l) = l$. *Thus,* $\mathsf{H}_\mathrm{q}(n)$ *is affine cellular, with ideals* lJ_kl*, and* B_k *of the form* $\mathscr{Z}[X_1, \ldots, X_{m_k}, X_{i_1}^{-1}, \ldots, X_{i_{n_k}}^{-1}]$ *for some* $\{i_1, \ldots, i_{n_k}\} \subseteq \{1, \ldots, m_k\}$.

Proof. Let $l = e_{\mathrm{diag}(1,\ldots,1,0,\ldots,0)}$. Then, as $\mathtt{P}(1, \ldots, 1, 0, \ldots, 0) = \varnothing$, we have that l is the identity map on the $x_\varnothing \mathsf{H}_\mathrm{q}(n)$ summand. This is idempotent, and $l\mathsf{S}_\mathrm{q}(n)l$ is precisely the $\mathsf{H}_\mathrm{q}(n)$-endomorphisms of $x_\varnothing \mathsf{H}_\mathrm{q}(n)$, which we canonically identified with $\mathsf{H}_\mathrm{q}(n)$.

Furthermore, by Lemma 4.6.3, l is fixed by the involution i.

Thus, by Proposition 2.11.2 and Theorem 4.6.4, $\mathsf{H}_\mathrm{q}(n)$ is affine cellular with ideals lJ_kl and B_k of the stated form. □

For every $\mathtt{P} \subseteq \mathtt{S}_f$, the Poincare series of $\mathtt{P}$ is defined to be $A_{\mathtt{P}}(\mathrm{q}) = \sum_{w \in W_{\mathtt{P}}} \mathrm{q}^{l(w)}$, where $W_{\mathtt{P}}$ is the subgroup of W_f generated by $\mathtt{P}$. We write $A(\mathrm{q})$ for $A_{\mathtt{S}_f}(\mathrm{q})$. For every $\mathtt{P}$, $A_{\mathtt{P}}(\mathrm{q})$ is a factor of $A(\mathrm{q})$ (Humphreys [1990], Section 5.12).

Proposition 4.6.8. *Suppose $A(q)$ is invertible in R and that R is a Noetherian domain. Then*

1. *$\mathsf{H}_R(n)$ and $\mathsf{S}_R(n)$ are Morita equivalent.*
2. *The two-sided ideal lJ_kl of $\mathsf{H}_R(n)/lJ_{k-1}l$ is generated by the idempotent $x'_k = \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)} x_{\mathtt{P}(\lambda^{(k)})}$.*
3. *$\mathsf{H}_R(n)$ is idempotent affine cellular.*
4. *If furthermore $\mathrm{gl}\dim(R) < \infty$, then $\mathrm{gl}\dim(\mathsf{H}_R(n)) < \infty$.*

Proof. The first claim is Deng and Yang [2016], Appendix Lemma 1.4, which they prove via Claim (2) of Proposition 2.11.2 by showing that $\mathsf{S}_R(n) l \mathsf{S}_R(n) = \mathsf{S}_R(n)$.

We now show the second claim. We know that J_k is generated by l_k, that is, by the identity map on $x_{\mathtt{P}(\lambda^{(k)})}\mathsf{H}_R(n)$, by Lemma 4.6.2 and Claim (4) of Proposition 4.6.1. Consider the embedding $\phi_1 : x_{\mathtt{P}(\lambda^{(k)})}\mathsf{H}_R(n) \to \mathsf{H}_R(n)$, and the projection $\phi_2 : \mathsf{H}_R(n) \to x_{\mathtt{P}(\lambda^{(k)})}\mathsf{H}_R(n)$ given by left multiplication by $x_{\mathtt{P}(\lambda^{(k)})}$. Then the element $\phi_1 l_k \phi_2$ lives in both $\mathsf{H}_R(n)$ and J_k, and hence in lJ_kl. But this element is just $x_{\mathtt{P}(\lambda^{(k)})}$. Hence x'_k also lives in lJ_kl.

Now Deng and Yang [2016] Equation 8.9 gives that $\phi_2\phi_1 = A_{\mathtt{P}(\lambda^{(k)})}(q) l_k$. Hence

$$\begin{aligned} x'^2_k &= \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)^2} x^2_{\mathtt{P}(\lambda^{(k)})} \\ &= \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)^2} \phi_1 l_k \phi_2 \phi_1 l_k \phi_2 \\ &= \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)} \phi_1 l_k^3 \phi_2 \\ &= \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)} \phi_1 l_k \phi_2 \\ &= \frac{1}{A_{\mathtt{P}(\lambda^{(k)})}(q)} x_{\mathtt{P}(\lambda^{(k)})} \\ &= x'_k \end{aligned} \tag{4.6.1}$$

so x'_k is idempotent. Furthermore,

$$\begin{aligned}\phi_2 x'_k \phi_1 &= \frac{1}{A_{\mathsf{P}(\lambda^{(k)})}(q)} \phi_2 \phi_1 l_k \phi_2 \phi_1 \\ &= A_{\mathsf{P}(\lambda^{(k)})}(q) l_k\end{aligned} \tag{4.6.2}$$

and hence, as l_k generates all of J'_k as a 2-sided ideal in $\mathsf{S}_R(n)/J_{k-1}$, so does x'_k. Thus we are done by Proposition 2.11.2.

The third and fourth claims follow from the second and Lemma 4.6.7, together with Lemma 4.6.5. □

When $A(q)$ is not invertible, we cannot define all the idempotents x'_k. It seems unlikely that $\mathsf{H}_R(n)$ is idempotent affine cellular in this case.

4.7 A Classical Generator for p-adic GL_n

We continue to assume $\mathrm{G} = \mathrm{GL}_n$, and further assume that R is an algebraically closed field of characteristic different from p. We now have everything we need to prove the first part of the main theorem, in the same manner as we did in the finite case.

Lemma 4.7.1. $D^b_{fg}(\mathsf{H}'_1(G)) = \mathrm{per}(\mathsf{H}'_1(G))$.

Proof. By Proposition 4.2.2, $\mathcal{B}'_1(G)$ has progenerator Q. Thus, $\mathsf{H}'_1(G)$ is Morita equivalent to $\mathsf{S}_R(n)$. Hence it suffices to show $D^b_{fg}(\mathsf{S}_R(n)) = \mathrm{per}(\mathsf{S}_R(n))$. Now, $\mathsf{S}_R(n)$ has finite global dimension by Theorem 4.5.6, and it is Noetherian by Corollary 4.3.4. Thus by Proposition 2.10.7 and Proposition 2.10.8, every object of $D^b_{fg}(\mathsf{S}_R(n))$ is isomorphic to an object in the subcategory $\mathrm{per}(\mathsf{S}_R(n))$. But as these are both isomorphism-closed subcategories of $D(\mathsf{S}_R(n))$, they must agree. □

Lemma 4.7.2. $D^b_{fg}(\mathsf{H}_1(G)) = \langle D^b_{fg}(\mathsf{H}'_1(G)) \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$

Proof. Inclusion of the right side in the left is immediate as all $\mathsf{H}'_1(G)$-modules are $\mathsf{H}_1(G)$-modules.

Let $M^\bullet$ be an object in $D^b_{fg}(\mathsf{H}_1(G))$. Without loss of generality we can thus take it to be a complex of finitely generated unipotent representations. Then by Proposition 4.2.2, we have some finite N such that $\mathcal{I}^N M^\bullet = 0$. Observe that, as $M^\bullet$ is a complex of unipotent representations, we have that $\mathcal{I}^i M^\bullet = \mathcal{I}_1^i M^\bullet$ for any $i \geq 0$. Now, by Corollary 4.3.4, we have that $\mathcal{I}_1$ is finitely generated, and $M^\bullet$ is a complex of finitely generated representations, so the $\mathcal{I}^i M^\bullet$ are also complexes of finitely generated representations for all $i \geq 0$. Hence the quotients $\mathcal{I}^i M^\bullet / \mathcal{I}^{i+1} M^\bullet$ are objects in $D^b_{fg}(\mathsf{H}'_1(G))$. Thus $M^\bullet$ is a repeated extension of complexes in $D^b_{fg}(\mathsf{H}'_1(G))$, and so is in $\langle D^b_{fg}(\mathsf{H}'_1(G))\rangle_{D^b_{fg}(\mathsf{H}_1(G))}$. $\square$

Theorem 4.7.3. $D^b_{fg}(\mathsf{H}_1(G)) = \langle Q \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$.

Proof. By Proposition 4.2.2, Q is a progenerator for $\mathcal{B}'_1(G)$. Thus by Proposition 2.10.6 we have that $\mathrm{per}(\mathsf{H}'_1(G)) = \langle Q \rangle_{\mathrm{per}(\mathsf{H}'_1(G))}$. Thus we are done by Lemma 4.7.1 and Lemma 4.7.2. $\square$

5

Describing the Derived l-Modular Unipotent Block

Synopsis

We finish the proof of the main theorem by establishing that V is a classical generator of $D^b_{fg}(\mathsf{H}_1(G))$ for $G = \mathrm{GL}_n(F)$, and hence that $D^b_{fg}(\mathsf{H}_1(G))$ is equivalent to perfect complexes over $\mathrm{dg\text{-}End}_G(V^\bullet)$. We do this by combining the first part of the main theorem from Chapter 4 with the finite version of the main theorem from Chapter 3. Lifting the finite version of the theorem to the p-adic setting requires the additional result from Chapter 3 as well a further result showing, essentially, that $\mathcal{I}_f$ is a subset of $\mathcal{I}$. This involves careful coset calculations in the global Hecke algebra. We then explore the structure of the dg Schur algebra $\mathrm{dg\text{-}End}_G(V^\bullet)$, giving a composition formula in terms of resolutions on the finite group $\mathrm{GL}_n(k)$.

5.1 A Second Classical Generator for p-adic GL_n

We retain the notation of the previous chapter. Furthermore, we assume throughout that R is an algebraically closed field of characteristic l different from p, and that $\mathrm{G} = \mathrm{GL}_n$.

Definition 5.1.1. Let $V = \mathrm{I}^G_{G_f,K} V_f$. Thus, $V = \bigoplus_{J \in \mathcal{J}} \mathrm{ind}^G_J \mathbb{1}$.

We seek to establish that $D^b_{fg}(\mathsf{H}_1(G)) = \langle V \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$. The idea is to lift the finite version of this result, that is, the second claim of Theorem 3.1.4, from G_f to G. To do this, we first relate $\mathcal{B}_1(G_f)$ and $\mathcal{B}_1(G)$:

Proposition 5.1.2. *Let $\pi_f \in \mathrm{Mod}(G_f)$. If $\pi_f \in \mathcal{B}_1(G_f)$, then $\mathrm{I}^G_{G_f,K}\pi_f \in \mathcal{B}_1(G)$. Conversely, if $\pi_f \in \mathcal{B}_{\neq 1}(G_f)$, then $\mathrm{I}^G_{G_f,K}\pi_f \in \mathcal{B}_{\neq 1}(G)$.*

Proof. This is Vignéras [2003], Lemma D14 (a_1) and (a_2), noting that Conjecture H_3 in said paper is stated to hold for $G = \mathrm{GL}_n(F)$. □

Corollary 5.1.3. *V and $\mathrm{I}^G_{G_f,K}Q_f$ are both finitely generated and unipotent.*

Proof. V and $\mathrm{I}^G_{G_f,K}Q_f$ are the image under parahoric induction of V_f and Q_f respectively. Both V_f and Q_f are unipotent and finitely generated, and parahoric induction preserves both properties. □

Thus both V and $\mathrm{I}^G_{G_f}(Q_f)$ are in $D^b_{fg}(\mathsf{H}_1(G_f))$. Now we are ready to lift the finite version of the main theorem to G.

Corollary 5.1.4. $\langle \mathrm{I}^G_{G_f}(Q_f)\rangle_{D^b_{fg}(\mathsf{H}_1(G))} = \langle V\rangle_{D^b_{fg}(\mathsf{H}_1(G))}$.

Proof. Parahoric induction is exact, so this is immediate from the second claim of Theorem 3.1.4. □

We want to conclude by Theorem 4.7.3, which says that $D^b_{fg}(\mathsf{H}_1(G)) = \langle Q\rangle_{D^b_{fg}(\mathsf{H}_1(G))}$. Unfortunately, $\mathrm{I}^G_{G_f,K}Q_f = \Gamma/(\mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f)$ is not a priori equal to $Q = \Gamma/\mathcal{I}\Gamma$. It thus remains to show that $\mathrm{I}^G_{G_f,K}Q_f$ and Q are in fact equal, that is, that $\mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f = \mathcal{I}\Gamma$. The first claim of Theorem 3.1.4 is exactly enough to show one inclusion:

Lemma 5.1.5. $\mathcal{I}\mathrm{I}^G_{G_f,K}Q_f = 0$.

Proof. By the first claim of Theorem 3.1.4, Q_f is a direct sum of subrepresentations of P_f. Hence $\mathrm{I}^G_{G_f,K}Q_f$ is a direct sum of subrepresentations of $\mathrm{I}^G_{G_f,K}P_f = P$, and subrepresentations of P are annihilated by $\mathcal{I} = \mathrm{Ann}(P)$. □

Corollary 5.1.6. $\mathcal{I}\Gamma \subseteq \mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f$.

Proof.

$$\begin{aligned} 0 &= \mathcal{I}\mathrm{I}_{G_f,K}^G Q_f \\ &= \mathcal{I}(\Gamma/(\mathrm{I}_{G_f,K}^G \mathcal{I}_f \Gamma_f)) \\ &= (\mathcal{I}\Gamma + \mathrm{I}_{G_f,K}^G \mathcal{I}_f \Gamma_f)/(\mathrm{I}_{G_f,K}^G \mathcal{I}_f \Gamma_f) \end{aligned}$$

□

To show the reverse inclusion, it is simplest to work with $\mathsf{H}(G)$-modules. Recall from Section 2.7 that we may view $\mathsf{H}(K)$ as the subalgebra of $\mathsf{H}(G)$ consisting of functions supported on K, and that the image of the endomorphism of $\mathsf{H}(K)$ given by $f \mapsto 1_{K^1} f 1_{K^1}$ may be identified with $\mathsf{H}(G_f)$ via $g \leftrightarrow 1_{gK^1}$. Also recall that, viewing representations of G_f, K and G as modules over $\mathsf{H}(G_f)$, $\mathsf{H}(K)$ and $\mathsf{H}(G)$ respectively, parahoric induction is then just $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} -$, where $\mathsf{H}(G)$ and $\mathsf{H}(G_f)$ are viewed as $\mathsf{H}(K)$-algebras via the aforementioned maps.

Using this, we may rewrite $\mathrm{I}_{G_f,K}^G \mathcal{I}_f \Gamma_f$ and $\mathcal{I}\Gamma$ as

$$\mathrm{I}_{G_f,K}^G \mathcal{I}_f \Gamma_f = \mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathcal{I}_f \Gamma_f \tag{5.1.1}$$

and

$$\mathcal{I}\Gamma = \mathcal{I}\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \Gamma_f = \mathcal{I} \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)\Gamma_f \tag{5.1.2}$$

respectively.

Hence, it will suffice to show that $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathcal{I}_f$ lies inside $\mathcal{I} \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$, as subsets of $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$.

There are a series of simplifications that can be made to this picture. First, recall that, by the Iwahori decomposition, $P = \mathrm{ind}_I^G \mathbb{1}$ is generated by elements of the form 1_{iwI} for $w \in W$ and $i \in I$, and $\mathsf{H}(G)$ acts on these elements by convolution on the left. Thus we may view P as a left ideal in $\mathsf{H}(G)$. Hence, if $Z \in \mathsf{H}(G)$, then $Z \in \mathcal{I}$ if and only if, for all $i \in I$ and $w \in W$, we have that $Z 1_{iwI} = 0$.

Similarly, by the Bruhat decomposition, $P_f = \mathrm{ind}_{I_f}^{G_f} \mathbb{1}$ is generated by the elements $iw_f I_f$ for $w_f \in W_f$ and $i \in I_f$, and $\mathsf{H}(G_f)$ acts on these elements by left multiplication, so P_f can be analogously viewed as a left ideal in $\mathsf{H}(G_f)$. Thus, if $Z \in \mathsf{H}(G_f)$, then $Z \in \mathcal{I}_f$ if and only if, for all $i \in I_f$ and $w_f \in W_f$, we have that $Ziw_fI_f = 0$.

Next, observe that the map $g \mapsto 1_{gK^1}$ identifies $\mathsf{H}(G_f)$ with a subalgebra of $\mathsf{H}(K)$, and thus of $\mathsf{H}(G)$. This identifies the element $iw_fI_f \in P_f$ with $1_{iw_fI} \in \mathsf{H}(G)$, and furthermore identifies $\mathcal{I}_f \subseteq \mathsf{H}(G_f)$ as a subset of $\mathsf{H}(G)$. Therefore, if $Z \in \mathcal{I}_f$, then, for all $i \in I_f$ and $w_f \in W_f$, we have that $Z1_{iw_fI} = 0$.

Furthermore, the above map is a splitting of the quotient map $\mathsf{H}(K) \to \mathsf{H}(G_f)$, and so in $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$ the element $f \otimes g$ is equal to the element $f1_{gK^1} \otimes 1$. Hence, $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathcal{I}_f = \mathsf{H}(G)\mathcal{I}_f \otimes 1$. Now, $\mathcal{I}$ is a left ideal in $\mathsf{H}(G)$, and so to show $\mathsf{H}(G)\mathcal{I}_f \otimes 1 \subseteq \mathcal{I} \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$ it is enough to show that $\mathcal{I}_f \otimes 1 \subseteq \mathcal{I} \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$ as subsets of $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$. Thus, in particular, it suffices to show that $\mathcal{I}_f \subseteq \mathcal{I}$.

Lemma 5.1.7. $\mathsf{H}(G) \otimes_{\mathsf{H}(K)} \mathcal{I}_f \subseteq \mathcal{I} \otimes_{\mathsf{H}(K)} \mathsf{H}(G_f)$.

Proof. By the previous remarks, it suffices to show that for any $Z \in \mathcal{I}_f$, $w \in W$, and $i \in I$, we have $Z1_{iwI} = 0$.

First, observe that $Z \in \mathsf{H}(G_f)$, and so Z is a linear combination of terms of the form 1_{gK^1}. Now, as $\mu(K^1) = 1$ by our convention for normalisation, we have $Z = Z1_{K^1}$, and so $Z1_{iwI} = Z1_{K^1}1_{iwI}$.

Now, by the convolution formula,

$$\begin{aligned} 1_{K^1}1_{iwI}(x) &= \mu(K^1 \cap iwIw^{-1}i^{-1}) \sum_{k \in K^1/(K^1 \cap iwIw^{-1}i^{-1})} 1_{K^1}(k)1_{iwI}(k^{-1}x) \\ &= [K^1 : K^1 \cap iwIw^{-1}i^{-1}]^{-1}1_{K^1iwI}. \end{aligned}$$

As K^1 is a pro-p group, $c = [K^1 : K^1 \cap iwIw^{-1}i^{-1}]^{-1}$ is well-defined and nonzero in R, so $Z1_{K^1}1_{iwI} = cZ1_{K^1iwI}$.

Let w_0 be a minimal length coset representative for w in $W_f \backslash W$. Hence $w = w_fw_0$ for some $w_f \in W_f$.

As $w_f \in W_f \subseteq K$, w_f normalises K^1. Similarly, as $i \in I \subseteq K$, we have that i also normalises K^1. Hence $Z1_{K^1iwI} = Z1_{iw_fK^1w_0I}$.

Now, by Lemma 3.19 and Variant 3.22 of Morris [1993], we have that $K^1(K \cap w_0Iw_0^{-1})$ is a standard parahoric subgroup, and so in particular contains I.

Multiplying by $w_0 I$ on the right thus gives

$$\begin{aligned} Iw_0I &\subseteq K^1(K \cap w_0 I w_0^{-1}) w_0 I \\ &\subseteq K^1(Kw_0I \cap w_0 I) \\ &= K^1 w_0 I. \end{aligned}$$

Thus we have $K^1 w_0 I \subseteq I w_0 I \subseteq K^1 w_0 I$, so we have equality $Iw_0I = K^1w_0I$. Hence $Z1_{iw_f K^1 w_0 I} = Z1_{iw_f I w_0 I}$.

Now, we write $Z = \sum_{g \in G_f} r_g 1_{gK^1}$ for some $r_g \in R$. Recall that i and w_f normalise $K^1 \subseteq I$, so $iw_f I w_0 I$ is left-K^1-invariant. Thus, as $\mu(K^1) = 1$, the convolution formula gives $Z1_{iw_f I w_0 I} = \sum_{g \in G_f} r_g 1_{giw_f I w_0 I}$.

By definition, $Z \in \mathcal{I}_f$. But recall that, by the previous remarks, this means that $Z1_{iw_f I} = 0$, that is, that $\sum_{g \in G_f} r_g 1_{giw_f I} = 0$. In particular, for any fixed coset kI of I in K, we have that

$$\sum_{\substack{g \in G_f \\ kI = giw_f I}} r_g = 0.$$

Hence

$$\begin{aligned} \sum_{g \in G_f} r_g 1_{giw_f I w_0 I} &= \sum_{k \in K/I} \left(\sum_{\substack{g \in G_f \\ kI = giw_f I}} r_g \right) 1_{kIw_0I} \\ &= \sum_{k \in K/I} 0 \\ &= 0 \end{aligned}$$

Putting this all together, we obtain $Z1_{iwI} = 0$. $\square$

Thus we get our desired equality.

Corollary 5.1.8. $Q = \mathrm{I}^G_{G_f,K} Q_f$.

Proof. By Lemma 5.1.7, together with Equation 5.1.1 and Equation 5.1.2, we have that $\mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f \subseteq \mathcal{I}\Gamma$. But by Corollary 5.1.6, the converse is true. Thus $\mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f = \mathcal{I}\Gamma$, and so $\mathrm{I}^G_{G_f,K}Q_f = \Gamma/(\mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f) = \Gamma/\mathcal{I}\Gamma = Q$. $\square$

We now have all the ingredients to prove our main theorem.

Theorem 5.1.9. $D^b_{fg}(\mathsf{H}_1(G)) = \langle V \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$

Proof. We already know from Theorem 4.7.3 that $D^b_{fg}(\mathsf{H}_1(G)) = \langle Q \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$, and from Corollary 5.1.4 that $\langle \mathrm{I}^G_{G_f,K}(Q_f) \rangle_{D^b_{fg}(\mathsf{H}_1(G))} = \langle V \rangle_{D^b_{fg}(\mathsf{H}_1(G))}$. But by Corollary 5.1.8 we have $Q = \mathrm{I}^G_{G_f,K} Q_f$. □

Corollary 5.1.10. *Let $V^\bullet$ be a projective resolution of V in $\mathrm{Mod}(G)$. There is a triangulated equivalence $D^b_{fg}(\mathsf{H}_1(G)) \simeq \mathrm{per}(\text{dg-End}_G(V^\bullet))$.*

Proof. This follows applying Theorem 2.10.4 to the category $D^b_{fg}(\mathsf{H}_1(G))$ and its classical generator V. □

5.1.1 Example: $n = 1$

When $n = 1$ and $l \mid q - 1$, we can be more explicit. In this case we have that $\Gamma = \mathrm{I}^G_{G_f,K}\Gamma_f = \mathrm{ind}^G_{K^1}\mathbb{1}$, and $V = P = \mathrm{I}^G_{G_f,K} Q_f = \mathrm{ind}^G_K \mathbb{1}$. In particular, we can see directly that $\mathcal{I}\mathrm{I}^G_{G_f,K} Q_f = \mathcal{I}P = 0$, so $\mathcal{I}\Gamma \subseteq \mathrm{I}^G_{G_f,K}\mathcal{I}_f\Gamma_f$.

Recall that $\mathcal{I}_f = \left\{ \sum_{g \in G_f} r_g g \middle| \sum_{g \in G_f} r_g = 0 \right\}$. This case is made simpler as $G = F^\times$ is abelian. Observe that $K = \mathcal{O}^\times$ and $W = \{\varpi^i | i \in \mathbb{Z}\}$. Hence, by the Iwahori decomposition, a basis for P is given by $x_i = 1_{\varpi^i \mathcal{O}^\times}$ for $i \in \mathbb{Z}$. By commutativity, $gx_i = x_i$ for any $g \in K$. Hence for $r = \sum_{g \in G_f} r_g g \in \mathcal{I}_f$, we have $rx_i = \sum_{g \in G_f} r_g g x_i = x_i \sum_{g \in G_f} r_g = 0$. Thus we indeed have $\mathcal{I}_f \subseteq \mathcal{I}$, and hence the exact equality $Q = \mathrm{I}^G_{G_f,K} Q_f = V$.

5.2 The dg Schur Algebra

Now that we have our main theorem, we wish to describe the structure of $\text{dg-End}_G(V^\bullet)$, as this will give us information about the structure of $D^b_{fg}(\mathsf{H}_1(G))$. We start by unwinding the definition of $V^\bullet$.

Definition 5.2.1. For $J \in \mathcal{J}$, let $\mathbb{1}^\bullet_J$ be a projective resolution of $\mathbb{1}_J$ in the category of smooth J-representations.

Then we may put $V^\bullet = \bigoplus_{J\in\mathcal{J}} \mathrm{ind}_J^G \mathbb{1}_J^\bullet$, as this is indeed a projective resolution of $V = \bigoplus_{J\in\mathcal{J}} \mathrm{ind}_J^G \mathbb{1}_J$.

Definition 5.2.2. We define the *dg Schur Algebra* to be $\mathcal{S} = \text{dg-End}_G(V^\bullet)$.

The following proposition justifies the choice of name.

Proposition 5.2.3. $\mathsf{S}_R(n) = H^0(\mathcal{S})$.

Proof. $H^0(\mathcal{S})$ is just the endomorphism algebra of V, which is $\mathsf{S}_R(n)$ by Vignéras [2003], Theorem 2.3. □

$\mathsf{S}_R(n)$ has a natural basis, which can then be used to give a composition formula. We perform an analogous decomposition for $\mathcal{S}$.

Definition 5.2.4. We introduce, for $J, M \in \mathcal{J}$ and $g \in G$, the complex

$$\mathcal{S}_{MgJ} = \text{dg-Hom}_{J\cap g^{-1}Mg}(\mathrm{res}^J_{J\cap g^{-1}Mg}\mathbb{1}_J^\bullet, \mathrm{res}^{g^{-1}Mg}_{J\cap g^{-1}Mg}(\mathbb{1}_M^\bullet)^g).$$

Proposition 5.2.5. *There is an isomorphism of complexes*

$$\mathcal{F} : \mathcal{S} \xrightarrow{\sim} \bigoplus_{J,M\in\mathcal{J}} \bigoplus_{g\in M\backslash G/J} \mathcal{S}_{MgJ}$$

Proof. By factoring over the direct sums comprising $V^\bullet$, we have

$$\mathcal{S} \cong \bigoplus_{J,M\in\mathcal{J}} \text{dg-Hom}_G(\mathrm{ind}_J^G\mathbb{1}_J^\bullet, \mathrm{ind}_M^G\mathbb{1}_M^\bullet)$$

as complexes of R-modules. Now, via Frobenius reciprocity (which preserves the differential) and the Mackey Decomposition,

$$\begin{aligned}
&\text{dg-Hom}_G(\mathrm{ind}_J^G\mathbb{1}_J^\bullet, \mathrm{ind}_M^G\mathbb{1}_M^\bullet)\\
\cong\, &\text{dg-Hom}_J(\mathbb{1}_J^\bullet, \mathrm{res}_J^G\mathrm{ind}_M^G\mathbb{1}_M^\bullet)\\
\cong\, &\text{dg-Hom}_J(\mathbb{1}_J^\bullet, \bigoplus_{g\in M\backslash G/J} \mathrm{ind}^J_{J\cap g^{-1}Mg}\mathrm{res}^{g^{-1}Mg}_{J\cap g^{-1}Mg}(\mathbb{1}_M^\bullet)^g)\\
\cong\, &\bigoplus_{g\in M\backslash G/J} \text{dg-Hom}_J(\mathbb{1}_J^\bullet, \mathrm{ind}^J_{J\cap g^{-1}Mg}\mathrm{res}^{g^{-1}Mg}_{J\cap g^{-1}Mg}(\mathbb{1}_M^\bullet)^g)\\
\cong\, &\bigoplus_{g\in M\backslash G/J} \text{dg-Hom}_{J\cap g^{-1}Mg}(\mathrm{res}^J_{J\cap g^{-1}Mg}\mathbb{1}_J^\bullet, \mathrm{res}^{g^{-1}Mg}_{J\cap g^{-1}Mg}(\mathbb{1}_M^\bullet)^g)\\
\cong\, &\bigoplus_{g\in M\backslash G/J} \mathcal{S}_{MgJ}.
\end{aligned}$$

□

In particular, we obtain the isomorphism of R-modules

$$H^0(\mathcal{F}) : \mathsf{S}_R(n) \xrightarrow{\sim} \bigoplus_{J,M \in \mathcal{J}} \bigoplus_{g \in M \backslash G / J} H^0(\mathcal{S}_{MgJ}).$$

Furthermore, we have that

$$\begin{aligned} H^0(\mathcal{S}_{MgJ}) &= \mathrm{Hom}_{J \cap g^{-1}Mg}(\mathrm{res}^J_{J \cap g^{-1}Mg} \mathbb{1}_J, \mathrm{res}^{g^{-1}Mg}_{J \cap g^{-1}Mg} (\mathbb{1}_M)^g) \\ &= \mathrm{Hom}_{J \cap g^{-1}Mg}(\mathbb{1}_{J \cap g^{-1}Mg}, \mathbb{1}_{J \cap g^{-1}Mg}). \end{aligned}$$

Thus, every element of $H^0(\mathcal{S}_{MgJ})$ is a scalar multiple of the identity map id_{MgJ} on $\mathbb{1}_{J \cap g^{-1}Mg}$.

Continuing our analogy with $\mathsf{S}_R(n)$, we describe composition on the $\mathcal{S}_{MgJ}$. Given an element $f^\bullet \in \mathcal{S}_{MgJ}$, we can pass along the three-step isomorphism $\mathcal{F}^{-1}$ of Proposition 5.2.5 to find the element of $\mathcal{S}$ which is sent to it under $\mathcal{F}$:

$$\begin{aligned} & f^\bullet \\ & \xmapsto{\text{Frobenius Reciprocity}} \\ & v \in \mathbb{1}_J^\bullet \mapsto [j \in J \mapsto f^\bullet(jv)] \\ & \xmapsto{\text{Mackey Decomposition}} \\ & v \in \mathbb{1}_J^\bullet \mapsto [x \in G \mapsto m_x f(j_x v) 1_{MgJ}(x)] \\ & \xmapsto{\text{Frobenius Reciprocity}} \\ & \phi \in \mathrm{ind}_J^G \mathbb{1}_J^\bullet \mapsto \left[x \in G \mapsto \sum_{y \in J \backslash G} m_{xy^{-1}} f^\bullet(j_{xy^{-1}} \phi(y)) 1_{MgJ}(xy^{-1}) \right] \end{aligned} \tag{5.2.1}$$

where, for $x \in MgJ$, we fix a decomposition $x = m_x g j_x$. Note that the choice of decomposition does not affect the isomorphism.

Definition 5.2.6. For $f_1^\bullet \in \mathcal{S}_{LgJ}$ and $f_2^\bullet \in \mathcal{S}_{Mg'L}$, write $f_2^\bullet \star f_1^\bullet$ for the element $\mathcal{F}(\mathcal{F}^{-1}(f_2^\bullet)\mathcal{F}^{-1}(f_1^\bullet))$ of $\bigoplus_{M \backslash G / J} \mathcal{S}_{M\bar{g}J}$.

To describe $\star$, we shall need the following sets.

Definition 5.2.7. Let $M, L, J \in \mathcal{J}$, and fix double cosets $LgJ, Mg'L, M\bar{g}J$ in G. Then we set

$$\mathcal{C}^{M\bar{g}J}_{Mg'L,LgJ} = \left\{ j \in (J \cap g^{-1}Lg)\backslash J \middle| \bar{g} \in Mg'Lgj \right\}.$$

We may also symmetrically define

$$\hat{\mathcal{C}}^{M\bar{g}J}_{Mg'L,LgJ} = \left\{ m \in M/(M \cap g'Lg'^{-1}) \middle| \bar{g} \in mg'LgJ \right\}.$$

With these we can give a general formula for composition in the dg-case:

Lemma 5.2.8. *Let $f_1^\bullet \in \mathcal{S}_{LgJ}$ and $f_2^\bullet \in \mathcal{S}_{Mg'L}$. Then the projection of $f_2^\bullet \star f_1^\bullet$ in $\mathcal{S}_{M\bar{g}J}$ is the map*

$$\left[v \in \mathbb{1}_J^\bullet \mapsto \sum_{j \in \mathcal{C}^{M\bar{g}J}_{Mg'L,LgJ}} m f_2^\bullet(l f_1^\bullet(jv)) \right]$$

where $\bar{g} = mg'lgj$.

Proof. By applying Equation 5.2.1 to $f_1^\bullet$ and $f_2^\bullet$, we know that $\mathcal{F}^{-1}(f_2^\bullet)\mathcal{F}^{-1}(f_1^\bullet)$ is the following morphism in $\mathcal{S}$:

$$\phi \in \operatorname{ind}_J^G \mathbb{1}_J^\bullet \mapsto x \in G \mapsto$$
$$\sum_{y \in L\backslash G} m_{xy^{-1}} f_2^\bullet \left(l_{xy^{-1}} \sum_{z \in J\backslash G} l_{yz^{-1}} f_1^\bullet \left(j_{yz^{-1}} \phi(z) \right) 1_{LgJ}(yz^{-1}) \right) 1_{Mg'L}(xy^{-1})$$

where $xy^{-1} = m_{xy^{-1}} g' l_{xy^{-1}}$ and $yz^{-1} = l_{yz^{-1}} g j_{yz^{-1}}$.

Now, to get the projection of $f_2^\bullet \star f_1^\bullet = \mathcal{F}(\mathcal{F}^{-1}(f_2^\bullet)\mathcal{F}^{-1}(f_1^\bullet))$ in $\mathcal{S}_{M\bar{g}J}$, we apply Equation 5.2.1 in reverse:

$$\xmapsto{\text{Frobenius Reciprocity}}$$

$$v \in \mathbb{1}_J^\bullet \mapsto x \in G \mapsto$$
$$\sum_{y \in L\backslash G} \sum_{z \in J\backslash G} m_{xy^{-1}} f_2^\bullet \left(l_{xy^{-1}} l_{yz^{-1}} f_1^\bullet \left(j_{yz^{-1}} z v 1_J(z) \right) \right) 1_{LgJ}(yz^{-1}) 1_{Mg'L}(xy^{-1})$$
$$= v \in \mathbb{1}_J^\bullet \mapsto x \in G \mapsto$$
$$\sum_{y \in L\backslash G} m_{xy^{-1}} f_2^\bullet \left(l_{xy^{-1}} l_y f_1^\bullet \left(j_y v \right) \right) 1_{LgJ}(y) 1_{Mg'L}(xy^{-1})$$
$$\xmapsto{\text{Mackey Decomposition, project onto } \mathcal{S}_{M\bar{g}J}}$$
$$v \in \mathbb{1}_J^\bullet \mapsto j \in J \mapsto$$
$$\sum_{y \in L\backslash G} m_{\bar{g}jy^{-1}} f_2^\bullet \left(l_{\bar{g}jy^{-1}} l_y f_1^\bullet \left(j_y v \right) \right) 1_{LgJ}(y) 1_{Mg'L}(\bar{g}jy^{-1})$$
$$\xmapsto{\text{Frobenius Reciprocity}}$$
$$\left[v \in \mathbb{1}_J^\bullet \mapsto \sum_{y \in L\backslash G} m_{\bar{g}y^{-1}} f_2^\bullet \left(l_{\bar{g}y^{-1}} l_y f_1^\bullet \left(j_y v \right) \right) 1_{LgJ}(y) 1_{Mg'L}(\bar{g}y^{-1}) \right]$$
$$= \left[v \in \mathbb{1}_J^\bullet \mapsto \sum_{j \in \mathcal{C}^{M\bar{g}J}_{Mg'L,LgJ}} m f_2^\bullet (l f_1^\bullet (jv)) \right]$$

□

Note that the above map may also be written as

$$\left[v \in \mathbb{1}_J^\bullet \mapsto \sum_{m \in \hat{\mathcal{C}}^{M\bar{g}J}_{Mg'L,LgJ}} m f_2^\bullet (l f_1^\bullet (jv)) \right]$$

where $\bar{g} = mg'lgj$.

In particular, taking zeroth cohomology shows that the composition map on $\mathsf{S}_R(n)$ is exactly convolution of the id_{MgJ}, so we recover the composition formula for $\mathsf{S}_R(n)$.

5.2.1 Example: $n = 1$

In this case $\mathcal{J}$ contains the single element $K = \mathcal{O}^\times$. Recall also that $W = \{\varpi^i | i \in \mathbb{Z}\}$, and that, by the Iwahori decomposition, W indexes the double cosets KgK. We thus fix W as the indexing set for K-double cosets going forward. Furthermore, as $G = F^\times$ is abelian, we have that $K \cap g^{-1}Kg = K$, and hence that

$$\mathcal{S}_{KgK} = \text{dg-Hom}_K(\mathbb{1}_K^\bullet, \mathbb{1}_K^\bullet).$$

In particular, for all g, we have that $\mathcal{S}_{KgK}$ is isomorphic to $\mathcal{S}_{K1K}$.

G being abelian also implies that

$$\mathcal{C}_{Kg'K,KgK}^{K\bar{g}K} = \begin{cases} \{1\}, & \bar{g} \in Kg'g \\ \varnothing, & \bar{g} \notin Kg'g \end{cases}.$$

As we are using W as our indexing set, we can see that $\bar{g} \in Kg'g$ if and only if $\bar{g} = gg'$. Thus, the composition $f_2^\bullet \star f_1^\bullet$ of $f_1^\bullet \in \mathcal{S}_{KgK}$ and $f_2^\bullet \in \mathcal{S}_{Kg'K}$ has zero projection in all $\mathcal{S}_{K\bar{g}K}$ except for $\mathcal{S}_{Kg'gK}$. Again using the abelian property and that $g'g = 1g'1g1$, we get that the projection of $f_2^\bullet \star f_1^\bullet$ in $\mathcal{S}_{Kg'gK}$ is the map $f_2 f_1$.

Thus we may define a map $\mathcal{S}_{K1K} \otimes_R \mathsf{S}_R(n) \to \mathcal{S}$ via $f \otimes \mathrm{id}_{KgK} \mapsto f \in \mathcal{S}_{KgK}$, and our previous work shows that, equipping the left hand side with componentwise composition, this is an isomorphism of dg algebras. Furthermore, $\mathsf{S}_R(n)$ is isomorphic to the polynomial ring $R[\varpi]$. It remains to describe $\mathcal{S}_{K1K}$.

By Ackermann [2006], Proposition 4.24, the projective cover $\hat{P}_f$ of $\mathbb{1}_{G_f}$ has a totally ordered submodule lattice of length l^r, where r is maximal such that $l^r \mid q - 1$, with all subquotients isomorphic to $\mathbb{1}_{G_f}$. Fix an endomorphism α of $\hat{P}_f$ with kernel $\mathbb{1}$. Then, inflating to K without changing notation, a minimal projective resolution $\mathbb{1}_K^\bullet$ of $\mathbb{1}_K$ has period 2 and is as follows:

$$\cdots \xrightarrow{\alpha} \hat{P}_f \xrightarrow{\alpha^{l^r-1}} \hat{P}_f \longrightarrow 0$$

Thus, an element of $\mathcal{S}_{K1K}$ of degree i is zero in degree $j > -i$, and in each degree $j \le -i$ is a linear combination of $\left\{\alpha^k \middle| 0 \le k < l^r\right\}$. These compose subject to the relation $\alpha^{l^r} = 0$, and no other relations.

Bibliography

Bernd Ackermann. The Loewy series of the Steinberg-PIM of finite general linear groups. *Proc. London Math. Soc. (3)*, 92(1):62–98, 2006. ISSN 0024-6115,1460-244X. doi: 10.1017/S0024611505015443. URL https://doi.org/10.1017/S0024611505015443.

Jeffrey D. Adler, Jessica Fintzen, Manish Mishra, and Kazuma Ohara. Reduction to depth zero for tame p-adic groups via hecke algebra isomorphisms, 2024a. URL https://arxiv.org/abs/2408.07805.

Jeffrey D. Adler, Jessica Fintzen, Manish Mishra, and Kazuma Ohara. Structure of hecke algebras arising from types, 2024b. URL https://arxiv.org/abs/2408.07801.

P. N. Ánh and L. Márki. Morita equivalence for rings without identity. *Tsukuba J. Math.*, 11(1):1–16, 1987. ISSN 0387-4982,2423-821X. doi: 10.21099/tkbjm/1496160500. URL https://doi.org/10.21099/tkbjm/1496160500.

Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-58423-4; 978-0-521-58631-3; 0-521-58631-3. doi: 10.1017/CBO9780511614309. URL https://doi.org/10.1017/CBO9780511614309. Techniques of representation theory.

J. N. Bernstein. Le "centre" de Bernstein. In P. Deligne, editor, *Representations of reductive groups over a local field*, Travaux en Cours, pages 1–32. Hermann, Paris, 1984. ISBN 2-7056-5989-7.

Colin J. Bushnell and Philip C. Kutzko. Semisimple types in GL_n. *Compositio*

Math., 119(1):53–97, 1999. ISSN 0010-437X,1570-5846. doi: 10.1023/A:1001773929735. URL `https://doi.org/10.1023/A:1001773929735`.

Charlotte Chan and Alexander Ivanov. Affine Deligne-Lusztig varieties at infinite level. *Math. Ann.*, 380(3-4):1801–1890, 2021. ISSN 0025-5831,1432-1807. doi: 10.1007/s00208-020-02092-4. URL `https://doi.org/10.1007/s00208-020-02092-4`.

Gianmarco Chinello. Blocks of the category of smooth ℓ-modular representations of $\mathrm{GL}(n, F)$ and its inner forms: reduction to level 0. *Algebra Number Theory*, 12(7):1675–1713, 2018. ISSN 1937-0652,1944-7833. doi: 10.2140/ant.2018.12.1675. URL `https://doi.org/10.2140/ant.2018.12.1675`.

E. Cline, B. Parshall, and L. Scott. Integral and graded quasi-hereditary algebras. I. *J. Algebra*, 131(1):126–160, 1990. ISSN 0021-8693,1090-266X. doi: 10.1016/0021-8693(90)90169-O. URL `https://doi.org/10.1016/0021-8693(90)90169-O`.

David A. Cox. *Primes of the form $x^2 + ny^2$—Fermat, class field theory, and complex multiplication*. AMS Chelsea Publishing, Providence, RI, third edition, 2022. ISBN [9781470470289]; [9781470471835]. With contributions by Roger Lipsett.

Peiyi Cui. Category decomposition of $\mathrm{Rep}_k(\mathrm{SL}_n(F))$. *J. Algebra*, 602:130–153, 2022. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2022.02.016. URL `https://doi.org/10.1016/j.jalgebra.2022.02.016`.

Weideng Cui. Affine cellularity of BLN algebras. *J. Algebra*, 441:582–600, 2015. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2015.06.031. URL `https://doi.org/10.1016/j.jalgebra.2015.06.031`.

Jean-François Dat. Finitude pour les représentations lisses de groupes p-adiques. *J. Inst. Math. Jussieu*, 8(2):261–333, 2009. ISSN 1474-7480,1475-3030. doi: 10.1017/S1474748008000054. URL `https://doi.org/10.1017/S1474748008000054`.

Jean-François Dat. A functoriality principle for blocks of p-adic linear groups. In *Around Langlands correspondences*, volume 691 of *Contemp. Math.*, pages 103–131. Amer. Math. Soc., Providence, RI, 2017. ISBN 978-1-4704-3573-8. doi: 10.1090/conm/691/13895. URL `https://doi.org/10.1090/conm/691/13895`.

Jean-François Dat. Equivalences of tame blocks for p-adic linear groups. *Math. Ann.*, 371(1-2):565–613, 2018a. ISSN 0025-5831,1432-1807. doi: 10.1007/s00208-018-1648-1. URL https://doi.org/10.1007/s00208-018-1648-1.

Jean-François Dat. Simple subquotients of big parabolically induced representations of p-adic groups. *J. Algebra*, 510:499–507, 2018b. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2018.05.022. URL https://doi.org/10.1016/j.jalgebra.2018.05.022.

Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss. Finiteness for Hecke algebras of p-adic groups. *J. Amer. Math. Soc.*, 37(3):929–949, 2024a. ISSN 0894-0347,1088-6834. doi: 10.1090/jams/1034. URL https://doi.org/10.1090/jams/1034.

Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss. Local langlands in families: The banal case, 2024b. URL https://arxiv.org/abs/2406.09283.

Bangming Deng and Guiyu Yang. Affine quasi-heredity of affine Schur algebras. *Algebr. Represent. Theory*, 19(2):435–462, 2016. ISSN 1386-923X,1572-9079. doi: 10.1007/s10468-015-9582-3. URL https://doi.org/10.1007/s10468-015-9582-3.

Fran¸cois Digne and Jean Michel. *Representations of finite groups of Lie type*, volume 21 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991. ISBN 0-521-40117-8; 0-521-40648-X. doi: 10.1017/CBO9781139172417. URL https://doi.org/10.1017/CBO9781139172417.

Richard Dipper. On the decomposition numbers of the finite general linear groups. II. *Trans. Amer. Math. Soc.*, 292(1):123–133, 1985. ISSN 0002-9947,1088-6850. doi: 10.2307/2000173. URL https://doi.org/10.2307/2000173.

Richard Dipper and Peter Fleischmann. Modular Harish-Chandra theory. I. *Math. Z.*, 211(1):49–71, 1992. ISSN 0025-5874,1432-1823. doi: 10.1007/BF02571417. URL https://doi.org/10.1007/BF02571417.

Richard Dipper and Gordon James. Identification of the irreducible modular representations of $\mathrm{GL}_n(q)$. *J. Algebra*, 104(2):266–288, 1986. ISSN 0021-

8693. doi: 10.1016/0021-8693(86)90215-2. URL https://doi.org/10.1016/0021-8693(86)90215-2.

Richard Dipper and Gordon James. The q-Schur algebra. *Proc. London Math. Soc. (3)*, 59(1):23–50, 1989. ISSN 0024-6115,1460-244X. doi: 10.1112/plms/s3-59.1.23. URL https://doi.org/10.1112/plms/s3-59.1.23.

Richard Dipper and Gordon James. q-tensor space and q-Weyl modules. *Trans. Amer. Math. Soc.*, 327(1):251–282, 1991. ISSN 0002-9947,1088-6850. doi: 10.2307/2001842. URL https://doi.org/10.2307/2001842.

J. Du, B. Parshall, and L. Scott. Cells and q-Schur algebras. *Transform. Groups*, 3(1):33–49, 1998. ISSN 1083-4362,1531-586X. doi: 10.1007/BF01237838. URL https://doi.org/10.1007/BF01237838.

Laurent Fargues and Peter Scholze. Geometrization of the local langlands correspondence, 2024. URL https://arxiv.org/abs/2102.13459.

Jessica Fintzen. On the construction of tame supercuspidal representations. *Compos. Math.*, 157(12):2733–2746, 2021a. ISSN 0010-437X,1570-5846. doi: 10.1112/S0010437X21007636. URL https://doi.org/10.1112/S0010437X21007636.

Jessica Fintzen. Types for tame p-adic groups. *Ann. of Math. (2)*, 193(1):303–346, 2021b. ISSN 0003-486X,1939-8980. doi: 10.4007/annals.2021.193.1.4. URL https://doi.org/10.4007/annals.2021.193.1.4.

Jessica Fintzen. Tame cuspidal representations in non-defining characteristics. *Michigan Math. J.*, 72:331–342, 2022. ISSN 0026-2285,1945-2365. doi: 10.1307/mmj/20217217. URL https://doi.org/10.1307/mmj/20217217.

Paul Fong and Bhama Srinivasan. The blocks of finite general linear and unitary groups. *Invent. Math.*, 69(1):109–153, 1982. ISSN 0020-9910,1432-1297. doi: 10.1007/BF01389188. URL https://doi.org/10.1007/BF01389188.

Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. ISBN 3-540-43583-2. doi: 10.1007/978-3-662-12492-5. URL https://doi.org/10.1007/978-3-662-12492-5.

David Helm. The Bernstein center of the category of smooth $W(k)[\mathrm{GL}_n(F)]$-modules. *Forum Math. Sigma*, 4:Paper No. e11, 98, 2016. ISSN 2050-5094. doi: 10.1017/fms.2016.10. URL `https://doi.org/10.1017/fms.2016.10`.

David Helm, Robert Kurinczuk, Daniel Skodlerack, and Shaun Stevens. Block decompositions for p-adic classical groups and their inner forms, 2024. URL `https://arxiv.org/abs/2405.13713`.

Gerhard Hiss. Harish-Chandra series of Brauer characters in a finite group with a split BN-pair. *J. London Math. Soc. (2)*, 48(2):219–228, 1993. ISSN 0024-6107,1469-7750. doi: 10.1112/jlms/s2-48.2.219. URL `https://doi.org/10.1112/jlms/s2-48.2.219`.

James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990. ISBN 0-521-37510-X. doi: 10.1017/CBO9780511623646. URL `https://doi.org/10.1017/CBO9780511623646`.

Naoki Imai. Finite langlands correspondence, 2025. URL `https://arxiv.org/abs/2508.15101`.

Naoki Imai and David A. Vogan Jr. Langlands parameters for reductive groups over finite fields, 2025. URL `https://arxiv.org/abs/2506.06961`.

Gordon James. The irreducible representations of the finite general linear groups. *Proceedings of the London Mathematical Society*, s3-52(2):236–268, 1986. doi: https://doi.org/10.1112/plms/s3-52.2.236. URL `https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/s3-52.2.236`.

Tasho Kaletha. Representations of reductive groups over local fields. In *ICM—International Congress of Mathematicians. Vol. 4. Sections 5–8*, pages 2948–2975. EMS Press, Berlin, 2023. ISBN 978-3-98547-062-4; 978-3-98547-562-9; 978-3-98547-058-7.

Tasho Kaletha and Gopal Prasad. *Bruhat-Tits theory—a new approach*, volume 44 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2023. ISBN 978-1-108-83196-3.

Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006. ISBN 978-3-03719-022-7.

Steffen Koenig and Changchang Xi. Affine cellular algebras. *Adv. Math.*, 229 (1):139–182, 2012. ISSN 0001-8708,1090-2082. doi: 10.1016/j.aim.2011.08.010. URL https://doi.org/10.1016/j.aim.2011.08.010.

Robert Kurinczuk and Shaun Stevens. Cuspidal ℓ-modular representations of p-adic classical groups. *J. Reine Angew. Math.*, 764:23–69, 2020. ISSN 0075-4102,1435-5345. doi: 10.1515/crelle-2019-0009. URL https://doi.org/10.1515/crelle-2019-0009.

R. P. Langlands. Problems in the theory of automorphic forms. In *Lectures in Modern Analysis and Applications, III*, volume Vol. 170 of *Lecture Notes in Math.*, pages 18–61. Springer, Berlin-New York, 1970.

Markus Linckelmann. *The block theory of finite group algebras. Vol. I*, volume 91 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2018. ISBN 978-1-108-42591-9; 978-1-108-44183-4; 978-1-108-44190-2.

G. Lusztig. On the representations of reductive groups with disconnected centre. In *Orbites unipotentes et représentations, I*, number 168 in Astérisque, pages 10, 157–166. Société Mathématique de France, 1988. URL http://www.numdam.org/book-part/AST_1988__168__157_0/.

G. Lusztig. Aperiodicity in quantum affine $\mathfrak{gl}_n$. *Asian J. Math.*, 3(1):147–177, 1999. ISSN 1093-6106,1945-0036. doi: 10.4310/AJM.1999.v3.n1.a7. URL https://doi.org/10.4310/AJM.1999.v3.n1.a7. Sir Michael Atiyah: a great mathematician of the twentieth century.

George Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984. ISBN 0-691-08350-9; 0-691-08351-7. doi: 10.1515/9781400881772. URL https://doi.org/10.1515/9781400881772.

J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. ISBN 0-8218-2169-5. doi: 10.1090/gsm/030. URL https://doi.org/10.1090/gsm/030. With the cooperation of L. W. Small.

Kevin McGerty. Cells in quantum affine $\mathfrak{sl}_n$. *Int. Math. Res. Not.*, 2003(24):1341–1361, 2003. ISSN 1073-7928,1687-0247. doi: 10.1155/S107379280321120X. URL https://doi.org/10.1155/S107379280321120X.

J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. ISBN 978-1-107-16748-3. doi: 10.1017/9781316711736. URL https://doi.org/10.1017/9781316711736. The theory of group schemes of finite type over a field.

Alberto Mínguez and Vincent Sécherre. Représentations lisses modulo ℓ de $\mathrm{GL}_m(D)$. *Duke Math. J.*, 163(4):795–887, 2014. ISSN 0012-7094,1547-7398. doi: 10.1215/00127094-2430025. URL https://doi.org/10.1215/00127094-2430025.

Michitaka Miyauchi and Shaun Stevens. Semisimple types for p-adic classical groups. *Math. Ann.*, 358(1-2):257–288, 2014. ISSN 0025-5831,1432-1807. doi: 10.1007/s00208-013-0953-y. URL https://doi.org/10.1007/s00208-013-0953-y.

Lawrence Morris. Tamely ramified intertwining algebras. *Invent. Math.*, 114(1): 1–54, 1993. ISSN 0020-9910,1432-1297. doi: 10.1007/BF01232662. URL https://doi.org/10.1007/BF01232662.

Allen Moy and Gopal Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv.*, 71(1):98–121, 1996. ISSN 0010-2571,1420-8946. doi: 10.1007/BF02566411. URL https://doi.org/10.1007/BF02566411.

Hiraku Nakajima. Affine cellularity of quantum affine algebras: an appendix to "Affine cellularity of BLN-algebras" by Weideng Cui. *J. Algebra*, 441: 601–608, 2015. ISSN 0021-8693,1090-266X. doi: 10.1016/j.jalgebra.2015.07.017. URL https://doi.org/10.1016/j.jalgebra.2015.07.017.

David Paige. The projective envelope of a cuspidal representation of a finite linear group. *J. Number Theory*, 136:354–374, 2014. ISSN 0022-314X,1096-1658. doi: 10.1016/j.jnt.2013.10.014. URL https://doi.org/10.1016/j.jnt.2013.10.014.

Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009. ISBN 978-0-387-24527-0. doi: 10.1007/b98977. URL `https://doi.org/10.1007/b98977`.

Vincent Sécherre and Shaun Stevens. Smooth representations of $GL_m(D)$ VI: semisimple types. *Int. Math. Res. Not. IMRN*, 2012(13):2994–3039, 2012. ISSN 1073-7928,1687-0247. doi: 10.1093/imrn/rnr122. URL `https://doi.org/10.1093/imrn/rnr122`.

Vincent Sécherre and Shaun Stevens. Block decomposition of the category of ℓ-modular smooth representations of $\mathrm{GL}_n(\mathrm{F})$ and its inner forms. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):669–709, 2016. ISSN 0012-9593,1873-2151. doi: 10.24033/asens.2293. URL `https://doi.org/10.24033/asens.2293`.

Mitsuhiro Takeuchi. The group ring of $\mathrm{GL}_n(q)$ and the q-Schur algebra. *J. Math. Soc. Japan*, 48(2):259–274, 1996. ISSN 0025-5645,1881-1167. doi: 10.2969/jmsj/04820259. URL `https://doi.org/10.2969/jmsj/04820259`.

Marie-France Vignéras. *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*, volume 137 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1996. ISBN 0-8176-3929-2.

Marie-France Vignéras. Induced R-representations of p-adic reductive groups. *Selecta Math. (N.S.)*, 4(4):549–623, 1998. ISSN 1022-1824,1420-9020. doi: 10.1007/s000290050040. URL `https://doi.org/10.1007/s000290050040`.

Marie-France Vignéras. Schur algebras of reductive p-adic groups. I. *Duke Math. J.*, 116(1):35–75, 2003. ISSN 0012-7094,1547-7398. doi: 10.1215/S0012-7094-03-11612-9. URL `https://doi.org/10.1215/S0012-7094-03-11612-9`.

Marie-France Vignéras. Algèbres de Hecke affines génériques. *Represent. Theory*, 10:1–20, 2006. ISSN 1088-4165. doi: 10.1090/S1088-4165-06-00185-3. URL `https://doi.org/10.1090/S1088-4165-06-00185-3`.

Marie-France Vignéras. The pro-p-Iwahori Hecke algebra of a reductive p-adic group I. *Compos. Math.*, 152(4):693–753, 2016. ISSN 0010-437X,1570-5846. doi: 10.1112/S0010437X15007666. URL `https://doi.org/10.1112/S0010437X15007666`.

Guiyu Yang. Affine cellular algebras and Morita equivalences. *Arch. Math. (Basel)*, 102(4):319–327, 2014. ISSN 0003-889X,1420-8938. doi: 10.1007/s00013-014-0634-4. URL https://doi.org/10.1007/s00013-014-0634-4.